Compact quantum groupoids in the setting of C^*-algebras

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Abstract

We propose a definition of compact quantum groupoids in the setting of C^*-algebras, associate to such a quantum groupoid a regular C^*-pseudo-multiplicative unitary, and use this unitary to construct a dual Hopf C^*-bimodule and to pass to a measurable quantum groupoid in the sense of Enock and Lesieur. Moreover, we discuss examples related to compact and to étale groupoids and study principal compact C^*-quantum groupoids.

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1 Introduction

Overview In the setting of von Neumann algebras, measurable quantum groupoids were studied by Lesieur and Enock [6, 7, 4, 9], building Vallin's notions of Hopf-von Neumann bimodules and pseudo-multiplicative unitaries [20, 21] and Haagerup's operator-valued weights.

In this article, we propose a definition of compact quantum groupoids in the setting of C^*-algebras, building on the notions of Hopf-C^*-bimodules and C^*-pseudo-multiplicative unitaries introduced in [17, 16]. To each compact C^*-quantum groupoid, we associate a regular C^*-pseudo-multiplicative unitary, a von-Neumann algebraic completion, and a dual

Hopf C^*-bimodule. Moreover, we extend this C^*-pseudo-multiplicative unitary to a weak C^*-pseudo-Kac system; hence, the results of [16] can be applied to the study of coactions of compact C^*-quantum groupoids on C^*-algebras.

To illustrate the general theory, we discuss several examples of compact C^*-quantum groupoids: the C^*-algebra of continuous functions on a compact groupoid, the reduced C^*-algebra of an étale groupoid with compact base, and principal compact C^*-quantum groupoids.

Let us mention that many constructions and results seem to extend to a more general notion of quantum C^*-groupoids where the Haar weights are still assumed to be bounded but where the C^*-algebra of units need no longer be unital and where the KMS-state on this C^*-algebra is replaced by a proper KMS-weight.

Plan Let us outline the contents and organization of this article in some more detail.

In the first part of this article (Sections 2,3,4), we introduce the definition of a compact C^*-quantum groupoid. Recall that a measured compact groupoid consists of a base space G^0, a total space G, range and source maps $r, s\colon G \to G^0$, a multiplication $G_s \times_r G \to G$, a left and a right Haar system, and a quasi-invariant measure on G^0. Roughly, the corresponding ingredients of a compact C^*-quantum groupoid are unital C^*-algebras B and A, representations $r, s\colon B^{(op)} \to A$, a comultiplication $\Delta\colon A \to A * A$, a left and a right Haar weight $\phi, \psi\colon A \to B^{(op)}$, and a KMS-state on B, subject to several axioms. These ingredients are introduced in several steps. In Section 2, we focus on the tuple (B, A, r, ϕ, s, ψ), which can be considered as a compact C^*-quantum graph, and review some related GNS-constructions. In Section 3, we recall from [17, 16] the definition of the fiber product $A * A$ and of the underlying relative tensor product of Hilbert modules over C^*-algebras. Finally, in Section 4, we give the definition of a compact C^*-quantum groupoid and obtain first properties like uniqueness of the Haar weights and the existence of an invariant state on the basis.

In the second part of this article (Sections 5,6,7), we associate to every compact C^*-quantum groupoid a fundamental unitary, a von-Neumann-algebraic completion, and a dual Hopf C^*-bimodule. This fundamental unitary satisfies a pentagon equation, generalizes the multiplicative unitaries of Baaj and Skandalis [1], and can be considered as a pseudo-multiplicative unitary in the sense of Vallin [21] equipped with additional data. The unitary and the completion will be constructed in Section 5. In Section 6, we study a particular feature of this unitary — the existence of fixed or cofixed elements — and show that for a general C^*-pseudo-multiplicative unitary, such (co)fixed elements yield invariant conditional expectations and bounded counits on the associated Hopf C^*-bimodules. In Section 7, we return to compact C^*-quantum groupoids and discuss their duals.

The last part of this article (Sections 8,9) is devoted to examples of compact C^*-quantum groupoids which are obtained from compact and from étale groupoids one side and from center-valued traces on C^*-algebras on the other side. For these examples, we give a detailed description of the ingredients, the associated fundamental unitaries, and the dual objects.

Preliminaries Let us fix some general notation and concepts used in this article.

Given a subset Y of a normed space X, we denote by $[Y] \subseteq X$ the closed linear span of Y. Given a C^*-algebra A and a C^*-subalgebra $B \subseteq M(A)$, we denote by $A \cap B'$ the relative commutant $\{a \in A \mid ab = ba \text{ for all } b \in B\}$. Given a Hilbert space H and a subset $X \subseteq \mathcal{L}(H)$, we denote by X' the commutant of X. All sesquilinear maps like inner products of Hilbert spaces are assumed to be conjugate-linear in the first component and linear in the second one.

We shall make extensive use of Hilbert C^*-modules. A standard reference is [8].

Let A and B be C^*-algebras. Given Hilbert C^*-modules E and F over B, we denote the space of all adjointable operators $E \to F$ by $\mathcal{L}_B(E, F)$. Let E and F be C^*-modules over A and B, respectively, and let $\pi\colon A \to \mathcal{L}_B(F)$ be a $*$-homomorphism. Then one can form the internal tensor product $E \otimes_\pi F$, which is a Hilbert C^*-module over B [8, Chapter 4]. This Hilbert C^*-module is the closed linear span of elements $\eta \otimes_A \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \eta \otimes_\pi \xi | \eta' \otimes_\pi \xi' \rangle = \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\eta \otimes_\pi \xi)b = \eta \otimes_\pi \xi b$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. We denote the internal tensor product by "$\ominus$"; thus, for example, $E \ominus_\pi F = E \otimes_\pi F$. If the representation π or both π and A are understood, we write "$\ominus_A$" or "$\ominus$", respectively, instead of "$\ominus_\pi$".

Given A, B, E, F and π as above, we define a *flipped internal tensor product* $F{}_\pi\oslash E$ as follows. We equip the algebraic tensor product $F \odot E$ with the structure maps $\langle \xi \odot \eta | \xi' \odot \eta' \rangle := \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$, $(\xi \odot \eta)b := \xi b \odot \eta$, and by factoring out the null-space of the semi-norm $\zeta \mapsto \|\langle \zeta | \zeta \rangle\|^{1/2}$ and taking completion, we obtain a Hilbert C^*-B-module $F{}_\pi\oslash E$. This is the closed linear span of elements $\xi{}_\pi\oslash\eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \xi{}_\pi\oslash\eta | \xi'{}_\pi\oslash\eta' \rangle = \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\xi{}_\pi\oslash\eta)b = \xi b{}_\pi\oslash\eta$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. As above, we write "${}_A\oslash$" or simply "$\oslash$" instead of "${}_\pi\oslash$" if the representation π or both π and A are understood, respectively. Evidently, the usual and the flipped internal tensor product are related by a unitary map $\Sigma \colon F \ominus E \xrightarrow{\cong} E \oslash F$, $\eta \ominus \xi \mapsto \xi \oslash \eta$.

2 Compact C^*-quantum graphs

The first basic ingredient in the definition of a compact C^*-quantum groupoids are compact C^*-quantum graphs. Roughly, such a compact C^*-quantum graph consists of a C^*-algebra B (of units) with a faithful KMS-state, a C^*-algebra A (of arrows), and two compatible module structures consisting of representations $B, B^{(op)} \to A$ and conditional expectations $A \to B, B^{(op)}$. Thinking of (the underlying graph of) a groupoid, the representations correspond to the range and the source map, and the conditional expectations to the left and the right Haar weight.

Throughout the following sections, we will use several GNS- and Rieffel-constructions for compact C^*-quantum graphs. We first recall the GNS-construction for KMS-states and present the Rieffel-construction for a single module structure, before we turn to compact C^*-quantum graphs. To prepare for the definition of the unitary antipode of a compact C^*-quantum groupoid, we finally discuss coinvolutions on compact C^*-quantum graphs.

KMS-states on C^*-algebras and associated GNS-constructions We shall use the theory of KMS-states on C^*-algebras, see [3, §5], [12, §8.12], and adopt the following conventions. Let μ be a faithful KMS-state on a C^*-algebra B. We denote by σ^μ the modular automorphism group, by H_μ the GNS-space, by $\Lambda_\mu \colon B \to H_\mu$ the GNS-map, by $\zeta_\mu = \Lambda_\mu(1_B)$ the cyclic vector, and by $J_\mu \colon H_\mu \to H_\mu$ the modular conjugation associated to μ. We shall frequently use the formula

$$J_\mu \Lambda_\mu(b) = \Lambda_\mu(\sigma^\mu_{i/2}(b)^*) \quad \text{for all } b \in \mathrm{Dom}(\sigma^\mu_{i/2}). \tag{1}$$

We omit explicit mentioning of the GNS-representation $\pi_\mu \colon B \to \mathcal{L}(H_\mu)$ and identify B with $\pi_\mu(B)$; thus, $b\Lambda_\mu(x) = \pi_\mu(b)\Lambda_\mu(x) = \Lambda_\mu(bx) = bx\zeta_\mu$ for all $b, x \in B$.

We denote by B^{op} the opposite C^*-algebra of B, which coincides with B as a Banach space with involution but has the reversed multiplication, and by $\mu^{op} \colon B^{op} \to \mathbb{C}$ the opposite state of μ, given by by $\mu^{op}(b^{op}) := \mu(b)$ for all $b \in B$. Using formula (1), one easily verifies that μ^{op} is a KMS-state, that the modular automorphism group $\sigma^{\mu^{op}}$ is given by $\sigma^{\mu^{op}}_t(b^{op}) = \sigma^\mu_{-t}(b)^{op}$ for all $b \in B$, $t \in \mathbb{R}$, and that one can always choose the GNS-space and GNS-map for μ^{op} such that $H_{\mu^{op}} = H_\mu$ and $\Lambda_{\mu^{op}}(b^{op}) = J_\mu \Lambda_\mu(b^*)$ for all $b \in B$. Then $\zeta_{\mu^{op}} = \zeta_\mu$, $J_{\mu^{op}} = J_\mu$, $\pi_{\mu^{op}}(b) = J_\mu \pi_\mu(b)^* J_\mu$ for all $b \in B$, and

$$\Lambda_{\mu^{op}}(b^{op}) = \Lambda_\mu(\sigma^\mu_{-i/2}(b)), \quad b^{op}\Lambda_\mu(x) = \Lambda_\mu(x\sigma^\mu_{-i/2}(b)) \quad \text{for all } b \in \mathrm{Dom}(\sigma^\mu_{-i/2}),\ x \in B.$$

Module structures and associated Rieffel constructions We shall use the following kind of module structures on C^*-algebras relative to KMS-states:

Definition 2.1. *Let μ be faithful KMS-state on a unital C^*-algebra B. A* μ-module structure *on a unital C^*-algebra A is pair (r, ϕ) consisting of a unital embedding $r \colon B \to A$ and a completely positive map $\phi \colon A \to B$ such that $r \circ \phi \colon A \to r(B)$ is a unital conditional expectation, $\nu := \mu \circ \phi$ is a KMS-state, and $\sigma_t^\nu(r(B)) \subseteq r(B)$ for all $t \in \mathbb{R}$.*

Given a module structure as above, we can form a GNS-Rieffel-construction as follows:

Lemma 2.2. *Let μ be a faithful KMS-state on a unital C^*-algebra B, let (r, ϕ) be a μ-module structure on a unital C^*-algebra A, and put $\nu := \mu \circ \phi$.*

i) $\sigma_t^\nu \circ r = r \circ \sigma_t^\mu$ for all $t \in \mathbb{R}$.

ii) There exists a unique isometry $\zeta \colon H_\mu \hookrightarrow H_\nu$ such that $\zeta \Lambda_\mu(b) = \Lambda_\nu(r(b))$ for all $b \in B$.

iii) $\zeta J_\mu = J_\nu \zeta$, $\zeta b = r(b)\zeta$, $\zeta^* \Lambda_\nu(a) = \Lambda_\mu(\phi(a))$, $\zeta^* a = \phi(a)\zeta^*$ for all $b \in B$, $a \in A$.

iv) There exists a μ^{op}-module structure (r^{op}, ϕ^{op}) on A^{op} such that $r^{op}(b^{op}) = r(b)^{op}$ and $\phi^{op}(a^{op}) = \phi(a)^{op}$ for all $b \in B$, $a \in A$. For all $b \in B$, $\zeta \Lambda_{\mu^{op}}(b^{op}) = \Lambda_{\nu^{op}}(r^{op}(b^{op}))$.

Proof. i) This follows easily from the uniqueness of the modular automorphism group of a faithful KMS-state.

ii) Straightforward.

iii) $\zeta J_\mu = J_\nu \zeta$ because $\mathrm{Dom}(\sigma_{i/2}^\mu)$ is dense in B and $\zeta J_\mu \Lambda_\mu(b) = \zeta \Lambda_\mu(\sigma_{i/2}^\mu(b)^*) = \Lambda_\nu(r(\sigma_{i/2}^\mu(b)^*)) = \Lambda_\nu(\sigma_{i/2}^\nu(r(b))^*) = J_\nu \zeta \Lambda_\mu(b)$ for all $b \in \mathrm{Dom}(\sigma_{i/2}^\mu)$ by i). The proof of the remaining assertions is routine.

iv) Straightforward. □

Compact C^*-quantum graphs The definition of a compact C^*-quantum graph involves the following simple variant of a Radon-Nikodym derivative for KMS-states:

Lemma 2.3. *Let A be a unital C^*-algebra with a KMS-state ν and a positive invertible element δ that satisfies $\nu(\delta) = 1$ and $\sigma_t^\nu(\delta) = \delta$ for all $t \in \mathbb{R}$.*

i) The state ν_δ on A given by $\nu_\delta(a) = \nu(\delta^{1/2} a \delta^{1/2})$ for all $a \in A$ is a faithful KMS-state and $\sigma_t^{\nu_\delta} = \mathrm{Ad}_{\delta^{it}} \circ \sigma_t^\nu = \sigma_t^\nu \circ \mathrm{Ad}_{\delta^{it}}$ for all $t \in \mathbb{R}$.

ii) The map $\Lambda_{\nu_\delta} \colon A \to H_\nu$, $a \mapsto \Lambda_\nu(a\delta^{1/2})$, is a GNS-map for ν_δ, and the associated modular conjugation J_{ν_δ} is equal to J_ν.

iii) If $\tilde{\delta} \in A$ is another positive invertible element satisfying $\nu(\tilde{\delta}) = 1$, $\sigma_t^\nu(\tilde{\delta}) = \delta$ for all $t \in \mathbb{R}$, and $\nu_{\tilde{\delta}} = \nu_\delta$, then $\delta = \tilde{\delta}$. □

Definition 2.4. *A* compact C^*-quantum graph *is a tuple $(B, \mu, A, r, \phi, s, \psi, \delta)$, where*

i) B is a unital C^*-algebra with a faithful KMS-state μ,

ii) A is a unital C^*-algebra,

iii) (r, ϕ) and (s, ψ) are a μ-module structure and a μ^{op}-module structure on A, respectively, such that $r(B)$ and $s(B^{op})$ commute,

iv) $\delta \in A \cap r(B)' \cap s(B^{op})'$ is a positive, invertible, σ^ν-invariant element such that $\nu(\delta) = 1$ and $\mu^{op} \circ \psi = (\mu \circ \phi)_\delta$.

Given a compact C^*-quantum graph $(B, \mu, A, r, \phi, s, \psi)$, we put $\nu := \mu \circ \phi$, $\nu^{-1} := \mu^{op} \circ \psi$, and denote by $\zeta_\phi, \zeta_\psi \colon H_\mu \to H_\nu$ the isometries associated to (r, ϕ), (s, ψ) as in Lemma 2.2.

For every compact C^*-quantum graph $(B, \mu, A, r, \phi, s, \psi)$, we have $\nu \circ r = \mu \circ \phi \circ r = \mu$ and $\nu^{-1} \circ s = \mu^{op} \circ \psi \circ s = \mu^{op}$. The compositions $\nu \circ s$ and $\nu^{-1} \circ r$ are related to μ^{op} and μ, respectively, as follows.

Lemma 2.5. *Let $(B, \mu, A, r, \phi, s, \psi, \delta)$ be a compact C^*-quantum graph.*

i) $\phi(\delta) \in B$ *and* $\psi(\delta^{-1}) \in B^{op}$ *are positive, invertible, central, invariant with respect to* σ^{μ} *and* $\sigma^{\mu^{op}}$*, respectively, and satisfy* $\mu(\phi(\delta)) = 1 = \mu^{op}(\psi(\delta^{-1}))$.

ii) $\nu^{-1} \circ r = \mu_{\phi(\delta)}$ *and* $\nu \circ s = \mu^{op}_{\psi(\delta^{-1})}$.

Proof. i) We only prove the assertions concerning $\phi(\delta)$. Since δ is positive, there exists an $\epsilon > 0$ such that $\delta > \epsilon 1_A$, and since ϕ is positive, we can conclude $\phi(\delta) > \epsilon\phi(1_A) = \epsilon 1_B$. Therefore, $\phi(\delta)$ is positive and invertible. It is central because $b\phi(\delta) = \phi(r(b)\delta) = \phi(\delta r(b)) = \phi(\delta)b$ for all $b \in B$, and invariant under σ^{μ} because $\sigma^{\mu}_t(\phi(\delta)) = \phi(\sigma^{\nu}_t(\delta)) = \phi(\delta)$ for all $t \in \mathbb{R}$.

ii) The first equation holds because for all $b \in B$, $\nu^{-1}(r(b)) = \mu(\phi(\delta^{1/2} r(b)\delta^{1/2})) = \mu(b\phi(\delta)) = \mu(\phi(\delta)^{1/2} b\phi(\delta)^{1/2})$. The second equation follows similarly. □

Let $(B, \mu, A, r, \phi, s, \psi, \delta)$ be a compact C^*-quantum graph. Then for all $b, c \in B$, $\psi(r(b))c^{op} = \psi(r(b)s(c^{op})) = \psi(s(c^{op})r(b)) = c^{op}\psi(r(b))$ and similarly $\phi(s(b^{op}))c = c\phi(s(b^{op}))$, so that we can define completely positive maps

$$\tau := \psi \circ r \colon B \to Z(B^{op}) \qquad \text{and} \qquad \tau^{\dagger} := \phi \circ s \colon B^{op} \to Z(B). \tag{2}$$

We identify $Z(B)$ and $Z(B^{op})$ with $B \cap B^{op} \subseteq \mathcal{L}(H_{\mu})$ in the natural way.

Coinvolutions on compact C^*-quantum graphs The following concept will be used to define the unitary antipode of a compact C^*-quantum groupoid:

Definition 2.6. *A* coinvolution *for a compact* C^**-quantum graph* $(B, \mu, A, r, \phi, s, \psi, \delta)$ *is an antiautomorphism* $R \colon A \to A$ *satisfying* $R \circ R = \mathrm{id}_A$ *and* $R(r(b)) = s(b^{op})$, $\phi(R(a)) = \psi(a)^{op}$ *for all* $b \in B$, $a \in A$.

Lemma 2.7. *Let* R *be a coinvolution for a compact* C^**-quantum graph* $(B, \mu, A, r, \phi, s, \psi, \delta)$.

i) $\sigma^{\nu}_t \circ R = R \circ \sigma^{\nu^{-1}}_{-t}$ *for all* $t \in \mathbb{R}$*, and* $R(\delta) = \delta^{-1}$*. In particular,* $\phi(\delta) = \psi(\delta^{-1})$.

ii) $\tau(b) = \tau^{\dagger}(b^{op})$ *for all* $b \in B$.

iii) There exists a unique antiunitary $I \colon H_{\nu} \to H_{\nu}$, $\Lambda_{\nu^{-1}}(a) \mapsto \Lambda_{\nu}(R(a)^*)$*, and*

$$I\Lambda_{\nu}(a) = \Lambda_{\nu}(R(a\delta^{1/2})^*), \quad Ia^*I = R(a), \quad I^2 = \mathrm{id}_H, \quad I\zeta_{\psi}J_{\mu} = \zeta_{\phi}, \quad IJ_{\nu} = J_{\nu}I.$$

Proof. i) The first equation follows from the fact that R is an antiautomorphism and that $\nu \circ R = \nu^{-1}$. To prove the second equation, put $\delta' := R(\delta^{-1})$. Then

$$\nu(\delta') = \nu^{-1}(\delta^{-1}) = \nu(1) = 1,$$

$$\sigma^{\nu}_t(\delta') = R(\sigma^{\nu^{-1}}_{-t}(\delta^{-1})) = R\big(\sigma^{\nu}_{-t} \circ \mathrm{Ad}_{\delta^{it}}(\delta^{-1})\big) = R(\sigma^{\nu}_{-t}(\delta^{-1})) = R(\delta^{-1}) = \delta',$$

$$\nu_{\delta}(a) = \nu^{-1}(a) = \nu(R(a)) = \nu^{-1}(\delta^{-1/2}R(a)\delta^{-1/2}) = \nu(\delta'^{1/2}a\delta'^{1/2}) = \nu_{\delta'}(a)$$

for all $a \in A$, and by Lemma 2.3 iii), $\delta = \delta'$.

ii) $(\phi \circ s)(b^{op}) = (\phi \circ R \circ R \circ s)(b^{op}) = (\psi \circ r)(b)^{op}$ for all $b \in B$.

iii) The formula for I defines an antiunitary because for all $a \in A$,

$$\big\|\Lambda_{\nu}(R(a)^*)\big\|^2 = (\mu \circ \phi)\big(R(a)R(a)^*\big) = (\mu \circ \phi \circ R)(a^*a) = (\mu^{op} \circ \psi)(a^*a) = \big\|\Lambda_{\nu^{-1}}(a)\big\|^2.$$

The first two equations given in ii) follow immediately. Next, $I^2 = \mathrm{id}_H$ because

$$I^2\Lambda_{\nu}(a) = \Lambda_{\nu}\big(R(R(a\delta^{1/2})^*\delta^{1/2})^*\big) = \Lambda_{\nu}\big(a\delta^{1/2}\delta^{-1/2}\big) = \Lambda_{\nu}(a)$$

for all $a \in A$, and $I\zeta_{\psi}J_{\mu} = \zeta_{\phi}$ because

$$I\zeta_{\psi}J_{\mu}\Lambda_{\mu}(b^*) = I\zeta_{\psi}\Lambda_{\mu^{op}}(b^{op}) = I\Lambda_{\nu^{-1}}(s(b^{op})) = \Lambda_{\nu}(r(b)^*) = \zeta_{\phi}\Lambda_{\mu}(b^*)$$

for all $b \in B$. The relation $\sigma^{\nu^{-1}}_t = R \circ \sigma^{\nu}_{-t} \circ R$ ($t \in \mathbb{R}$) implies that for all $a \in \mathrm{Dom}(\sigma^{\nu^{-1}}_{i/2})$,

$$J_{\nu}I\Lambda_{\nu^{-1}}(a) = \Lambda_{\nu}\big(\sigma^{\nu}_{i/2}(R(a)^*)^*\big) = \Lambda_{\nu}\big(R(\sigma^{\nu^{-1}}_{i/2}(a)^*)^*\big) = IJ_{\nu^{-1}}\Lambda_{\nu^{-1}}(a).$$

Since $J_{\nu^{-1}} = J_{\nu}$, we can conclude $J_{\nu}I = IJ_{\nu}$. □

3 The relative tensor product and the fiber product

Fundamental to the following development is the general language of C^*-modules and C^*-algebras over KMS-states, the relative tensor product of such C^*-modules, and the fiber product of such C^*-algebras: The fiber product is needed to define the target of the comultiplication of a compact C^*-quantum groupoid, and the relative tensor product is needed to define this fiber product and the domain and the range of the fundamental unitary.

We proceed as follows. First, we introduce the language of C^*-modules and C^*-algebras over KMS-states. Next, we describe the C^*-module structures that arise from a compact C^*-quantum graph and which are needed later. Finally, we present the relative tensor product and the fiber product. Except for the second paragraph, the reference is [17]

C^*-modules and C^*-algebras over KMS-states We adapt the framework of C^*-modules and C^*-algebras over C^*-bases introduced in [17] to our present needs, replacing C^*-bases by KMS-states as follows. A *C^*-base* is a triple $(\mathfrak{B}, \mathfrak{H}, \mathfrak{B}^\dagger)$ consisting of a Hilbert space $\mathfrak{H}$ and two commuting nondegenerate C^*-algebras $\mathfrak{B}, \mathfrak{B}^\dagger \subseteq \mathcal{L}(\mathfrak{H})$. We restrict ourselves to C^*-bases of the form (H_μ, B, B^{op}), where H_μ is the GNS-space of a faithful KMS-state μ on a C^*-algebra B, and where B and B^{op} act on $H_\mu = H_{\mu^{op}}$ via the GNS-representations, and obtain the following notions of C^*-modules over μ.

Definition 3.1. *Let μ be a faithful KMS-state on a C^*-algebra B. A* C^*-μ-module *is a pair (H, α), briefly written H_α, where H is a Hilbert space and $\alpha \subseteq \mathcal{L}(H_\mu, H)$ is a closed subspace satisfying $[\alpha H_\mu] = H$, $[\alpha B] = \alpha$, and $[\alpha^*\alpha] = B \subseteq \mathcal{L}(H_\mu)$. A* morphism *between C^*-μ-modules H_α and K_β is an operator $T \in \mathcal{L}(H, K)$ satisfying $T\alpha \subseteq \beta$ and $T^*\beta \subseteq \alpha$. We denote the set of such morphisms by $\mathcal{L}(H_\alpha, K_\beta)$.*

Lemma 3.2. *Let μ be a faithful KMS-state on a C^*-algebra B and let H_α be a C^*-μ-module.*

i) α is a right Hilbert C^*-B-module with inner product given by $\langle\xi|\xi'\rangle = \xi^*\xi$ for all $\xi, \xi' \in \alpha$.

ii) There exist isomorphisms $\alpha \otimes H_\mu \to H$, $\xi \otimes \zeta \mapsto \xi\zeta$, and $H_\mu \otimes \alpha \to H$, $\zeta \otimes \xi \mapsto \xi\zeta$.

iii) There exists a nondegenerate representation $\rho_\alpha \colon B^{op} \to \mathcal{L}(H)$ such that $\rho_\alpha(b^{op})(\xi\zeta) = \xi b^{op}\zeta$ for all $b \in B$, $\xi \in \alpha$, $\zeta \in H_\mu$.

iv) Let K_β be a C^*-μ-module and $T \in \mathcal{L}(H_\alpha, K_\beta)$. Then left multiplication by T defines an operator in $\mathcal{L}_B(\alpha, \beta)$, again denoted by T, and $T\rho_\alpha(b^{op}) = \rho_\beta(b^{op})T$ for all $b \in B$. □

Definition 3.3. *Let $\mu_1, \ldots, \mu_n$ be faithful KMS-states on C^*-algebras $B_1, \ldots, B_n$. A* C^*-$(\mu_1, \ldots, \mu_n)$-module *is a tuple $(H, \alpha_1, \ldots, \alpha_n)$, where H is a Hilbert space and (H, α_i) is a C^*-μ_i-module for each $i = 1, \ldots, n$ such that $[\rho_{\alpha_i}(B_i^{op})\alpha_j] = \alpha_j$ whenever $i \neq j$. The set of morphisms of C^*-$(\mu_1, \ldots, \mu_n)$-modules $(H, \alpha_1, \ldots, \alpha_n)$ and $(K, \beta_1, \ldots, \beta_n)$ is $\mathcal{L}((H, \alpha_1, \ldots, \alpha_n), (K, \beta_1, \ldots, \beta_n)) := \bigcap_{i=1}^n \mathcal{L}(H_{\alpha_i}, K_{\beta_i}) \subseteq \mathcal{L}(H, K)$.*

Remark 3.4. Let $\mu_1, \ldots, \mu_n$ be faithful KMS-states on C^*-algebras $B_1, \ldots, B_n$ and let $(H, \alpha_1, \ldots, \alpha_n)$ be a C^*-$(\mu_1, \ldots, \mu_n)$-module. Then $\rho_{\alpha_i}(B_i^{op}) \subseteq \mathcal{L}(H_{\alpha_j})$ whenever $i \neq j$; in particular, $[\rho_{\alpha_i}(B_i^{op}), \rho_{\alpha_j}(B_j^{op})] = 0$ whenever $i \neq j$.

Next, we define C^*-algebras over KMS-states.

Definition 3.5. *Let $\mu_1, \ldots, \mu_n$ be faithful KMS-states on C^*-algebras $B_1, \ldots, B_n$. A* C^*-$(\mu_1, \ldots, \mu_n)$-algebra *consists of a C^*-$(\mu_1, \ldots, \mu_n)$-module $(H, \alpha_1, \ldots, \alpha_n)$ and a nondegenerate C^*-algebra $A \subseteq \mathcal{L}(H)$ such that $[\rho_{\alpha_i}(B_i^{op})A] \subseteq A$ for each $i = 1, \ldots, n$. In the cases $n = 1, 2$, we abbreviate $A_H^\alpha := (H_\alpha, A)$, $A_H^{\alpha,\beta} := ((H, \alpha, \beta), A)$. A* morphism *between C^*-$(\mu_1, \ldots, \mu_n)$-algebras $((H, \alpha_1, \ldots, \alpha_n), A)$ and $((K, \gamma_1, \ldots, \gamma_n), C)$ is a nondegenerate $*$-homomorphism $\phi \colon A \to M(C)$ such that for each $i = 1, \ldots, n$, we have $[I_{\phi,i}\alpha_i] = \gamma_i$, where $I_{\phi,i} := \{T \in \mathcal{L}(H_{\alpha_i}, K_{\gamma_i}) \mid Ta = \phi(a)T \text{ for all } a \in A\}$. We denote the set of all such morphisms by* $\mathrm{Mor}(((H, \alpha_1, \ldots, \alpha_n), A), ((K, \gamma_1, \ldots, \gamma_n), C))$.

Remark 3.6. If ϕ is a morphism between C^*-μ-algebras A_H^α and C_K^γ, then $\phi(\rho_\alpha(b^{op})) = \rho_\gamma(b^{op})$ for all $b \in B$; see [16, Lemma 2.2].

The C^*-module of a compact C^*-quantum graph To proceed from compact C^*-quantum graphs to compact C^*-quantum groupoids, we need several C^*-module structures arising from the GNS-Rieffel-construction in Lemma 2.2.

Lemma 3.7. *Let μ be a faithful KMS-state on a unital C^*-algebra B, let (r, ϕ) be a C^*-μ-module structure on a unital C^*-algebra A, and put $\nu := \mu \circ \phi$, $H := H_\nu$, $\widehat{\alpha} := [A\zeta]$, $\beta := [A^{op}\zeta]$.*

i) ${}_{\widehat{\alpha}}H_\beta$ is a C^*-(μ, μ^{op})-module and $\rho_{\widehat{\alpha}} = r^{op}$, $\rho_\beta = r$.

ii) A_H^β is a C^*-μ^{op}-algebra.

iii) $a^{op}\zeta = \sigma^\nu_{-i/2}(a)\zeta$ for all $a \in \mathrm{Dom}(\sigma^\nu_{-i/2}) \cap r(B)'$.

iv) $A + (A \cap r(B)')^{op} \subseteq \mathcal{L}(H_{\widehat{\alpha}})$ and $A^{op} + (A \cap r(B)') \subseteq \mathcal{L}(H_\beta)$.

Proof. i) Lemma 2.2 immediately implies that $H_{\widehat{\alpha}}$ is a C^*-μ-module and that H_β is a C^*-μ^{op}-module. The equations for $\rho_{\widehat{\alpha}}$ and ρ_β follow from the fact that by Lemma 2.2, $\rho_{\widehat{\alpha}}(b^{op})a\zeta = a\zeta b^{op} = ar(b)^{op}\zeta = r(b)^{op}a\zeta$ and $\rho_\beta(b)a^{op}\zeta = a^{op}\zeta b = a^{op}r(b)\zeta = r(b)a^{op}\zeta$ for all $b \in B$, $a \in A$. In particular, $[\rho_{\widehat{\alpha}}(B^{op})\beta] = [r(B)^{op}A^{op}\zeta] = \beta$ and $[\rho_\beta(B)\widehat{\alpha}] = [r(B)A\zeta] = \widehat{\alpha}$, whence ${}_{\widehat{\alpha}}H_\beta$ is a C^*-(μ, μ^{op})-module.

ii) By i), $[\rho_\beta(B)A] = A$.

iii) For all $a \in r(B)' \cap \mathrm{Dom}(\sigma^\nu_{-i/2})$ and $b \in B$,

$$a^{op}\zeta\Lambda_\mu(b) = \Lambda_\nu\big(r(b)\sigma^\nu_{-i/2}(a)\big) = \Lambda_\nu\big(\sigma^\nu_{-i/2}(a)r(b)\big) = \sigma^\nu_{-i/2}(a)\zeta\Lambda_\mu(b).$$

iii) We only prove the first inclusion, the second one follows similarly. Clearly, $[A\widehat{\alpha}] = \widehat{\alpha}$. Since $\sigma^\nu_t(r(B)) \subseteq r(B)$ for all $t \in \mathbb{R}$, the subspace $C := \mathrm{Dom}(\sigma^\nu_{-i/2}) \cap r(B)'$ is dense in $A \cap r(B)'$, and by iii), $[(A \cap r(B)')^{op}\widehat{\alpha}] = [CA\zeta] = [AC\zeta] \subseteq [A\zeta] = \widehat{\alpha}$. ☐

Proposition 3.8. *Let $(B, \mu, A, r, \phi, s, \psi, \delta)$ be a compact C^*-quantum graph. Put $\nu := \mu \circ \phi$, $\nu^{-1} := \mu^{op} \circ \psi = \nu_\delta$ and*

$$H := H_\nu, \qquad \widehat{\alpha} := [A\zeta_\phi], \qquad \beta := [A^{op}\zeta_\phi], \qquad \widehat{\beta} := [A\zeta_\psi], \qquad \alpha := [A^{op}\zeta_\psi]. \tag{3}$$

i) $(H, \widehat{\alpha}, \beta, \widehat{\beta}, \alpha)$ is a C^*-$(\mu, \mu^{op}, \mu^{op}, \mu)$-module and $\rho_{\widehat{\alpha}} = r^{op}$, $\rho_\beta = r$, $\rho_{\widehat{\beta}} = s^{op}$, $\rho_\alpha = s$.

ii) $A_H^{\alpha,\beta}$ is a C^*-(μ, μ^{op})-algebra.

iii) Let R be a coinvolution for $(B, \mu, A, r, \phi, s, \psi, \delta)$ and let $I \colon H_\nu \to H_\nu$ be given by $\Lambda_{\nu^{-1}}(a) \mapsto \Lambda_\nu(R(a)^*)$. Then $I\zeta_\phi J_\mu = \zeta_\psi$, $I\zeta_\psi J_\mu = \zeta_\phi$ and $I\widehat{\beta}J_\mu = \widehat{\alpha}$, $I\beta J_\mu = \alpha$.

Proof. i), ii) Immediate from Lemma 3.7.

iii) We have $I\zeta_\psi J_\mu = \zeta_\phi$ because for all $b \in B$,

$$I\zeta_\psi J_\mu\Lambda_\mu(b^*) = I\zeta_\psi\Lambda_{\mu^{op}}(b^{op}) = I\Lambda_{\nu^{-1}}(s(b^{op})) = \Lambda_\nu(R(s(b^{op}))^*) = \Lambda_\nu(r(b^*)) = \zeta_\phi\Lambda_\mu(b^*).$$

The remaining assertions follow easily. ☐

The relative tensor product of C^*-modules The relative tensor product of C^*-modules over KMS-states is a symmetrized version of the internal tensor product of Hilbert C^*-modules and a C^*-algebraic analogue of the relative tensor product of Hilbert spaces over a von Neumann algebra. We briefly summarize the definition and the main properties; for details, see [17, Section 2.2].

Let μ be a faithful KMS-state on a C^*-algebra B, let H_β be a C^*-μ-module, and let K_γ be a C^*-μ^{op}-module. The *relative tensor product* of H_β and K_γ is the Hilbert space $H_\beta {\otimes}_\gamma K := \beta \otimes H_\mu \otimes \gamma$. It is spanned by elements $\xi \otimes \zeta \otimes \eta$, where $\xi \in \beta$, $\zeta \in H_\mu$, $\eta \in \gamma$, and the inner product is given by $\langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta' \rangle = \langle \zeta | \xi^*\xi'\eta^*\eta'\zeta' \rangle = \langle \zeta | \eta^*\eta'\xi^*\xi'\zeta' \rangle$ for all $\xi, \xi' \in \beta$, $\zeta, \zeta' \in H_\mu$, $\eta, \eta' \in \gamma$.

Obviously, there exists *flip* isomorphism

$$\Sigma\colon H_{\beta}{\otimes}_{\gamma}K \to K_{\gamma}{\otimes}_{\beta}H, \quad \xi \ogreaterthan \zeta \olessthan \eta \mapsto \eta \ogreaterthan \zeta \olessthan \xi. \tag{4}$$

The isomorphisms $\beta \ogreaterthan H_\mu \cong H$, $\xi \ogreaterthan \zeta \equiv \xi\zeta$, and $H_\mu \olessthan \gamma \cong K$, $\zeta \olessthan \zeta \equiv \eta\zeta$, (see Lemma 3.2) induce the following isomorphisms which we use without further notice:

$$H_{\rho_\beta}{\olessthan}\gamma \cong H_{\beta}{\otimes}_{\gamma}K \cong \beta \ogreaterthan_{\rho_\gamma} K, \qquad \xi\zeta \olessthan \eta \equiv \xi \ogreaterthan \zeta \olessthan \eta \equiv \xi \ogreaterthan \eta\zeta \quad (\xi \in \beta,\, \zeta \in H_\mu,\, \eta \in \gamma).$$

Using these isomorphisms, we define the following tensor products of operators:

$$\begin{aligned} S_{\beta}{\otimes}_{\gamma}T &:= S \ogreaterthan T \in \mathcal{L}(\beta \ogreaterthan_{\rho_\gamma} K) = \mathcal{L}(H_{\beta}{\otimes}_{\gamma}K) \quad \text{for all } S \in \mathcal{L}(H_\alpha),\ T \in \rho_\gamma(B)' \subseteq \mathcal{L}(K), \\ S_{\beta}{\otimes}_{\gamma}T &:= S \olessthan T \in \mathcal{L}(H_{\rho_\beta}{\olessthan}\gamma) = \mathcal{L}(H_{\beta}{\otimes}_{\gamma}K) \quad \text{for all } S \in \rho_\beta(B^{op})' \subseteq \mathcal{L}(H),\ T \in \mathcal{L}(K_\beta). \end{aligned}$$

Note that $S \ogreaterthan T = S \ogreaterthan \mathrm{id} \olessthan T = S \olessthan T$ for all $S \in \mathcal{L}(H_\beta)$, $T \in \mathcal{L}(K_\gamma)$.

For each $\xi \in \beta$, $\eta \in \gamma$, there exist bounded linear operators

$$\begin{aligned} &|\xi\rangle_1\colon K \to H_{\beta}{\otimes}_{\gamma}K,\ \omega \mapsto \xi \ogreaterthan \omega, && \langle\xi|_1 := |\xi\rangle_1^*\colon \xi' \ogreaterthan \omega \mapsto \rho_\gamma(\xi^*\xi')\omega, \\ &|\eta\rangle_2\colon H \to H_{\beta}{\otimes}_{\gamma}K,\ \omega \mapsto \omega \olessthan \eta, && \langle\eta|_2 := |\eta\rangle_2^*\colon \omega \olessthan \eta \mapsto \rho_\beta(\eta^*\eta')\omega. \end{aligned}$$

We put $|\beta\rangle_1 := \{|\xi\rangle_1 \,|\, \xi \in \beta\}$ and similarly define $\langle\beta|_1$, $|\gamma\rangle_2$, $\langle\gamma|_2$.

Assume that $\mathfrak{H} = (H, \alpha_1, \ldots, \alpha_m, \beta)$ is a C^*-$(\sigma_1, \ldots, \sigma_m, \mu)$-module and that $\mathfrak{K} = (K, \gamma, \delta_1, \ldots, \delta_n)$ is a C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n)$-module, where $\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n$ are faithful KMS-states on C^*-algebras $A_1, \ldots, A_m, C_1, \ldots, C_n$. For $i = 1, \ldots, m$ and $j = 1, \ldots, n$, put

$$\alpha_i \triangleleft \gamma := [|\gamma\rangle_2\alpha_i] \subseteq \mathcal{L}(H_{\sigma_i}, H_{\beta}{\otimes}_{\gamma}K), \qquad \beta \triangleright \delta_j := [|\beta\rangle_1\delta_j] \subseteq \mathcal{L}(H_{\tau_j}, H_{\beta}{\otimes}_{\gamma}K).$$

Then the tuple $\mathfrak{H} \underset{\mu}{\otimes} \mathfrak{K} := (H_{\beta}{\otimes}_{\gamma}K, \alpha_1 \triangleleft \gamma, \ldots, \alpha_m \triangleleft \gamma, \beta \triangleright \delta_1, \ldots, \beta \triangleright \delta_n)$ is a C^*-$(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n)$-module, called the *relative tensor product* of $\mathfrak{H}$ and $\mathfrak{K}$. For all $i = 1, \ldots, m$, $a \in A_i$ and $j = 1, \ldots, n$, $c \in C_j$,

$$\rho_{(\alpha_i \triangleleft \gamma)}(a^{op}) = \rho_{\alpha_i}(a^{op})_{\beta}{\otimes}_{\gamma}\,\mathrm{id}, \qquad \rho_{(\beta \triangleright \delta_j)}(c^{op}) = \mathrm{id}\,_{\beta}{\otimes}_{\gamma}\rho_{\delta_j}(c^{op}).$$

The C^*-relative tensor product is bifunctorial: If $\tilde{\mathfrak{H}} = (\tilde{H}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, \tilde{\beta})$ is a C^*-$(\sigma_1, \ldots, \sigma_m, \mu)$-module, $\tilde{\mathfrak{K}} = (\tilde{K}, \tilde{\gamma}, \tilde{\delta}_1, \ldots, \tilde{\delta}_n)$ a C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n)$-module, and $S \in \mathcal{L}(\mathfrak{H}, \tilde{\mathfrak{H}})$, $T \in \mathcal{L}(\mathfrak{K}, \tilde{\mathfrak{K}})$, then there exists a unique operator $S \underset{\mu}{\otimes} T \in \mathcal{L}(\mathfrak{H} \underset{\mu}{\otimes} \mathfrak{K}, \tilde{\mathfrak{H}} \underset{\mu}{\otimes} \tilde{\mathfrak{K}})$ such that

$$(S \underset{\mu}{\otimes} T)(\xi \ogreaterthan \zeta \olessthan \eta) = S\xi \ogreaterthan \zeta \olessthan T\eta \quad \text{for all } \xi \in \beta,\, \zeta \in H_\mu,\, \eta \in \gamma.$$

The C^*-relative tensor product is unital in the following sense. If we consider B, B^{op} embedded in $\mathcal{L}(H_\mu)$ via the GNS-representations, then the tuple $\mathfrak{U} := (H_\mu, B, B^{op})$ is a C^*-(μ, μ^{op})-module, and the maps

$$H_{\beta}{\otimes}_{B^{op}}H_\mu \to H,\ \xi \ogreaterthan \zeta \olessthan b^{op} \mapsto \xi b^{op}\zeta, \quad H_{\mu\ B}{\otimes}_{\gamma}K \to K,\ b \ogreaterthan \zeta \olessthan \eta \mapsto \eta b\zeta,$$

are isomorphisms of C^*-$(\sigma_1, \ldots, \sigma_m, \mu)$- and C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n)$-modules $\mathfrak{H} \underset{\mu}{\otimes} \mathfrak{U} \cong \mathfrak{H}$ and $\mathfrak{U} \underset{\mu}{\otimes} \mathfrak{K} \cong \mathfrak{K}$, natural in $\mathfrak{H}$ and $\mathfrak{K}$, respectively.

The C^*-relative tensor product is associative in the following sense. Assume that $\nu, \rho_1, \ldots, \rho_l$ are faithful KMS-states on C^*-algebras $D, E_1, \ldots, E_l$, that $\hat{\mathfrak{K}} = (K, \gamma, \delta_1, \ldots, \delta_n, \epsilon)$ is a C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n, \nu)$-module, and $\mathfrak{L} = (L, \phi, \psi_1, \ldots, \psi_l)$ a C^*-$(\nu^{op}, \rho_1, \ldots, \rho_l)$-module. Then the isomorphisms of Hilbert spaces

$$(H_{\beta}{\otimes}_{\gamma}K)_{\beta \triangleright \epsilon}{\otimes}_{\phi}L \cong \beta \ogreaterthan_{\rho_\gamma} K_{\rho_\epsilon}{\olessthan}\,\phi \cong H_{\beta}{\otimes}_{\gamma \triangleleft \phi}(K_{\epsilon}{\otimes}_{\phi}L) \tag{5}$$

are isomorphisms of C^*-$(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n, \rho_1, \ldots, \rho_l)$-modules $(\mathfrak{H} \underset{\mu}{\otimes} \hat{\mathfrak{K}}) \underset{\nu}{\otimes} \mathfrak{L} \cong \mathfrak{H} \underset{\mu}{\otimes} (\hat{\mathfrak{K}} \underset{\nu}{\otimes} \mathfrak{L})$. We shall identify the Hilbert spaces in (5) without further notice and denote these Hilbert spaces by $H_{\beta}{\otimes}_{\gamma}K_{\epsilon}{\otimes}_{\phi}L$.

We shall need the following simple construction not mentioned in [17]:

Lemma 3.9. *Let H_β, $\tilde{H}_{\tilde{\beta}}$ be C^*-μ-modules, K_γ, $\tilde{K}_{\tilde{\gamma}}$ C^*-μ^{op}-modules, and $I\colon H \to \tilde{H}$, $J\colon K \to \tilde{K}$ anti-unitaries such that $I\beta J_\mu = \tilde{\beta}$ and $J\gamma J_\mu = \tilde{\gamma}$.*

i) There exists a unique anti-unitary $I_\beta \underset{J_\mu}{\otimes}_\gamma J\colon H_\beta{\otimes}_\gamma K \to \tilde{H}_{\tilde{\beta}}{\otimes}_{\tilde{\gamma}}\tilde{K}$ such that

$$(I_\beta \underset{J_\mu}{\otimes}_\gamma J)(\xi \otimes \zeta \otimes \eta) = I\xi J_\mu \otimes J_\mu\zeta \otimes J\eta J_\mu \quad \text{for all } \xi \in \beta,\ \zeta \in H_\mu,\ \eta \in \gamma.$$

ii) $(I_\beta \underset{J_\mu}{\otimes}_\gamma J)|\xi\rangle_1 = |I\xi J_\mu\rangle_1 J$ and $(I_\beta \underset{J_\mu}{\otimes}_\gamma J)|\eta\rangle_2 = |J\eta J_\mu\rangle_2 I$ for all $\xi \in \beta$, $\eta \in \gamma$.

iii) $(I_\beta \underset{J_\mu}{\otimes}_\gamma J)(S_\beta{\otimes}_\gamma T) = (ISI^*{}_{\tilde{\beta}}{\otimes}_{\tilde{\gamma}} JTJ^*)(I_\beta \underset{J_\mu}{\otimes}_\gamma J)$ for all $S \in \mathcal{L}(H_\beta)$, $T \in \mathcal{L}(K_\gamma)$.

Proof. Straightforward. □

The fiber product of C^*-algebras The fiber product of C^*-algebras over KMS-states is an analogue of the fiber product of von Neumann algebras. We briefly summarize the definition and main properties; for details, see [17, Section 3].

Let μ be a faithful KMS-state on a C^*-algebra B, let A^β_H be a C^*-μ-algebra, and let C^γ_K be a C^*-μ^{op}-algebra. The *fiber product* of A^β_H and C^γ_K is the C^*-algebra

$$A_\beta{*}_\gamma C := \{x \in \mathcal{L}(H_\beta{\otimes}_\gamma K) \mid x|\beta\rangle_1, x^*|\beta\rangle_1 \subseteq [|\beta\rangle_1 C] \text{ and } x|\gamma\rangle_2, x^*|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A]\}.$$

If A and C are unital, so is $A_\beta{*}_\gamma C$, but otherwise, $A_\beta{*}_\gamma C$ may be degenerate.

Conjugation by the flip $\Sigma\colon H_\beta{\otimes}_\gamma K \to K_\gamma{\otimes}_\beta H$ in (4) yields an isomorphism

$$\mathrm{Ad}_\Sigma\colon A_\beta{*}_\gamma C \to C_\gamma{*}_\beta A. \tag{6}$$

Assume that $\mathfrak{A} = (H, \alpha_1, \ldots, \alpha_m, \beta, A)$ is a C^*-$(\sigma_1, \ldots, \sigma_m, \mu)$-algebra and $\mathfrak{C} = (K, \gamma, \delta_1, \ldots, \delta_n, C)$ a C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n)$-algebra, where $\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n$ are faithful KMS-states on C^*-algebras $A_1, \ldots, A_m, C_1, \ldots, C_n$. If $A_\beta{*}_\gamma C$ is nondegenerate, then

$$\mathfrak{A} * \mathfrak{C} := ((H_\beta{\otimes}_\gamma K, \alpha_1 \triangleleft \gamma, \ldots, \alpha_m \triangleleft \gamma, \beta \triangleright \delta_1, \ldots, \beta \triangleright \delta_n), A_\beta{*}_\gamma C)$$

is a C^*-$(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n)$-algebra, called the *fiber product* of $\mathfrak{A}$ and $\mathfrak{C}$.

Assume furthermore that $\tilde{\mathfrak{A}} = (\tilde{H}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, \tilde{\beta}, \tilde{A})$ is a C^*-$(\sigma_1, \ldots, \sigma_m, \mu)$-algebra and $\tilde{\mathfrak{C}} = (\tilde{K}, \tilde{\gamma}, \tilde{\delta}_1, \ldots, \tilde{\delta}_n, \tilde{C})$ is a C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n)$-algebra. Then for each $\phi \in \mathrm{Mor}(\mathfrak{A}, \tilde{\mathfrak{A}})$ and $\psi \in \mathrm{Mor}(\mathfrak{C}, \tilde{\mathfrak{C}})$, there exists a unique morphism

$$\phi * \psi \in \mathrm{Mor}(\mathfrak{A} * \mathfrak{C}, \tilde{\mathfrak{A}} * \tilde{\mathfrak{C}})$$

such that $(\phi{*}\psi)(x)(S_\beta{\otimes}_\gamma T) = (S_\beta{\otimes}_\gamma T)x$ for all $x \in A_\beta{*}_\gamma C$, $S \in \mathcal{L}(H_\beta, \tilde{H}_{\tilde{\beta}})$, $T \in \mathcal{L}(K_\gamma, \tilde{K}_{\tilde{\gamma}})$ satisfying $Sa = \phi(a)S$ and $Tc = \psi(c)T$ for all $a \in A$, $c \in C$.

A fundamental deficiency of the fiber product is that it need not be associative. In our applications, however, the fiber product will only appear as the target of a comultiplication, and the non-associativity of the former will be compensated by the coassociativity of the latter.

We shall need the following simple construction not mentioned in [17]:

Lemma 3.10. *Let A^β_H, $\tilde{A}^{\tilde{\beta}}_{\tilde{H}}$ be C^*-μ-algebras, C^γ_K, $\tilde{C}^{\tilde{\gamma}}_{\tilde{K}}$ C^*-μ^{op}-algebras, and $R\colon A \to \tilde{A}^{op}$, $S\colon C \to \tilde{C}^{op}$ $*$-homomorphisms. Assume that $I\colon H \to \tilde{H}$ and $J\colon K \to \tilde{K}$ are anti-unitaries such that $I\beta J_\mu = \tilde{\beta}$, $R(a) = I^*a^*I$ for all $a \in A$, and $J\gamma J_\mu = \tilde{\gamma}$, $S(c) = J^*c^*J$ for all $c \in C$. Then there exists a $*$-homomorphism $R_\beta{*}_\gamma S\colon A_\beta{*}_\gamma C \to (\tilde{A}_{\tilde{\beta}}{*}_{\tilde{\gamma}}\tilde{C})^{op}$ such that $(R_\beta{*}_\gamma S)(x) := (I_\beta \underset{J_\mu}{\otimes}_\gamma J)^* x^* (I_\beta \underset{J_\mu}{\otimes}_\gamma J)$ for all $x \in A_\beta{*}_\gamma C$. This $*$-homomorphism does not depend on the choice of I or J.*

Proof. Evidently, the formula defines a $*$-homomorphism $R_\beta{*}_\gamma S$. The definition does not depend on the choice of J because $\langle\xi|_1(R_\beta{*}_\gamma S)(x)|\xi'\rangle_1 = J^*\langle I\xi J_\mu|_1 x^*|I\xi' J_\mu\rangle_1 J = S(\langle I\xi' J_\mu|_1 x|I\xi J_\mu\rangle_1)$ for all $x \in A_\beta{*}_\gamma C$ by Lemma 3.9 ii), and a similar argument shows that it does not depend on the choice of I. □

4 Compact C^*-quantum groupoids

In this section, we introduce the main object of study of this article — compact C^*-quantum groupoids. Roughly, a compact C^*-quantum groupoid is a compact C^*-quantum graph equipped with a coinvolution and a comultiplication subject to several relations. Most importantly, we assume left- and right-invariance of the Haar weights, the existence of a modular element, and a strong invariance condition relating the coinvolution to the Haar weights and to the comultiplication.

We proceed as follows. First, we discuss the appropriate notion of a comultiplication and recall the notion of a Hopf C^*-bimodule, of bounded invariant Haar weights, and of bounded counits. Then, we introduce and study the precise definition of a compact C^*-quantum groupoid. Finally, we show that the modular element can always be assumed to be trivial, and that the Haar weights are unique up to scaling.

Hopf C^*-bimodules over KMS-states Throughout this paragraph, let μ be a faithful KMS-state on a C^*-algebra B.

Definition 4.1 ([17])**.** *A* comultiplication *on a C^*-(μ, μ^{op})-algebra $A_H^{\alpha,\beta}$ is a morphism $\Delta \in \mathrm{Mor}(A_H^{\alpha,\beta}, A_H^{\alpha,\beta} * A_H^{\alpha,\beta})$ that makes the following diagram commute:*

$$\begin{array}{ccccc}
A & & \xrightarrow{\quad\Delta\quad} & & A_\alpha *_\beta A \\
 & & & & \big\downarrow \mathrm{id} * \Delta \\
\big\downarrow \Delta & & & & A_\alpha *_{\beta \triangleleft \beta} (A_\alpha *_\beta A) \\
 & & & & \big\downarrow \\
A_\alpha *_\beta A & \xrightarrow{\Delta * \mathrm{id}} & (A_\alpha *_\beta A)_{\alpha \triangleright \alpha} *_\beta A & \hookrightarrow & \mathcal{L}(H_\alpha *_\beta H_\alpha *_\beta H).
\end{array}$$

A Hopf C^*-bimodule over μ *is a C^*-(μ, μ^{op})-algebra together with a comultiplication.*

The following important invariance conditions will be imposed on the Haar weights of a compact C^*-quantum groupoid:

Definition 4.2. *Let $(A_H^{\alpha,\beta}, \Delta)$ be a Hopf C^*-bimodule over μ. A* bounded left Haar weight *for $(A_H^{\alpha,\beta}, \Delta)$ is a non-zero completely positive contraction $\phi\colon A \to B$ satisfying*

i) $\phi(\rho_\beta(b) a \rho_\beta(c)) = b\phi(a)c$ for all $a \in A$ and $b, c \in B$,

ii) $\phi(\langle \xi|_1 \Delta(a) |\xi'\rangle_1) = \xi^* \rho_\beta(\phi(a))\xi'$ for all $a \in A$ and $\xi, \xi' \in \alpha$.

A bounded right Haar weight *for $(A_H^{\alpha,\beta}, \Delta)$ is a non-zero completely positive contraction $\psi\colon A \to B^{op}$ satisfying*

i)' $\psi(\rho_\alpha(b^{op}) a \rho_\alpha(c^{op})) = b^{op}\psi(a)c^{op}$ for all $a \in A$ and $b, c \in B$,

ii)' $\psi(\langle \eta|_2 \Delta(a)|\eta'\rangle_2) = \eta^* \rho_\alpha(\psi(a))\eta'$ for all $a \in A$ and $\eta, \eta' \in \beta$.

Remarks 4.3. Let $(A_H^{\alpha,\beta}, \Delta)$ be a Hopf C^*-bimodule over μ.

i) If ϕ is a bounded left Haar weight for $(A_H^{\alpha,\beta}, \Delta)$, then $\rho_\beta \circ \phi\colon A \to \rho_\beta(\mathfrak{B})$ is a conditional expectation.

ii) If $\phi\colon A \to B$ satisfies condition ii) and if $[\langle \alpha|_1 \Delta(A)|\alpha\rangle_1] = A$, then ϕ also satisfies condition i) because $\phi(\rho_\beta(b)\langle \xi|_1\Delta(a)|\xi'\rangle_1 \rho_\beta(c)) = \phi(\langle \xi b|_1 \Delta(a)|\xi' c\rangle_1) = b^*\xi^*\rho_\beta(\phi(a))\xi' c = b^*\phi(\langle \xi|_1\Delta(a)|\xi'\rangle_1)c$ for all $a \in A$, $b, c \in B$, $\xi, \xi' \in \alpha$.

Similar remarks apply to bounded right Haar weights.

The notion of a counit of a Hopf algebra extends to Hopf C^*-bimodules as follows.

Definition 4.4. *Let $(A_H^{\alpha,\beta}, \Delta)$ be a Hopf C^*-bimodule over μ. A* bounded (left/right) counit *for $(A_H^{\alpha,\beta}, \Delta)$ is a morphism $\epsilon \in \mathrm{Mor}\left(A_H^{\alpha,\beta}, \mathcal{L}(H_\mu)_{H_\mu}^{B,B^{op}}\right)$ satisfying (the first/second of) the following conditions:*

i) $\epsilon(\langle\eta|_2\Delta(a)|\eta'\rangle_2) = \eta^* a\eta'$ *for all* $a \in A$ *and* $\eta, \eta' \in \beta$,

ii) $\epsilon(\langle\xi|_1\Delta(a)|\xi'\rangle_1) = \xi^* a\xi'$ *for all* $a \in A$ *and* $\xi, \xi' \in \alpha$.

Remark 4.5. i) Condition i) and ii), respectively, hold if and only if the left/the right square of the following diagram commute:

$$\begin{array}{ccccccccc}
A_\alpha *_\beta A & & \xleftarrow{\quad\Delta\quad} & & A & & \xrightarrow{\quad\Delta\quad} & & A_\alpha *_\beta A \\
\downarrow{\scriptstyle \epsilon * \mathrm{id}} & & & & \downarrow & & & & \downarrow{\scriptstyle \mathrm{id} * \epsilon} \\
\mathcal{L}(H_\mu)_B *_\beta A & \longrightarrow & \mathcal{L}(H_\mu {}_B\otimes_\beta H) & \xrightarrow{\cong} & \mathcal{L}(H) & \xleftarrow{\cong} & \mathcal{L}(H_\alpha \otimes_{B^{op}} H_\mu) & \longleftarrow & A_\alpha *_{B^{op}} \mathcal{L}(H_\mu).
\end{array}$$

ii) A standard argument shows that if a bounded left and a bounded right counit exist, then they are equal and a counit.

Compact C^*-quantum groupoids Given a compact C^*-quantum graph $(B, \mu, A, r, \phi, s, \psi, \delta)$ with coinvolution R, we use the notation introduced in Proposition 3.8 and put $\nu := \mu \circ \phi$, $\nu^{-1} := \mu^{op} \circ \psi = \nu_\delta$, $J := J_\nu = J_{\nu^{-1}}$,

$$H := H_\nu, \qquad \widehat{\alpha} := [A\zeta_\phi], \qquad \beta := [A^{op}\zeta_\phi], \qquad \widehat{\beta} := [A\zeta_\psi], \qquad \alpha := [A^{op}\zeta_\psi], \tag{7}$$

and define an antiunitary $I: H \to H$ by $I\Lambda_{\nu^{-1}}(a) = \Lambda_\nu(R(a)^*)$ for all $a \in A$. Since $I\alpha J_\mu = \beta$, $I\beta J_\mu = \alpha$, and $R(a) = Ia^*I$ for all $a \in A$, we can define a $*$-antihomomorphism $R_\alpha *_\beta R$: $A_\alpha *_\beta A \to A_\beta *_\alpha A$ by $x \mapsto (I_\alpha \underset{J_\mu}{\otimes}{}_\beta I)^* x^* (I_\alpha \underset{J_\mu}{\otimes}{}_\alpha I)$ (Lemma 3.10).

The definition of a compact C^*-quantum groupoid involves the following conditions that are analogues of the strong invariance property known from quantum groups:

Lemma 4.6. *Let* $(B, \mu, A, r, \phi, s, \psi, \delta)$ *be a compact* C^**-quantum graph with a coinvolution* R *and a comultiplication* Δ *for* $A_H^{\alpha,\beta}$ *such that* $(R_\alpha *_\beta R) \circ \Delta = \mathrm{Ad}_\Sigma \circ \Delta \circ R$. *Then*

$$R\big(\langle\zeta_\psi|_1\Delta(a)(d^{op}{}_\alpha\otimes_\beta 1)|\zeta_\psi\rangle_1\big) = \langle\zeta_\phi|_2(1_\alpha\otimes_\beta R(d)^{op})\Delta(R(a))|\zeta_\phi\rangle_2 \quad \text{for all } a, d \in A.$$

Proof. Let $a, d \in A$. By Lemmas 3.9 and 2.7,

$$\begin{aligned}
\langle\zeta_\phi|_2(1_\alpha\otimes_\beta R(d)^{op})\Delta(R(a))|\zeta_\phi\rangle_2 &= \langle\zeta_\phi|_2(1_\alpha\otimes_\beta I(d^{op})^*I)\Sigma(I_\alpha\otimes_\beta I)\Delta(a)^*(I_\alpha \underset{J_\mu}{\otimes}{}_\beta I)^*\Sigma|\zeta_\phi\rangle_2 \\
&= \langle\zeta_\phi|_1(I_\alpha \underset{J_\mu}{\otimes}{}_\beta I)((d^{op})^*{}_\alpha\otimes_\beta 1)\Delta(a)^*|I\zeta_\phi J_\mu\rangle_1 I \\
&= I\langle\zeta_\psi|_1((d^{op})^*{}_\alpha\otimes_\beta 1)\Delta(a)^*|\zeta_\psi\rangle_1 I \\
&= R\big(\langle\zeta_\psi|_1\Delta(a)(d^{op}{}_\alpha\otimes_\beta 1)|\zeta_\psi\rangle_1\big).
\end{aligned}$$

□

As a direct consequence, we obtain the following result:

Lemma 4.7. *Let* $(B, \mu, A, r, \phi, s, \psi, \delta)$ *be a compact* C^**-quantum graph with a coinvolution* R *and a comultiplication* Δ *for* $A_H^{\alpha,\beta}$ *such that* $(R_\alpha *_\beta R) \circ \Delta = \mathrm{Ad}_\Sigma \circ \Delta \circ R$. *Then the following two conditions are equivalent:*

i) $R\big(\langle\zeta_\psi|_1\Delta(a)(d^{op}{}_\alpha\otimes_\beta 1)|\zeta_\psi\rangle_1\big) = \langle\zeta_\psi|_1(a^{op}{}_\alpha\otimes_\beta 1)\Delta(d)|\zeta_\psi\rangle_1$ *for all* $a, d \in A$.

ii) $R\big(\langle\zeta_\phi|_2\Delta(a)(1_\alpha\otimes_\beta d^{op})|\zeta_\phi\rangle_2\big) = \langle\zeta_{\phi|_2}(1_\alpha\otimes_\beta a^{op})\Delta(d)|\zeta_\phi\rangle_2$ *for all* $a, d \in A$. □

Now we come to the main definition of this article:

Definition 4.8. *A* compact C^*-quantum groupoid *is a compact* C^**-quantum graph* $(B, \mu, A, r, \phi, s, \psi, \delta)$ *with a coinvolution* R *and a comultiplication* Δ *for* $A_H^{\alpha,\beta}$ *such that*

i) $[\Delta(A)|\alpha\rangle_1] = [|\alpha\rangle_1 A] = [\Delta(A)|\zeta_\psi\rangle_1 A]$ *and* $[\Delta(A)|\beta\rangle_2] = [|\beta\rangle_2 A] = [\Delta(A)|\zeta_\phi\rangle_2 A]$;

ii) ϕ *is a bounded left Haar weight and* ψ *a bounded right Haar weight for* $(A_H^{\alpha,\beta}, \Delta)$;

iii) $R\big(\langle\zeta_\psi|_1\Delta(a)(d^{op}{}_\alpha\otimes_\beta 1)|\zeta_\psi\rangle_1\big) = \langle\zeta_\psi|_1(a^{op}{}_\alpha\otimes_\beta 1)\Delta(d)|\zeta_\psi\rangle_1$ *for all* $a, d \in A$.

Let us briefly comment on this definition. The coinvolution R is uniquely determined by condition iii). The Haar weights are unique up to some scaling, as we shall see at the end of this section. At the end of the next section, we will see that $(R_{\alpha}*_{\beta}R)\circ\Delta = \mathrm{Ad}_{\Sigma}\circ\Delta\circ R$; in particular, the modified strong invariance condition in Lemma 4.7 ii) holds by Lemma 4.7.

From now on, let $(B,\mu,A,r,\phi,s,\psi,\delta,R,\Delta)$ be a compact C^*-quantum groupoid.

Lemma 4.9. *$\{a\in A\cap r(B)' \mid \Delta(a) = 1_{\alpha}\otimes_{\beta}a\} = s(B^{op})$ and $\{a\in A\cap s(B^{op})' \mid \Delta(a) = a_{\alpha}\otimes_{\beta}1\} = r(B)$.*

Proof. We only prove the first equation. Clearly, the right hand side is contained in the left hand side. Conversely, if $a\in A\cap r(B)'$ and $\Delta(a) = 1_{\alpha}\otimes_{\beta}a$, then $a = \langle\zeta_{\psi}|_1\Delta(a)|\zeta_{\psi}\rangle_1 = s(\psi(a))$ by right-invariance of ψ. □

The conditional expectation onto the C^*-algebra of orbits Let us study the maps $\tau = \psi\circ r\colon B\to Z(B^{op})\cong Z(B)$ and $\tau^{\dagger} = \phi\circ s\colon B^{op}\to Z(B)\cong Z(B^{op})$ introduced in (2)). First, note that $\tau(b) = \tau^{\dagger}(b^{op})$ for all $b\in B$ by Lemma 2.7 ii).

Proposition 4.10. *The maps τ and $\tau^{\dagger}$ are conditional expectations onto a C^*-subalgebra of $Z(B) = B\cap B^{op}$ and satisfy*

$$s\circ\tau = r\circ\tau, \qquad \sigma^{\mu}_t\circ\tau = \tau \text{ for all } t\in\mathbb{R},$$
$$\tau\circ\phi = \tau^{\dagger}\circ\psi, \qquad \tau(b\sigma^{\mu}_{-i/2}(d)) = \tau(d\sigma^{\mu}_{-i/2}(d)) \text{ for all } b,d\in\mathrm{Dom}(\sigma^{\mu}_{-i/2}).$$

The proof involves the following equation:

Lemma 4.11. *For all $b,c,e\in B$ and $d\in\mathrm{Dom}(\sigma^{\mu}_{-i/2})$,*

$$\langle\zeta_{\psi}|_1\Delta(r(b)s(c^{op}))((r(d)s(e^{op}))^{op}{}_{\alpha}\otimes_{\beta}1)|\zeta_{\psi}\rangle_1 = r(\tau(b\sigma^{\mu}_{-i/2}(d)))r(e)s(c^{op}).$$

Proof. Let b,c,d,e as above. Then

$$\begin{aligned}\langle\zeta_{\psi}|_1\Delta(r(b)s(c^{op}))((r(d)s(e^{op}))^{op}{}_{\alpha}\otimes_{\beta}1)|\zeta_{\psi}\rangle_1 &= \langle\zeta_{\psi}|_1(r(b)(r(d)s(e^{op}))^{op}{}_{\alpha}\otimes_{\beta}s(c^{op}))|\zeta_{\psi}\rangle_1\\ &= \rho_{\beta}(\zeta_{\psi}^*r(b)r(d)^{op}s(e^{op})^{op}\zeta_{\psi})s(c^{op})\\ &= r(\zeta_{\psi}^*r(b)r(d)^{op}\zeta_{\psi}e)s(c^{op}),\end{aligned}$$

and by Lemma 3.7 iii), $\zeta_{\psi}^*r(b)r(d)^{op}\zeta_{\psi} = \zeta_{\psi}^*r(b\sigma^{\mu}_{-i/2}(d))\zeta_{\psi} = \tau(b\sigma^{\mu}_{-i/2}(d))$. □

Proof of Proposition 4.10. The left- and right-invariance of ϕ and ψ imply that for all $a\in A$,

$$\begin{aligned}\phi(s(\psi(a))) = \zeta_{\phi}^*s(\psi(a))\zeta_{\phi} &= \zeta_{\psi}^*\langle\zeta_{\phi}|_2\Delta(a)|\zeta_{\phi}\rangle_2\zeta_{\psi}\\ &= \zeta_{\phi}^*\langle\zeta_{\psi}|_1\Delta(a)|\zeta_{\psi}\rangle_1\zeta_{\phi}^* = \zeta_{\psi}^*r(\phi(a))\zeta_{\psi} = \psi(r(\phi(a))).\end{aligned}$$

Therefore, $\tau^{\dagger}\circ\psi = \tau\circ\phi$ and $\tau\circ\tau = \tau^{\dagger}\circ\tau = \tau^{\dagger}\circ(\psi\circ r) = \tau\circ\phi\circ r = \tau$. Next, $s\circ\tau = r\circ\tau$ because for all $b\in B$,

$$s(\psi(r(b))) = \langle\zeta_{\psi}|_1\Delta(r(b))|\zeta_{\psi}\rangle_1 = \langle\zeta_{\psi}|_1(r(b)_{\alpha}\otimes_{\beta}1)|\zeta_{\psi}\rangle_1 = \rho_{\beta}(\zeta_{\psi}^*r(b)\zeta_{\psi}) = r(\psi(r(b))).$$

In particular, we find that for all $b,c,d\in B$,

$$\tau(b)\tau(c)\tau(d) = \tau(b)\psi(r(c))\tau(d) = \psi(s(\tau(b))r(c)s(\tau(d))) = \psi(r(\tau(bcd))) = \tau(bcd).$$

Therefore, τ is a conditional expectation onto its image.

Let $t\in\mathbb{R}$. Then $\sigma^{\mu}_t(\tau(B))\subseteq\tau(B)$ because $\sigma^{\mu}_t\circ\tau = \sigma^{\mu^{op}}_{-t}\circ\psi\circ r = \psi\circ\sigma^{\nu^{-1}}_{-t}\circ r = \psi\circ\sigma^{\nu}_t\circ r = \psi\circ r\circ\sigma^{\mu}_t$. Since $\upsilon := \mu|_{\tau(B)}$ is a trace, we can conclude from Lemma 2.2 i) that $\sigma^{\mu}_t\circ\tau = \tau\circ\sigma^{\upsilon}_t = \tau$.

Finally, let $b,d\in\mathrm{Dom}(\sigma^{\mu}_{-i/2})$. By Lemma 4.11 and condition iii) in Definition 4.8,

$$\begin{aligned}r(\tau(b\sigma^{\mu}_{-i/2}(d))) &= \langle\zeta_{\psi}|_1\Delta(r(b))(r(d)^{op}{}_{\alpha}\otimes_{\beta}1)|\zeta_{\psi}\rangle_1\\ &= R(\langle\zeta_{\psi}|_1\Delta(r(d))(r(b)^{op}{}_{\alpha}\otimes_{\beta}1)|\zeta_{\psi}\rangle_1) = s(\tau(d\sigma^{\mu}_{-i/2}(b))).\end{aligned}$$

Since $s\circ\tau = r\circ\tau$ and r is injective, we can conclude $\tau(b\sigma^{\mu}_{-i/2}(d)) = \tau(d\sigma^{\mu}_{-i/2}(b))$. □

The modular element The modular element of a compact C^*-quantum groupoid can be described in terms of the element

$$\theta := \phi(\delta) = \psi(\delta^{-1}) \in B \cap B^{op}$$

(see Lemma 2.5 and Lemma 2.7 i)) as follows.

Proposition 4.12. *$\delta = r(\theta)s(\theta)^{-1}$ and $\Delta(\delta) = \delta_{\alpha}\otimes_{\beta}\delta$.*

Proof. By Lemma 2.5 i), the element $\tilde{\delta} := r(\theta)s(\theta)^{-1}$ is positive, invertible, and invariant with respect to σ^{ν}. Moreover, $\nu^{-1}(a) = \nu(\tilde{\delta}^{1/2}a\tilde{\delta}^{1/2})$ for all $a \in A$ because

$$\begin{aligned}\nu^{-1}\big(s(\theta)^{1/2}as(\theta)^{1/2}\big) &= \mu^{op}(\theta^{1/2}\psi(a)\theta^{1/2}) = (\mu \circ \phi \circ s \circ \psi)(a) \\ &= (\mu^{op} \circ \psi \circ r \circ \phi)(a) = \mu(\theta^{1/2}\phi(a)\theta^{1/2}) = \nu\big(r(\theta)^{1/2}ar(\theta)^{1/2}\big)\end{aligned}$$

for all $a \in A$ by Proposition 4.10 and Lemma 2.5 ii). Now, $\delta = \tilde{\delta}$ by Lemma 2.3 iii), and $\Delta(\delta) = r(\theta)_{\alpha}\otimes_{\beta}s(\theta)^{-1} = r(\theta)\rho_{\alpha}(\theta^{-1})_{\alpha}\otimes_{\beta}\rho_{\beta}(\theta)s(\theta)^{-1} = \delta_{\alpha}\otimes_{\beta}\delta$ because $\theta \in Z(B)$. □

An important consequence of the preceding result is that for every compact C^*-quantum groupoid, there exists a faithful invariant KMS-state on the basis:

Corollary 4.13. $\mu_{\theta} \circ \phi = (\mu_{\theta})^{op} \circ \psi$.

Proof. For all $a \in A$, we have $\mu\big(\theta^{1/2}\phi(a)\theta^{1/2}\big) = \nu\big(r(\theta)^{1/2}ar(\theta)^{1/2}\big) = \nu^{-1}\big(s(\theta)^{1/2}as(\theta)^{1/2}\big) = \mu^{op}\big(\theta^{1/2}\psi(a)\theta^{1/2}\big)$. □

This result implies that in principle, we could restrict to compact C^*-quantum groupoids with trivial modular element $\delta = 1_A$. We shall not do so for several reasons. First, the treatment of a nontrivial modular element does not require substantially more work. Second, the freedom to choose the state μ might be useful in applications. Finally, we hope to prepare the ground for a more general theory of locally compact quantum groupoids, where the modular element can no longer be assumed to be trivial.

The KMS-state μ can be factorized into a state υ on the commutative C^*-algebra $\tau(B) \subseteq Z(B)$ and a perturbation of τ as follows. We define maps

$$\tau_{\theta^{-1}} : B \to \tau(B),\ b \mapsto \tau(\theta^{-1/2}b\theta^{-1/2}), \qquad \upsilon = \mu_{\theta}|_{\tau(B)} : \tau(B) \to \mathbb{C},\ b \mapsto \mu(\theta^{1/2}b\theta^{1/2}).$$

Note that $\tau(\theta^{-1}) = 1$ because $\theta = \phi(\delta) = \phi(r(\theta)s(\theta)^{-1}) = \theta\tau(\theta^{-1})$.

Proposition 4.14. $\mu = \upsilon \circ \tau_{\theta^{-1}}$.

Proof. By Propositions 4.10 and 4.12, we have $\mu(b) = \nu(r(b)) = \nu^{-1}(\delta^{-1/2}r(b)\delta^{-1/2}) = \mu^{op}\big(\theta^{1/2}\psi(r(\theta^{-1/2}b\theta^{-1/2}))\theta^{1/2}\big) = (\upsilon \circ \tau_{\theta^{-1}})(b)$ for all $b \in B$. □

Uniqueness of the Haar weights A central result in the theory of locally compact (quantum) groups is the uniqueness of the Haar weights up to scaling. In this paragraph, we prove a similar uniqueness result for the Haar weights of a compact C^*-quantum groupoid.

The Haar weights of a compact C^*-quantum groupoid can be rescaled by elements of B as follows: For every positive element $\gamma \in B^{op}$, the completely positive contraction

$$\phi_{s(\gamma)} : A \to B,\ a \mapsto \phi\big(s(\gamma)^{1/2}as(\gamma)^{1/2}\big),$$

is a bounded left Haar weight for $(A_H^{\alpha,\beta}, \Delta)$ because for all $a \in A$ and $\xi, \xi' \in \alpha$

$$\begin{aligned}\phi_{s(\gamma)}\big(\langle\xi|_1\Delta(a)|\xi'\rangle_1\big) &= \phi\big(\langle\xi|_1(1_{\alpha}\otimes_{\beta}s(\gamma)^{1/2})\Delta(a)(1_{\alpha}\otimes_{\beta}s(\gamma)^{1/2})|\xi'\rangle_1\big) \\ &= \phi\big(\langle\xi|_1\Delta\big(s(\gamma)^{1/2}as(\gamma^{1/2})\big)|\xi'\rangle_1\big) = \xi^*\phi_{s(\gamma)}(a)\xi'.\end{aligned}$$

Likewise, for every positive element $\gamma \in B$, the completely positive contraction

$$\psi_{r(\gamma)} : A \to B^{op},\ a \mapsto \psi\big(r(\gamma)^{1/2}ar(\gamma)^{1/2}\big),$$

is a bounded right Haar weight for $(A_H^{\alpha,\beta}, \Delta)$.

Theorem 4.15. *Let $\tilde\phi$, $\tilde\psi$, $\tilde\delta$ be such that $(B,\mu,A,r,\tilde\phi,s,\tilde\psi,\tilde\delta)$ is a compact C^*-quantum graph.*

i) If $\tilde\phi$ is a bounded left Haar weight for $(A_H^{\alpha,\beta},\Delta)$, then $\tilde\phi=\phi_\gamma$, where $\gamma=\tilde\psi(\tilde\delta^{-1})\theta^{-1}$.

ii) If $\tilde\psi$ is a bounded right Haar weight for $(A_H^{\alpha,\beta},\Delta)$, then $\tilde\psi=\psi_\gamma$, where $\gamma=\tilde\phi(\tilde\delta)\theta^{-1}$.

Proof. We only prove i), the proof of ii) is similar. Put $\tilde\nu:=\mu\circ\tilde\phi$, $\tilde\nu^{-1}:=\mu^{op}\circ\tilde\psi$, $\tilde\theta:=\tilde\psi(\tilde\delta^{-1})$. Let $a\in A$. Then

$$\tilde\phi(s(\psi(a)))=\tilde\phi(\langle\zeta_\psi|_1\Delta(a)|\zeta_\psi\rangle_1)=\psi(r(\tilde\phi(a))). \tag{8}$$

We apply μ to the left hand side and find, using Lemma 2.5 ii),

$$\tilde\nu(s(\psi(a)))=\mu^{op}_{\tilde\theta}(\psi(a))=\nu^{-1}\big(s(\tilde\theta)^{1/2}as(\tilde\theta)^{1/2}\big)=\nu\big(\delta^{1/2}s(\tilde\theta)^{1/2}as(\tilde\theta)^{1/2}\delta^{1/2}\big).$$

Next, we apply μ to the right hand side of equation (8) and find

$$\nu^{-1}(r(\tilde\phi(a)))=\mu_\theta(\tilde\phi(a))=\tilde\nu\big(r(\theta)^{1/2}ar(\theta)^{1/2}\big).$$

Since the left hand side and the right hand side of equation (8) are equal and $\delta=r(\theta)s(\theta)^{-1}$, we can conclude $\tilde\nu(d)=\nu\big(s(\gamma)^{1/2}ds(\gamma)^{1/2}\big)$ for all $d\in A$ and in particular

$$\mu(b^*\tilde\phi(a))=\tilde\nu\big(r(b)^*a\big)=\nu\big(s(\gamma)^{1/2}r(b)^*as(\gamma)^{1/2}\big)=\mu\big(b^*\phi(s(\gamma)^{1/2}as(\gamma)^{1/2})\big)$$

for all $b\in B$, $a\in A$. Since μ is faithful, we have $\tilde\phi(a)=\phi(s(\gamma)^{1/2}as(\gamma)^{1/2})$ for all $a\in A$. ☐

5 The fundamental unitary

In the theory of locally compact quantum groups, a fundamental rôle is played by the multiplicative unitaries of Baaj, Skandalis [1] and Woronowicz [22]: To every locally compact quantum group, one can associate a manageable multiplicative unitary, and to every manageable multiplicative unitary two Hopf C^*-algebras called the "legs" of the unitary. One of these legs coincides with the initial quantum group, and the other one is its generalized Pontrjagin dual. Moreover, the multiplicative unitary can be used to switch between the reduced C^*-algebra and the von Neumann algebra of the quantum group.

Similarly, we associate to every compact C^*-quantum groupoid a generalized multiplicative unitary. More precisely, this unitary is a regular C^*-pseudo-multiplicative unitary in the sense of [17]. The first application of this unitary will be to prove that the coinvolution of a compact C^*-quantum groupoid reverses the comultiplication. The second application will be to associate to every compact C^*-quantum groupoid a measured quantum groupoid in the sense of Enock and Lesieur [5, 9]. The third application, given in the next section, will be to construct a generalized Pontrjagin dual of the compact C^*-quantum groupoid in form of a Hopf C^*-bimodule. Finally, one can use this unitary to define reduced crossed products for coactions of the compact C^*-quantum groupoid as in [16].

C^*-pseudo-multiplicative unitaries The notion of a C^*-pseudo-multiplicative unitary extends the notion of a multiplicative unitary [1], of a continuous field of multiplicative unitaries [2], and of a pseudo-multiplicative unitary on C^*-modules [10, 18], and is closely related to pseudo-multiplicative unitaries on Hilbert spaces [21]; see [17, Section 4.1]. The precise definition is as follows. Let μ be a faithful KMS-state on a C^*-algebra B.

Definition 5.1 ([17])**.** *A* C^*-pseudo-multiplicative unitary over μ *consists of a* C^*-(μ^{op},μ,μ^{op})-*module $(H,\widehat\beta,\alpha,\beta)$ and a unitary $V\colon H_{\widehat\beta}\otimes_\alpha H\to H_\alpha\otimes_\beta H$ such that*

$$V(\alpha\triangleleft\alpha)=\alpha\triangleright\alpha,\quad V(\widehat\beta\triangleright\beta)=\widehat\beta\triangleleft\beta,\quad V(\widehat\beta\triangleright\widehat\beta)=\alpha\triangleright\widehat\beta,\quad V(\beta\triangleleft\alpha)=\beta\triangleleft\beta \tag{9}$$

and the following diagram commutes:

$$
\begin{array}{ccccc}
H_{\hat{\beta}}\otimes_{\alpha} H_{\hat{\beta}}\otimes_{\alpha} H & \xrightarrow{V_{\hat{\beta}\triangleright\hat{\beta}}\otimes_{\alpha}\mathrm{id}} & H_{\alpha}\otimes_{\beta} H_{\hat{\beta}}\otimes_{\alpha} H & \xrightarrow{\mathrm{id}_{\alpha}\otimes_{\beta\triangleleft\alpha} V} & H_{\alpha}\otimes_{\beta} H_{\alpha}\otimes_{\beta} H, \\
{\scriptstyle \mathrm{id}_{\hat{\beta}}\otimes_{\alpha\triangleleft\alpha} V}\downarrow & & & & \uparrow{\scriptstyle V_{\alpha\triangleleft\alpha}\otimes_{\beta}\mathrm{id}} \\
H_{\hat{\beta}}\otimes_{\alpha\triangleright\alpha}(H_{\alpha}\otimes_{\beta} H) & & & & (H_{\hat{\beta}}\otimes_{\alpha} H)_{\alpha\triangleleft\alpha}\otimes_{\beta} H \\
{\scriptstyle \mathrm{id}_{\hat{\beta}}\otimes_{\alpha\triangleright\alpha}\Sigma}\downarrow & & & & \uparrow{\scriptstyle \Sigma_{23}} \\
H_{\hat{\beta}}\otimes_{\alpha} H_{\beta}\otimes_{\alpha} H & & \xrightarrow{\qquad V_{\hat{\beta}\triangleright\beta}\otimes_{\alpha}\mathrm{id}\qquad} & & \big(H_{\alpha}\otimes_{\beta} H\big)_{\hat{\beta}\triangleleft\beta}\otimes_{\alpha} H
\end{array}
\tag{10}
$$

where Σ_{23} *denotes the isomorphism*

$$\big(H_{\alpha}\otimes_{\beta} H\big)_{\hat{\beta}\triangleleft\beta}\otimes_{\alpha} H \cong (H_{\rho_{\alpha}} \otimes \beta)_{\rho_{\hat{\beta}\triangleleft\beta}} \otimes \alpha \xrightarrow{\cong} (H_{\rho_{\hat{\beta}}} \otimes \alpha)_{\rho_{\alpha\triangleleft\alpha}} \otimes \beta \cong (H_{\hat{\beta}}\otimes_{\alpha} H)_{\alpha\triangleleft\alpha}\otimes_{\beta} H,$$

$$(\zeta \otimes \xi) \otimes \eta \mapsto (\zeta \otimes \eta) \otimes \xi.$$

Given a C^**-pseudo-multiplicative unitary* $(H, \hat{\beta}, \alpha, \beta, V)$*, we adopt the following leg notation. We abbreviate the operators* $V_{\hat{\beta}\triangleright\hat{\beta}}\otimes_{\alpha}\mathrm{id}$ *and* $V_{\alpha\triangleleft\alpha}\otimes_{\beta}\mathrm{id}$ *by* V_{12}*, the operators* $\mathrm{id}_{\alpha}\otimes_{\beta\triangleleft\alpha} V$ *and* $\mathrm{id}_{\hat{\beta}}\otimes_{\alpha\triangleleft\alpha} V$ *by* V_{23}*, and* $(\mathrm{id}_{\hat{\beta}}\otimes_{\alpha\triangleright\alpha}\Sigma)V_{12}\Sigma_{23}$ *by* V_{13}*. Thus, the indices indicate those positions in a relative tensor product where the operator acts like* V*.*

Let $(H, \hat{\beta}, \alpha, \beta, V)$ be a C^*-pseudo-multiplicative unitary. We put

$$\hat{A}(V) := \big[\langle\beta|_2 V|\alpha\rangle_2\big] \subseteq \mathcal{L}(H), \qquad A(V) := \big[\langle\alpha|_1 V|\hat{\beta}\rangle_1\big] \subseteq \mathcal{L}(H).$$

These spaces satisfy $\hat{A}(V) \subseteq \mathcal{L}(H_{\beta})$ and $A(V) \subseteq \mathcal{L}(H_{\hat{\beta}})$, so that we can define maps

$$\hat{\Delta}_V : \hat{A} \to \mathcal{L}\big(H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha} H\big),\ \hat{a} \mapsto V^*(1_{\alpha}\otimes_{\beta}\hat{a})V, \quad \Delta_V : A \to \mathcal{L}\big(H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta} H\big),\ a \mapsto V(a_{\hat{\beta}}\otimes_{\alpha} 1)V^*.$$

Definition 5.2 ([17])**.** *We call a* C^**-pseudo-multiplicative unitary* $(H, \hat{\beta}, \alpha, \beta, V)$ regular *if* $[\langle\alpha|_1 V|\alpha\rangle_2] = [\alpha\alpha^*]$*, and* well-behaved *if* $(\hat{A}(V)^{\hat{\beta},\alpha}_{H}, \hat{\Delta}_V)$ *and* $(A(V)^{\alpha,\beta}_{H}, \Delta_V)$ *are Hopf* C^**-bimodules over* μ^{op} *and* μ*, respectively.*

Theorem 5.3 ([17])**.** *Every regular* C^**-pseudo-multiplicative unitary is well-behaved.*

Let $(H, \hat{\beta}, \alpha, \beta, V)$ be a C^*-pseudo-multiplicative unitary. We put

$$V^{op} := \Sigma V^* \Sigma : H_{\beta}\otimes_{\alpha} H \xrightarrow{\Sigma} H_{\alpha}\otimes_{\beta} H \xrightarrow{V^*} H_{\hat{\beta}}\otimes_{\alpha} H \xrightarrow{\Sigma} H_{\alpha}\otimes_{\hat{\beta}} H.$$

Then $(H, \beta, \alpha, \hat{\beta}, V^{op})$ is a C^*-pseudo-multiplicative unitary over μ^{op}, called the *opposite* of $(H, \hat{\beta}, \alpha, \beta, V)$ [17, Remark 4.3]. One easily checks that V^{op} is regular if V is regular.

The fundamental unitary of a compact C^*-quantum groupoid Throughout this section, let $(B, \mu, A, r, \phi, s, \psi, \delta, R, \Delta)$ be a compact C^*-quantum groupoid. We use the same notation as in the preceding section.

The main result of this paragraph is the following theorem.

Theorem 5.4. *There exists a regular* C^**-pseudo-multiplicative unitary* $(H, \hat{\beta}, \alpha, \beta, V)$ *such that* $V|a\zeta_{\psi}\rangle_1 = \Delta(a)|\zeta_{\psi}\rangle_1$ *for all* $a \in A$*.*

We prove this result in several steps. Til the end of this section, we fix a compact C^*-quantum groupoid $(B, \mu, A, r, \phi, s, \psi, \delta, R, \Delta)$ and use the same notation as in Section 4.

Proposition 5.5. *i) There exists a unique unitary* $V : H_{\hat{\beta}}\otimes_{\alpha} H \to H_{\alpha}\otimes_{\beta} H$ *such that* $V|a\zeta_{\psi}\rangle_1 = \Delta(a)|\zeta_{\psi}\rangle_1$ *for all* $a \in A$*.*

ii) $V(a\zeta_\nu \olessthan d^{op}\zeta_\psi) = \Delta(a)(\zeta_\nu \olessthan d^{op}\zeta_\phi)$ *for all* $a, d \in A$.

iii) $V(\widehat{\beta} \triangleright \beta) = \widehat{\beta} \triangleleft \beta$, $V(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}$, $V(\widehat{\beta} \triangleright \widehat{\alpha}) = \alpha \triangleright \widehat{\alpha}$, $V(\widehat{\alpha} \triangleleft \alpha) = \widehat{\alpha} \triangleleft \beta$.

Proof. i) Let $a \in A$, $\eta \in \beta$, $\zeta \in H_\mu$. Since ψ is a bounded right Haar weight for $(A_H^{\alpha,\beta}, \Delta)$,

$$\begin{aligned} \big\langle \Delta(a)(\zeta_\psi \ogreaterthan \zeta \ogreaterthan \eta) \big| \langle \Delta(a)(\zeta_\psi \ogreaterthan \zeta \ogreaterthan \eta) \big\rangle_{(H_\alpha \otimes_\beta H)} &= \big\langle \zeta \big| \zeta_\psi^* \langle \eta |_2 \Delta(a^* a) | \eta \rangle_2 \zeta_\psi \zeta \big\rangle \\ &= \big\langle \zeta \big| \eta^* \rho_\alpha(\zeta_\psi^* a^* a \zeta_\psi) \eta \zeta \big\rangle \\ &= \big\langle a\zeta_\psi \ogreaterthan \eta\zeta \big| a\zeta_\psi \ogreaterthan \eta\zeta \big\rangle_{(H_{\widehat{\beta}} \otimes_\alpha H)}. \end{aligned}$$

Therefore, there exists an isometry $V \colon H_{\widehat{\beta}} \otimes_\alpha H \to H_\alpha \otimes_\beta H$ such that $V|a\zeta_\psi\rangle_1 = \Delta(a)|\zeta_\psi\rangle_1$ for all $a \in A$. Since $[\Delta(A)|\beta\rangle_2] = [|\beta\rangle_2 A]$,

$$V(\widehat{\beta} \triangleleft \beta) = [V|A\zeta_\psi\rangle_1 \beta] = [\Delta(A)|\zeta_\psi\rangle_1 \beta] = [\Delta(A)|\beta\rangle_2 \zeta_\psi] = [|\beta\rangle_2 A\zeta_\psi] = \widehat{\beta} \triangleleft \beta.$$

In particular, V is surjective and hence a unitary.

ii) By Proposition 4.12, we have for all $a, d \in A$

$$\begin{aligned} V(a\zeta_\nu \olessthan d^{op}\zeta_\psi) &= V(a\delta^{-1/2}\zeta_\psi\zeta_\mu \olessthan d^{op}\zeta_\psi) \\ &= V(a\delta^{-1/2}\zeta_\psi \ogreaterthan d^{op}\zeta_{\nu^{-1}}) \\ &= \Delta(a\delta^{-1/2})(\zeta_\psi \ogreaterthan d^{op}\delta^{1/2}\zeta_\nu) \\ &= \Delta(a)(\delta^{-1/2}{}_\alpha\otimes_\beta \delta^{-1/2})(\zeta_{\nu^{-1}} \olessthan d^{op}\delta^{1/2}\zeta_\phi) = \Delta(a)(\zeta_\nu \olessthan d^{op}\zeta_\phi). \end{aligned}$$

iii) The first relation was already proven above. Since $[\Delta(A)|\zeta_\psi\rangle_1 A] = [|\alpha\rangle_1 A]$,

$$V(\widehat{\beta} \triangleright \widehat{\beta}) = [V|\widehat{\beta}\rangle_1 A\zeta_\psi] = [V|A\zeta_\psi\rangle_1 A\zeta_\psi] = [\Delta(A)|\zeta_\psi\rangle_1 A\zeta_\psi] = [|\alpha\rangle_1 A\zeta_\psi] = \alpha \triangleright \widehat{\beta}$$

and similarly $V(\widehat{\beta} \triangleright \widehat{\alpha}) = \alpha \triangleright \widehat{\alpha}$. Finally, by ii), for all $b \in B$ and $a, d \in A$

$$\begin{aligned} V|a^{op}\zeta_\psi\rangle_2 d\zeta_\phi b\zeta_\mu = V\big(dr(b)\zeta_\nu \olessthan a^{op}\zeta_\psi\big) &= \Delta(dr(b))(\zeta_\nu \olessthan a^{op}\zeta_\phi) \\ &= \Delta(d)\big(r(b)\zeta_\nu \olessthan a^{op}\zeta_\phi\big) = \Delta(d)|a^{op}\zeta_\phi\rangle_2 \zeta_\phi b\zeta_\mu \end{aligned}$$

and hence $[V|\widehat{\alpha}\rangle_2\alpha] = [V|A^{op}\zeta_\psi\rangle_2 A\zeta_\phi] = [\Delta(A)|A^{op}\zeta_\phi\rangle_2\zeta_\phi] = [\Delta(A)|\beta\rangle_2\zeta_\phi] = [|\beta\rangle_2 A\zeta_\phi] = \widehat{\alpha} \triangleleft \beta$. □

The strong invariance condition on the coinvolution yields the following important inversion formula for the unitary V constructed above.

Theorem 5.6. $V^* = (J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)V(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)$.

Proof. Put $\tilde{V} := (J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)V(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)$. Then for all $a, b, c, d \in A$

$$\begin{aligned} \big\langle a\zeta_\psi \ogreaterthan b^{op}\zeta_{\nu^{-1}} \big| V^*(c^{op}\zeta_{\nu^{-1}} \olessthan d^{op}\zeta_\phi)\big\rangle &= \big\langle \Delta(a)(\zeta_\psi \ogreaterthan b^{op}\zeta_{\nu^{-1}}) \big| c^{op}\zeta_\psi \ogreaterthan d^{op}\zeta_\nu \big\rangle \\ &= \big\langle \zeta_\psi \ogreaterthan \zeta_{\nu^{-1}} \big| \Delta(a^*)(c^{op}{}_\alpha\otimes_\beta 1)(\zeta_\psi \ogreaterthan (b^{op})^* d^{op}\zeta_\nu)\big\rangle \\ &= \big\langle \zeta_{\nu^{-1}} \big| \langle \zeta_\psi|_1 \Delta(a^*)(c^{op}{}_\alpha\otimes_\beta 1)|\zeta_\psi\rangle_1 (db^*)^{op}\zeta_\nu \big\rangle, \\ \big\langle a\zeta_\psi \ogreaterthan b^{op}\zeta_{\nu^{-1}} \big| \tilde{V}(c^{op}\zeta_{\nu^{-1}} \olessthan d^{op}\zeta_\phi)\big\rangle &= \overline{\big\langle (a^*)^{op}\zeta_\psi \ogreaterthan Ib^{op}\zeta_{\nu^{-1}} \big| V(c^*\zeta_{\nu^{-1}} \olessthan Id^{op}\zeta_\phi)\big\rangle} \\ &= \overline{\big\langle \zeta_\psi \ogreaterthan Ib^{op}\zeta_{\nu^{-1}} \big| (a^{op}{}_\alpha\otimes_\beta 1)\Delta(c^*)(\zeta_\psi \ogreaterthan Id^{op}\zeta_\nu)\big\rangle} \\ &= \overline{\big\langle \zeta_\psi \ogreaterthan I\zeta_{\nu^{-1}} \big| (a^{op}{}_\alpha\otimes_\beta 1)\Delta(c^*)(\zeta_\psi \ogreaterthan I(b^{op})^* d^{op}\zeta_\nu)\big\rangle} \\ &= \overline{\big\langle I\zeta_{\nu^{-1}} \big| \langle \zeta_\psi|_1 (a^{op}{}_\alpha\otimes_\beta 1)\Delta(c^*)|\zeta_\psi\rangle_1 I(db^*)^{op}\zeta_\nu \big\rangle} \\ &= \big\langle \zeta_{\nu^{-1}} \big| I\langle \zeta_\psi|_1 (a^{op}{}_\alpha\otimes_\beta 1)\Delta(c^*)|\zeta_\psi\rangle_1 I(db^*)^{op}\zeta_\nu \big\rangle. \end{aligned}$$

Now, the claim follows from condition iii) in Definition 4.8. □

Proof of Theorem 5.4. By Lemma 3.9 ii) and Propositions 3.8 iii) and 5.5 iii), left multiplication by $(J_{\hat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)V^*(J_{\hat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)$ acts on subspaces of $\mathcal{L}(H_\mu, H_{\hat{\beta}}\otimes_\alpha H)$ below as follows:

$$
\begin{aligned}
&[|\alpha\rangle_2\alpha] \xrightarrow{(J_{\hat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)} [|\beta\rangle_2 J\alpha] = |\beta\rangle_2 \hat{\beta} J_\mu \xrightarrow{V^*} [|\hat{\beta}\rangle_1 \beta J_\mu] \xrightarrow{(J_{\hat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)} [|\alpha\rangle_1 I\beta J_\mu] = [|\alpha\rangle_1\alpha],\\
&[|\alpha\rangle_2\beta] \xrightarrow{(J_{\hat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)} [|\beta\rangle_2 J\beta] = [|\beta\rangle_2 \hat{\alpha} J_\mu] \xrightarrow{V^*} [|\alpha\rangle_2 \hat{\alpha} J_\mu] \xrightarrow{(J_{\hat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)} [|\beta\rangle_2 J\hat{\alpha} J_\mu] = [|\beta\rangle_2\beta],\\
&[|\hat{\beta}\rangle_1\hat{\beta}] \xrightarrow{(J_{\hat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)} [|\alpha\rangle_1 I\hat{\beta}] = [|\alpha\rangle_1 \hat{\alpha} J_\mu] \xrightarrow{V^*} [|\hat{\beta}\rangle_1 \hat{\alpha} J_\mu] \xrightarrow{(J_{\hat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)} [|\alpha\rangle_1 I\hat{\alpha} J_\mu] = [|\alpha\rangle_1\hat{\beta}].
\end{aligned}
$$

Now, Theorem 5.6 implies $V(\alpha \triangleleft \alpha) = \alpha \triangleright \alpha$, $V(\beta \triangleleft \alpha) = \beta \triangleleft \beta$, $V(\hat{\beta} \triangleright \hat{\beta}) = \alpha \triangleright \hat{\beta}$. These relations and the relations in Proposition 5.5 iii) are precisely (9).

Let us show that diagram (10) commutes. Let $a, d \in A$ and $\omega \in H$. Then

$$V_{23}V_{12}(a\zeta_\psi \otimes d\zeta_\psi \otimes \omega) = V_{23}(\Delta(a)_{\alpha\triangleright\hat{\beta}}\otimes_\alpha \mathrm{id})(\zeta_\psi \otimes d\zeta_\psi \otimes \omega) = \Delta^{(2)}(a)\big(\zeta_\psi \otimes \Delta(d)(\zeta_\psi \otimes \omega)\big),$$

where $\Delta^{(2)} = (\Delta * \mathrm{id}) \circ \Delta = (\mathrm{id} * \Delta) \circ \Delta$. On the other hand,

$$
\begin{aligned}
V_{12}V_{13}V_{23}(a\zeta_\psi \otimes d\zeta_\psi \otimes \omega) &= V_{12}V_{13}\big(a\zeta_\psi \otimes \Delta(d)(\zeta_\psi \otimes \omega)\big)\\
&= V_{12}\Delta_{13}(a)\big(\zeta_\psi \otimes \Delta(d)(\zeta_\psi \otimes \omega)\big) = \Delta^{(2)}(a)\big(\zeta_\psi \otimes \Delta(d)(\zeta_\psi \otimes \omega)\big)\big),
\end{aligned}
$$

where $\Delta_{13}(a) = \Sigma_{23}(\Delta(a)_{\hat{\beta}\triangleleft\beta}\otimes_\alpha \mathrm{id})\Sigma_{23}$. Since a and d were arbitrary, we can conclude $V_{23}V_{12} = V_{12}V_{13}V_{23}$.

Finally, V is regular because by Theorem 5.6, Lemma 3.9 ii) and Proposition 3.8 iii),

$$
\begin{aligned}
[\langle\alpha|_1 V|\alpha\rangle_2] &= [\langle\alpha|_1(J_{\hat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)V^*(J_{\hat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)|\alpha\rangle_2]\\
&= [I\langle\hat{\beta}|_1 V^*|\beta\rangle_2 J]\\
&= [I\langle\zeta_\psi|_1\Delta(A)|\beta\rangle_2 J]\\
&= [I\langle\zeta_\psi|_1|\beta\rangle_2 AJ] = [I\beta J_\mu \cdot J_\mu\zeta_\psi^* AJ] = [\alpha\alpha^*].
\end{aligned}
$$
□

By Theorem 5.3, the regular C^*-pseudo-multiplicative unitary $(H, \hat{\beta}, \alpha, \beta, V)$ constructed above yields two Hopf C^*-bimodules $(A(V)_H^{\alpha,\beta}, \Delta_V)$ and $(\hat{A}(V)_H^{\hat{\beta},\alpha}, \hat{\Delta}_V)$.

Proposition 5.7. $(A(V)_H^{\alpha,\beta}, \Delta_V) = (A_H^{\alpha,\beta}, \Delta)$.

Proof. We have $A(V) = [\langle\alpha|_1 V|\hat{\beta}\rangle_1] = [\langle\alpha|_1\Delta(A)|\zeta_\psi\rangle_1] = [A\langle\alpha|_1|\zeta_\psi\rangle_1] = [A\rho_\alpha(\alpha^*\zeta_\psi)] = [As(B^{op})] = A$ and $\Delta_V(a) = V(a_{\hat{\beta}}\otimes_\alpha 1)V^* = \Delta(a)$ for all $a \in A$. □

The Hopf C^*-bimodule $(\hat{A}(V)_H^{\hat{\beta},\alpha}, \hat{\Delta}_V)$ will be studied in the next section.

Our first application of the fundamental unitary is to prove that the coinvolution reverses the comultiplication.

Theorem 5.8. $(R_\alpha *_\beta R) \circ \Delta = \mathrm{Ad}_\Sigma \circ \Delta \circ R$.

The proof involves the following formulas:

Lemma 5.9. *i)* $\Delta(\langle\xi|_1 V|\xi'\rangle_1) = \langle\xi|_1 V_{12}V_{13}|\xi'\rangle_1$ *for all* $\xi \in \alpha, \xi' \in \hat{\beta}$.

ii) $R(\langle\xi|_1 V|\xi'\rangle_1) = \langle J\xi' J_\mu|_1 V|J\xi J_\mu\rangle_1$ *for all* $\xi \in \alpha, \xi' \in \hat{\beta}$.

Proof. i) For all $\xi \in \alpha$, $\xi' \in \hat{\beta}$, we have that $\Delta(\langle\xi|_1 V|\xi'\rangle_1) = V((\langle\xi|_1 V|\xi\rangle_1)_{\hat{\beta}}\otimes_\alpha 1)V^* = \langle\xi|_1 V_{23}^* V_{12}V_{23}|\xi'\rangle_1 = \langle\xi|_1 V_{12}V_{13}|\xi'\rangle_1$; see also [17, Lemma 4.13].

ii) By Lemma 3.9 and Theorem 5.6, we have that $R(\langle\xi|_1 V|\xi'\rangle_1) = I\langle\xi'|_1 V^*|\xi\rangle_1 I = \langle J\xi' J_\mu|_1(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)^* V^*(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)^*|J\xi J_\mu\rangle_1 = \langle J\xi' J_\mu|_1 V|J\xi J_\mu\rangle_1$ for all $\xi \in \alpha$, $\xi' \in \hat{\beta}$ □

Proof of Theorem 5.8. Let $\xi \in \alpha$ and $\xi' \in \widehat{\beta}$. By Lemma 5.9 i),

$$\begin{aligned}(\mathrm{Ad}_{\Sigma} \circ (R_{\alpha} *_{\beta} R) \circ \Delta)\big(\langle \xi|_1 V|\xi'\rangle_1\big) &= (\mathrm{Ad}_{\Sigma} \circ (R_{\alpha} *_{\beta} R))\big(\langle \xi|_1 V_{12}V_{13}|\xi'\rangle_1\big) \\ &= \mathrm{Ad}_{\Sigma}\big((I_{\alpha} \underset{J_\mu}{\otimes}{}_{\beta} I)^* \langle \xi'|_1 V_{13}^* V_{12}^* |\xi\rangle_1 (I_{\alpha} \underset{J_\mu}{\otimes}{}_{\beta} I)\big).\end{aligned}$$

By Lemma 3.9 ii), we can rewrite this expression in the form

$$\mathrm{Ad}_{\Sigma}\big(\langle J\xi' J_\mu|_1 (J_{\widehat{\beta}} \underset{J_\mu}{\otimes}{}_{\alpha \triangleright \alpha} (I_{\alpha} \underset{J_\mu}{\otimes}{}_{\beta} I)) V_{13}^* V_{12}^* (J_{\alpha} \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} I_{\alpha} \underset{J_\mu}{\otimes}{}_{\beta} I)^* |J\xi J_\mu\rangle_1\big).$$

Two applications of Lemma 3.9 iii) and Theorem 5.6 and an application of Lemma 5.9 ii) show that this expression is equal to

$$\begin{aligned}\mathrm{Ad}_{\Sigma}\big(\langle J\xi' J_\mu|_1 V_{13} V_{12} |J\xi J_\mu\rangle_1\big) &= \langle J\xi' J_\mu|_1 V_{12} V_{13} |J\xi J_\mu\rangle_1 \\ &= \Delta\big(\langle J\xi' J_\mu|_1 V|J\xi J_\mu\rangle_1\big) = \Delta(R(\langle \xi|_1 V|\xi'\rangle_1)).\end{aligned}$$ □

A second fundamental unitary Like in the theory of locally compact quantum groups, we can associate to a given compact C^*-quantum groupoid besides $(H, \widehat{\beta}, \alpha, \beta, V)$ a second C^*-pseudo-multiplicative unitary $(H, \widehat{\alpha}, \beta, \alpha, W)$ as follows.

Theorem 5.10. *There exists a regular C^*-pseudo-multiplicative unitary $(H, \alpha, \beta, \widehat{\alpha}, W)$ such that $W^*|a\zeta_\phi\rangle_2 = \Delta(a)|\zeta_\phi\rangle_2$ for all $a \in A$. Moreover,*

$$W = \Sigma(I_{\widehat{\beta}} \underset{J_\mu}{\otimes}{}_{\alpha} I) V^* (I_{\beta} \underset{J_\mu}{\otimes}{}_{\alpha} I)\Sigma = (I_{\alpha} \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} I) V^{op} (I_{\alpha} \underset{J_\mu}{\otimes}{}_{\beta} I).$$

Proof. Let $a \in A$ and $\xi \in H$. Since $I\zeta_\psi J_\mu = \zeta_\phi$ and $\Delta(R(a)^*) = \Sigma(I_{\alpha} \underset{J_\mu}{\otimes}{}_{\beta} I)\Delta(a)(I_{\beta} \underset{J_\mu}{\otimes}{}_{\alpha} I)\Sigma$,

$$\begin{aligned}\Sigma(I_{\beta} \underset{J_\mu}{\otimes}{}_{\alpha} I)^* V (I_{\widehat{\beta}} \underset{J_\mu}{\otimes}{}_{\alpha} I)^* \Sigma(\xi \circledcirc a\zeta_\phi) &= \Sigma(I_{\alpha} \underset{J_\mu}{\otimes}{}_{\beta} I) V (Ia\zeta_\phi J_\mu \circledcirc I\xi) \\ &= \Sigma(I_{\alpha} \underset{J_\mu}{\otimes}{}_{\beta} I) V (R(a)^* \zeta_\psi \circledcirc I\xi) \\ &= \Sigma(I_{\alpha} \underset{J_\mu}{\otimes}{}_{\beta} I) \Delta(R(a)^*)(\zeta_\psi \circledcirc I\xi) \\ &= \Delta(a)(I_{\alpha} \underset{J_\mu}{\otimes}{}_{\beta} I)\Sigma(\zeta_\psi \circledcirc I\xi) = \Delta(a)(\xi \circledcirc \zeta_\phi).\end{aligned}$$

Therefore, the unitary $W = \Sigma(I_{\widehat{\beta}} \underset{J_\mu}{\otimes}{}_{\alpha} I) V^* (I_{\beta} \underset{J_\mu}{\otimes}{}_{\alpha} I)\Sigma$ satisfies $W^*|a\zeta_\phi\rangle_2 = \Delta(a)|\zeta_\phi\rangle_2$ for all $a \in A$. Since $(H, \beta, \alpha, \widehat{\beta}, V^{op})$ is a regular C^*-pseudo-multiplicative unitary, so is $(H, \alpha, \beta, \widehat{\alpha}, W)$. □

The passage to the setting of von Neumann algebras In this paragraph, we indicate how every compact C^*-quantum groupoid can be completed to a measurable quantum groupoid in the sense of Lesieur [9] and Enock [5]. We assume some familiarity with [9] or [5].

Let μ be a faithful KMS-state on a unital C^*-algebra B. Then the state $\tilde{\mu}$ on $N := B'' \subseteq \mathcal{L}(H_\mu)$ given by $y \mapsto \langle \zeta_\mu | y\zeta_\mu\rangle$ is the unique normal extension of μ and is faithful because ζ_μ is cyclic for $\pi_{\mu^{op}}(B^{op}) \subseteq N'$. Evidently, the Hilbert space $H_{\tilde{\mu}} := H_\mu$ and the map $\Lambda_{\tilde{\mu}} \colon N \to H_{\tilde{\mu}}$, $y \mapsto y\zeta_\mu$, form a GNS-representation for $\tilde{\mu}$.

Lemma 5.11. *Let μ be a faithful KMS-state on a unital C^*-algebra B and let (r, ϕ) be a μ-module structure on a unital C^*-algebra A. We put $N := B'' \subseteq \mathcal{L}(H_\mu)$, $M := A'' \subseteq \mathcal{L}(H_\nu)$, and use the notation of Lemma 2.2.*

i) r extends uniquely to a normal embedding $\tilde{r} \colon N \to M$.

ii) ϕ extends uniquely to a normal completely positive map $\tilde{\phi} \colon M \to N$, and $\tilde{\nu} = \tilde{\mu} \circ \tilde{\phi}$. If ϕ is faithful, so is $\tilde{\phi}$.

iii) $\zeta y = \tilde{r}(y)\zeta$, $\zeta^* x = \tilde{\phi}(x)\zeta^*$, $\tilde{\phi}(x\tilde{r}(y)) = \tilde{\phi}(x)\tilde{r}(y)$ *for all* $x \in M$, $y \in N$.

Proof. i) Uniqueness is clear. Put $\widehat{\alpha} := [A\zeta]$. By Lemmas 3.2 and 3.7, $H_\nu \cong \widehat{\alpha} \ominus H_\mu$. Hence, we can define a $*$-homomorphism $\tilde{r}\colon N \to \mathcal{L}(\alpha \ominus H_\mu) \cong \mathcal{L}(H_\nu)$ by $x \mapsto \mathrm{id}_{\widehat{\alpha}} \ominus x$. By Lemma 3.7 i), $\tilde{r}$ extends r, and routine arguments show that $\tilde{r}$ is normal and injective.

ii) $\tilde{\phi}$ is uniquely determined by $\tilde{\phi}(x) = \zeta^* x \zeta$ for all $x \in M$, and clearly $\tilde{\nu}(x) = \langle \zeta_\nu | x\zeta_\nu \rangle = \langle \zeta_\mu | \zeta^* x \zeta \zeta_\mu \rangle = (\tilde{\mu} \circ \tilde{\phi})(x)$ for all $x \in M$. If ϕ is faithful, so is ν and, since $\tilde{\mu}$ is faithful and $\tilde{\nu} = \tilde{\mu} \circ \tilde{\phi}$, also $\tilde{\phi}$ is faithful.

iii) Use Lemma 2.2 iii) and the fact that $\tilde{r}$, $\tilde{\phi}$ are normal extensions of r, ϕ. ☐

Let $(B, \mu, A, r, \phi, s, \psi, \delta, R, \Delta)$ be a compact C^*-quantum groupoid. We keep the notation introduced before and put

$$N := B'' \subseteq \mathcal{L}(H_\mu), \qquad N^{op} := (B^{op})'' = N' \subseteq \mathcal{L}(H_\mu), \qquad M := A'' \subseteq \mathcal{L}(H).$$

By the previous remarks, the maps μ, r, ϕ, s, ψ have unique normal extensions

$$\tilde{\mu}\colon N \to \mathbb{C}, \quad \tilde{r}\colon N \to M, \quad \tilde{\phi}\colon M \to N, \quad \tilde{s}\colon N' \to M, \quad \tilde{\psi}\colon M \to N'.$$

Before we can extend the comultiplication Δ from A to M, we need to recall the definition of the fiber product of von Neumann algebras [14] and the underlying relative tensor product of Hilbert spaces [15]; a reference is also [19, §10]. The relative tensor product of H with itself, taken with respect to $\tilde{s}, \tilde{r}$ and $\tilde{\mu}$, is defined as follows. Put

$$D(H_{\tilde{r}}; \tilde{\mu}) := \{\eta \in H \,|\, \exists C > 0 \forall y \in N : \|\tilde{r}(y)\eta\| \leqslant C\|y\zeta_\mu\|\}.$$

Evidently, an element $\eta \in H$ belongs to $D(H_{\tilde{r}}; \tilde{\mu})$ if and only if the map $N\zeta_\mu \to H$ given by $y\zeta_\mu \mapsto \tilde{r}(y)\eta$ extends to a bounded linear map $L(\eta)\colon H_\mu \to H$, and $L(\eta)^* L(\eta') \in N'$ for all $\eta, \eta' \in D(H_{\tilde{r}}; \tilde{\mu})$. The relative tensor product $H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H$ is the separated completion of the algebraic tensor product $H \odot D(H_{\tilde{r}}; \tilde{\mu})$ with respect to the sesquilinear form defined by

$$\langle \omega \odot \eta | \omega' \odot \eta' \rangle = \langle \omega | \tilde{s}(L(\eta)^* L(\eta'))\omega' \rangle \quad \text{for all } \omega, \omega' \in H,\ \eta, \eta' \in D(H_{\tilde{r}}; \tilde{\mu}).$$

We denote the image of an element $\omega \odot \eta$ in $H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H$ by $\omega {}_{\tilde{s}}\otimes_{\tilde{r}} \eta$.

Lemma 5.12. *i)* $a^{op}\zeta_\nu \in D(H_{\tilde{r}}; \tilde{\mu})$ *and* $L(a^{op}\zeta_\nu) = a^{op}\zeta_\phi \in \beta$ *for all* $a \in A$.

ii) There exist inverse isomorphisms

$$\Phi_{\alpha,\beta}\colon H {}_\alpha\otimes_\beta H \cong H_{\rho_\alpha} \oslash \beta \to H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H, \qquad \Psi_{\alpha,\beta}\colon H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H \to \alpha \ominus_{\rho_\beta} H \cong H {}_\alpha\otimes_\beta H$$

such that for all $\omega \in H$, $a \in A$, $\xi \in \alpha$, $\eta \in D(H_{\tilde{r}}, \tilde{\mu})$, $\zeta \in H_\mu$

$$\Phi_{\alpha,\beta}(\omega \oslash a^{op}\zeta_\phi) = \omega {}_{\tilde{s}}\otimes_{\tilde{r}} a^{op}\zeta_\nu, \qquad \Psi_{\alpha,\beta}(\xi\zeta {}_{\tilde{s}}\otimes_{\tilde{r}} \eta) = \xi \ominus L(\eta)\zeta.$$

Proof. i) For all $a \in A$, $y \in N$, $\tilde{r}(y)a^{op}\zeta_\nu = a^{op}\tilde{r}(y)\zeta_\phi\zeta_\mu = a^{op}\zeta_\phi y\zeta_\mu$. The claims follow.

ii) The formulas for $\Phi_{\alpha,\beta}$ and $\Psi_{\alpha,\beta}$ define isometries because for all $\omega, a, \xi, \eta, \zeta$ as above,

$$\|\omega \oslash a^{op}\zeta_\phi\|^2 = \langle \omega | \rho_\alpha(\zeta_\phi^* (a^{op})^* a^{op}\zeta_\phi)\omega \rangle = \langle \omega | \tilde{s}(L(a^{op}\zeta_\nu)^* L(a^{op}\zeta_\nu))\omega \rangle = \|\omega {}_{\tilde{s}}\otimes_{\tilde{r}} a^{op}\zeta_\nu\|^2$$

and

$$\begin{aligned}\|\xi\zeta {}_{\tilde{s}}\otimes_{\tilde{r}} \eta\|^2 = \langle \xi\zeta | \tilde{s}(L(\eta)^* L(\eta))\xi\zeta \rangle &= \langle \zeta | \xi^*\xi L(\eta)^* L(\eta)\zeta \rangle \\ &= \langle \zeta | L(\eta)^* \rho_\beta(\xi^*\xi^*) L(\eta)\zeta \rangle = \|\xi \ominus L(\eta)\zeta\|^2.\end{aligned}$$

Moreover, $\Psi_{\alpha,\beta} \circ \Phi_{\alpha,\beta} = \mathrm{id}$ because for all a, ξ, ζ as above,

$$(\Psi_{\alpha,\beta} \circ \Phi_{\alpha,\beta})(\xi\zeta \oslash a^{op}\zeta_\phi) = \xi \ominus L(a^{op}\zeta_\nu)\zeta = \xi \ominus a^{op}\zeta_\phi\zeta \equiv \xi\zeta \oslash a^{op}\zeta_\phi. \qquad ☐$$

We identify $H_\alpha \otimes_\beta H$ with $H_{\tilde{s}} \underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H$ via $\Phi_{\alpha,\beta}$ and $\Psi_{\alpha,\beta}$ without further notice.

The fiber product $M_{\tilde{s}} \underset{\tilde{\mu}}{*}{}_{\tilde{r}} M$ is defined as follows. One has $\tilde{r}(N)' D(H_{\tilde{r}}; \tilde{\mu}) \subseteq D(H_{\tilde{r}}; \tilde{\mu})$, and for each $x, x' \in M'$, there exists a well-defined operator $x_{\tilde{s}} \underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} x' \subseteq \mathcal{L}(H_{\tilde{s}} \underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H)$ such that $(x_{\tilde{s}} \underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} x')(\omega_{\tilde{s}} \otimes_{\tilde{r}} \eta) = x\omega_{\tilde{s}} \otimes_{\tilde{r}} x\eta$ for all $\omega \in H$, $\eta \in D(H_{\tilde{r}}; \tilde{\mu})$. Now,

$$M_{\tilde{s}} \underset{\tilde{\mu}}{*}{}_{\tilde{r}} M = (M'_{\tilde{s}} \otimes_{\tilde{r}} M')' \subseteq \mathcal{L}(H_{\tilde{s}} \underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H).$$

Lemma 5.13. *Δ extends to a normal $*$-homomorphism $\tilde{\Delta}\colon M \to M_{\tilde{s}} \underset{\tilde{\mu}}{*}{}_{\tilde{r}} M$.*

Proof. Simply define $\tilde{\Delta}$ by $\tilde{\Delta}(x) := V(x \otimes \mathrm{id}_\alpha)V^*$ for all $x \in M$. □

The notion of a measurable quantum groupoid was first defined in [9]; later, the definition was changed in [5, §6].

Theorem 5.14. *$(N, M, \tilde{r}, \tilde{s}, \tilde{\Delta}, \tilde{\phi}, \tilde{\psi}, \tilde{\mu})$ is a measurable quantum groupoid.*

Proof. First, one has to check that $(N, M, \tilde{r}, \tilde{s}, \tilde{\Delta})$ is a Hopf-bimodule; this follows from the definition of $\tilde{\Delta}$ and the fact that V is a C^*-pseudo-multiplicative unitary.

Second, one has to check that $\tilde{\phi}$ and $\tilde{\psi}$ are left- and right-invariant, respectively. This follows from the fact that these maps are normal extensions of ϕ and ψ, which are left- and right-invariant, respectively.

Finally, one has to check that the modular automorphism groups of $\tilde{\nu} = \tilde{\mu} \circ \tilde{\phi}$ and $\tilde{\nu}^{-1} = \tilde{\mu}^{op} \circ \tilde{\psi}$ commute, but this follows from the fact that $\tilde{\nu}^{-1} = \tilde{\nu}_{\delta^{1/2}}$. □

6 Supplements on C^*-pseudo-multiplicative unitaries

In this section, we interrupt our discussion of compact C^*-quantum groupoids and study several properties C^*-pseudo-multiplicative unitaries that shall prove useful later. The corresponding properties for multiplicative unitaries were introduced and studied in [1]. Throughout this section, let μ be a faithful KMS-state on a unital C^*-algebra B.

Fixed and cofixed elements for a C^*-pseudo-multiplicative unitary We shall study elements with the following property:

Definition 6.1. *Let $(H, \hat{\beta}, \alpha, \beta, V)$ be a C^*-pseudo-multiplicative unitary over μ. A* fixed element *for V is an element $\eta \in \widehat{\beta} \cap \alpha$ satisfying $V|\eta\rangle_1 = |\eta\rangle_1 \in \mathcal{L}(H, H_\alpha \otimes_\beta H)$. A* cofixed element *for V is an element $\xi \in \alpha \cap \beta$ satisfying $V|\xi\rangle_2 = |\xi\rangle_2 \in \mathcal{L}(H, H_\alpha \otimes_\beta H)$. We denote the set of all fixed/cofixed elements for V by $\mathrm{Fix}(V)/\mathrm{Cofix}(V)$.*

Til the end of this paragraph, let $(H, \hat{\beta}, \alpha, \beta, V)$ be a C^*-pseudo-multiplicative unitary over μ.

Remarks 6.2. i) $\mathrm{Fix}(V) = \mathrm{Cofix}(V^{op})$ and $\mathrm{Cofix}(V) = \mathrm{Fix}(V^{op})$.

ii) $\mathrm{Fix}(V)^*\mathrm{Fix}(V)$ and $\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)$ are contained in $B \cap B^{op} = Z(B)$.

iii) Since $\mathrm{Fix}(V) \subseteq \widehat{\beta} \cap \alpha$, we have $\rho_\alpha(B^{op})\mathrm{Fix}(V) = \mathrm{Fix}(V)B^{op} \subseteq \widehat{\beta}$ and $\rho_{\widehat{\beta}}(B)\mathrm{Fix}(V) = \mathrm{Fix}(V)B \subseteq \alpha$. Likewise, $\rho_\beta(B)\mathrm{Cofix}(V) \subseteq \alpha$ and $\rho_\alpha(B^{op})\mathrm{Cofix}(V) \subseteq \beta$.

Lemma 6.3. *i) $\langle\xi|_2 V|\xi'\rangle_2 = \rho_\alpha(\xi^*\xi') = \rho_{\widehat{\beta}}(\xi^*\xi')$ for all $\xi, \xi' \in \mathrm{Cofix}(V)$, and $\langle\eta|_1 V|\eta'\rangle_1 = \rho_\beta(\eta^*\eta') = \rho_\alpha(\eta^*\eta')$ for all $\eta, \eta' \in \mathrm{Fix}(V)$.*

ii) $\rho_{\widehat{\beta}}(B)\mathrm{Cofix}(V) \subseteq \mathrm{Cofix}(V)$ and $\rho_\beta(B)\mathrm{Fix}(V) \subseteq \mathrm{Fix}(V)$.

iii) $[EE^*E] = E$ for $E \in \{\mathrm{Cofix}(V), \mathrm{Fix}(V)\}$.

iv) $[\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)]$ and $[\mathrm{Fix}(V)^*\mathrm{Fix}(V)]$ are C^*-subalgebras of $Z(B)$.

Proof. We only prove the assertions on $\mathrm{Cofix}(V)$; the other assertions follow similarly.

i) For all $\xi, \xi' \in \mathrm{Cofix}(V)$ and $\zeta \in H$, $\langle \xi|_2 V|\xi'\rangle_2 \zeta = \langle \xi|_2|\xi'\rangle_2\zeta = \rho_\alpha(\xi^*\xi')\zeta$ and $(\langle\xi|_2 V|\xi'\rangle_2)^*\zeta = \langle \xi'|_2|\xi\rangle_2\zeta = \rho_{\widehat{\beta}}(\xi^*\xi')^*\zeta$.

ii) Let $b \in B$ and $\xi \in \mathrm{Cofix}(V)$. Then $\rho_{\widehat{\beta}}(b)\xi \in \rho_{\widehat{\beta}}(B)\beta \cap \rho_{\widehat{\beta}}(B)\alpha \subseteq \beta \cap \alpha$, and $V|\rho_{\widehat{\beta}}(b)\xi\rangle_2 = V\rho_{(\widehat{\beta} \triangleright \widehat{\beta})}(b)|\xi\rangle_2 = \rho_{(\alpha \triangleright \widehat{\beta})}(b)V|\xi\rangle_2 = \rho_{(\alpha\triangleright\widehat{\beta})}(b)|\xi\rangle_2 = |\rho_{\widehat{\beta}}(b)\xi\rangle_2$ because $V(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}$.

iii) Let $\xi, \xi', \xi'' \in \mathrm{Cofix}(V)$. Then $\rho_\alpha(\xi'^*\xi'') = \rho_{\widehat{\beta}}(\xi'^*\xi'')$ by i) and hence $V|\xi\xi'^*\xi''\rangle_2 = V|\xi\rangle_2\rho_{\widehat{\beta}}(\xi'^*\xi'') = |\xi\rangle_2\rho_\alpha(\xi'^*\xi'') = |\xi\xi'^*\xi''\rangle_2$ in $\mathcal{L}(H, H_\alpha \otimes_\beta H)$.

iv) Immediate from iii). ☐

Definition 6.4. *We call $(H, \hat{\beta}, \alpha, \beta, V)$ or briefly V* étale *if $\eta^*\eta = \mathrm{id}_{\mathfrak{K}}$ for some $\eta \in \mathrm{Fix}(V)$, and* compact *if $\xi^*\xi = \mathrm{id}_{\mathfrak{K}}$ for some $\xi \in \mathrm{Cofix}(V)$.*

Remarks 6.5. i) By Remark 6.2, V is étale/compact if and only if V^{op} is compact/étale.

ii) If V is compact, then $\mathrm{id}_H \in \widehat{A}(V)$; if V is étale, then $\mathrm{id}_H \in A(V)$. This follows directly from Lemma 6.3.

The following observation supports the plausibility of the assumptions in condition i) of Definition 4.8:

Remark 6.6. Let $(H, \hat{\beta}, \alpha, \beta, V)$ be a regular C^*-pseudo-multiplicative unitary over μ. If $\xi_0 \in \mathrm{Fix}(V)$ and $\widehat{\beta} = [A(V)\xi_0]$, then by [16, Lemma 5.8],

$$\begin{aligned}[\Delta_V(A(V))|\xi_0\rangle_1 A(V)] &= [V(A(V)\,{}_{\widehat{\beta}}\!\otimes_\alpha 1)V^*|\xi_0\rangle_1 A(V)] \\ &= [V|A(V)\xi_0\rangle_1 A(V)] = [V|\widehat{\beta}\rangle_1 A(V)] = [|\alpha\rangle_1 A(V)].\end{aligned}$$

Likewise, if $\eta_0 \in \mathrm{Cofix}(V)$ and $\alpha = [\widehat{A}(V)\eta_0]$, then $[\widehat{\Delta}_V(\widehat{A}(V))|\eta_0\rangle_2\widehat{A}(V)] = [|\beta\rangle_2\widehat{A}(V)]$.

The (co)fixed vectors of the C^*-pseudo-multiplicative unitaries introduced in Theorems 5.4 and 5.10 are easily determined:

Proposition 6.7. *Let $(B, \mu, A, r, \phi, s, \psi, \delta, R, \Delta)$ be a compact C^*-quantum groupoid.*

i) The associated C^*-pseudo-multiplicative unitary $(H, \widehat{\beta}, \alpha, \beta, V)$ is compact and $\mathrm{Fix}(V) = [r(B)\zeta_\psi]$.

ii) The associated C^*-pseudo-multiplicative unitary $(H, \alpha, \beta, \widehat{\alpha}, W)$ is étale and $\mathrm{Cofix}(W) = [s(B^{op})\zeta_\phi]$.

Proof. i) Evidently, $\zeta_\psi \in \mathrm{Fix}(V)$, and by Lemma 6.3 ii), $[r(B)\zeta_\psi] = [\rho_\beta(B)\zeta_\psi] \subseteq \mathrm{Fix}(V)$. Conversely, if $\eta_0 \in \mathrm{Fix}(V)$, then $\eta_0 \in \widehat{\beta} = [A\zeta_\psi]$ and therefore

$$\begin{aligned}\eta_0 = \rho_\alpha(\zeta_\phi^*\zeta_\phi)\eta_0 = \langle\zeta_\phi|_2|\eta_0\rangle_1\zeta_\phi &= \langle\zeta_\phi|_2 V|\eta_0\rangle_1\zeta_\phi \\ &\in [\langle\zeta_\phi|_2\Delta(A)|\zeta_\psi\rangle_1\zeta_\phi] \\ &= [\langle\zeta_\phi|_2\Delta(A)|\zeta_\phi\rangle_2\zeta_\psi] = [r(\phi(A))\zeta_\psi] = [r(B)\zeta_\psi].\end{aligned}$$

ii) This follows easily from i) and the relation $W = (I_\alpha \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} I)V^{op}(I_\alpha \underset{J_\mu}{\otimes}{}_\beta I)$. ☐

Haar weights and counits obtained from (co)fixed elements

Fixed and cofixed elements for a C^*-pseudo-multiplicative unitary yield bounded Haar weights and bounded counits on the legs as follows:

Theorem 6.8. *Let $(H, \hat{\beta}, \alpha, \beta, V)$ be a well-behaved C^*-pseudo-multiplicative unitary over μ.*

i) Assume that $(H, \hat{\beta}, \alpha, \beta, V)$ is étale and that $\eta_0 \in \mathrm{Fix}(V)$ satisfies $\eta_0^*\eta_0 = \mathrm{id}_{H_\mu}$.

(a) *A bounded left counit* $\hat{\epsilon}$ *for* $(\hat{A}(V)_H^{\hat{\beta},\alpha}, \hat{\Delta}_V)$ *is given by* $\hat{\epsilon}(\hat{a}) := \eta_0^* \hat{a} \eta_0$. *For all* $\eta \in \beta, \xi \in \alpha$, *we have* $\hat{\epsilon}(\langle \eta|_2 V|\xi\rangle_2) = \eta^*\xi$. *In particular,* $\hat{\epsilon}$ *does not depend on the choice of* η_0, $\hat{\epsilon}(\hat{A}(V)) = [\beta^*\alpha]$, *and* $[\beta^*\alpha]$ *is a* C^**-algebra. If* V *is regular, then* $\hat{\epsilon}$ *is a bounded counit.*

(b) *A bounded right Haar weight* ψ *for* $(A(V)_H^{\alpha,\beta}, \Delta_V)$ *is given by* $\psi(a) := \eta_0^* a \eta_0$.

ii) *Assume that* $(H, \hat{\beta}, \alpha, \beta, V)$ *is compact and that* $\xi_0 \in \mathrm{Cofix}(V)$ *satisfies* $\xi_0^*\xi_0 = \mathrm{id}_{H_\mu}$.

(a) *A bounded right counit* ϵ *for* $(A(V)_H^{\alpha,\beta}, \Delta)$ *is given by* $\epsilon(a) := \xi_0^* a \xi_0$. *For all* $\eta \in \alpha, \xi \in \hat{\beta}$, *we have* $\epsilon(\langle \eta|_1 V|\xi\rangle_1) = \eta^*\xi$. *In particular,* ϵ *does not depend on the choice of* η_0, $\epsilon(A(V)) = [\alpha^*\hat{\beta}]$, *and* $[\alpha^*\hat{\beta}]$ *is a* C^**-algebra. If* V *is regular, then* $\hat{\epsilon}$ *is a bounded counit.*

(b) *A bounded left Haar weight* $\hat{\phi}$ *for* $(\hat{A}(V)_H^{\hat{\beta},\alpha}, \hat{\Delta}_V)$ *is given by* $\hat{\phi}(\hat{a}) := \xi_0^* \hat{a} \xi_0$.

Proof. We only prove the assertions concerning $(\hat{A}(V)_H^{\hat{\beta},\alpha}, \hat{\Delta}_V)$, the corresponding assertions for $(A(V)_H^{\alpha,\beta}, \Delta_V)$ follow by replacing V by V^{op}.

i) (a) Evidently, $\hat{\epsilon}$ is a completely positive contraction. Let $\eta, \eta' \in \beta$ and $\xi, \xi' \in \alpha$. Then

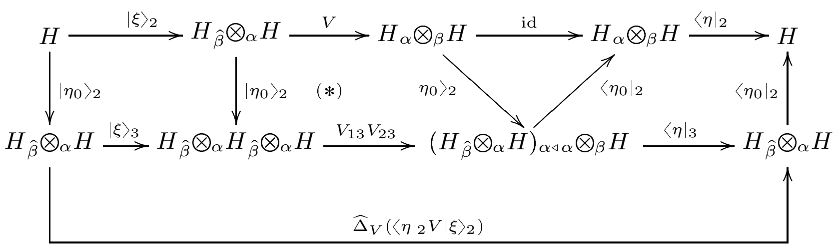
$$\langle \eta|_2 V|\xi\rangle_2 \eta_0 = \langle \eta|_2 |\eta_0\rangle_1 \xi = \eta_0 \eta^* \xi = \eta_0 \eta_0^* \langle \eta|_2 V|\xi\rangle_2 \eta_0 = \eta_0 \hat{\epsilon}(\eta|_2 V|\xi\rangle_2). \tag{11}$$

Now, $\hat{\epsilon}$ is a $*$-homomorphism and $\hat{\epsilon}(\hat{A}(V)) = [\beta^*\alpha]$ because

$$\eta_0^* \langle \eta|_2 V|\xi\rangle_2 \eta_0 \eta_0^* \langle \eta'|_2 V|\xi'\rangle_2 \eta_0 = \eta_0^* \langle \eta|_2 V|\xi\rangle_2 \langle \eta'|_2 V|\xi'\rangle_2 \eta_0, \quad \hat{\epsilon}(\langle \eta|_2 V|\xi\rangle_2) = \eta_0^* \eta_0 \eta^* \xi = \eta^*\xi.$$

Since $[\eta_0^*\alpha] = B$ and $[\eta_0^*\hat{\beta}] = B^{op}$, the map $\hat{\epsilon}$ is morphism of C^*-(μ, μ^{op})-algebras $\hat{A}(V)_H^{\hat{\beta},\alpha}$ and $[\beta^*\alpha]_{H_\mu}^{B^{op},B}$. It is a left counit because $(\hat{\epsilon} * \mathrm{id})(\hat{\Delta}(\hat{a})) = \langle \eta_0|_1 V^*(1_\alpha \otimes_\beta \hat{a}) V|\eta_0\rangle_1 = \langle \eta_0|_1 (1_\alpha \otimes_\beta \hat{a})|\eta_0\rangle_1 = \hat{a}$ for all $\hat{a} \in \hat{A}(V)$.

Assume that V is regular, and consider the following diagram:



The lower cell commutes by the proof of [17, Lemma 4.13], cell (*) commutes because $V_{23}|\eta_0\rangle_2 = |\eta_0\rangle_2$, and the other cells commute as well. Since $\eta \in \beta$ and $\xi \in \alpha$ were arbitrary, $\hat{\epsilon}$ is a bounded right counit.

ii) (b) By Remark 6.2 i), $[\xi_0^* \hat{A}(V) \xi_0] = [\xi_0^* \rho_\alpha(B^{op}) \hat{A}(V) \rho_\alpha(B^{op}) \xi_0] \subseteq [\beta^* \hat{A}(V) \beta] \subseteq B^{op}$. Hence, the given formula defines a completely positive contraction $\hat{\phi}\colon \hat{A}(V) \to B^{op}$. Since $\rho_\alpha(b^{op})\xi_0 = \xi_0 b^{op}$ for all $b^{op} \in B^{op}$, condition i) of Definition 4.2 holds. Condition ii) holds because for all $\hat{a} \in \hat{A}(V)$ and $\eta, \eta' \in \hat{\beta}$,

$$\begin{aligned} \xi_0^* \langle \eta|_1 \hat{\Delta}_V(\hat{a}) |\eta'\rangle_1 \xi_0 &= \eta^* \langle \xi_0|_2 V^* (\mathrm{id}_{\,\alpha} \otimes_\beta \hat{a}) V |\xi_0\rangle_2 \eta' \\ &= \eta^* \langle \xi_0|_2 (\mathrm{id}_{\,\alpha} \otimes_\beta \hat{a}) |\xi_0\rangle_2 \eta' = \eta^* \rho_\alpha(\xi_0^* \hat{a} \xi_0) \eta'. \end{aligned}$$

□

Balanced C^*-pseudo-multiplicative unitaries and C^*-pseudo-Kac systems

Weak C^*-pseudo-Kac systems were introduced in [16] as a framework to construct reduced crossed products for coactions of Hopf C^*-bimodules. Let us briefly recall the definition.

Definition 6.9 ([16])**.** *A* balanced C^*-pseudo-multiplicative unitary over μ *is a tuple* $(H, \alpha, \hat{\alpha}, \beta, \hat{\beta}, U, V)$, *where* $(H, \alpha, \hat{\alpha}, \beta, \hat{\beta})$ *is a* C^**-*$(\mu, \mu, \mu^{op}, \mu^{op})$*-module,* $V\colon H_{\hat{\beta}}\otimes_\alpha H \to H_\alpha \otimes_\beta H$ *is a unitary and* $U\colon H \to H$ *is a symmetry satisfying the following conditions:*

i) $U\alpha = \widehat{\alpha}$ *and* $U\beta = \widehat{\beta}$*;*

ii) $(H, \widehat{\beta}, \alpha, \beta, V)$*,* $(H, \widehat{\alpha}, \widehat{\beta}, \alpha, \check{V})$*,* $(H, \alpha, \beta, \widehat{\alpha}, \widehat{V})$ *are well-behaved* C^**-pseudo-multiplicative unitaries, where* $\check{V}$ *and* $\widehat{V}$ *are defined by*

$$\check{V} := \Sigma(1_\alpha \otimes_\beta U)V(1_{\widehat{\beta}} \otimes_{\widehat{\alpha}} U)\Sigma \colon H_{\widehat{\alpha}} \otimes_{\widehat{\beta}} H \to H_{\widehat{\beta}} \otimes_\alpha H,$$
$$\widehat{V} := \Sigma(U_\alpha \otimes_\beta 1)V(U_\beta \otimes_\alpha 1)\Sigma \colon H_\alpha \otimes_\beta H \to H_\beta \otimes_{\widehat{\alpha}} H.$$

A weak C^*-pseudo-Kac system over μ *is a balanced* C^**-pseudo-multiplicative unitary* $(H, \alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ *such that* $(H, \beta, \alpha, \widehat{\beta}, V)$ *is well-behaved and* $[\widehat{A}(V), U\widehat{A}(V)U] = 0 = [A(V), UA(V)U]$*. A weak* C^**-pseudo-Kac system* $(H, \alpha, \beta, \widehat{\alpha}, \widehat{\beta}, V, U)$ *is a* C^*-pseudo-Kac system *if* $(H, \widehat{\beta}, \alpha, \beta, V)$*,* $(H, \widehat{\alpha}, \widehat{\beta}, \alpha, \check{V})$*,* $(H, \alpha, \beta, \widehat{\alpha}, \widehat{V})$ *are regular and* $\big(\Sigma(1_\alpha \otimes_\beta U)V\big)^3 = \mathrm{id} \in \mathcal{L}(H_{\widehat{\beta}} \otimes_\alpha H)$*.*

Let $(H, \alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ be a balanced C^*-pseudo-multiplicative unitary over μ. Then by [16, Proposition 3.3],

$$\begin{aligned} \widehat{A}(\check{V}) &= \mathrm{Ad}_U(A(V)), \quad \widehat{\Delta}_{\check{V}} = \mathrm{Ad}_{(U \underset{\mathfrak{H}}{\otimes} U)} \circ \Delta_V \circ \mathrm{Ad}_U, \quad A(\check{V}) = \widehat{A}(V), \quad \Delta_{\check{V}} = \widehat{\Delta}_V, \\ A(\widehat{V}) &= \mathrm{Ad}_U(\widehat{A}(V)), \quad \Delta_{\widehat{V}} = \mathrm{Ad}_{(U \underset{\mathfrak{H}}{\otimes} U)} \circ \widehat{\Delta}_V \circ \mathrm{Ad}_U, \quad \widehat{A}(\widehat{V}) = A(V), \quad \widehat{\Delta}_{\widehat{V}} = \Delta_V. \end{aligned} \tag{12}$$

In particular, $\check{V}$ and $\widehat{V}$ are well-behaved if V is well-behaved.

Lemma 6.10. *If* $(H, \alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ *is a balanced* C^**-pseudo-multiplicative unitary, then* $\mathrm{Fix}(V) = U\mathrm{Cofix}(\widehat{V}) = \mathrm{Cofix}(\check{V})$ *and* $\mathrm{Cofix}(V) = \mathrm{Fix}(\widehat{V}) = U\mathrm{Fix}(\check{V})$*.* □

Corollary 6.11. *Let* $(H, \alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ *be a balanced* C^**-pseudo-multiplicative unitary over* μ*, where* V *is well-behaved.*

i) Assume that $(H, \hat{\beta}, \alpha, \beta, V)$ *is étale and that* $\eta_0 \in \mathrm{Fix}(V)$ *satisfies* $\eta_0^* \eta_0 = \mathrm{id}_{H_\mu}$*.*

(a) A bounded counit for $(\widehat{A}(V)_H^{\widehat{\beta}, \alpha}, \widehat{\Delta}_V)$ *is given by* $\widehat{a} \mapsto \eta_0^* \widehat{a} \eta_0$*.*

(b) A bounded left Haar weight for $(A(V)_H^{\alpha, \beta}, \Delta_V)$ *is given by* $a \mapsto \eta_0^* U^* a U \eta_0$*.*

ii) Assume that $(H, \hat{\beta}, \alpha, \beta, V)$ *is proper and* $\xi_0 \in \mathrm{Cofix}(V)$ *satisfies* $\xi_0^* \xi_0 = \mathrm{id}_{H_\mu}$*.*

(a) A bounded counit for $(A(V)_H^{\alpha, \beta}, \Delta_V)$ *is given by* $a \mapsto \xi_0^* a \xi_0$*.*

(b) A bounded right Haar weight for $(\widehat{A}(V)_H^{\widehat{\beta}, \alpha}, \widehat{\Delta}_V)$ *is given by* $\widehat{a} \mapsto \xi_0^* U^* \widehat{a} U \xi_0$*.*

Proof. Apply Theorem 6.8 to $\check{V}$ or $\widehat{V}$, respectively, and use Remark 6.2 ii) and (12). □

7 The dual Hopf C^*-bimodule

In the preceding section, we saw that the fundamental unitary associated to a compact C^*-quantum groupoid gives rise to two Hopf C^*-bimodules and that one of these two coincides with the underlying Hopf C^*-bimodule of the initial C^*-quantum groupoid. In this short section, we study the other Hopf C^*-bimodule, which can be considered as (the underlying Hopf C^*-bimodule of) the generalized Pontrjagin dual of the initial C^*-quantum groupoid.

In principle, the dual Hopf C^*-bimodules of compact C^*-quantum groupoids should precisely exhaust the class of étale C^*-quantum groupoids with compact base, but a precise definition of étale C^*-quantum groupoids is not yet available. However, we can describe some important ingredients like the underlying Hopf C^*-bimodule, the unitary antipode, and the counits of the dual of a compact C^*-quantum groupoid.

Throughout this section, let $(B, \mu, A, r, \phi, s, \psi, \delta, R, \Delta)$ be a compact C^*-quantum groupoid. We use the notation introduced in the preceding sections.

The dual Hopf C^*-bimodule In Theorem 5.4 and Proposition 5.7, we associated to the compact C^*-quantum groupoid a regular C^*-pseudo-multiplicative unitary $(H, \widehat{\beta}, \alpha, \beta, V)$. Now, we determine the C^*-algebra of the associated Hopf C^*-bimodule $(\widehat{A}(V)_H^{\widehat{\beta},\alpha}, \widehat{\Delta}_V)$.

Proposition 7.1. *i) For each $a \in A$, there exists an operator $\lambda(a) \in \mathcal{L}(H)$ such that $\lambda(a)\Lambda_\nu(d) = \Lambda_\nu\big(\langle\langle\zeta_\phi|_2\Delta(d)|a^{op}\zeta_\phi\rangle_2\big)$ for all $d \in A$, and $\lambda(a)^* = J\lambda(R(a))J$.*

ii) $\langle x^{op}\zeta_\phi|_2V|y^{op}\zeta_\psi\rangle_2 = \lambda(yx^*)$ for all $x, y \in A$.

iii) $\widehat{A}(V) = [\lambda(A)]$.

Proof. By definition, $\widehat{A}(V)$ is the closed linear span of all operators of the form $\langle x^{op}\zeta_\phi|_2V|y^{op}\zeta_\psi\rangle_2$, where $x, y \in A$. But for all $x, y, d \in A$,

$$\begin{aligned}\langle x^{op}\zeta_\phi|_2V|y^{op}\zeta_\psi\rangle_2 d\zeta_\nu &= \langle x^{op}\zeta_\phi|_2V(d\zeta_\nu \circledcirc y^{op}\zeta_\psi) \\ &= \langle x^{op}\zeta_\phi|_2\Delta(d)(\zeta_\nu \circledcirc y^{op}\zeta_\phi) = \Lambda_\nu\big(\langle\langle\zeta_\phi|_2\Delta(d)|(x^{op})^*y^{op})\zeta_\phi\rangle_2\big).\end{aligned}$$

This calculation proves the existence of the operators $\lambda(a)$ for all $a \in A$ and that $\widehat{A}(V) = [\lambda(A)]$. Finally, by Theorem 5.6, Lemma 3.9 and Proposition 3.8,

$$\begin{aligned}\lambda(yx^*)^* &= \big(\langle x^{op}\zeta_\phi|_2V|y^{op}\zeta_\psi\rangle_2\big)^* \\ &= \langle y^{op}\zeta_\psi|_2(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)V(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)|x^{op}\zeta_\phi\rangle_2 \\ &= J\langle Iy^{op}\zeta_\psi J_\mu|_2V|Ix^{op}\zeta_\phi J_\mu\rangle_2 J \\ &= J\langle R(y^*)^{op}\zeta_\phi|_2V|R(x^*)^{op}\zeta_\psi\rangle_2 J = J\lambda(R(x)^*R(y))J = J\lambda(R(yx^*))J.\end{aligned}$$

□

The associated weak C^*-pseudo-Kac system Put $U := IJ = JI$.

Theorem 7.2. *$(H, \alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ is a weak C^*-pseudo-Kac system.*

The proof involves the following formula:

Lemma 7.3. *$I\lambda(a)I\Lambda_{\nu^{-1}}(d) = \Lambda_{\nu^{-1}}\big(\langle\langle\zeta_\psi|_1\Delta(d)|R(a^*)^{op}\zeta_\psi\rangle_1\big)$ for all $a, d \in A$.*

Proof. By Lemma 4.6, we have for all $a, d \in A$

$$\begin{aligned}I\lambda(a)I\Lambda_{\nu^{-1}}(d) &= I\lambda(a)\Lambda_\nu(R(d)^*) \\ &= I\langle\zeta_\phi|_2\Delta(R(d)^*)|a^{op}\zeta_\phi\rangle_2 I\zeta_{\nu^{-1}} = \big(\langle\zeta_\psi|_1\Delta(d)|R(a^*)^{op}\zeta_\psi\rangle_1\big)\zeta_{\nu^{-1}}.\end{aligned}$$

□

Lemma 7.4. *$\widehat{V} = W$ and $\check{V} = (J_\alpha \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} J)V^{op}(J_\alpha \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} J)$.*

Proof. Theorems 5.6 and 5.10 imply $\check{V} = (U_\beta \otimes_{\widehat{\alpha}} U)W(U_{\widehat{\alpha}} \otimes_{\widehat{\beta}} U) = (J_\alpha \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} J)V^{op}(J_\alpha \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} J)$ and $\widehat{V} = \Sigma(U_\alpha \otimes_\beta 1)(J_{\widehat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)V^*(J_{\widehat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)(U_\beta \otimes_\alpha 1)\Sigma = \Sigma(I_{\widehat{\beta}} \underset{J_\mu}{\otimes}{}_\alpha I)V^*(I_\beta \underset{J_\mu}{\otimes}{}_\alpha I)\Sigma = W$. □

Proof of Theorem 7.2. By Lemma 7.4, $(H, \alpha, \beta, \widehat{\alpha}, \widehat{V})$ and $(H, \widehat{\alpha}, \widehat{\beta}, \alpha, \check{V})$ are regular C^*-pseudo-multiplicative unitaries. Clearly, we have $[A(V), UA(V)U] = [A(V), JIA(V)IJ] = [A(V), JA(V)J] = 0$. It remains to show that $[\widehat{A}(V), U\widehat{A}(V)U] = 0$. But for all $x, y, d \in A$,

$$\begin{aligned}I\lambda(x)I\lambda(y)d\zeta_\nu &= I\lambda(x)I\langle\zeta_\phi|_2\Delta(d)|y^{op}\zeta_\phi\rangle_2\delta^{-1/2}\zeta_{\nu^{-1}} \\ &= \langle\zeta_\psi|_1\Delta\big(\langle\zeta_\phi|_2\Delta(d)|y^{op}\zeta_\phi\rangle_2\delta^{-1/2}\big)|R(x^*)^{op}\zeta_\psi\rangle_1\zeta_{\nu^{-1}} \\ &= \langle\zeta_\psi|_1\langle\zeta_\phi|_3\Delta^{(2)}(d)|y^{op}\zeta_\phi\rangle_3|\delta^{-1/2}R(x^*)^{op}\zeta_\psi\rangle_1\delta^{-1/2}\zeta_{\nu^{-1}} \\ &= \langle\zeta_\phi|_2\langle\zeta_\psi|_1\Delta^{(2)}(d)|\delta^{-1/2}R(x^*)^{op}\zeta_\psi\rangle_1|y^{op}\zeta_\phi\rangle_2\zeta_\nu \\ &= \lambda(y)\langle\zeta_\psi|_1\Delta(d)|\delta^{-1/2}R(x^*)^{op}\zeta_\psi\rangle_1\zeta_\nu \\ &= \lambda(y)\langle\zeta_\psi|_1\Delta(d\delta^{-1/2})|R(x^*)^{op}\zeta_\psi\rangle_1\delta^{1/2}\zeta_\nu \\ &= \lambda(y)I\lambda(x)Id\delta^{-1/2}\zeta_{\nu^{-1}} = \lambda(y)I\lambda(x)Id\zeta_\nu.\end{aligned}$$

Therefore, $[\widehat{A}(V), U\widehat{A}(V)U] = [\widehat{A}(V), IJ\widehat{A}(V)JI] = [\widehat{A}(V), I\widehat{A}(V)I] = 0$. □

Coinvolution and counit on the dual Hopf C^*-bimodule Proposition 7.1 immediately implies:

Corollary 7.5. *There exists a $*$-antiautomorphism $\widehat{R}\colon \widehat{A}(V) \to \widehat{A}(V)$, $\hat{a} \mapsto J\hat{a}^*J$.* □

This $*$-antiautomorphism is a coinvolution of the Hopf C^*-bimodule $\big(\widehat{A}(V)_H^{\hat{\beta},\alpha}, \widehat{\Delta}_V\big)$ in the sense that it reverses the comultiplication:

Proposition 7.6. $\widehat{\Delta} \circ \widehat{R} = \mathrm{Ad}_\Sigma \circ (\widehat{R}_{\hat{\beta}} *_\alpha \widehat{R}) \circ \widehat{\Delta}$.

Proof. By (12) and Lemma 7.4, we have for all $\hat{a} \in \widehat{A}(V)$

$$\begin{aligned}\widehat{\Delta}_V(\hat{a}) = \check{V}(\hat{a}\,{}_{\hat{\alpha}}\!\otimes_{\hat{\beta}} 1)\check{V}^* &= (J_\alpha \underset{J_\mu}{\otimes}{}_{\hat{\beta}} J)\Sigma V^*\Sigma(J_{\hat{\alpha}} \underset{J_\mu}{\otimes}{}_{\hat{\beta}} J)(\hat{a}_{\hat{\alpha}}\!\otimes_{\hat{\beta}} 1)(J_\alpha \underset{J_\mu}{\otimes}{}_{\hat{\beta}} J)^*\Sigma V\Sigma(J_\alpha \underset{J_\mu}{\otimes}{}_{\hat{\beta}} J)^* \\ &= (\mathrm{Ad}_\Sigma \circ(\widehat{R}_{\hat{\beta}} *_\alpha \widehat{R}) \circ \widehat{\Delta}_V)(\widehat{R}(\hat{a})).\end{aligned}$$ □

The constructions in Section 6 yield a counit on $\widehat{A}(V)$:

Proposition 7.7. *i) The Hopf C^*-bimodule $(\widehat{A}(V)_H^{\hat{\beta},\alpha}, \widehat{\Delta}_V)$ has a bounded counit $\hat{\epsilon}$, given by $\hat{\epsilon}(\lambda(y^*x)) = \zeta_\psi^*\lambda(y^*x)\zeta_\psi = J_\mu\zeta_\phi^*x^*y\zeta_\psi J_\mu$ for all $x, y \in A$.*

ii) $\hat{\epsilon}(\widehat{R}(\hat{a})) = J_\mu\hat{\epsilon}(\hat{a})^*J_\mu$ for all $\hat{a} \in \widehat{A}(V)$.

Proof. i) By Proposition 6.7, Theorem 6.8 i), and Corollary 6.11 i), the map $\hat{\epsilon}\colon \widehat{A}(V) \to \mathcal{L}(H_\mu)$, $\hat{a} \mapsto \zeta_\psi^*\hat{a}\zeta_\psi$, is a bounded counit, and by Theorem 6.8 i) and Proposition 7.1, $\hat{\epsilon}(\lambda(y^*x)) = \hat{\epsilon}\big(\langle JxJ\zeta_\phi|_2 V|JyJ\zeta_\psi\rangle_2\big) = J_\mu\zeta_\phi^*x^*y\zeta_\psi J_\mu$ for all $x, y \in A$.

ii) For all $\hat{a} \in \widehat{A}(V)$, we have $\hat{\epsilon}(\widehat{R}(\hat{a})) = \zeta_\psi^*J\hat{a}^*J\zeta_\psi = J_\mu(\zeta_\psi^*\hat{a}\zeta_\psi)^*J_\mu = J_\mu\hat{\epsilon}(\hat{a})^*J_\mu$. □

8 Principal compact C^*-quantum groupoids

In this section, we study compact C^*-quantum groupoids that are principal. Most importantly, we show that a principal compact C^*-quantum groupoid is essentially determined by the conditional expectation $\tau\colon B \to \tau(B) \subseteq Z(B)$ and the state $\mu|_{\tau(B)}$, and that the dual of a principal compact C^*-quantum groupoid is the C^*-algebra of compact operators on a certain C^*-module.

Principal compact C^*-quantum groupoids Recall that a compact groupoid G is principal if the map $G \to G^0 \times G^0$ given by $x \mapsto (r(x), s(x))$ is injective or, equivalently, if $C(G) = [r^*(C(G^0))s^*(C(G^0))]$. The second condition suggests the following definition:

Definition 8.1. *A compact C^*-quantum graph $(B, \mu, A, r, \phi, s, \psi, \delta)$ is* principal *if $A = [r(B)s(B^{op})]$, and a compact C^*-quantum groupoid $(B, \mu, A, r, \phi, s, \psi, \delta, R, \Delta)$ is* principal *if $A = [r(B)s(B^{op})]$.*

To simplify the following discussion, we only consider the case where $\delta = 1_A$. Corollary 4.13 shows that this is not a serious restriction.

Let $(B, \mu, A, r, \phi, s, \psi, 1_A)$ be a principal compact C^*-quantum graph. Then there exist at most one coinvolution R for $(B, \mu, A, r, \phi, s, \psi, 1_A)$ and at most one comultiplication Δ for $A_H^{\alpha,\beta}$ because necessarily $R(r(b)s(c^{op})) = s(b^{op})r(c)$ and $\Delta(r(b)s(c^{op})) = r(b)_\alpha\!\otimes_\beta s(c^{op})$ for all $b, c \in B$. We shall give conditions for the existence of such a coinvolution and a comultiplication, and determine when $(B, \mu, A, r, \phi, s, \psi, 1_A, R, \Delta)$ is a principal compact C^*-quantum groupoid. These conditions involve the completely positive contractions $\tau = \psi \circ r\colon B \to Z(B^{op}) \cong Z(B)$ and $\tau^\dagger = \phi \circ s\colon B^{op} \to Z(B) \cong Z(B^{op})$ introduced in (2).

Theorem 8.2. *Let $(B, \mu, A, r, \phi, s, \psi, 1_A)$ be a principal compact C^*-quantum graph. Then the following two conditions are equivalent:*

i) There exist R, Δ such that $(B, \mu, A, r, \phi, s, \psi, 1_A, R, \Delta)$ is a compact C^*-quantum groupoid.

ii) $\tau(b) = \tau^\dagger(b^{op})$ for all $b \in B$, $\tau\colon B \to \tau(B)$ is a conditional expectation, $\mu \circ \tau = \mu$, $r \circ \tau = s \circ \tau$, and $\tau(b\sigma^\mu_{-i/2}(d)) = \tau(d\sigma^\mu_{-i/2}(b))$ for all $b, d \in \mathrm{Dom}(\sigma^\mu_{-i/2})$.

Before we prove this result, let us give an application: every compact C^*-quantum groupoid has an underlying principal compact C^*-quantum groupoid. The nontrivial part of this assertion is that the comultiplication restricts to a morphism of C^*-(μ, μ^{op})-algebras.

Corollary 8.3. *Let $(B, \mu, A, r, \phi, s, \psi, 1_A, R, \Delta)$ be a compact C^*-quantum groupoid. Put $\tilde{A} := [r(B)s(B^{op})] \subseteq A$, $\tilde{\phi} := \phi|_{\tilde{A}}$, $\tilde{\psi} := \psi|_{\tilde{A}}$, $\tilde{R} := R|_{\tilde{A}}$. Then there exists a unique $\tilde{\Delta}$ such that $(B, \mu, \tilde{A}, r, \tilde{\phi}, s, \tilde{\psi}, 1_{\tilde{A}}, \tilde{R}, \tilde{\Delta})$ is a compact C^*-quantum groupoid.* □

The proof of Theorem 8.2 is divided into several steps. First, note that for all $b, c \in B$,

$$\begin{aligned} \phi(s(b^{op})r(c)) &= \tau^\dagger(b^{op})c, & \psi(r(b)s(c^{op})) &= \tau(b)c^{op}, \\ \nu(s(b^{op})r(c)) &= \mu(\tau^\dagger(b^{op})c), & \nu^{-1}(r(b)s(c^{op})) &= \mu^{op}(\tau(b)c^{op}). \end{aligned} \tag{13}$$

Lemma 8.4. *Let $(B, \mu, A, r, \phi, s, \psi, 1_A)$ be a principal compact C^*-quantum graph. There exists a coinvolution R for $(B, \mu, A, r, \phi, s, \psi, 1_A)$ if and only if $\tau(b) = \tau^\dagger(b^{op})$ for all $b \in B$.*

Proof. The only if part is Lemma 2.7 ii). So, assume that $\tau(b) = \tau^\dagger(b^{op})$ for all $b \in B$. Then there exists an antiunitary $I\colon H \to H$ such that $Ir(b)s(c^{op})\zeta_{\nu^{-1}} = s(b^{op})^* r(c)^* \zeta_\nu$ for all $b, c \in B$ because by (13),

$$\left\|s(b^{op})^* r(c)^* \zeta_\nu\right\|^2 = \nu\big(s((b^*b)^{op})r(cc^*)\big) = \nu^{-1}\big(r(b^*b)s((cc^*)^{op})\big) = \left\|r(b)s(c^{op})\zeta_{\nu^{-1}}\right\|^2.$$

A short calculation shows that $Ir(b)^* s(c^{op})^* I = s(b^{op})r(c)$ for all $b, c \in B$. Therefore, we can define a $*$-homomorphism $R\colon A \to A$ by $a \mapsto Ia^*I$, and $R(r(b)s(c^{op})) = s(c^{op})r(b)$ for all $b, c \in B$. Finally, (13) implies that $(\phi \circ R)(a) = \psi(a)^{op}$ for all $a \in A$. □

Lemma 8.5. *Let $(B, \mu, A, r, \phi, s, \psi, 1_A)$ be a principal compact C^*-quantum graph such that $\tau(b) = \tau^\dagger(b^{op})$ for all $b \in B$, $\tau\colon B \to \tau(B)$ is a conditional expectation, and $\mu \circ \tau = \mu$.*

i) For all $d, e \in B$, where e is analytic for σ^μ, there exists an operator $T_{d,e} \in \mathcal{L}(H, H_\alpha \otimes_\beta H)$ such that for all $x \in r(B) \cup r(B)^{op}$ and $y \in s(B^{op}) \cup s(B^{op})^{op}$,

$$T_{d,e}xy\zeta_\nu = x\zeta_\psi \otimes de^{op}\zeta_\mu \otimes y\zeta_\phi, \tag{14}$$

and for all $b, c, b', c', d', e' \in B$, where e' is analytic for σ^μ,

$$T^*_{d,e}\big(r(b')\zeta_\psi \otimes d'e'^{op}\zeta_\mu \otimes s(c'^{op})\zeta_\phi\big) = r(\tau(d^*d'\sigma^\mu_{-i/2}(e'e^*))r(b')s(c'^{op})\zeta_\nu. \tag{15}$$

ii) Put $\mathcal{T} := \{T_{d,e} \mid d, e \in B,\ e$ analytic for $\sigma^\mu\}$. Then $[\mathcal{T}\alpha] = \alpha \triangleright \alpha$, $[\mathcal{T}^*(\alpha \triangleright \alpha)] = \alpha$ and $[\mathcal{T}\beta] = \beta \triangleleft \beta$, $[\mathcal{T}^*(\beta \triangleleft \beta)] = \beta$.

iii) There exists a comultiplication Δ for $A_H^{\alpha,\beta}$.

Proof. i) Let d, e be as in i). Then there exists a $T_{d,e} \in \mathcal{L}(H, H_\alpha \otimes_\beta H)$ such that equations (14) and (15) hold for all $x \in r(B)$, $y \in s(B^{op})$ because

$$\begin{aligned} &\big\langle r(b)\zeta_\psi \otimes de^{op}\zeta_\mu \otimes s(c^{op})\zeta_\phi \big| r(b')\zeta_\psi d' \otimes \zeta_\mu \otimes s(c'^{op})\zeta_\phi e'^{op}\big\rangle_{H_\alpha \otimes_\beta H} \\ &\quad = \big\langle \zeta_\mu \big| d^*(e^{op})^* \psi(r(b^*b'))\phi(s((c'c^*)^{op}))d'e'^{op}\zeta_\mu\big\rangle \\ &\quad = \big\langle \zeta_\mu \big| \tau(b^*b')\tau(c'c^*)d^*d'(e'e^*)^{op}\zeta_\mu\big\rangle \\ &\quad = \mu\big(\tau(b^*b')\tau(c'c^*)d^*d'\sigma^\mu_{-i/2}(e'e^*)\big) \\ &\quad = \mu\big(\tau(b^*b')\tau(c'c^*)\tau(d^*d'\sigma^\mu_{-i/2}(e'e^*))\big) \\ &\quad = \mu\big(\tau(b^*b'\tau(d^*d'\sigma^\mu_{-i/2}(e'e^*)))\tau(c'c^*)\big) \\ &\quad = \big\langle r(b)s(c^{op})\zeta_\nu \big| r(\tau(d^*d'\sigma^\mu_{-i/2}(e'e^*))r(b')s(c'^{op})\zeta_\nu\big\rangle \end{aligned}$$

for all $b, c, b', c', d', e' \in B$, where e' is analytic for σ^μ. Using Lemma 3.7 iii), one easily concludes that $T_{d,e}xy\zeta_\nu = x\zeta_\psi \oslash de^{op}\zeta_\mu \oslash y\zeta_\phi$ for all $x \in r(B)^{op}$ and $y \in s(B^{op})^{op}$.

ii) Let $b, c \in B$ and d, e as in i). Then for all $f \in B$,

$$\begin{aligned} T_{d,e}r(b)^{op}s(c^{op})^{op}\zeta_\phi f^{op}\zeta_\mu &= T_{d,e}r(fb)^{op}s(c^{op})^{op}\zeta_\nu \\ &= r(fb)^{op}\zeta_\psi \oslash de^{op}\zeta_\mu \oslash s(c^{op})^{op}\zeta_\phi \\ &= |s(c^{op})^{op}r(e)^{op}\zeta_\phi\rangle_2 r(b)^{op}s(d^{op})^{op}\zeta_\phi f^{op}\zeta_\mu. \end{aligned}$$

This relation implies $[\mathcal{T}\beta] = \beta \triangleleft \beta$, and the remaining assertions follow similarly.

iii) For all $b, c, d, e \in B$, where e is analytic for σ^μ, we have $T_{d,e}r(b)s(c^{op}) = r(b)_\alpha\otimes_\beta s(c^{op})$. Now, the claim follows from ii). □

Proof of Theorem 8.2. i) implies ii) by Proposition 4.10. Conversely, assume that ii) holds. Then the preceding lemmas imply that there exist a coinvolution R for $(B, \mu, A, r, \phi, s, \psi, 1_A)$ and a comultiplication Δ for $A_H^{\alpha,\beta}$. We show that the conditions in Definition 4.8 hold.

First, we check condition 4.8 i). Since $\rho_\beta = r$ and $\rho_\alpha = s$,

$$[\Delta(A)|\alpha\rangle_1] = [|\rho_\beta(B)\alpha\rangle_1\rho_\alpha(B^{op})] = [|\alpha B\rangle_1\rho_\alpha(B^{op})] = [|\alpha\rangle_1\rho_\beta(B)\rho_\alpha(B^{op})] = [|\alpha\rangle_1 A],$$
$$[\Delta(A)|\zeta_\psi\rangle_1 A] = [|\rho_\beta(B)\zeta_\psi\rangle_1\rho_\beta(B)A] = [|\rho_\beta(B)\zeta_\phi B\rangle_1 A] = [|\rho_\beta(B)s(B^{op})^{op}\zeta_\psi\rangle_1 A] = [|\alpha\rangle_1 A].$$

Similar calculations show that $[\Delta(A)|\beta\rangle_2] = [|\beta\rangle_2 A]$ and $[\Delta(A)|\zeta_\phi\rangle_2 A] = [|\beta\rangle_2 A]$.

Next, ϕ is a bounded left Haar weight for $(A_H^{\alpha,\beta}, \Delta)$ because for all $b, c \in B$,

$$\begin{aligned} \langle\zeta_\phi|_2\Delta(r(b)s(c^{op}))|\zeta_\phi\rangle_2 &= \langle\zeta_\phi|_2(r(b)_\alpha\otimes_\beta s(c^{op}))|\zeta_\phi\rangle_2 \\ &= r(b)\rho_\alpha(\zeta_\phi^* s(c^{op})\zeta_\phi) \\ &= r(b)s(\phi(s(c^{op}))) = r(b)s(\tau(c)) = r(b\tau(c)) = r(\phi(r(b)s(c^{op}))). \end{aligned}$$

A similar calculation shows that ψ is a bounded right Haar weight for $(A_H^{\alpha,\beta}, \Delta)$.

Finally, we prove that ϕ, ψ and R satisfy the strong invariance condition 4.8 iii). By Lemma 4.11, we have for all $b, c, d, e \in \mathrm{Dom}(\sigma^\mu_{-i/2})$

$$\begin{aligned} \langle\zeta_\psi|_1\Delta(r(b)s(c^{op}))((r(d)s(e^{op}))^{op}\,{}_\alpha\otimes_\beta 1)|\zeta_\psi\rangle_1 &= r(e)s(c^{op})r(\tau(b\sigma^\mu_{-i/2}(d))) \\ &= R\big(s(e^{op})r(c)s(\tau(d\sigma^\mu_{-i/2}(b))\big) \\ &= R\big(\langle\zeta_\psi|_1\Delta(r(d)s(e^{op}))((r(b)s(c^{op}))^{op}\,{}_\alpha\otimes_\beta 1)|\zeta_\psi\rangle_1\big). \end{aligned}$$

Since $\mathrm{Dom}(\sigma^\mu_{-i/2}) \subseteq B$ is dense, condition 4.8 iii) holds. □

The reconstruction of a principal compact C^*-quantum groupoid A principal compact C^*-quantum groupoid is completely determined by the conditional expectation $\tau\colon B \to \tau(B) \subseteq Z(B)$ and can be reconstructed from τ as follows. Assume that

- C is a commutative unital C^*-algebra with a faithful state υ,
- B is a unital C^*-algebra with a υ-module structure (ι, τ) such that $\iota(C) \subseteq Z(B)$.

We put $\mu := \upsilon \circ \tau$ and identify C with $\iota(C)$ via ι.

Lemma 8.6. *$\tau(b\sigma^\mu_{-i/2}(d)) = \tau(d\sigma^\mu_{-i/2}(b))$ for all $b, d \in \mathrm{Dom}(\sigma^\mu_{-i/2})$.*

Proof. For all $c \in C$, we have $\sigma^\mu_t(c) = \sigma^\upsilon_t(c) = c$ for all $t \in \mathbb{R}$ by Lemma 2.2 and hence

$$\begin{aligned} \upsilon(c^*\tau(b\sigma^\mu_{-i/2}(d)) = \mu(c^* b\sigma^\mu_{-i/2}(d)) &= \langle\Lambda_\mu(b^*c)|J\Lambda_\mu(d^*)\rangle \\ &= \langle\Lambda_\mu(d^*)|J\Lambda_\mu(b^*c)\rangle \\ &= \mu(d\sigma^\mu_{-i/2}(c^*b)) = \mu(dc^*\sigma^\mu_{-i/2}(b)) = \upsilon(c^*\tau(d\sigma^\mu_{-i/2}(b)). \end{aligned}$$

Since $c \in C$ was arbitrary and υ faithful, the claim follows. □

As in Proposition 3.7, we define an isometry $\zeta_\tau \colon H_\upsilon \to H_\mu$ by $\Lambda_\upsilon(c) \mapsto \Lambda_\mu(c)$, identify B, B^{op} with C^*-subalgebras of $\mathcal{L}(H_\mu)$ via the GNS-representations, and put

$$\gamma := [B\zeta_\tau] \subseteq \mathcal{L}(H_\upsilon, H_\mu), \qquad \gamma^{op} := [B^{op}\zeta_\tau] \subseteq \mathcal{L}(H_\upsilon, H_\mu).$$

Proposition 8.7. *There exists a unique principal compact C^*-quantum groupoid $(B, \mu, A, r, \phi, s, \psi, 1_A, R, \Delta)$ such that $A = B_{\gamma^{op}} \otimes_\gamma B^{op} \subseteq \mathcal{L}\big((H_\mu)_{\gamma^{op}} \otimes_\gamma (H_\mu)\big)$ and for all $b, c \in B$,*

$$r(b) = b_{\gamma^{op}} \otimes_\gamma 1^{op}, \quad \phi(b_{\gamma^{op}} \otimes_\gamma c^{op}) = b\tau(c), \quad s(c^{op}) = 1_{\gamma^{op}} \otimes_\gamma c^{op}, \quad \psi(b_{\gamma^{op}} \otimes_\gamma c^{op}) = \tau(b)c^{op}.$$

Proof. Routine arguments show that there exists a unique principal compact C^*-quantum graph $(B, \mu, A, r, \phi, s, \psi, 1_A)$ with A, r, s, ϕ, ψ as above; let us only note that the completely positive contractions $\phi \colon A \to B$ and $\psi \to B^{op}$ are well-defined because they are given by $x \mapsto \langle \zeta_\tau |_2 x | \zeta_\tau \rangle_2$ and $x \mapsto \langle \zeta_\tau |_1 x | \zeta_\tau \rangle_1$, respectively. Now, the assertion follows from Theorem 8.2. ☐

Every principal compact C^*-quantum groupoid is of the form constructed above:

Proposition 8.8. *Let $(B, \mu, A, r, \phi, s, \psi, 1_A, R, \Delta)$ be a principal compact C^*-quantum groupoid and put $\tau = \psi \circ r$.*

i) $C := \tau(B)$ is a commutative unital C^*-algebra, $\upsilon := \mu|_C$ is a faithful state on C, (id, τ) is a υ-module structure on B, and $\mu = \upsilon \circ \tau$.

Denote by $\zeta_\tau \colon H_\upsilon \to H_\tau$ the isometry $c\zeta_\upsilon \mapsto c\zeta_\tau$ and put $\gamma := [B\zeta_\tau]$, $\gamma^{op} := [B^{op}\zeta_\tau]$.

ii) There exists a unitary $\Xi \colon H_\nu \to (H_\mu)_{\gamma^{op}} \otimes_\gamma (H_\mu)$ such that for all $b, c \in B$,

$$\Xi\big(r(b)^{op} s(c^{op})^{op} \zeta_\nu\big) = b^{op}\zeta_\tau \otimes \zeta_\upsilon \otimes c\zeta_\tau \quad \textit{and} \quad \Xi\big(r(b)s(c^{op})\zeta_\nu\big) = b\zeta_\tau \otimes \zeta_\upsilon \otimes c^{op}\zeta_\tau.$$

Moreover, $\Xi\widehat{\beta} = [|\gamma\rangle_1 B^{op}]$ and $\Xi\alpha = [|\gamma^{op}\rangle_1 B]$.

iii) Ad_Ξ restricts to an isomorphism $A \to B_{\gamma^{op}} \otimes_\gamma B^{op}$, $r(b)s(c^{op}) \mapsto b_{\gamma^{op}} \otimes_\gamma c^{op}$.

Proof. i) This follows directly from Proposition 4.10 and Proposition 4.14.

ii) There exists an isomorphism $\Xi \colon H_\nu \to (H_\mu)_{\gamma^{op}} \otimes_\gamma (H_\mu)$ satisfying the first equation in ii) because by Proposition 4.14, (13), and i) $\|r(b)^{op}s(c^{op})^{op}\zeta_\nu\|^2 = \nu(r(bb^*)s((c^*c)^{op})) = \upsilon(\tau(bb^*)\tau(c^*c)) = \|b^{op}\zeta_\tau \otimes \zeta_\upsilon \otimes c\zeta_\tau\|^2$ for all $b, c \in B$. From Lemma (3.2) iii), one easily deduces $Tr(b)s(c^{op})\zeta_\nu = b\zeta_\tau \otimes \zeta_\upsilon \otimes c^{op}\zeta_\tau$ for all $b, c \in B$. Finally, $\Xi\widehat{\beta} = [|\gamma\rangle_1 B^{op}]$ and $\Xi\alpha = [|\gamma^{op}\rangle_1 B]$ because for all $b, c, d \in B$,

$$\Xi r(b)s(c^{op})\zeta_\psi d^{op}\zeta_\mu = \Xi r(b)s(c^{op}d^{op})\zeta_{\nu^{-1}} = b\zeta_\tau \otimes \zeta_\upsilon \otimes c^{op}d^{op}\zeta_\tau = |b\zeta_\tau\rangle_1 c^{op}d^{op}\zeta_\mu,$$
$$\Xi r(b)^{op}s(c^{op})^{op}\zeta_\psi d^{op}\zeta_\mu = r(b)^{op}s(c^{op})^{op}s(d^{op})\zeta_\nu = b^{op}\zeta_\tau \otimes \zeta_\upsilon \otimes cd^{op}\zeta_\tau = |b^{op}\zeta_\tau\rangle_1 cd^{op}\zeta_\mu.$$

iii) Straightforward. ☐

The dual Hopf C^*-bimodule Let $(B, \mu, A, r, \phi, s, \psi, \delta, R, \Delta)$ be a principal compact C^*-quantum groupoid and $(H, \widehat{\beta}, \alpha, \beta, V)$ the associated C^*-pseudo-multiplicative unitary (see Theorem 5.4). We show that the dual Hopf C^*-bimodule $(\widehat{A}(V)_H^{\widehat{\beta},\alpha}, \widehat{\Delta}_V)$ studied in Section 7 can be identified with the C^*-algebra of compact operators on a Hilbert C^*-module over $\tau(B)$. This result is a (reduced) analogue of the result that for every principal compact groupoid G, the irreducible representations of $C^*(G)$ are labelled by the orbits G^0/G and that each such representation is by all compact operators [13].

We use the notation of Proposition 8.8 and denote by $\mathcal{K}_\tau \subseteq \mathcal{L}(H_\mu)$ the C^*-algebra corresponding to $\mathcal{K}_C(\gamma) \otimes \mathrm{id} \subseteq \mathcal{L}(\gamma \otimes H_\upsilon)$ with respect to the natural isomorphism $\gamma \otimes H_\upsilon \cong H_\mu$, $\xi \otimes \zeta \mapsto \xi\zeta$. Thus, $\mathcal{K}_\tau = [\{k_{b,c} \mid b, c \in B\}]$, where $k_{b,c} \colon H_\mu \to H_\mu$ is given by $d\zeta_\mu \mapsto b\tau(c^*d)\zeta_\mu$ for all $b, c \in B$. Note that $\mathcal{K}_\tau \subseteq \mathcal{L}((H_\mu)_\gamma)$.

Lemma 8.9. $(\mathcal{K}_\tau)_{H_\mu}^{B^{op},B}$ *is a C^*-(μ^{op}, μ)-algebra.*

Proof. Clearly, (H_μ, B^{op}, B) is a C^*-(μ^{op}, μ)-module. We have $[\rho_{B^{op}}(B)\mathcal{K}_\tau] = \mathcal{K}_\tau = [\rho_B(B^{op})\mathcal{K}_\tau]$ because for all $a, b, c, d \in B$, $a' \in \mathrm{Dom}(\sigma^\mu_{-i/2})$, $\rho_{B^{op}}(a)k_{b,c}d\zeta_\mu = ab\tau(c^*d)\zeta_\mu = k_{ab,c}d\zeta_\mu$ and $\rho_B(a'^{op})k_{b,c}d\zeta_\mu = a'^{op}b\tau(c^*d)\zeta_\mu = b\sigma^\mu_{-i/2}(a')\tau(c^*d)\zeta_\nu = k_{b\sigma^\mu_{-i/2}(a'),c}d\zeta_\mu$. □

The comultiplication $\widehat{\Delta}_V$ can be described in terms of the isomorphism

$$\Upsilon\colon (H_\mu)_{B^{op}} \otimes_B (H_\mu) = B^{op} \otimes H_\mu \otimes B \xrightarrow{\cong} H_\mu, \quad b^{op} \otimes \zeta \otimes c \mapsto b^{op} c\zeta.$$

Note that $\Upsilon^*\mathcal{K}_\tau\Upsilon \subseteq (\mathcal{K}_\tau)_{B^{op}} *_B (\mathcal{K}_\tau)$ because $[\Upsilon^*\mathcal{K}_\tau\Upsilon|B^{op}\rangle_1] = [\Upsilon^*\mathcal{K}_\tau B^{op}] = [\Upsilon^* B^{op}\mathcal{K}_\tau] = [|B^{op}\rangle_1\mathcal{K}_\tau]$ and similarly $[\Upsilon^*\mathcal{K}_\tau\Upsilon|B\rangle_2] = [|B\rangle_1\mathcal{K}_\tau]$.

Theorem 8.10. *Let $(B, \mu, A, r, \phi, s, \psi, 1_A, R, \Delta)$ be a principal compact C^*-quantum groupoid and $(\widehat{A}(V)^{\widehat{\beta},\alpha}_H, \widehat{\Delta}_V)$ the dual Hopf C^*-bimodule.*

- *i) There exists an isomorphism of C^*-(μ^{op}, μ)-algebras $j\colon (\mathcal{K}_\tau)^{B^{op},B}_{H_\mu} \to \widehat{A}(V)^{\widehat{\beta},\alpha}_H$, given by $k \mapsto \Xi^*(\mathrm{id}_{\gamma^{op}} \otimes_\gamma k)\Xi$.*
- *ii) $\widehat{\Delta}_V \circ j = (j * j) \circ \mathrm{Ad}^{-1}_\Upsilon$.*
- *iii) $\widehat{R}(j(k_{b,c})) = j(k_{c',b'})$, where $c' = \sigma^\mu_{i/2}(c)^*$ and $b' = \sigma^\mu_{i/2}(b)^*$ for all $b, c \in \mathrm{Dom}(\sigma^\mu_{i/2})$.*
- *iv) $\widehat{\epsilon} \circ j = \mathrm{id}_{\mathcal{K}_\tau}$.*

Proof. i) Let $b, c \in B$ be analytic for σ^μ and put $a := r(b)s(c^{op})$. Then the operator $\lambda(a)$ defined in Proposition 7.1 acts as follows. For all $d, e \in B$,

$$\begin{aligned}\lambda(a)r(d)s(e^{op})\zeta_\nu &= \Lambda_\nu\big(\langle\zeta_\phi|_2(r(d)\,{}_\alpha\!\otimes_\beta s(e^{op})r(b)^{op}s(c^{op})^{op})|\zeta_\phi\rangle_2\big) \\ &= \rho_\alpha\big(b^{op}\zeta^*_\phi s(e^{op})s(c^{op})^{op}\zeta_\phi\big)r(d)\zeta_\nu = s\big(b^{op}\phi(s((\sigma^\mu_{i/2}(c)e)^{op}))\big)r(d)\zeta_\nu,\end{aligned}$$

and hence $\Xi\lambda(a)\Xi^*(d\zeta_\tau \otimes \zeta_\upsilon \otimes e^{op}\zeta_\tau) = d\zeta_\tau \otimes \zeta_\upsilon \otimes b^{op}\tau(\sigma^\mu_{i/2}(c)e)\zeta_\tau$. Assume that $e \in \mathrm{Dom}(\sigma^\mu_{-i/2})$. Then by Proposition 4.10, Lemma 3.7 iii), and σ^μ-invariance of τ,

$$\begin{aligned}\Xi\lambda(a)\Xi^*(d\zeta_\tau \otimes \zeta_\upsilon \otimes \sigma^\mu_{-i/2}(e)\zeta_\tau) &= d\zeta_\tau \otimes \zeta_\upsilon \otimes \sigma^\mu_{-i/2}(b)\tau(\sigma^\mu_{i/2}(c)e)\zeta_\tau \\ &= d\zeta_\tau \otimes \zeta_\upsilon \otimes \sigma^\mu_{-i/2}(b)\tau(c\sigma^\mu_{-i/2}(e))\zeta_\tau.\end{aligned}$$

Therefore, $\Xi\lambda(a)\Xi^* = (\mathrm{id}_{\gamma^{op}} \otimes_\gamma k_{b',c^*})$, where $b' = \sigma^\mu_{-i/2}(b)$, and $\widehat{A}(V) = \Xi^*(\mathrm{id}_{\gamma^{op}} \otimes_\gamma \mathcal{K}_\tau)\Xi$. Since υ is faithful and Ξ unitary, the map $j\colon \mathcal{K}_\tau \to \widehat{A}(V)$ given by $k \mapsto \Xi^*(\mathrm{id}_{\gamma^{op}} \otimes_\gamma k)\Xi$ is an isomorphism of C^*-algebras.

It remains to show that j is a morphism of C^*-(μ^{op}, μ)-algebras. Evidently, $tk = j(k)t$ for all $k \in \mathcal{K}_\tau$ and all $t \in [\Xi^*|\gamma^{(op)}\rangle_1]$. By Proposition 8.8 ii), $[\Xi^*|\gamma\rangle_1 B^{op}] = \widehat{\beta}$, $[\Xi^*|\gamma^{op}\rangle_1 B] = \alpha$, $[\langle\gamma|_1\Xi\widehat{\beta}] = [\langle\gamma|_1|\gamma\rangle_1 B^{op}] = [CB^{op}] = B^{op}$, and $[\langle\gamma^{op}|_1\Xi\alpha] = [\langle\gamma^{op}|_1|\gamma^{op}\rangle_1 B] = [CB] = B$. The claim follows.

ii) By definition of $\widehat{\Delta}_V$ and j, we have for all $x \in \alpha$, $y \in \gamma$, $k \in \mathcal{K}_\tau$

$$\widehat{\Delta}_V(j(k))V^*|x\rangle_1\Xi^*|y\rangle_1 = V^*(1_\alpha \otimes_\beta j(k))|x\rangle_1\Xi^*|y\rangle_1 = V^*|x\rangle_1 j(k)\Xi^*|y\rangle_1 = V^*|x\rangle_1\Xi^*|y\rangle_1 k.$$

Likewise, by definition of $j * j$ and Υ, we have for all $u \in \gamma^{op}$, $v \in \gamma$, $k \in \mathcal{K}_\tau$

$$\begin{aligned}((j * j)(\Upsilon^* k\Upsilon))(\Xi^*|u\rangle_{1\,B^{op}} \otimes_B \Xi^*|v\rangle_1)\Upsilon^* &= (\Xi^*|u\rangle_{1\,B^{op}} \otimes_B \Xi^*|v\rangle_1)\Upsilon^* k\Upsilon\Upsilon^* \\ &= (\Xi^*|u\rangle_{1\,B^{op}} \otimes_B \Xi^*|v\rangle_1)\Upsilon^* k.\end{aligned}$$

Now, $[V^*|\alpha\rangle_1\Xi^*|\gamma\rangle_1] = [(\Xi^*|\gamma^{op}\rangle_{1\,B^{op}} \otimes_B \Xi^*|\gamma\rangle_1)\Upsilon^*]$ because for all $b, c, d, e \in B$,

$$\begin{aligned}V^*|r(b)^{op}s(c^{op})^{op}\zeta_\psi\rangle_1\Xi^*|d\zeta_\tau\rangle_1 e^{op}\zeta_\mu &= V^*\big(r(b)^{op}s(c^{op})^{op}\zeta_\psi \otimes r(d)s(e^{op})\zeta_\nu\big) \\ &= V^*\big(r(b)^{op}\zeta_\psi \otimes r(cd)s(e^{op})\zeta_\nu\big) \\ &= r(b)^{op}\zeta_\psi \otimes r(cd)s(e^{op})\zeta_\nu \\ &= (\Xi^*|b^{op}\zeta_\tau\rangle_{1\,B^{op}} \otimes_B \Xi^*|cd\zeta_\tau\rangle_1)(1 \otimes \zeta_\mu \otimes e^{op}) \\ &= (\Xi^*|b^{op}\zeta_\tau\rangle_{1\,B^{op}} \otimes_B \Xi^*|cd\zeta_\tau\rangle_1)\Upsilon^* e^{op}\zeta_\mu.\end{aligned}$$

Since $[V^*|\alpha\rangle_1\Xi^*|\gamma\rangle_1 H_\mu] = H$, we can conclude $\widehat{\Delta}_V(j(k)) = (j * j)(\Upsilon^* k \Upsilon)$ for all $k \in \mathcal{K}_\tau$.

iii) Let $e \in \mathrm{Dom}(\sigma^\mu_{i/2})$ and $b, c \in \mathrm{Dom}(\sigma^\mu_{i/2})$. Since $Js(f^{op})^{op}\zeta_\nu = \sigma^{\nu^{op}}_{i/2}(s(f^{op})^{op})^*\zeta_\nu = s(\sigma^\mu_{-i/2}(f^*)^{op})^{op}\zeta_\nu$ for all $f \in \mathrm{Dom}(\sigma^\mu_{i/2})$ and $\tau(b^*\sigma^\mu_{-i/2}(e^*)) = \tau(e^*\sigma^\mu_{-i/2}(b^*))$,

$$\begin{aligned}\widehat{R}(j(k_{b,c}))s(e^{op})^{op}\zeta_\nu &= Jj(k_{b,c})^* Js(e^{op})^{op}\zeta_\nu \\ &= Jj(k_{c,b})s(\sigma^\mu_{-i/2}(e^*)^{op})^{op}\zeta_\nu \\ &= Js((c\tau(b^*\sigma^\mu_{-i/2}(e^*)))^{op})^{op}\zeta_\nu \\ &= s(\sigma^\mu_{-i/2}(c^*\tau(\sigma^\mu_{-i/2}(b^*)^*e))^{op})^{op}\zeta_\nu = s((c'\tau(b'^*e))^{op})^{op}\zeta_\nu,\end{aligned}$$

where $b' = \sigma^\mu_{i/2}(b)^*$ and $c' = \sigma^\mu_{i/2}(c)^*$. The claim follows.

iv) For all $b, c, d \in B$,

$$\begin{aligned}\widehat{\epsilon}(j(k_{b,c}))d\zeta_\mu = \zeta_\psi^* j(k_{b,c})\zeta_\psi d\zeta_\mu &= \zeta_\psi^* j(k_{b,c})s(d^{op})^{op}\zeta_\nu \\ &= \zeta_\psi^* s((b\tau(c^*d))^{op})^{op}\zeta_\nu = b\tau(c^*d)\zeta_\mu = k_{b,c}d\zeta_\mu. \qquad \square\end{aligned}$$

The C^*-pseudo-Kac system Recall that in Theorem 7.2, we associated to every compact C^*-quantum groupoid a weak C^*-pseudo-Kac system.

Theorem 8.11. *Let $(B, \mu, A, r, \phi, s, \psi, 1_A, R, \Delta)$ be a principal compact C^*-quantum groupoid. Then the weak C^*-pseudo-Kac system $(H, \alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ is a C^*-pseudo-Kac system.*

Proof. The C^*-pseudo-multiplicative unitaries $(H, \widehat{\beta}, \alpha, \beta, V)$, $(H, \widehat{\alpha}, \widehat{\beta}, \alpha, \check{V})$, $(H, \alpha, \beta, \widehat{\alpha}, \widehat{V})$ are regular by Theorems 5.4, 5.10 and Lemma 7.4, and the operator $X := \Sigma(1_\alpha \otimes_\beta U)V \in \mathcal{L}(H_{\widehat{\beta}}\otimes_\alpha H)$ satisfies $X^3 = \mathrm{id}$ because for all $b, c, d, e \in B$,

$$\begin{aligned}X^3\big(r(b)s(c^{op})\zeta_\psi \otimes r(d)s(e^{op})\zeta_\nu\big) &= X^2\Sigma(1 \otimes U)\big(r(b)\zeta_\psi \otimes r(d)s(c^{op})s(e^{op})\zeta_\nu\big) \\ &= X^2\big(s(d^{op})r(ec)\zeta_\nu \otimes r(b)\zeta_\psi\big) \\ &= X\Sigma(1 \otimes U)\big(r(ec)\zeta_\nu \otimes s(d^{op})r(b)\zeta_\phi\big) \\ &= X\big(r(d)s(b^{op})\zeta_\psi \otimes r(ec)\zeta_\nu\big) \\ &= \Sigma(1 \otimes U)\big(r(d)\zeta_\psi \otimes s(b^{op})r(ec)\zeta_\nu\big) \\ &= r(b)s((ec)^{op})\zeta_\nu \otimes r(d)\zeta_\psi \\ &= r(b)s(c^{op})\zeta_\psi e^{op} \otimes r(d)\zeta_\nu = r(b)s(c^{op})\zeta_\psi \otimes r(d)s(e^{op})\zeta_\nu. \qquad \square\end{aligned}$$

9 Compact and étale groupoids

Prototypical examples of compact C^*-quantum groupoids are the function algebra of a compact groupoid and the reduced groupoid C^*-algebra of an étale groupoid with compact space of units. In this section, we construct these examples, determine the associated dual Hopf C^*-bimodules, and show that the associated weak C^*-pseudo-Kac systems are C^*-pseudo Kac systems. We shall use some results from [17] and [16] which we recall first.

Preliminaries on locally compact groupoids Let us fix some notation and terminology related to locally compact groupoids; for details, see [13] or [11].

Throughout this section, let G be a locally compact, Hausdorff, second countable groupoid. We denote its unit space by G^0, its range map by r_G, its source map by s_G, and put $G^u := r_G^{-1}(\{u\})$, $G_u := s_G^{-1}(u)$ for each $u \in G^0$.

Let λ be a left Haar system on G and denote by λ^{-1} the associated right Haar system. Let μ_G be a probability measure on G^0 with full support and define measures ν_G, ν_G^{-1} on G by

$$\int_G f\, d\nu_G := \int_{G^0}\int_{G^u} f(x)\, d\lambda^u(x)\, d\mu_G(u), \qquad \int_G f d\nu_G^{-1} = \int_{G^0}\int_{G_u} f(x) d\lambda_u^{-1}(x)\, d\mu_G(u)$$

for all $f \in C_c(G)$. Thus, $\nu_G^{-1} = i_* \nu_G$, where $i\colon G \to G$ is given by $x \mapsto x^{-1}$. We assume that μ is quasi-invariant in the sense that ν_G and ν_G^{-1} are equivalent, and denote by $D := d\nu_G/d\nu_G^{-1}$ the Radon-Nikodym derivative.

In the following applications, we shall always assume that the unit space G^0 is compact and that the Radon-Nikodym derivative D is continuous.

The C^*-pseudo-Kac system of a locally compact groupoid In [17] and [16], we associated to G a C^*-pseudo-multiplicative unitary and a C^*-pseudo-Kac system as follows.

Denote by μ the trace on $C(G^0)$ given by $f \mapsto \int_{G^0} f d\mu_G$. Put $K := L^2(G, \nu_G)$ and define representations $r, s\colon C(G^0) \to \mathcal{L}(K)$ such that for all $x \in G$, $\xi \in C_c(G)$, and $f \in C(G^0)$,

$$\big(r(f)\xi\big)(x) := f\big(r_G(x)\big)\xi(x), \qquad \big(s(f)\xi\big)(x) := f\big(s_G(x)\big)\xi(x).$$

We define Hilbert C^*-$C(G^0)$-modules $L^2(G,\lambda)$ and $L^2(G,\lambda^{-1})$ as the respective completions of the pre-C^*-module $C_c(G)$, where for all $\xi, \xi' \in C_c(G)$, $u \in G^0$, $f \in C(G^0)$, $x \in G$,

$$\langle \xi' | \xi \rangle(u) = \int_{G^u} \overline{\xi'(x)}\xi(x) d\lambda^u(x), \qquad (\xi f)(x) = \xi(x) f(r_G(x)) \quad \text{in case of } L^2(G,\lambda),$$

$$\langle \xi' | \xi \rangle(u) = \int_{G_u} \overline{\xi'(x)}\xi(x) d\lambda_u^{-1}(x), \qquad (\xi f)(x) = \xi(x) f(s_G(x)) \quad \text{in case of } L^2(G,\lambda^{-1}).$$

There exist isometric embeddings $j\colon L^2(G,\lambda) \to \mathcal{L}(H_\mu, K)$ and $\hat{j}\colon L^2(G,\lambda^{-1}) \to \mathcal{L}(H_\mu, K)$ such that for all $\xi \in C_c(G)$, $\zeta \in H_\mu$, $x \in G$,

$$\big(j(\xi)\zeta\big)(x) = \xi(x)\zeta(r_G(x)), \qquad \big(\hat{j}(\xi)\zeta\big)(x) = \xi(x) D(x)^{-1/2}\zeta(s_G(x)). \tag{16}$$

Put $\rho := j(L^2(G,\lambda))$ and $\sigma := \hat{j}(L^2(G,\lambda^{-1}))$. Then (K,σ,ρ,ρ) is a C^*-(μ^{op},μ,μ^{op})-module, and j and $\hat{j}$ are unitary maps of Hilbert C^*-modules over $C(G^0)$.

Define measures $\nu^2_{s,r}$ on $G^2_{s,r} := \{(x,y) \in G \times G \mid s_G(x) = r_G(y)\}$ and $\nu^2_{r,r}$ on $G^2_{r,r} := \{(x,y) \in G^2 \mid r_G(x) = r_G(y)\}$ by

$$\int_{G^2_{s,r}} f\, d\nu^2_{s,r} := \int_{G^0} \int_{G^u} \int_{G^{s_G(x)}} f(x,y)\, d\lambda^{s_G(x)}(y)\, d\lambda^u(x)\, d\mu_G(u),$$

$$\int_{G^2_{r,r}} g\, d\nu^2_{r,r} := \int_{G^0} \int_{G^u} \int_{G^u} g(x,y)\, d\lambda^u(y)\, d\lambda^u(x)\, d\mu_G(u)$$

for all $f \in C_c(G^2_{s,r})$, $g \in C_c(G^2_{r,r})$. Then there exist isomorphisms

$$\Phi_{\sigma,\rho}\colon K_\sigma \otimes_\rho K \to L^2(G^2_{s,r}, \nu^2_{s,r}), \qquad \Phi_{\rho,\rho}\colon K_\rho \otimes_\rho K \to L^2(G^2_{r,r}, \nu^2_{r,r})$$

such that for all $\eta, \xi \in C_c(G)$, $\zeta \in C_c(G^0)$, $(x,y) \in G^2_{s,r}$, $(x',y') \in G^2_{r,r}$,

$$\Phi_{\sigma,\rho}\big(\hat{j}(\eta) \circledS \zeta \circledS j(\xi)\big)(x,y) = \eta(x) D(x)^{-1/2} \zeta(s_G(x)) \xi(y),$$

$$\Phi_{\rho,\rho}\big(j(\eta) \circledS \zeta \circledS j(\xi)\big)(x',y') = \eta(x') \zeta(r_G(x')) \xi(y').$$

From now on, we identify $K_\sigma \otimes_\rho K$ with $L^2(G^2_{s,r}, \nu^2_{s,r})$ via $\Phi_{\sigma,\rho}$ and $K_\rho \otimes_\rho K$ with $L^2(G^2_{r,r}, \nu^2_{r,r})$ via $\Phi_{\rho,\rho}$ without further notice.

Theorem 9.1 ([17],[16]). *There exists a C^*-pseudo-Kac system $(K,\rho,\sigma,\rho,\sigma,U_G,V_G)$ such that for all $\omega \in C_c(G^2_{s,r})$, $(x,y) \in G^2_{r,r}$, $\xi \in C_c(G)$, $z \in G$,*

$$(V_G\omega)(x,y) = \omega(x, x^{-1}y), \qquad (U_G\xi)(z) = \xi(z^{-1}) D(z)^{-1/2}.$$

Proposition 9.2. *i) If G is r-discrete and $1_{G^0} \in C(G)$ denotes the characteristic function of the unit space, then $j(1_{G^0}) = \hat{j}(1_{G^0}) \in \mathrm{Fix}(V_G)$ and (K,σ,ρ,ρ,V_G) is compact.*

ii) If G is compact, then $j(1_G) \in \mathrm{Cofix}(V_G)$ and (K,σ,ρ,ρ,V_G) is étale.

Proof. Straightforward. □

The concrete Hopf C^*-bimodules $\big(\widehat{A}(V_G)^{\sigma,\rho}_K, \widehat{\Delta}_{V_G}\big)$ and $\big(A(V_G)^{\rho,\rho}_K, \Delta_{V_G}\big)$ can be described as follows. Denote by $m\colon C_0(G) \to \mathcal{L}(L^2(G,\nu_G))$ the representation given by multiplication operators. Recall that for each $g \in C_c(G)$, there exists an operator $L(g) \in \mathcal{L}(K)$ such that

$$\big(L(g)f\big)(y) = \int_{G^{r_G(y)}} g(x)D(x)^{-1/2}f(x^{-1}y)d\lambda^{r_G(y)}(x) \quad \text{for all } f \in C_C(G), y \in G,$$

and that $L(g)^* = L(g^*)$, where $g^*(x) = \overline{g(x^{-1})}$ for all $x \in G$. The reduced groupoid C^*-algebra $C^*_r(G)$ is the closed linear span of all operators $L(g)$, where $g \in C_c(G)$ [13].

Theorem 9.3 ([17]). *i)* $\widehat{A}(V_G) = m(C_0(G))$ *and* $\big(\widehat{\Delta}_{V_G}(m(f))\omega\big)(x,y) = f(xy)\omega(x,y)$ *for each* $f \in C_0(G)$, $\omega \in L^2(G^2_{s,r}, \nu^2_{s,r})$, $(x,y) \in G^2_{s,r}$.

ii) $A(V_G) = C^*_r(G)$, *and for each* $g \in C_c(G)$, $\omega \in L^2(G^2_{r,r}, \nu^2_{r,r})$, $(x,y) \in G^2_{r,r}$,

$$\big(\Delta_{V_G}(L(g))\omega\big)(x,y) = \int_{G^{r_G(x)}} g(z)D(z)^{-1/2}\omega(z^{-1}x, z^{-1}y)d\lambda^{r_G(x)}(z).$$

The function algebra of a compact groupoid Let G be a locally compact groupoid as before but assume that G is compact. Then the following assertions are evident:

Lemma 9.4. *i) There exists a compact* C^**-quantum graph* $(C(G^0), \mu, C(G), r, \phi, s, \psi, D^{-1})$ *with coinvolution* R *such that*

$$(r(f))(x) = f(r_G(x)), \qquad (s(f))(x) = f(s_G(x)) \qquad \text{for all } f \in C(G^0),\ x \in G,$$

$$(\phi(g))(u) = \int_{G^u} g(y)d\lambda^u(y), \quad (\psi(g))(u) = \int_{G_u} g(y)d\lambda_u^{-1}(y) \quad \text{for all } g \in C(G),\ u \in G^0,$$

and $(R(g))(x) = g(x^{-1})$ *for all* $g \in C(G)$, $x \in G$. *The states* $\nu = \mu \circ \phi$ *and* $\nu^{-1} = \mu \circ \psi$ *are given by* $\nu(g) = \int_G g d\nu_G$ *and* $\nu^{-1}(g) = \int_G g d\nu_G^{-1}$ *for all* $g \in C(G)$.

ii) If we identify $H = H_\nu$ *with* $K = L^2(G,\nu)$ *via* $f\zeta_\nu \equiv f$ *for all* $f \in C(G)$, *then* $j(f) = f\zeta_\phi$ *and* $\hat{j}(f) = f\zeta_\psi$ *for all* $f \in C(G)$, *and* $(H, \hat{\alpha}, \beta, \hat{\beta}, \alpha) = (K, \rho, \rho, \sigma, \sigma)$. □

With respect to the canonical identification $H = H_\nu \cong L^2(G,\nu) = K$, the representation $m\colon C(G) \to \mathcal{L}(K) \cong \mathcal{L}(H)$ corresponds to the GNS-representation for ν. We identify $C(G)$ with $m(C(G)) \subseteq \mathcal{L}(K) = \mathcal{L}(H)$ via m.

Theorem 9.5. *i)* $(C(G^0), \mu, C(G), r, \phi, s, \psi, D^{-1}, \widehat{\Delta}_{V_G}, R)$ *is a compact* C^**-quantum groupoid.*

ii) The associated C^**-pseudo-multiplicative unitary* $(H, \alpha, \beta, \hat{\alpha}, W)$ *equals* $(K, \sigma, \rho, \rho, V_G)$.

iii) The associated weak C^**-pseudo-Kac system* $(H, \alpha, \hat{\alpha}, \beta, \hat{\beta}, U, V)$ *is a* C^**-pseudo-Kac system.*

iv) Ad_U *defines an isomorphism between the dual Hopf* C^**-bimodule* $(\widehat{A}(V)^{\hat{\beta},\alpha}_H, \widehat{\Delta}_V)$ *and* $(C^*_r(G)^{\rho,\rho}_K, \Delta_{V_G})$.

Proof. i) Put $\Delta := \widehat{\Delta}_{V_G}$. By Theorems 9.1, 9.3, [17, Theorem 4.14], and Lemma 9.4, $(C(G)^{\alpha,\beta}_H, \Delta) = (\widehat{A}(V_G)^{\sigma,\rho}_K, \widehat{\Delta}_{V_G})$ is a Hopf C^*-bimodule and $[\Delta(C(G))|\alpha\rangle_1] = [|\alpha\rangle_1 C(G)]$, $[\Delta(C(G))|\beta\rangle_2] = [|\beta\rangle_2 C(G)]$.

By Lemma 9.4 iii) and Proposition 9.2 ii), $\zeta_\phi = j(1_G) \in \mathrm{Cofix}(V_G)$. Remark 6.6 shows that $[\Delta(C(G))|\zeta_\phi\rangle_2 C(G)] = [|\beta\rangle_2 C(G)]$. Moreover, $\zeta_\psi = U\zeta_\phi \in \mathrm{Fix}(\check{V}_G)$ by Lemma 6.10, $C(G) = \widehat{A}(V_G) = A(\check{V}_G)$ by equation (12), and now a second application of Remark 6.6 shows that $[\Delta(C(G))|\zeta_\psi\rangle_1 C(G)] = [|\alpha\rangle_1 C(G)]$.

By Theorem 6.8 and Corollary 6.11, $\phi\colon a \mapsto \zeta_\phi^* a \zeta_\phi$ and $\psi\colon a \mapsto \zeta_\phi^* U a U \zeta_\phi$ are a bounded left and a bounded right Haar weight for $(C(G)^{\alpha,\beta}_H, \Delta)$, respectively.

Finally, we show that the strong invariance condition iii) in Definition 4.8 holds. For all $f, g \in C(G)$, the operator $\langle\zeta_\psi|_1\Delta(f)(g^{op}{}_\alpha\otimes_\beta 1)|\zeta_\psi\rangle_1$ is given by pointwise multiplication by the function

$$H_{f,g}\colon G \to \mathbb{C},\ y \mapsto \int_{G_{r_G(y)}} f(xy)g(x)d\lambda^{-1}_{r_G(y)}(x),$$

and by right-invariance of λ^{-1},

$$\begin{aligned}\big(R(H_{g,f})\big)(y) = H_{g,f}(y^{-1}) &= \int_{G_{r_G(y^{-1})}} g(xy^{-1})f(x)d\lambda^{-1}_{r_G(y^{-1})}(x) \\ &= \int_{G_{r_G(y)}} g(x')f(x'y)d\lambda^{-1}_{r_G(y)}(x') = H_{f,g}(y) \quad \text{for all } y \in G.\end{aligned}$$

ii) With respect to the identifications $H_\alpha\otimes_\beta H = K_\sigma\otimes_\rho K \cong L^2(G^2_{s,r}, \nu^2_{s,r})$ and $H_\beta\otimes_{\hat\alpha} H = K_\rho\otimes_\rho K \cong L^2(G^2_{r,r}, \nu^2_{r,r})$,

$$\big(W^*|j(g)\rangle_2 f\big)(x,y) = \big(\Delta(g)|\zeta_\phi\rangle_2 f\big)(x,y) = f(x)g(xy)$$

for all $(x,y) \in G^2_{s,r}$ and $f, g \in C(G)$ and hence $(W^*\omega)(x,y) = \omega(x,xy) = (V_G^*\omega)(x,y)$ for all $\omega \in L^2(G^2_{s,r}, \nu^2_{s,r})$.

iii) Since $C(G)$ is commutative, $J\xi = \overline{\xi}$, and $(U\xi)(x) = (IJ\xi)(x) = \xi(x^{-1})D(x)^{-1/2} = (U_G\xi)(x)$ for all $\xi \in L^2(G,\nu_G)$ and $x \in G$. By Theorem 9.1, $(K, \rho, \sigma, \rho, \sigma, U_G, V_G)$ is a C^*-pseudo-Kac system, and since $V_G = W = \widehat{V}$ by ii) and Lemma 7.4, we can use [16, Proposition 5.5] to conclude that $(H, \alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ is a C^*-pseudo-Kac system.

iv) This assertion follows from the relation $\widehat{V} = V_G$ (see above), equation (12), and Theorem 9.3. ☐

The groupoid C^*-algebra of an étale groupoid Let G be a locally compact groupoid as above but assume that G is r-discrete and that λ is the family of counting measures. Since $G^0 \subseteq G$ is closed and open, we can embed $C(G^0)$ in $C(G)$ by extending each function by 0 outside of G^0. Thus, 1_{G^0} gets identified with the characteristic function of G^0. Denote by $r, s\colon C(G^0) \to C(G)$ the transpose of the range and the source map r_G and s_G, respectively.

Lemma 9.6. *i) There exists a compact C^*-quantum graph $(C(G^0), \mu, C^*_r(G), \iota, \phi, \iota, \phi, 1)$ with coinvolution R such that*

$$\iota(f) = L(f) \text{ for each } f \in C(G^0), \quad \big(\phi(L(g))\big)(u) = g(u) \text{ for each } g \in C_c(G),\ u \in G^0,$$
$$RL(f) = L(f^\dagger), \text{ where } f^\dagger(x) = f(x^{-1}) \text{ for all } x \in G,\ f \in C_c(G).$$

The state $\nu = \mu \circ \phi$ is given by $\nu(a) = \langle 1_{G^0}|a 1_{G^0}\rangle$ for all $a \in C^*_r(G)$, and its modular automorphism group is given by $\sigma^\nu_t(L(f)) = L(D^{it}f)$ for all $f \in C_c(G)$, $t \in \mathbb{R}$.

ii) There exists an isomorphism $\Xi\colon H_\nu \to K$, $L(f)\zeta_\nu \mapsto fD^{-1/2}$, and $\Xi L(f)^{op}\zeta_\nu = f$, $\Xi L(f)\zeta_\phi = \hat{j}(f)$, $\Xi L(f)^{op}\zeta_\phi = j(f)$ for all $f \in C_c(G)$. In particular, $\Xi\widehat{\alpha} = \Xi\widehat{\beta} = \sigma$ and $\Xi\alpha = \Xi\beta = \rho$.

Proof. i) The $*$-homomorphism ι clearly is well-defined. Denote by $\zeta\colon L^2(G^0,\mu_G) \to L^2(G,\nu_G)$ the embedding that extends each function outside of G^0 by 0. Then for each $g \in C_c(G)$, the operator $\zeta^*L(g)\zeta \subseteq \mathcal{L}(L^2(G^0,\mu_G))$ is given by pointwise multiplication by the function $g|_{G^0}$, and we can define $\phi\colon C^*_r(G) \to C(G^0) \subseteq \mathcal{L}(L^2(G^0,\mu_G))$ by $a \mapsto \zeta^*a\zeta$. Clearly, $\iota\circ\phi\colon C^*_r(G) \to \iota(C(G^0))$ is a conditional expectation. Since $\zeta\zeta_\mu = 1_{G^0}$, the state

$\mu \circ \phi$ is given by $\nu(a) = \mu(\zeta^* a\zeta) = \langle 1_{G^0}|a 1_{G^0}\rangle$ for all $a \in C_r^*(G)$. By [13, §II.5], this is a KMS-state with modular automorphism group as stated above — indeed, for all $f \in C_c(G)$,

$$\begin{aligned}\nu(L(f)^* L(f)) &= \int_{G^0}\int_{G^u} \overline{f(x^{-1})} f(x^{-1}) d\lambda^u(x) d\mu_G(u) \\ &= \int_G \overline{f(x)} f(x) D(x)^{-1} d\nu_G(x) = \nu(L(D^{-1/2}f)L(D^{-1/2}f)^*).\end{aligned}$$

Finally, $\sigma_t^\nu \circ \iota = \iota$ and $\phi \circ \sigma_t^\nu = \phi$ for all $t \in \mathbb{R}$ because $D|_{G^0} = 1_{G^0}$.

ii) First, observe that for all $f \in C_c(G)$ and $x \in G$

$$\big(L(f)1_{G^0}\big)(x) = f(x)D^{-1/2}(x) \tag{17}$$

and hence $\|L(f)\zeta_\nu\|_H^2 = \nu(L(f)^*L(f)) = \langle L(f)1_{G^0}|L(f)1_{G^0}\rangle = \|fD^{-1/2}\|_K^2$. The existence of Ξ follows. The remaining assertions hold because for all $f \in C_c(G)$, $g \in C(G^0)$,

$$\begin{aligned}\Xi L(f)^{op}\zeta_\nu &= \Xi\sigma_{-i/2}^\nu(L(f))\zeta_\nu = \Xi L(D^{1/2}f)\zeta_\nu = f, \\ \Xi L(f)\zeta_\phi g\zeta_\mu &= \Xi L(f)L(g)\zeta_\nu = \Xi L(fs(g))\zeta_\nu = fs(g)D^{-1/2} = \hat{\jmath}(f)g, \\ \Xi L(f)^{op}\zeta_\phi g\zeta_\mu &= \Xi L(g)L(f)^{op}\zeta_\nu = \Xi r(g)L(D^{1/2}f)\zeta_\nu = r(g)f = j(f)g.\end{aligned}$$

□

From now on, we identify $H = H_\nu$ with K via Ξ without further notice.

Lemma 9.7. *$\big(L(f)^{op}g\big)(x) = \int_{G^{s_G(x)}} g(xy)f(y^{-1})d\lambda^{s_G(x)}(y)$ for all $f, g \in C_c(G)$, $x \in G$.*

Proof. Let $f, g \in C_c(G)$. Then

$$\begin{aligned}L(f)^{op}g = \Xi L(f)^{op}L(gD^{1/2})\zeta_\nu &= \Xi L(D^{1/2}g)\sigma_{-i/2}^\nu(L(f))\zeta_\nu \\ &= \Xi L(D^{1/2}g)L(D^{1/2}f)\zeta_\nu = \Xi L(h)\zeta_\nu = hD^{-1/2},\end{aligned}$$

where for all $x \in G$,

$$\begin{aligned}h(x) &= \int_{G^{r_G(x)}} D^{1/2}(z)g(z)D^{1/2}(z^{-1}x)f(z^{-1}x)d\lambda^{r_G(x)}(z) \\ &= D^{1/2}(x)\int_{G^{s_G(x)}} g(xy)f(y^{-1})d\lambda^{s_G(x)}(y).\end{aligned}$$

□

Theorem 9.8. *i) $(C(G^0), \mu, C_r^*(G), \iota, \phi, \iota, \phi, 1, R, \Delta_{V_G})$ is a compact C^*-quantum groupoid.*

ii) The associated C^*-pseudo-multiplicative unitary is V_G.

iii) The associated weak C^*-pseudo-Kac system is a C^*-pseudo-Kac system.

iv) The dual Hopf C^*-bimodule is $(C(G)_K^{\sigma,\rho}, \widehat{\Delta}_{V_G})$.

Proof. i) Put $\Delta := \Delta_{V_G}$. By Theorems 9.1, 9.3, [17, Theorem 4.14], and Lemma 9.6, $(C_r^*(G)_H^{\alpha,\beta}, \Delta) = (C_r^*(G)_K^{\rho,\rho}, \Delta_{V_G})$ is a Hopf C^*-bimodule and $[\Delta(C_r^*(G))|\alpha\rangle_1] = [|\alpha\rangle_1 C_r^*(G)]$, $[\Delta(C_r^*(G))|\beta\rangle_2] = [|\beta\rangle_2 C_r^*(G)]$.

By Lemma 9.6 and Proposition 9.2, $1_{C_r^*(G)}\zeta_\phi = L(1_{G^0})\zeta_\phi = j(1_{G^0}) = \hat{\jmath}(1_{G^0}) \in \mathrm{Fix}(V_G)$. Remark 6.6 shows that $[\Delta(C_r^*(G))|\zeta_\phi\rangle_1 C_r^*(G)] = [|\alpha\rangle_1 C_r^*(G)]$. Moreover, $\zeta_\phi \in \mathrm{Cofix}(\widehat{V}_G)$ by Lemma 6.10, $C_r^*(G) = A(V_G) = \widehat{A}(\widehat{V}_G)$ by equation (12), and now a second application of Remark 6.6 shows that $[\Delta(C_r^*(G))|\zeta_\phi\rangle_2 C_r^*(G)] = [|\beta\rangle_2 C_r^*(G)]$.

The map $\phi\colon a \mapsto \zeta_\phi^* a\zeta_\phi$ is a bounded left and a bounded right Haar weight for $(C_r^*(G)_H^{\alpha,\beta}, \Delta)$ by Theorem 6.8 and Corollary 6.11.

Finally, let us prove that the strong invariance condition iii) in Definition 4.8 holds. Let $f, g, \xi \in C_c(G)$. By the previous Lemma,

$$\big((L(g^{op})_\alpha \otimes_\beta 1)|\zeta_\psi\rangle_1\xi\big)(x, y) = \xi(y)g(x) \quad \text{for all } (x, y) \in G_{r,r}^2$$

and hence

$$\big(\langle\zeta_\psi|_1\Delta(L(f))(L(g^{op})_\alpha\otimes_\beta 1)|\zeta_\psi\rangle_1\xi\big)(y) = \int_{G^{r_G(y)}} f(z)D^{-1/2}(z)g(z^{-1})\xi(z^{-1}y)d^{r_G(y)}(z)$$
$$= \big(L(h)\xi\big)(y),$$

where $h(z) = f(z)g(z^{-1})$ for all $z \in G$. Switching f, g, we find

$$\big(\langle\zeta_\psi|_1\Delta(L(g))(L(f^{op})_\alpha\otimes_\beta 1)|\zeta_\psi\rangle_1\big) = L(h^\dagger) = R(L(h)).$$

Since f, g were arbitrary, condition iii) in Definition 4.8 holds.

ii) By Lemma 9.6, we have for all $f, g \in C_c(G)$, $(x, y) \in G^2_{r,r}$,

$$\begin{aligned}\big(V(L(f)\zeta_\psi \circledcirc L(g)\zeta_\nu)\big)(x,y) &= \big(\Delta(L(f))(\zeta_\psi \circledcirc L(g)\zeta_\nu)\big)(x,y)\\ &= \int_{G^{r_G(x)}} f(z)D^{-1/2}(z)\big(\zeta_\psi \circledcirc L(g)\zeta_\nu\big)(z^{-1}x, z^{-1}y)d\lambda^{r_G(x)}(z)\\ &= \int_{G^{r_G(x)}} f(z)D^{-1/2}(z)1_{G^0}(z^{-1}x)g(z^{-1}y)D^{-1/2}(z^{-1}y)d\lambda^{r_G(x)}(z)\\ &= f(x)D^{-1/2}(x)g(x^{-1}y)D^{-1/2}(x^{-1}y)\\ &= (\hat{j}(f) \circledcirc gD^{-1/2})(x, x^{-1}y) = \big(L(f)\zeta_\psi \circledcirc L(g)\zeta_\nu\big)(x, x^{-1}y).\end{aligned}$$

iii), iv) The proof is similar to the proof of Theorem 9.5 iii), iv). □

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