C^*-pseudo-multiplicative unitaries and Hopf C^*-bimodules

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Abstract

We introduce C^*-pseudo-multiplicative unitaries and concrete Hopf C^*-bimodules for the study of quantum groupoids in the setting of C^*-algebras. These unitaries and Hopf C^*-bimodules generalize multiplicative unitaries and Hopf C^*-algebras and are analogues of the pseudo-multiplicative unitaries and Hopf–von Neumann-bimodules studied by Enock, Lesieur and Vallin. To each C^*-pseudo-multiplicative unitary, we associate two Fourier algebras with a duality pairing, a C^*-tensor category of representations, and in the regular case two reduced and two universal Hopf C^*-bimodules. The theory is illustrated by examples related to locally compact Hausdorff groupoids. In particular, we obtain a continuous Fourier algebra for a locally compact Hausdorff groupoid.

1 Introduction

Multiplicative unitaries, which were first systematically studied by Baaj and Skandalis [3], are fundamental to the theory of quantum groups in the setting of operator algebras and to generalizations of Pontrjagin duality [36]. First, one can associate to every locally compact quantum group a multiplicative unitary [15, 16, 21]. Out of this unitary, one can construct two Hopf C^*-algebras, where one coincides with the initial quantum group, while the other is the generalized Pontrjagin dual of the quantum group. The duality manifests itself by a pairing on dense Fourier subalgebras of the two Hopf C^*-algebras. These Hopf C^*-algebras can be completed to Hopf–von Neumann algebras and are reduced in the sense that they correspond to the regular representations of the quantum group and of its dual, respectively. Considering arbitrary representations, one can also construct out of the associated unitary two universal Hopf C^*-algebras with morphisms onto the reduced ones. In the study of coactions of quantum groups on algebras, the unitary is an essential tool for the construction of dual coactions on the reduced crossed products and in the proof of biduality [3] which generalizes the Takesaki–Takai duality.

Much of the theory of quantum groups has been generalized for quantum groupoids in a variety of settings, for example, for finite quantum groupoids in the setting of finite-dimensional C^*-algebras by Böhm, Szlachányi, Nikshych and others [5, 6, 7, 22] and for measurable quantum groupoids in the setting of von Neumann algebras by Enock, Lesieur and Vallin [10, 11, 12, 19]. Fundamental for the second theory are the Hopf–von Neumann bimodules and pseudo-multiplicative unitaries introduced by Vallin [37, 38].

In this article, we introduce generalizations of multiplicative unitaries and Hopf C^*-algebras that are suited for the study of locally compact quantum groupoids in the setting of C^*-algebras, and extend many of the results on multiplicative unitaries that were obtained by Baaj and Skandalis in [3]. In particular, we associate to every regular C^*-pseudo-multiplicative unitary two Hopf C^*-bimodules and two Fourier algebras with a duality pairing, and construct universal Hopf C^*-bimodules from a C^*-tensor category of representations of the unitary. The theory presented here was applied already in [30] to the definition and study of compact C^*-quantum groupoids, and will be applied in a forthcoming article to the study of reduced crossed products for coactions of Hopf C^*-bimodules on C^*-algebras and to an extension of the Baaj-Skandalis duality theorem; see also [32].

Our concepts are related to their von Neumann-algebraic counterparts as follows. In the theory of quantum groups, one can use the multiplicative unitary to pass between the setting of von Neumann algebras and the setting of C^*-algebras and thus obtains a bijective correspondence between measurable and locally compact quantum groups. This correspondence breaks down for quantum groupoids — already for ordinary spaces, considered as groupoids consisting entirely of units, a measure does not determine a topology. In particular, one can not expect to pass from a measurable quantum groupoid in the setting of von Neumann algebras to a locally compact quantum groupoid in the setting of C^*-algebras in a canonical way. The reverse passage, however, is possible, at least on the level of the unitaries and the Hopf bimodules.

Fundamental to our approach is the framework of modules, relative tensor products and fiber products in the setting of C^*-algebras introduced in [31]. That article also explains in detail how the theory developed here can be reformulated in the setting of von Neumann algebras, where we recover Vallin's notions of a pseudo-multiplicative unitary and a Hopf–von Neumann bimodule, and how to pass from the level of C^*-algebras to the setting of von Neumann algebras by means of various functors.

The theory presented here overcomes several restrictions of our former generalizations of multiplicative unitaries and Hopf C^*-algebras [34]; see also [33].

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Organization This article is organized as follows. We start with preliminaries, summarizing notation, terminology and some background on Hilbert C^*-modules.

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In Section 2, we recall the notion of a multiplicative unitary and define C^*-pseudo-multiplicative unitaries. This definition involves C^*-modules over C^*-bases and their relative tensor product, which were introduced in [31] and which we briefly recall. As an example, we construct the C^*-pseudo-multiplicative unitary of a locally compact Hausdorff groupoid. We shall come back to this example frequently.

In Section 3, we associate to every well-behaved C^*-pseudo-multiplicative unitary two Hopf C^*-bimodules. These Hopf C^*-bimodules are generalized Hopf C^*-algebras, where the target of the comultiplication is no longer a tensor product but a fiber product that is taken relative to an underlying C^*-base. Inside these Hopf C^*-bimodules, we identify dense convolution subalgebras which can be considered as generalized Fourier algebras, and construct a dual pairing on these subalgebras. To illustrate the theory, we apply all constructions to the unitary associated to a groupoid G, where one recovers the reduced groupoid C^*-algebra of G on one side and the function algebra of G on the other side.

In Section 4, we study representations and corepresentations of C^*-pseudo-multiplicative unitaries. These (co)representations form a C^*-tensor category and lead to the construction of universal variants of the Hopf C^*-bimodules introduced in Section 3. For the unitary associated to a groupoid, we establish a categorical equivalence between corepresentations of the unitary and representations of the groupoid.

In Section 5, we show that every C^*-pseudo-multiplicative unitary satisfying a certain regularity condition is well-behaved. This condition is satisfied, for example, by the unitaries associated to groupoids and by the unitaries associated to compact quantum groupoids. Furthermore, we collect some results on proper and étale C^*-pseudo-multiplicative unitaries.

Terminology and notation Given a subset Y of a normed space X, we denote by $[Y] \subset X$ the closed linear span of Y. We call a linear map ϕ between normed spaces *contractive* or a *linear contraction* if $\|\phi\| \leq 1$.

All sesquilinear maps like inner products of Hilbert spaces are assumed to be conjugate-linear in the first component and linear in the second one. Let H,K be Hilbert spaces. We canonically identify $\mathcal{L}(H,K)$ with a subspace of $\mathcal{L}(H \oplus K)$. Given subsets $X \subseteq \mathcal{L}(H)$ and $Y \subseteq \mathcal{L}(H,K)$, we denote by X' the commutant of X and by $[\![Y]\!]$ the σ-weak closure of Y.

Given a C^*-subalgebra $A \subseteq \mathcal{L}(H)$ and a $*$-homomorphism $\pi\colon A \to \mathcal{L}(K)$, we put

$$\mathcal{L}^{\pi}(H,K) := \{T \in \mathcal{L}(H,K) \mid Ta = \pi(a)T \text{ for all } a \in A\}. \tag{1}$$

We use the ket-bra notation and define for each $\xi \in H$ operators $|\xi\rangle\colon \mathbb{C} \to H$, $\lambda \mapsto \lambda\xi$, and $\langle\xi| = |\xi\rangle^*\colon H \to \mathbb{C}$, $\xi' \mapsto \langle\xi|\xi'\rangle$.

We shall use some theory of groupoids; for background, see [26] or [24]. Given a groupoid G, we denote its unit space by G^0, its range map by r, its source map by s, and let $G_r\times_r G = \{(x,y) \in G \times G \mid r(x) = r(y)\}$, $G_s\times_r G = \{(x,y) \in G \times G \mid s(x) = r(y)\}$ and $G^u = r^{-1}(u)$, $G_u = s^{-1}(u)$ for each $u \in G^0$.

We shall make extensive use of (right) Hilbert C^*-modules and the internal tensor product; a standard reference is [17]. Let A and B be C^*-algebras. Given Hilbert C^*-modules E and F over B, we denote by $\mathcal{L}_B(E,F)$ the space of all adjointable operators from E to F. Let E and F be C^*-modules over A and B, respectively, and let $\pi\colon A \to \mathcal{L}_B(F)$ be a $*$-homomorphism. Recall that the internal tensor product $E \otimes_\pi F$ is the Hilbert C^*-module over B which is the closed linear span of elements $\eta \otimes_\pi \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary and $\langle \eta \otimes_\pi \xi | \eta' \otimes_\pi \xi' \rangle = \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\eta \otimes_\pi \xi) b = \eta \otimes_\pi \xi b$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$ [17, §4]. We denote the internal tensor product by "$\otimes$" and drop the index π if the representation is understood; thus, for example, $E \otimes F = E \otimes_\pi F = E \otimes_\pi F$.

We also define a *flipped internal tensor product* $F_\pi\otimes E$ as follows. We equip the algebraic tensor product $F \odot E$ with the structure maps $\langle \xi \odot \eta | \xi' \odot \eta' \rangle := \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$, $(\xi \odot \eta) b := \xi b \odot \eta$, form the separated completion, and obtain a Hilbert C^*-module $F_\pi\otimes E$ over B which is the closed linear span of elements $\xi_\pi\otimes\eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary and $\langle \xi_\pi\otimes\eta | \xi'_\pi\otimes\eta' \rangle = \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\xi_\pi\otimes\eta) b = \xi b_\pi\otimes\eta$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$. As above, we drop the index π and simply write "$\otimes$" instead of "${}_\pi\otimes$" if the representation π is understood. Evidently, the usual and the flipped internal tensor product are related by a unitary map $\Sigma\colon F \otimes E \xrightarrow{\cong} E \otimes F$, $\eta \otimes \xi \mapsto \xi \otimes \eta$.

For each $\xi \in E$, the maps

$$l_F^\pi(\xi)\colon F \to E \otimes F,\ \eta \mapsto \xi \otimes \eta, \qquad r_F^\pi(\xi)\colon F \to F \otimes E,\ \eta \mapsto \eta \otimes \xi, \tag{2}$$

are adjointable operators, and for all $\eta \in F$, $\xi' \in E$,

$$l_F^\pi(\xi)^*(\xi' \otimes \eta) = \pi(\langle \xi | \xi' \rangle)\eta = r_F^\pi(\xi)^*(\eta \otimes \xi').$$

Again, we drop the supscript π in $l_F^\pi(\xi)$ and $r_F^\pi(\xi)$ if this representation is understood.

Finally, let E_1, E_2 be Hilbert C^*-modules over A, let F_1, F_2 be Hilbert C^*-modules over B with representations $\pi_i\colon A \to \mathcal{L}_B(F_i)$ $(i = 1,2)$, and let $S \in \mathcal{L}_A(E_1, E_2)$, $T \in \mathcal{L}_B(F_1, F_2)$ such that $T\pi_1(a) = \pi_2(a)T$ for all $a \in A$. Then there exists a unique operator $S \otimes T \in \mathcal{L}_B(E_1 \otimes F_1, E_2 \otimes F_2)$ such that $(S \otimes T)(\eta \otimes \xi) = S\eta \otimes T\xi$ for all $\eta \in E_1$, $\xi \in F_1$, and $(S \otimes T)^* = S^* \otimes T^*$ [9, Proposition 1.34].

2 C^*-pseudo-multiplicative unitaries

Recall that a multiplicative unitary on a Hilbert space H is a unitary $V\colon H \otimes H \to H \otimes H$ that satisfies the *pentagon equation* $V_{12}V_{13}V_{23} = V_{23}V_{12}$ (see [3]). Here, V_{12}, V_{13}, V_{23} are operators on $H \otimes H \otimes H$ defined by $V_{12} = V \otimes \mathrm{id}$, $V_{23} = \mathrm{id} \otimes V$, $V_{13} = (\Sigma \otimes \mathrm{id})V_{23}(\Sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \Sigma)V_{12}(\mathrm{id} \otimes \Sigma)$, where $\Sigma \in \mathcal{L}(H \otimes H)$ denotes the flip $\eta \otimes \xi \mapsto \xi \otimes \eta$. As an example, consider a locally compact group G with left Haar measure λ. Then the formula

$$(Vf)(x,y) = f(x, x^{-1}y) \tag{3}$$

defines a linear bijection of $C_c(G \times G)$ that extends to a unitary on $L^2(G \times G, \lambda \otimes \lambda) \cong L^2(G, \lambda) \otimes L^2(G, \lambda)$. This unitary is multiplicative, and the pentagon equation amounts to associativity of the multiplication in G.

In this section, we generalize the notion of a multiplicative unitary so that it covers the example above if we replace the group G by a locally compact Hausdorff groupoid

G. In that case, formula (3) only makes sense for $(x,y) \in G_r{}_{\times_r}G$ and defines a linear bijection from $C_c(G_s{}_{\times_r}G)$ to $C_c(G_r{}_{\times_r}G)$. If the groupoid G is finite, that bijection is a unitary from $l^2(G_s{}_{\times_r}G)$ to $l^2(G_r{}_{\times_r}G)$, and these two Hilbert spaces can be identified with tensor products of $l^2(G)$ with $l^2(G)$, considered as a module over the algebra $C(G^0)$ with respect to representations that are naturally induced by the maps $s,r\colon G \to G^0$. For a general groupoid, the simple algebraic tensor product of modules has to be replaced by a refined version. In the setting of von Neumann algebras, Vallin used the relative tensor product of Hilbert modules introduced by Connes, also known as Connes' fusion of correspondences, to define pseudo-multiplicative unitaries [38] which include as a main example the unitary of a measurable groupoid. To take the topology of G into account, we shall work in the setting of C^*-algebras and use the relative tensor product of C^*-modules over C^*-bases introduced in [31].

2.1 The relative tensor product of C^*-modules over C^*-bases

In this subsection, we recall the relative tensor product of C^*-modules over C^*-bases which is fundamental to the definition of a C^*-pseudo-multiplicative unitary, and generalize the theory presented in [31, Section 2] in two respects. First, we introduce the notion of a semi-morphism between C^*-modules which will be important in subsection 4.4. Second, the definition of a C^*-pseudo-multiplicative unitaries forces us to consider C^*-n-modules for $n \geq 2$ and not only C^*-bimodules. We shall not give separate proofs of statements that are only mild generalizations of statements found in [31]. For additional motivation and details, we refer to [31]; an extended example can be found in subsection 2.3.

C^*-modules over C^*-bases A C^*-*base* is a triple $(\mathfrak{K},\mathfrak{B},\mathfrak{B}^\dagger)$ consisting of a Hilbert space $\mathfrak{K}$ and two commuting nondegenerate C^*-algebras $\mathfrak{B},\mathfrak{B}^\dagger \subseteq \mathcal{L}(\mathfrak{K})$. A C^*-base should be thought of as a C^*-algebraic counterpart to pairs consisting of a von Neumann algebra and its commutant. As an example, one can associate to every faithful KMS-state μ on a C^*-algebra B the C^*-base (H_μ, B, B^{op}), where H_μ is the GNS-space for μ and B and B^{op} act on $H_\mu = H_{\mu^{op}}$ via the GNS-representations [31, Example 2.9]. If $\mathfrak{b} = (\mathfrak{K},\mathfrak{B},\mathfrak{B}^\dagger)$ is a C^*-base, then so is $\mathfrak{b}^\dagger := (\mathfrak{K},\mathfrak{B}^\dagger,\mathfrak{B})$ and $M(\mathfrak{b}) := (\mathfrak{K}, M(\mathfrak{B}), M(\mathfrak{B}^\dagger))$, where $M(\mathfrak{B})$ and $M(\mathfrak{B}^\dagger)$ are naturally represented of $\mathfrak{K}$.

From now on, let $\mathfrak{b} = (\mathfrak{K},\mathfrak{B},\mathfrak{B}^\dagger)$ be a C^*-base. We shall use the following notion of a C^*-module. A C^*-$\mathfrak{b}$-*module* is a pair $H_\alpha = (H,\alpha)$, where H is a Hilbert space and $\alpha \subseteq \mathcal{L}(\mathfrak{K},H)$ is a closed subspace satisfying $[\alpha\mathfrak{K}] = H$, $[\alpha\mathfrak{B}] = \alpha$, and $[\alpha^*\alpha] = \mathfrak{B} \subseteq \mathcal{L}(\mathfrak{K})$. If H_α is a C^*-$\mathfrak{b}$-module, then α is a Hilbert C^*-module over B with inner product $(\xi,\xi') \mapsto \xi^*\xi'$ and there exist isomorphisms

$$\alpha \otimes \mathfrak{K} \to H,\ \xi \otimes \zeta \mapsto \xi\zeta, \qquad\qquad \mathfrak{K} \otimes \alpha \to H,\ \zeta \otimes \xi \mapsto \xi\zeta, \tag{4}$$

and a nondegenerate representation

$$\rho_\alpha\colon \mathfrak{B}^\dagger \to \mathcal{L}(H), \quad \rho_\alpha(b^\dagger)(\xi\zeta) = \xi b^\dagger \zeta \quad \text{for all } b^\dagger \in \mathfrak{B}^\dagger, \xi \in \alpha, \zeta \in \mathfrak{K}.$$

A *semi-morphism* between C^*-$\mathfrak{b}$-modules H_α and K_β is an operator $T \in \mathcal{L}(H,K)$ satisfying $T\alpha \subseteq \beta$. If additionally $T^*\beta \subseteq \alpha$, we call T a *morphism*. We denote the set of all (semi-)morphisms by $\mathcal{L}_{(s)}(H_\alpha, K_\beta)$. If $T \in \mathcal{L}_s(H_\alpha, K_\beta)$, then $T\rho_\alpha(b^\dagger) = \rho_\beta(b^\dagger)T$ for all $b^\dagger \in \mathfrak{B}^\dagger$, and if additionally $T \in \mathcal{L}(H_\alpha, K_\beta)$, then left multiplication by T defines an operator in $\mathcal{L}_{\mathfrak{B}}(\alpha, \beta)$ which we again denote by T.

We shall use the following notion of C^*-bi- and C^*-n-modules. Let $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ be C^*-bases, where $\mathfrak{b}_i = (\mathfrak{K}_i, \mathfrak{B}_i, \mathfrak{B}_i^\dagger)$ for each i. A C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-*module* is a tuple $(H, \alpha_1, \ldots, \alpha_n)$, where H is a Hilbert space and (H, α_i) is a C^*-$\mathfrak{b}_i$-module for each i such that $[\rho_{\alpha_i}(\mathfrak{B}_i^\dagger)\alpha_j] = \alpha_j$ whenever $i \neq j$. In the case $n = 2$, we abbreviate ${}_\alpha H_\beta := (H, \alpha, \beta)$. We note that if $(H, \alpha_1, \ldots, \alpha_n)$ is a C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-module, then $[\rho_{\alpha_i}(\mathfrak{B}_i^\dagger), \rho_{\alpha_j}(\mathfrak{B}_j^\dagger)] = 0$ whenever $i \neq j$. The set of *(semi-)morphisms* between C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-modules $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_n)$, $\mathcal{K} = (K, \gamma_1, \ldots, \gamma_n)$ is $\mathcal{L}_{(s)}(\mathcal{H}, \mathcal{K}) := \bigcap_{i=1}^n \mathcal{L}_{(s)}(H_{\alpha_i}, K_{\gamma_i}) \subseteq \mathcal{L}(H, K)$.

The relative tensor product Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, H_β C^*-$\mathfrak{b}$-module, and K_γ a C^*-$\mathfrak{b}^\dagger$-module. The *relative tensor product* of H_β and K_γ is the Hilbert space

$$H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K := \beta \otimes \mathfrak{K} \otimes \gamma.$$

It is spanned by elements $\xi \otimes \zeta \otimes \eta$, where $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, and

$$\langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta' \rangle = \langle \zeta | \xi^* \xi' \eta^* \eta' \zeta' \rangle = \langle \zeta | \eta^* \eta' \xi^* \xi' \zeta' \rangle$$

for all $\xi, \xi' \in \beta$, $\zeta, \zeta' \in \mathfrak{K}$, $\eta, \eta' \in \gamma$. Obviously, there exists a unitary *flip*

$$\Sigma \colon H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K \to K_\gamma \underset{\mathfrak{b}^\dagger}{\otimes}_\beta H, \quad \xi \otimes \zeta \otimes \eta \mapsto \eta \otimes \zeta \otimes \xi.$$

Using the unitaries in (4) on H_β and K_γ, respectively, we shall make the following identifications without further notice:

$$H_{\rho_\beta} \otimes \gamma \cong H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K \cong \beta \otimes_{\rho_\gamma} K, \quad \xi\zeta \otimes \eta \equiv \xi \otimes \zeta \otimes \eta \equiv \xi \otimes \eta\zeta.$$

For all $S \in \rho_\beta(\mathfrak{B}^\dagger)'$ and $T \in \rho_\gamma(\mathfrak{B})'$, we have operators

$$S \otimes \mathrm{id} \in \mathcal{L}(H_{\rho_\beta} \otimes \gamma) = \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K), \qquad \mathrm{id} \otimes T \in \mathcal{L}(\beta \otimes_{\rho_\gamma} K) = \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K).$$

If $S \in \mathcal{L}_s(H_\beta)$ or $T \in \mathcal{L}_s(K_\gamma)$, then $(S \otimes \mathrm{id})(\xi \otimes \eta\zeta) = S\xi \otimes \eta\zeta$ or $(\mathrm{id} \otimes T)(\xi\zeta \otimes \eta) = \xi\zeta \otimes T\eta$, respectively, for all $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, so that we can define

$$S \underset{\mathfrak{b}}{\otimes} T := (S \otimes \mathrm{id})(\mathrm{id} \otimes T) = (\mathrm{id} \otimes T)(S \otimes \mathrm{id}) \in \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K)$$
$$\text{for all } (S, T) \in \left(\mathcal{L}_s(H_\beta) \times \rho_\gamma(\mathfrak{B})'\right) \cup \left(\rho_\beta(\mathfrak{B}^\dagger)' \times \mathcal{L}_s(K_\gamma)\right).$$

For each $\xi \in \beta$ and $\eta \in \gamma$, there exist bounded linear operators

$$|\xi\rangle_1 \colon K \to H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K, \ \omega \mapsto \xi \otimes \omega, \qquad |\eta\rangle_2 \colon H \to H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K, \ \omega \mapsto \omega \otimes \eta,$$

whose adjoints $\langle\xi|_1 := |\xi\rangle_1^*$ and $\langle\eta|_2 := |\eta\rangle_2^*$ are given by

$$\langle\xi|_1 \colon \xi' \ominus \omega \mapsto \rho_\gamma(\xi^*\xi')\omega, \qquad \langle\eta|_2 \colon \omega \oslash \eta' \mapsto \rho_\beta(\eta^*\eta')\omega.$$

We put $|\beta\rangle_1 := \{|\xi\rangle_1 \,|\, \xi \in \beta\} \subseteq \mathcal{L}(K, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$ and similarly define $\langle\beta|_1$, $|\gamma\rangle_2$, $\langle\gamma|_2$.

Let $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_m, \beta)$ be a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{b})$-module and $\mathcal{K} = (K, \gamma, \delta_1, \ldots, \delta_n)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, where $\mathfrak{a}_i = (\mathfrak{H}_i, \mathfrak{A}_i, \mathfrak{A}_i^\dagger)$ and $\mathfrak{c}_j = (\mathfrak{L}_j, \mathfrak{C}_j, \mathfrak{C}_j^\dagger)$ are C^*-bases for all i, j. We put

$$\alpha_i \triangleleft \gamma := [|\gamma\rangle_2 \alpha_i] \subseteq \mathcal{L}(\mathfrak{H}_i, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K), \qquad \beta \triangleright \delta_j := [|\beta\rangle_1 \delta_j] \subseteq \mathcal{L}(\mathfrak{L}_j, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$$

for all i, j. Then $(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K, \alpha_1 \triangleleft \gamma, \ldots, \alpha_m \triangleleft \gamma, \beta \triangleright \delta_1, \ldots, \beta \triangleright \delta_n)$ is a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, called the *relative tensor product* of $\mathcal{H}$ and $\mathcal{K}$ and denoted by $\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}$. For all i, j and $a^\dagger \in \mathfrak{A}_i^\dagger$, $c^\dagger \in \mathfrak{C}_j^\dagger$,

$$\rho_{(\alpha_i \triangleleft \gamma)}(a^\dagger) = \rho_{\alpha_i}(a^\dagger) \underset{\mathfrak{b}}{\otimes} \mathrm{id}, \qquad \rho_{(\beta \triangleright \delta_j)}(c^\dagger) = \mathrm{id} \underset{\mathfrak{b}}{\otimes} \rho_{\delta_j}(c^\dagger).$$

The relative tensor product has nice categorical properties:

Functoriality Let $\tilde{\mathcal{H}} = (\tilde{H}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, \tilde{\beta})$ be a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{b})$-module, $\tilde{\mathcal{K}} = (\tilde{K}, \tilde{\gamma}, \tilde{\delta}_1, \ldots, \tilde{\delta}_n)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, and $S \in \mathcal{L}_{(s)}(\mathcal{H}, \tilde{\mathcal{H}})$, $T \in \mathcal{L}_{(s)}(\mathcal{K}, \tilde{\mathcal{K}})$. Then there exists a unique operator $S \underset{\mathfrak{b}}{\otimes} T \in \mathcal{L}_{(s)}(\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}, \tilde{\mathcal{H}} \underset{\mathfrak{b}}{\otimes} \tilde{\mathcal{K}})$ satisfying $(S \underset{\mathfrak{b}}{\otimes} T)(\xi \ominus \zeta \oslash \eta) = S\xi \ominus \zeta \oslash T\eta$ for all $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$.

Unitality The triple $\mathcal{U} := (\mathfrak{K}, \mathfrak{B}^\dagger, \mathfrak{B})$ is a C^*-$(\mathfrak{B}^\dagger, \mathfrak{B})$-module and the maps

$$\begin{aligned} l_{\mathcal{H}} \colon H_\beta \underset{\mathfrak{b}}{\otimes}{}_{\mathfrak{B}^\dagger} \mathfrak{K} \to H, &\quad \xi \ominus \zeta \oslash b^\dagger \mapsto \xi b^\dagger \zeta, \\ r_{\mathcal{K}} \colon \mathfrak{K}_{\mathfrak{B}} \underset{\mathfrak{b}}{\otimes}{}_\gamma K \to K, &\quad b \ominus \zeta \oslash \eta \mapsto \eta b \zeta, \end{aligned} \tag{5}$$

are isomorphisms of C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{b})$-modules and C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-modules $\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{U} \to \mathcal{H}$ and $\mathcal{U} \underset{\mathfrak{b}}{\otimes} \mathcal{K} \to \mathcal{K}$, respectively, natural in $\mathcal{H}$ and $\mathcal{K}$.

Associativity Let $\mathfrak{d}, \mathfrak{e}_1, \ldots, \mathfrak{e}_l$ be C^*-bases, $\hat{\mathcal{K}} = (K, \gamma, \delta_1, \ldots, \delta_n, \epsilon)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n, \mathfrak{d})$-module and $\mathcal{L} = (L, \phi, \psi_1, \ldots, \psi_l)$ a C^*-$(\mathfrak{d}^\dagger, \mathfrak{e}_1, \ldots, \mathfrak{e}_l)$-module. Then there exists a canonical isomorphism

$$a_{\mathcal{H},\mathcal{K},\mathcal{L}} \colon (H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)_{\beta \triangleright \epsilon} \underset{\mathfrak{d}}{\otimes}{}_\phi L \to \beta \ominus_{\rho_\gamma} K_{\rho_\epsilon} \oslash \phi \to H_\beta \underset{\mathfrak{b}}{\otimes}{}_{\gamma \triangleleft \phi} (K_\epsilon \underset{\mathfrak{d}}{\otimes}{}_\phi L) \tag{6}$$

which is an isomorphism of C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{c}_1, \ldots, \mathfrak{c}_n, \mathfrak{e}_1, \ldots, \mathfrak{e}_l)$-modules $(\mathcal{H} \underset{\mathfrak{b}}{\otimes} \hat{\mathcal{K}}) \underset{\mathfrak{d}}{\otimes} \mathcal{L} \to \mathcal{H} \underset{\mathfrak{b}}{\otimes} (\hat{\mathcal{K}} \underset{\mathfrak{d}}{\otimes} \mathcal{L})$. From now on, we identify the Hilbert spaces in (6) and denote them by $H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K_\epsilon \underset{\mathfrak{d}}{\otimes}{}_\phi L$.

Direct sums Let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ and $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be C^*-bases and let $(\mathcal{H}_i)_i$ be a family of C^*-$(\mathfrak{a}, \mathfrak{b})$-modules, where $\mathcal{H}_i = (H_i, \alpha_i, \beta_i)$ for each i. Denote by $\boxplus_i \alpha_i \subseteq \mathcal{L}(\mathfrak{H}, \oplus_i H_i)$ the norm-closed linear span of all operators of the form $\zeta \mapsto (\xi_i \zeta)_i$, where $(\xi_i)_i$ is contained in the algebraic direct sum $\bigoplus_i^{\mathrm{alg}} \alpha_j$, and similarly define $\boxplus_i \beta_i \subseteq \mathcal{L}(\mathfrak{K}, \oplus_i H_i)$. Then the triple $\boxplus_i \mathcal{H}_i := \big(\oplus_i H_i, \boxplus_i \alpha_i, \boxplus_i \beta_i\big)$ is a C^*-$(\mathfrak{a}, \mathfrak{b})$-module, and for each j, the canonical inclusions $\iota_j^{\mathcal{H}} \colon H_j \to \oplus_i H_i$ and projection $\pi_j^{\mathcal{H}} \colon \oplus_i H_i \to H_j$ are morphisms $\mathcal{H}_j \to \boxplus_i \mathcal{H}_i$ and $\boxplus_i \mathcal{H}_i \to \mathcal{H}_j$. With respect to these maps, $\boxplus_i \mathcal{H}_i$ is the direct sum of the family $(\mathcal{H}_i)_i$.

Let $\mathfrak{c}$ be a C^*-base and $(\mathcal{K}_j)_j$ a family of C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-modules, and define the direct sum $\boxplus_j \mathcal{K}_j$ similarly as above. Then there exist inverse isomorphisms $\boxplus_{i,j} (\mathcal{H}_i \underset{\mathfrak{b}}{\otimes} \mathcal{K}_j) \leftrightarrows (\boxplus_i \mathcal{H}_i) \underset{\mathfrak{b}}{\otimes} (\boxplus_j \mathcal{K}_j)$ given by $(\omega_{i,j})_{i,j} \mapsto \sum_{i,j} (\iota_i^{\mathcal{H}} \underset{\mathfrak{b}}{\otimes} \iota_j^{\mathcal{K}})(\omega_{i,j})$ and $\big((\pi_i^{\mathcal{H}} \underset{\mathfrak{b}}{\otimes} \pi_j^{\mathcal{K}})(\omega)\big)_{i,j} \leftarrow\!\shortmid \omega$, respectively.

Similar constructions apply to C^*-$\mathfrak{b}$-modules and C^*-$\mathfrak{b}^\dagger$-modules.

2.2 The definition of C^*-pseudo-multiplicative unitaries

Using the relative tensor product of C^*-modules introduced above, we generalize the notion of a multiplicative unitary as follows. Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, $(H, \widehat{\beta}, \alpha, \beta)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module, and $V \colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \to H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H$ a unitary satisfying

$$V(\alpha \triangleleft \alpha) = \alpha \triangleright \alpha, \quad V(\widehat{\beta} \triangleright \beta) = \widehat{\beta} \triangleleft \beta, \quad V(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}, \quad V(\beta \triangleleft \alpha) = \beta \triangleleft \beta \tag{7}$$

in $\mathcal{L}(\mathfrak{K}, H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H)$. Then all operators in the following diagram are well defined,

$$\begin{tikzcd}
H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[r, "V \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id}"] \arrow[d, "\mathrm{id} \underset{\mathfrak{b}^\dagger}{\otimes} V"] & H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[r, "\mathrm{id} \underset{\mathfrak{b}}{\otimes} V"] & H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H, \\
H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{(\alpha \triangleright \alpha)} (H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H) \arrow[d, "\mathrm{id} \underset{\mathfrak{b}^\dagger}{\otimes} \Sigma"] & & (H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H)_{(\alpha \triangleleft \alpha)} \underset{\mathfrak{b}}{\otimes} {}_\beta H \arrow[u, "V \underset{\mathfrak{b}}{\otimes} \mathrm{id}"'] \\
H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H_\beta \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[rr, "V \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id}"] & & \big(H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H\big)_{\widehat{\beta} \triangleleft \beta} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[u, "\Sigma_{23}"']
\end{tikzcd} \tag{8}$$

where Σ_{23} denotes the isomorphism

$$\big(H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H\big)_{\widehat{\beta} \triangleleft \beta} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \cong (H_{\rho_\alpha} \varolessthan \beta)_{\rho_{(\widehat{\beta} \triangleleft \beta)}} \varolessthan \alpha \xrightarrow{\cong} (H_{\rho_{\widehat{\beta}}} \varolessthan \alpha)_{\rho_{(\alpha \triangleleft \alpha)}} \varolessthan \beta \cong (H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H)_{(\alpha \triangleleft \alpha)} \underset{\mathfrak{b}}{\otimes} {}_\beta H$$

given by $(\zeta \varolessthan \xi) \varolessthan \eta \mapsto (\zeta \varolessthan \eta) \varolessthan \xi$. We adopt the leg notation [3] and write

$$V_{12} \text{ for } V \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id}, V \underset{\mathfrak{b}}{\otimes} \mathrm{id}; \qquad V_{23} \text{ for } V \underset{\mathfrak{b}}{\otimes} \mathrm{id}, V \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id}; \qquad V_{13} \text{ for } \Sigma_{23}(V \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})(\mathrm{id} \underset{\mathfrak{b}^\dagger}{\otimes} \Sigma).$$

Definition 2.1. *A* C^*-pseudo-multiplicative unitary *is a tuple* $(\mathfrak{b}, H, \widehat{\beta}, \alpha, \beta, V)$ *consisting of a* C^**-base* $\mathfrak{b}$*, a* C^**-*$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$*-module* $(H, \widehat{\beta}, \alpha, \beta)$*, and a unitary* $V : H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \to H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H$ *such that equation* (7) *holds and diagram* (8) *commutes. We frequently call just* V *a* C^**-pseudo-multiplicative unitary.*

This definition covers the following special cases:

Remarks 2.2. Let $(\mathfrak{b}, H, \hat{\beta}, \alpha, \beta, V)$ be a C^*-pseudo-multiplicative unitary.

i) If $\mathfrak{b}$ is the trivial C^*-base $(\mathbb{C}, \mathbb{C}, \mathbb{C})$, then $H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \cong H \otimes H \cong H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H$, and V is a multiplicative unitary.

ii) If we consider $\rho_{\widehat{\beta}}$ and ρ_β as representations $\rho_{\widehat{\beta}}, \rho_\beta : \mathfrak{B} \to \mathcal{L}(H_\alpha) \cong \mathcal{L}_{\mathfrak{B}}(\alpha)$, then the map $\alpha_{\rho_{\widehat{\beta}}} \otimes \alpha \cong \alpha \triangleleft \alpha \to \alpha \triangleright \alpha \cong \alpha \otimes_{\rho_\beta} \alpha$ given by $\omega \mapsto V\omega$ is a pseudo-multiplicative unitary on C^*-modules in the sense of [34].

iii) Assume that $\mathfrak{b} = \mathfrak{b}^\dagger$; then $\mathfrak{B} = \mathfrak{B}^\dagger$ is commutative. If $\widehat{\beta} = \alpha$, then the pseudo-multiplicative unitary in ii) is a pseudo-multiplicative unitary in the sense of O'uchi [23]. If additionally $\widehat{\beta} = \alpha = \beta$, then the unitary in ii) is a continuous field of multiplicative unitaries in the sense of Blanchard [4].

iv) Assume that $\mathfrak{b}$ is the C^*-base associated to a faithful proper KMS-weight μ on a C^*-algebra B (see [31, Example 2.9]). Then μ extends to a n.s.f. weight $\tilde{\mu}$ on $[\![\mathfrak{B}]\!]$, and with respect to the canonical isomorphisms $H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \cong H_{\rho_{\widehat{\beta}}} \underset{\tilde{\mu}^{op}}{\overline{\otimes}} {}_{\rho_\alpha} H$ and $H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H \cong H_{\rho_\alpha} \underset{\tilde{\mu}}{\overline{\otimes}} {}_{\rho_\beta} H$ (see [31, Corollary 2.21]), V is a pseudo-multiplicative unitary on Hilbert spaces in the sense of Vallin [38].

Let us give some examples and easy constructions:

Examples 2.3. i) To every locally compact, Hausdorff, second countable groupoid with a left Haar system, we shall associate a C^*-pseudo-multiplicative unitary in the next subsection.

ii) In [35], a C^*-pseudo-multiplicative unitary is associated to every compact C^*-quantum groupoid.

iii) The *opposite* of a C^*-pseudo-multiplicative unitary $(\mathfrak{b}, H, \hat{\beta}, \alpha, \beta, V)$ is the tuple $(\mathfrak{b}, H, \beta, \alpha, \hat{\beta}, V^{op})$, where V^{op} denotes the composition $\Sigma V^* \Sigma$: $H_\beta \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \xrightarrow{\Sigma} H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H \xrightarrow{V^*} H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \xrightarrow{\Sigma} H_\alpha \underset{\mathfrak{b}}{\otimes} {}_{\widehat{\beta}} H$. A tedious but straightforward calculation shows that this is a C^*-pseudo-multiplicative unitary.

iv) The *direct sum* of a family $((\mathfrak{b}_i, H^i, \widehat{\beta}_i, \alpha_i, \beta_i, V_i))_i$ of C^*-pseudo-multiplicative unitaries is defined as follows. Write $\mathfrak{b}^i = (\mathfrak{H}^i, \mathfrak{B}_i, \mathfrak{B}_i^\dagger)$ for each i, put $\mathfrak{H} := \bigoplus_i \mathfrak{H}^i$, $H := \bigoplus_i H^i$, denote by $\mathfrak{B}^{(\dagger)} := \bigoplus_i \mathfrak{B}_i^{(\dagger)} \subseteq \mathcal{L}(\mathfrak{H})$ the c_0-direct sum of C^*-algebras, and by $\widehat{\beta} := \bigoplus_i \widehat{\beta}_i$, $\alpha := \bigoplus_i \alpha_i$, $\beta := \bigoplus_i \beta_i$ the c_0-direct sum in $\mathcal{L}(\mathfrak{H}, H)$. Then $\mathfrak{b} := (\mathfrak{H}, \mathfrak{B}, \mathfrak{B}^\dagger)$ is a C^*-base, there exist natural isomorphisms $H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \cong \bigoplus_i H^i{}_{\widehat{\beta}_i} \underset{\mathfrak{b}_i^\dagger}{\otimes} {}_{\alpha_i} H^i$ and $H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H \cong \bigoplus_i H^i{}_{\alpha_i} \underset{\mathfrak{b}_i^\dagger}{\otimes} {}_{\beta_i} H^i$ [31, Proposition 2.17], and if V

denotes the unitary corresponding to $\bigoplus_i V_i$ with respect to these isomorphisms, then the tuple $(\mathfrak{b}, H, \widehat{\beta}, \alpha, \beta, V)$ is a C^*-pseudo-multiplicative unitary.

v) The *tensor product* of C^*-pseudo-multiplicative unitaries $(\mathfrak{b}, H, \hat{\beta}, \alpha, \beta, V)$ and $(\mathfrak{c}, K, \hat{\delta}, \gamma, \delta, W)$ is defined as follows. Denote by $\mathfrak{B}^{(\dagger)} \otimes \mathfrak{C}^{(\dagger)} \subseteq \mathcal{L}(\mathfrak{H} \otimes \mathfrak{K})$ and $\widehat{\beta} \otimes \widehat{\delta}, \alpha \otimes \gamma, \beta \otimes \delta \subseteq \mathcal{L}(\mathfrak{H} \otimes \mathfrak{K}, H \otimes K)$ the closed subspaces generated by elementary tensor products. Then $\mathfrak{b} \otimes \mathfrak{c} := (\mathfrak{H} \otimes \mathfrak{K}, \mathfrak{B} \otimes \mathfrak{C}, \mathfrak{B}^\dagger \otimes \mathfrak{C}^\dagger)$ is a C^*-base, there exist natural isomorphisms $(H \otimes K)_{\widehat{\beta} \otimes \widehat{\delta}} \underset{(\mathfrak{b} \otimes \mathfrak{c})^\dagger}{\otimes} {}_{\alpha \otimes \gamma}(H \otimes K) \cong (H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H) \otimes (K_{\widehat{\delta}} \underset{\mathfrak{c}^\dagger}{\otimes} {}_\gamma K)$ and $(H \otimes K)_{\alpha \otimes \gamma} \underset{\mathfrak{b} \otimes \mathfrak{c}}{\otimes} {}_{\beta \otimes \delta}(H \otimes K) \cong (H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H) \otimes (K_\gamma \underset{\mathfrak{c}}{\otimes} {}_\delta K)$, and if U denotes the unitary corresponding to $V \otimes W$ with respect to these isomorphisms, then $(\mathfrak{b} \otimes \mathfrak{c}, H \otimes K, \widehat{\beta} \otimes \widehat{\delta}, \alpha \otimes \gamma, \beta \otimes \delta, U)$ is a C^*-pseudo-multiplicative unitary.

2.3 The C^*-pseudo-multiplicative unitary of a groupoid

To every locally compact, Hausdorff, second countable groupoid with left Haar system, we shall associate a C^*-pseudo-multiplicative unitary. The underlying pseudo-multiplicative unitary was introduced by Vallin [38], and associated unitaries on C^*-modules were discussed in [23, 34]. We focus on the aspects that are new in the present setting.

Let G be a locally compact, Hausdorff, second countable groupoid with left Haar system λ and associated right Haar system λ^{-1}, and let μ be a measure on G^0 with full support. Define measures ν, ν^{-1} on G by

$$\int_G f \,\mathrm{d}\nu := \int_{G^0} \int_{G^u} f(x) \,\mathrm{d}\lambda^u(x) \,\mathrm{d}\mu(u), \quad \int_G f d\nu^{-1} = \int_{G^0} \int_{G_u} f(x) \,\mathrm{d}\lambda_u^{-1}(x) \,\mathrm{d}\mu(u)$$

for all $f \in C_c(G)$. Thus, $\nu^{-1} = i_* \nu$, where $i \colon G \to G$ is given by $x \mapsto x^{-1}$. We assume that μ is quasi-invariant in the sense that ν and ν^{-1} are equivalent, and denote by $D := \mathrm{d}\nu / \mathrm{d}\nu^{-1}$ the Radon-Nikodym derivative.

We identify functions in $C_b(G^0)$ and $C_b(G)$ with multiplication operators on the Hilbert spaces $L^2(G^0, \mu)$ and $L^2(G, \nu)$, respectively, and let

$$\mathfrak{K} := L^2(G^0, \mu), \quad \mathfrak{B} = \mathfrak{B}^\dagger := C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K}), \quad \mathfrak{b} := (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger), \quad H := L^2(G, \nu).$$

Pulling functions on G^0 back to G along r or s, we obtain representations

$$r^* \colon C_0(G^0) \to C_b(G) \hookrightarrow \mathcal{L}(H), \qquad s^* \colon C_0(G^0) \to C_b(G) \hookrightarrow \mathcal{L}(H).$$

We define Hilbert C^*-modules $L^2(G, \lambda)$ and $L^2(G, \lambda^{-1})$ over $C_0(G^0)$ as the respective completions of the pre-C^*-module $C_c(G)$, the structure maps being given by

$$\langle \xi' | \xi \rangle (u) = \int_{G^u} \overline{\xi'(x)} \xi(x) \,\mathrm{d}\lambda^u(x), \qquad \xi f = r^*(f) \xi \quad \text{in the case of } L^2(G, \lambda),$$
$$\langle \xi' | \xi \rangle (u) = \int_{G_u} \overline{\xi'(x)} \xi(x) \,\mathrm{d}\lambda_u^{-1}(x), \qquad \xi f = s^*(f) \xi \quad \text{in the case of } L^2(G, \lambda^{-1})$$

respectively, for all $\xi, \xi' \in C_c(G)$, $u \in G^0$, $f \in C_0(G^0)$.

Lemma 2.4. *There exist isometric embeddings*

$$j\colon L^2(G,\lambda)\to \mathcal{L}(\mathfrak{K},H), \qquad \hat{j}\colon L^2(G,\lambda^{-1})\to \mathcal{L}\big(\mathfrak{K},H\big)$$

such that for all $\xi\in C_c(G)$, $\zeta\in C_c(G^0)$

$$\big(j(\xi)\zeta\big)(x)=\xi(x)\zeta(r(x)), \qquad \big(\hat{j}(\xi)\zeta\big)(x)=\xi(x)D^{-1/2}(x)\zeta(s(x)).$$

Proof. Let $E:=L^2(G,\lambda)$, $\hat{E}:=L^2(G,\lambda^{-1})$, and $\xi,\xi'\in C_c(G)$, $\zeta,\zeta'\in C_c(G^0)$. Then

$$\begin{aligned}
\big\langle j(\xi')\zeta'\big|j(\xi)\zeta\big\rangle &= \int_{G^0}\int_{G^u}\overline{\xi'(x)\zeta'(r(x))}\xi(x)\zeta(r(x))\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u)=\big\langle\zeta'\big|\langle\xi'|\xi\rangle_E\zeta\big\rangle,\\
\big\langle \hat{j}(\xi')\zeta'\big|\hat{j}(\xi)\zeta\big\rangle &= \int_{G}\overline{\xi'(x)\zeta'(s(x))}\xi(x)\zeta(s(x))\underbrace{D^{-1}(x)\,\mathrm{d}\nu(x)}_{=\mathrm{d}\nu^{-1}(x)}\\
&= \int_{G^0}\int_{G_u}\overline{\zeta'(u)\xi'(x)}\xi(x)\zeta(u)\,\mathrm{d}\lambda_u^{-1}(x)\,\mathrm{d}\mu(u)=\big\langle\zeta'\big|\langle\xi'|\xi\rangle_{\hat{E}}\zeta\big\rangle. \qquad\square
\end{aligned}$$

Let $\alpha:=\beta:=j(L^2(G,\lambda))$ and $\widehat{\beta}:=\hat{j}(L^2(G,\lambda^{-1}))$. Easy calculations show:

Lemma 2.5. $(H,\widehat{\beta},\alpha,\beta)$ *is a* C^**-*$(\mathfrak{b}^\dagger,\mathfrak{b},\mathfrak{b}^\dagger)$*-module,* $\rho_\alpha=\rho_\beta=r^*$ *and* $\rho_{\widehat{\beta}}=s^*$*, and* j *and* $\hat{j}$ *are unitary maps of Hilbert* C^**-modules over* $C_0(G^0)\cong\mathfrak{B}$. $\square$

The Hilbert spaces $H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H$ and $H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H$ can be described as follows. We define measures $\nu^2_{s,r}$ on $G_s{\times}_rG$ and $\nu^2_{r,r}$ on $G_r{\times}_rG$ by

$$\begin{aligned}
\int_{G_s\times_rG} f\,\mathrm{d}\nu^2_{s,r} &:= \int_{G^0}\int_{G^u}\int_{G^{s(x)}} f(x,y)\,\mathrm{d}\lambda^{s(x)}(y)\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u),\\
\int_{G_r\times_rG} g\,\mathrm{d}\nu^2_{r,r} &:= \int_{G^0}\int_{G^u}\int_{G^u} g(x,y)\,\mathrm{d}\lambda^u(y)\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u)
\end{aligned}$$

for all $f\in C_c(G_s{\times}_rG)$, $g\in C_c(G_r{\times}_rG)$. Routine calculations show:

Lemma 2.6. *There exist unique isomorphisms*

$$\Phi_{\widehat{\beta},\alpha}\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H\to L^2(G_s{\times}_rG,\nu^2_{s,r}), \qquad \Phi_{\alpha,\beta}\colon H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H\to L^2(G_r{\times}_rG,\nu^2_{r,r})$$

such that for all $\eta,\xi\in C_c(G)$ *and* $\zeta\in C_c(G^0)$,

$$\begin{aligned}
\Phi_{\widehat{\beta},\alpha}\big(\hat{j}(\eta)\otimes\zeta\otimes j(\xi)\big)(x,y)&=\eta(x)D^{-1/2}(x)\zeta(s(x))\xi(y),\\
\Phi_{\alpha,\beta}\big(j(\eta)\otimes\zeta\otimes j(\xi)\big)(x,y)&=\eta(x)\zeta(r(x))\xi(y). \quad\square
\end{aligned}$$

From now on, we identify $H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H$ with $L^2(G_s{\times}_rG)$ and $H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H$ with $L^2(G_r{\times}_rG)$ via $\Phi_{\widehat{\beta},\alpha}$ and $\Phi_{\alpha,\beta}$, respectively, without further notice.

Theorem 2.7. *There exists a* C^**-pseudo-multiplicative unitary* $(\mathfrak{b},H,\hat{\beta},\alpha,\beta,V)$ *such that* $(V\omega)(x,y)=\omega(x,x^{-1}y)$ *for all* $\omega\in C_c(G_s{\times}_rG)$ *and* $(x,y)\in G_r{\times}_rG$.

Proof. Straightforward calculations show that $(H, \widehat{\beta}, \alpha, \beta)$ is a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module.

The homeomorphism $G_r {}_r\!\times_r G \to G_s {}_s\!\times_r G$, $(x,y) \mapsto (x, x^{-1}y)$, induces an isomorphism $V_0 \colon C_c(G_s {}_s\!\times_r G) \to C_c(G_r {}_r\!\times_r G)$ such that $(V_0\omega)(x,y) = \omega(x, x^{-1}y)$ for all $\omega \in C_c(G_s {}_s\!\times_r G)$ and $(x,y) \in G_r {}_r\!\times_r G$. Using left-invariance of λ, one finds that V_0 extends to a unitary $V \colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \cong L^2(G_s {}_s\!\times_r G) \to L^2(G_r {}_r\!\times_r G) \cong H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H$.

We claim that V is a C^*-pseudo-multiplicative unitary. First, we show that $V(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}$. For each $\xi, \xi' \in C_c(G)$, $\zeta \in C_c(G^0)$, and $(x,y) \in G_s {}_s\!\times_r G$,

$$\begin{aligned}
\big(V|\hat{j}(\xi)\rangle_1 \hat{j}(\xi')\zeta\big)(x,y) &= \big(|\hat{j}(\xi)\rangle_1 \hat{j}(\xi')\zeta\big)(x, x^{-1}y) \\
&= \xi(x)\xi'(x^{-1}y)D^{-1/2}(x)D^{-1/2}(x^{-1}y)\zeta(s(y)), \\
\big(|j(\xi)\rangle_1 \hat{j}(\xi')\zeta\big)(x,y) &= \xi(x)\xi'(y)D^{-1/2}(y)\zeta(s(y)).
\end{aligned}$$

Using standard approximation arguments and the fact that $D(x)D(x^{-1}y) = D(y)$ for $\nu^2_{r,r}$-almost all $(x,y) \in G_r {}_r\!\times_r G$ (see [13] or [24, p. 89]), we find that $V(\widehat{\beta} \triangleright \widehat{\beta}) = [T(C_c(G_r {}_r\!\times_r G))] = \alpha \triangleright \widehat{\beta}$, where for each $\omega \in C_c(G_r {}_r\!\times_r G)$,

$$(T(\omega)\zeta)(x,y) = \omega(x,y)D^{-1/2}(y)\zeta(s(y)) \quad \text{for all } \zeta \in C_c(G^0),\ (x,y) \in G_r {}_r\!\times_r G.$$

Similar calculations show that the remaining relations in (7) hold.

Tedious but straightforward calculations show that diagram (8) commutes; see also [38]. Therefore, V is a C^*-pseudo-multiplicative unitary. $\square$

3 Hopf C^*-bimodules and the legs of a C^*-pseudo multiplicative unitary

To every regular multiplicative unitary V on a Hilbert space H, Baaj and Skandalis associate two Hopf C^*-algebras $(\widehat{A}_V, \widehat{\Delta}_V)$ and (A_V, Δ_V) as follows [3]. They show for every multiplicative unitary V, the subspaces $\widehat{A}^0_V$ and A^0_V of $\mathcal{L}(H)$ defined by

$$\widehat{A}^0_V := \{(\mathrm{id} \mathbin{\overline{\otimes}} \omega)(V) \mid \omega \in \mathcal{L}(H)_*\}, \qquad A^0_V := \{(\upsilon \mathbin{\overline{\otimes}} \mathrm{id})(V) \mid \upsilon \in \mathcal{L}(H)_*\} \tag{9}$$

are closed under multiplication. In the regular case, their norm closures $\widehat{A}_V$ and A_V, respectively, are C^*-algebras, and the $*$-homomorphisms $\widehat{\Delta}_V \colon \widehat{A}_V \to \mathcal{L}(H \otimes H)$ and $\Delta_V \colon A_V \to \mathcal{L}(H \otimes H)$ given by

$$\widehat{\Delta}_V \colon \hat{a} \mapsto V^*(1 \otimes \hat{a}), \qquad\qquad \Delta_V \colon a \mapsto V(a \otimes 1)V^*, \tag{10}$$

map $\widehat{A}_V$ to $M(\widehat{A}_V \otimes \widehat{A}_V) \subseteq \mathcal{L}(H \otimes H)$ and A_V to $M(A_V \otimes A_V) \subseteq \mathcal{L}(H \otimes H)$, respectively, and form comultiplications on $\widehat{A}_V$ and A_V. Finally, there exists a perfect pairing

$$\widehat{A}^0_V \times A^0_V \to \mathbb{C}, \quad ((\mathrm{id} \mathbin{\overline{\otimes}} \omega)(V), (\upsilon \mathbin{\overline{\otimes}} \mathrm{id})(V)) \to (\upsilon \mathbin{\overline{\otimes}} \omega)(V), \tag{11}$$

which expresses the duality between $(\widehat{A}_V, \widehat{\Delta}_V)$ and (A_V, Δ_V).

Applied to the multiplicative unitary of a locally compact group G, this construction yields the C^*-algebras $C_0(G)$ and $C_r^*(G)$ with the comultiplications $\widehat{\Delta}\colon C_0(G) \to M(C_0(G) \otimes C_0(G)) \cong C_b(G \times G)$ and $\Delta\colon C_r^*(G) \to M(C_r^*(G) \otimes C_r^*(G))$ given by

$$\widehat{\Delta}(f)(x,y) = f(xy) \text{ for all } f \in C_0(G), \qquad \Delta(U_x) = U_x \otimes U_x \text{ for all } x \in G, \tag{12}$$

where $U\colon G \to M(C_r^*(G)), x \mapsto U_x$, is the canonical embedding.

To adapt these constructions to C^*-pseudo-multiplicative unitaries, we have to solve the following problems.

First, we have to find substitutes for the space $\mathcal{L}(H)_*$ and the slice maps $\mathrm{id} \mathbin{\overline{\otimes}} \omega$ and $\upsilon \mathbin{\overline{\otimes}} \mathrm{id}$ that were used in the definition of $\widehat{A}_V^0$ and A_V^0. It turns out that for a C^*-pseudo-multiplicative unitary, the norm closures $\widehat{A}_V$ and A_V are easier to define. Therefore, we first study these algebras, before we introduce the spaces $\widehat{A}_V^0$ and A_V^0 and the dual pairing. The spaces $\widehat{A}_V^0$ and A_V^0 carry the structure of Banach algebras and can be considered as a Fourier algebra and a dual Fourier algebra for V.

The main difficulty, however, is to find a suitable generalization of the notion of a Hopf C^*-algebra and, more precisely, to describe the targets of the comultiplications $\widehat{\Delta}_V$ and Δ_V. For example, if we replace the multiplicative unitary of a group G by the C^*-pseudo-multiplicative unitary of a groupoid G, we expect to obtain the C^*-algebras $\widehat{A}_V = C_0(G)$ and $A_V = C_r^*(G)$ with $*$-homomorphisms $\widehat{\Delta}$ and Δ given by the same formulas as in (12). Then the target of $\widehat{\Delta}$ would be $M(C_0(G_s{\times}_rG))$, and $C_0(G_s{\times}_rG)$ can be identified with the relative tensor product $C_0(G)_{s^*} \underset{C_0(G^0)}{\otimes} {}_{r^*}C_0(G)$ of $C_0(G^0)$-algebras [4]. But the target of Δ can not be described in a similar way, and in general, we need to replace the balanced tensor product by a fiber product relative to some base algebra which may even be non-commutative. If G is finite, then $C(G)$ and $C_r^*(G) = \mathbb{C}G$ can be considered as modules over $R = C(G^0)$ in several ways, and the targets of $\widehat{\Delta}$ and Δ can be written using the $\times_R$-product of Takeuchi [29] in the form $C(G) \times_R C(G)$ and $\mathbb{C}G \times_R \mathbb{C}G$, respectively. In the setting of von Neumann algebras, the targets of the comultiplications can be described using Sauvageot's fiber product [28, 37]. For the setting of C^*-algebras, a partial solution was proposed in [34], and a general fiber product construction which suits our purpose was introduced in [31].

We proceed as follows. First, we recall the fiber product of C^*-algebras over C^*-bases and systematically study slice maps and related constructions. These prerequisites are then used to define Hopf C^*-bimodules and associated convolution algebras. Finally, we adapt the constructions of Baaj and Skandalis to C^*-pseudo-multiplicative unitaries and apply them to the unitary associated to a groupoid.

Throughout this section, let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, $(H, \widehat{\beta}, \alpha, \beta)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module and $V\colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \to H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H$ a C^*-pseudo-multiplicative unitary.

3.1 The fiber product of C^*-algebras over C^*-bases

In this subsection, we recall the fiber product of C^*-algebras over C^*-bases [31], introduce several new notions of a morphism of such C^*-algebras, and show that the fiber product is also functorial with respect to these generalized morphisms. For additional motivation and details, we refer to [31]; two examples can be found in subsection 3.5.

Let $\mathfrak{b}_1,\dots,\mathfrak{b}_n$ be C^*-bases, where $\mathfrak{b}_i = (\mathfrak{K}_i,\mathfrak{B}_i,\mathfrak{B}_i^\dagger)$ for each i. A *(nondegenerate) C^*-$(\mathfrak{b}_1,\dots,\mathfrak{b}_n)$-algebra* consists of a C^*-$(\mathfrak{b}_1,\dots,\mathfrak{b}_n)$-module $(H,\alpha_1,\dots,\alpha_n)$ and a (nondegenerate) C^*-algebra $A \subseteq \mathcal{L}(H)$ such that $\rho_{\alpha_i}(\mathfrak{B}_i^\dagger)A$ is contained in A for each i. We shall only be interested in the cases $n = 1,2$, where we abbreviate $A_H^\alpha := (H_\alpha, A)$, $A_H^{\alpha,\beta} := ({}_\alpha H_\beta, A)$.

We need several natural notions of a morphism. Let $\mathcal{A} = (\mathcal{H},A)$ and $\mathcal{C} = (\mathcal{K},C)$ be C^*-$(\mathfrak{b}_1,\dots,\mathfrak{b}_n)$-algebras, where $\mathcal{H} = (H,\alpha_1,\dots,\alpha_n)$ and $\mathcal{K} = (K,\gamma_1,\dots,\gamma_n)$. A $*$-homomorphism $\pi\colon A \to C$ is called a *(semi-)normal morphism* or *jointly (semi-)normal morphism* from $\mathcal{A}$ to $\mathcal{C}$ if $[\mathcal{L}^\pi_{(s)}(H_{\alpha_i},K_{\gamma_i})\alpha_i] = \gamma_i$ or $[\mathcal{L}^\pi_{(s)}(\mathcal{H},\mathcal{K})\alpha_i] = \gamma_i$, respectively, for each i, where

$$\mathcal{L}^\pi_{(s)}(H_{\alpha_i},K_{\gamma_i}) = \mathcal{L}^\pi(H,K) \cap \mathcal{L}_{(s)}(H_{\alpha_i},K_{\gamma_i}), \quad \mathcal{L}^\pi_{(s)}(\mathcal{H},\mathcal{K}) = \mathcal{L}^\pi(H,K) \cap \mathcal{L}_{(s)}(\mathcal{H},\mathcal{K}).$$

One easily verifies that every semi-normal morphism π between C^*-$\mathfrak{b}$-algebras A_H^α and C_K^γ satisfies $\pi(\rho_\alpha(b^\dagger)) = \rho_\gamma(b^\dagger)$ for all $b^\dagger \in \mathfrak{B}^\dagger$.

We construct a fiber product of C^*-algebras over C^*-bases as follows. Let $\mathfrak{b}$ be a C^*-base, A_H^β a C^*-$\mathfrak{b}$-algebra, and B_K^γ a C^*-$\mathfrak{b}^\dagger$-algebra. The *fiber product* of A_H^β and B_K^γ is the C^*-algebra

$$\begin{aligned} A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B := \big\{ x \in \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K) \,\big|\, & x|\beta\rangle_1, x^*|\beta\rangle_1 \subseteq [|\beta\rangle_1 B] \text{ as subsets of } \mathcal{L}(K, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K), \\ & x|\gamma\rangle_2, x^*|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A] \text{ as subsets of } \mathcal{L}(H, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K) \big\}. \end{aligned}$$

If A and B are unital, so is $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$, but otherwise, $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$ may be degenerate. Clearly, conjugation by the flip $\Sigma\colon H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K \to K_\gamma \underset{\mathfrak{b}^\dagger}{\otimes}{}_\beta H$ yields an isomorphism

$$\mathrm{Ad}_\Sigma\colon A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \to B_\gamma \underset{\mathfrak{b}^\dagger}{*}{}_\beta A.$$

If $\mathfrak{a},\mathfrak{c}$ are C^*-bases, $A_H^{\alpha,\beta}$ is a C^*-$(\mathfrak{a},\mathfrak{b})$-algebra and $B_K^{\gamma,\delta}$ a C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-algebra, then

$$A_H^{\alpha,\beta} \underset{\mathfrak{b}}{*} B_K^{\gamma,\delta} = ({}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta, A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)$$

is a C^*-$(\mathfrak{a},\mathfrak{c})$-algebra, called the *fiber product* of $A_H^{\alpha,\beta}$ and $B_K^{\gamma,\delta}$.

The fiber product construction is functorial with respect to normal morphisms [31, Theorem 3.23], but also with respect to (jointly) semi-normal morphisms. For the proof, we slightly modify [31, Lemma 3.22].

Lemma 3.1. *Let π be a semi-normal morphism of C^*-$\mathfrak{b}$-algebras A_H^β and C_L^λ, let B_K^γ be a C^*-$\mathfrak{b}^\dagger$-algebra, and let $I := \mathcal{L}^\pi(H,L) \underset{\mathfrak{b}}{\otimes} \mathrm{id}$.*

i) $II^*I \subseteq I$ and there exists a unique $*$-homomorphism $\rho_I\colon (I^*I)' \to (II^*)'$ such that $\rho_I(x)y = yx$ for all $x \in (I^*I)'$ and $y \in I$.

ii) There exists a linear contraction j_π from the subspace $[|\gamma\rangle_2 A] \subseteq \mathcal{L}(H, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$ to $[|\gamma\rangle_2 C] \subseteq \mathcal{L}(L, L_\lambda \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$ given by $|\eta\rangle_2 a \mapsto |\eta\rangle_2 \pi(a)$.

iii) $\rho_I(A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B) \subseteq C_{\lambda}\underset{\mathfrak{b}}{*}{}_{\gamma}B$ *and* $\rho_I(x)|\eta\rangle_2 = j_\pi(x|\eta\rangle_2)$ *for all* $x \in A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B$, $\eta \in \gamma$.

Proof. i) The assertion on I is evident and the assertion on ρ_I is [31, Proposition 2.2].

ii) The existence of j_π follows from the fact that we have $(|\eta\rangle_2\pi(a))^*(|\eta'\rangle_2\pi(a')) = \pi(a)^*\rho_\lambda(\eta^*\eta')\pi(a') = \pi\big((|\eta\rangle_2 a)^*(|\eta'\rangle_2 a')\big)$ for all $\eta, \eta' \in \gamma$, $a, a' \in A$.

iii) Let $J := \mathcal{L}^{\pi}_{s}(H_\beta, L_\lambda)$. First, we have $\rho_I(A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B)|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 C]$ because $\rho_I(x)|\eta\rangle_2 = j_\pi(x|\eta\rangle_2) \in [|\gamma\rangle_2 C]$ for all $x \in A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B$, $\eta \in \gamma$. Indeed, for all $S \in J$,

$$\rho_I(x)|\eta\rangle_2 S = \rho_I(x)(S \underset{\mathfrak{b}}{\otimes} \mathrm{id})|\eta\rangle_2 = (S \underset{\mathfrak{b}}{\otimes} \mathrm{id})x|\eta\rangle_2 = j_\pi(x|\eta\rangle_2)S.$$

Second, $\rho_I(A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B)|\lambda\rangle_1 = \rho_I(A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B)[(J \underset{\mathfrak{b}}{\otimes} \mathrm{id})|\beta\rangle_1] \subseteq [(J \underset{\mathfrak{b}}{\otimes} \mathrm{id})(A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B)|\beta\rangle_1] \subseteq [(J \underset{\mathfrak{b}}{\otimes} \mathrm{id})|\beta\rangle_1 B] \subseteq [|\lambda\rangle_1 B]$. ☐

Theorem 3.2. *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be C^*-bases, ϕ a semi-normal morphism of C^*-$(\mathfrak{a},\mathfrak{b})$-algebras $\mathcal{A} = A^{\alpha,\beta}_H$ and $\mathcal{C} = C^{\kappa,\lambda}_L$, and ψ a semi-normal morphism of C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-algebras $\mathcal{B} = B^{\gamma,\delta}_K$ and $\mathcal{D} = D^{\mu,\nu}_M$. Then there exists a unique semi-normal morphism of C^*-$(\mathfrak{a},\mathfrak{c})$-algebras $\phi \underset{\mathfrak{b}}{*} \psi \colon \mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B} \to \mathcal{C} \underset{\mathfrak{b}}{*} \mathcal{D}$ such that*

$$(\phi \underset{\mathfrak{b}}{*} \psi)(x)R = Rx \quad \textit{for all } x \in A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B \textit{ and } R \in I_M J_H + J_L I_K, \tag{13}$$

where $I_X = \mathcal{L}^{\phi}(H,L) \underset{\mathfrak{b}}{\otimes} \mathrm{id}_X$, $J_Y = \mathrm{id}_Y \underset{\mathfrak{b}}{\otimes} \mathcal{L}^{\psi}(K,M)$ for $X \in \{K,M\}, Y \in \{H,L\}$. If both ϕ and ψ are normal, jointly semi-normal or jointly normal, then also $\phi \underset{\mathfrak{b}}{*} \psi$ is normal, jointly semi-normal or jointly normal, respectively.

Proof. This follows from Lemma 3.1 and a similar argument as in the proof of [31, Theorem 3.13]. ☐

Unfortunately, the fiber product need not be associative, but in our applications, it will only appear as the target of a comultiplication whose coassociativity will compensate the non-associativity of the fiber product.

3.2 Spaces of maps on C^*-algebras over C^*-bases

To define convolution algebras of Hopf C^*-bimodules and generalized Fourier algebras of C^*-pseudo-multiplicative unitaries, we need to consider several spaces of maps on C^*-algebras over C^*-bases.

Let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ and $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be C^*-bases, H a Hilbert space, H_α a C^*-$\mathfrak{a}$-module, H_β a C^*-$\mathfrak{b}$-module, and $A \subseteq \mathcal{L}(H)$ a C^*-algebra. We denote by α^∞ the space of all sequences $\eta = (\eta_k)_{k\in\mathbb{N}}$ in α for which the sum $\sum_k \eta_k^*\eta_k$ converges in norm, and put $\|\eta\| := \|\sum_k \eta_k^*\eta_k\|^{1/2}$ for each $\eta \in \alpha^\infty$. Similarly, we define β^∞. Then standard arguments show that for all $\eta \in \beta^\infty, \eta' \in \alpha^\infty$, there exists a bounded linear map

$$\omega_{\eta,\eta'} \colon A \to \mathcal{L}(\mathfrak{H},\mathfrak{K}), \quad T \mapsto \sum_{k\in\mathbb{N}} \eta_k^* T \eta'_k,$$

where the sum converges in norm and $\|\omega_{\eta,\eta'}\| \leq \|\eta\|\|\eta'\|$. We put

$$\Omega_{\beta,\alpha}(A) := \{\omega_{\eta,\eta'} \mid \eta \in \beta^\infty, \eta' \in \alpha^\infty\} \subseteq L(A, \mathcal{L}(\mathfrak{H}, \mathfrak{K}),$$

where $L(A, \mathcal{L}(\mathfrak{H}, \mathfrak{K}))$ denotes the space of bounded linear maps from A to $\mathcal{L}(\mathfrak{H}, \mathfrak{K})$. If $\beta = \alpha$, we abbreviate $\Omega_\beta(A) := \Omega_{\beta,\alpha}(A)$. It is easy to see that $\Omega_{\beta,\alpha}(A)$ is a subspace of $L(A, \mathcal{L}(\mathfrak{H}, \mathfrak{K}))$ and that the following formula defines a norm on $\Omega_{\beta,\alpha}(A)$:

$$\|\omega\| := \inf\left\{\|\eta\|\|\eta'\| \,\middle|\, \eta \in \beta^\infty, \eta' \in \alpha^\infty, \omega = \omega_{\eta,\eta'}\right\} \text{ for all } \omega \in \Omega_{\beta,\alpha}(A).$$

Lemma 3.3. *$\Omega_{\beta,\alpha}(A)$ is a Banach space.*

Proof. Let $(\omega^k)_k$ be a sequence in $\Omega_{\beta,\alpha}(A)$ such that $\|\omega^k\| \leq 4^{-k}$ for all $k \in \mathbb{N}$. We show that the sum $\sum_k \omega^k$ converges in norm in $\Omega_{\beta,\alpha}(A)$. For each $k \in \mathbb{N}$, we can choose $\eta^k \in \beta^\infty$ and $\eta'^k \in \alpha^\infty$ such that $\omega^k = \omega_{\eta^k,\eta'^k}$ and $\|\eta^k\|\|\eta'^k\| \leq 4^{1-k}$. Without loss of generality, we may assume $\|\eta^k\| \leq 2^{1-k}$ and $\|\eta'^k\| \leq 2^{1-k}$. Choose a bijection $i\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and let $\eta_{i(k,n)} = \eta^k_n$ and $\eta'_{i(k,n)} = \eta'^k_n$ for all $k, n \in \mathbb{N}$. Routine calculations show that $\eta \in \beta^\infty$, $\eta' \in \alpha^\infty$, and that the sum $\sum_k \omega^k$ converges in norm to $\omega_{\eta,\eta'} \in \Omega_{\beta,\alpha}(A)$. □

We have the following straightforward result:

Proposition 3.4. *There exists a linear isometry $\Omega_{\beta,\alpha}(A) \to \Omega_{\alpha,\beta}(A)$, $\omega \mapsto \omega^*$, such that $\omega^*(a) = \omega(a^*)^*$ for all $a \in A$ and $(\omega_{\eta,\eta'})^* = \omega_{\eta',\eta}$ for all $\eta \in \beta^\infty, \eta' \in \alpha^\infty$.* □

We can pull back maps of the form considered above via morphisms as follows:

Proposition 3.5. *i) Let π be a normal morphism of C^*-$\mathfrak{b}$-algebras A^α_H and B^γ_K. Then there exists a linear contraction $\pi^*\colon \Omega_\gamma(B) \to \Omega_\alpha(A)$ given by $\omega \mapsto \omega \circ \pi$.*

ii) Let π be a jointly normal morphism of C^*-$(\mathfrak{a}, \mathfrak{b})$-algebras $A^{\alpha,\beta}_H$ and $B^{\gamma,\delta}_K$. Then there exists a linear contraction $\pi^*\colon \Omega_{\delta,\gamma}(B) \to \Omega_{\beta,\alpha}(A)$ given by $\omega \mapsto \omega \circ \pi$.

Proof. We only prove ii), the proof of i) is similar. Let $I := \mathcal{L}^\pi({}_\alpha H_\beta, {}_\gamma K_\delta)$ and $\eta \in \delta^\infty$, $\eta' \in \gamma^\infty$. Then there exists a closed separable subspace $I_0 \subseteq I$ such that $\eta_n \in [I_0\beta]$ and $\eta'_n \in [I_0\alpha]$ for all $n \in \mathbb{N}$. We may also assume that $I_0 I_0^* I_0 \subseteq I_0$, and then $[I_0 I_0^*]$ is a σ-unital C^*-algebra and has a bounded sequential approximate unit $(u_k)_k$ of the form $u_k = \sum_{l=1}^k T_l T_l^*$, where $(T_l)_l$ is a sequence in I_0 [17, Proposition 6.7]. We choose a bijection $i\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and let $\xi_{i(l,n)} := T_l^*\eta_n \in \beta$ and $\xi'_{i(l,n)} := T_l^*\eta'_n \in \alpha$ for all $l, n \in \mathbb{N}$. Then the sum $\sum_l \xi^*_{i(l,n)}\xi_{i(l,n)} = \sum_l \eta_n^* T_l T_l^* \eta_n$ converges to $\eta_n^*\eta_n$ for each $n \in \mathbb{N}$ in norm because $\eta_n \in [I_0\beta]$. Therefore, $\xi \in \beta^\infty$ and $\|\xi\| = \|\eta\|$, and a similar argument shows that $\xi' \in \alpha^\infty$ and $\|\xi'\| = \|\eta'\|$. Finally,

$$\omega_{\xi,\xi'}(a) = \sum_{l,n} \eta_n^* T_l a T_l^* \eta'_n = \sum_{l,n} \eta_n^* \pi(a) T_l T_l^* \eta'_n = \sum_n \eta_n^* \pi(a) \eta'_n = \omega_{\eta,\eta'}(\pi(a))$$

for each $a \in A$, where the sum converges in norm, and hence $\omega_{\eta,\eta'} \circ \pi = \omega_{\xi,\xi'} \in \Omega_{\beta,\alpha}(A)$ and $\|\omega_{\eta,\eta'} \circ \pi\| \leq \|\xi\|\|\xi'\| = \|\eta\|\|\eta'\|$. □

For each map of the form considered above, we can form a slice map as follows.

Proposition 3.6. *Let A_H^β be a C^*-$\mathfrak{b}$-algebra and B_K^γ a C^*-$\mathfrak{b}^\dagger$-algebra.*

i) There exist linear contractions

$$\Omega_\beta(A) \to \Omega_{|\beta\rangle_1}(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B),\ \phi \mapsto \phi * \mathrm{id}, \quad \Omega_\gamma(B) \to \Omega_{|\gamma\rangle_2}(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B),\ \psi \mapsto \mathrm{id} * \psi,$$

such that for all $\xi, \xi' \in \beta^\infty$ and $\eta, \eta' \in \gamma^\infty$,

$$\omega_{\xi,\xi'} * \mathrm{id} = \omega_{\tilde{\xi},\tilde{\xi}'},\ \textit{where } \tilde{\xi}_n = |\xi_n\rangle_1,\ \tilde{\xi}'_n = |\xi'_n\rangle_1 \textit{ for all } n \in \mathbb{N},$$
$$\mathrm{id} * \omega_{\eta,\eta'} = \omega_{\tilde{\eta},\tilde{\eta}'},\ \textit{where } \tilde{\eta}_n = |\eta_n\rangle_2, \tilde{\eta}'_n = |\eta'_n\rangle_2 \textit{ for all } n \in \mathbb{N}.$$

ii) We have $\psi \circ (\phi * \mathrm{id}) = \phi \circ (\mathrm{id} * \psi)$ for all $\phi \in \Omega_\beta(A)$ and $\psi \in \Omega_\gamma(B)$.

Proof. i), ii) Straightforward; see [31, Proposition 3.30]. ☐

Finally, we need to consider fiber products of the linear maps considered above. We denote by "$\hat{\otimes}$" the projective tensor product of Banach spaces.

Proposition 3.7. *Let $A_H^{\alpha,\beta}$ be a C^*-$(\mathfrak{a},\mathfrak{b})$-algebra and $B_K^{\gamma,\delta}$ a C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-algebra.*

i) There exist linear contractions

$$\Omega_\alpha(A) \hat{\otimes} \Omega_\gamma(B) \to \Omega_{(\alpha\lhd\gamma)}(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B),\ \omega \otimes \omega' \mapsto \omega \boxtimes \omega' := \omega \circ (\mathrm{id} * \omega'),$$
$$\Omega_\beta(A) \hat{\otimes} \Omega_\delta(B) \to \Omega_{(\beta\rhd\delta)}(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B),\ \omega \otimes \omega' \mapsto \omega \boxtimes \omega' := \omega' \circ (\omega * \mathrm{id}).$$

ii) There exist linear contractions

$$\Omega_{\alpha,\beta}(A) \hat{\otimes} \Omega_{\gamma,\delta}(B) \to \Omega_{(\alpha\lhd\gamma),(\beta\rhd\delta)}(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B),\ \omega \otimes \omega' \mapsto \omega \boxtimes \omega',$$
$$\Omega_{\beta,\alpha}(A) \hat{\otimes} \Omega_{\delta,\beta}(B) \to \Omega_{(\beta\rhd\delta),(\alpha\lhd\gamma)}(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B),\ \omega \otimes \omega' \mapsto \omega \boxtimes \omega',$$

such that for all $\xi \in \alpha^\infty$, $\xi' \in \beta^\infty$, $\eta \in \gamma^\infty$, $\eta' \in \delta^\infty$ and each bijection $i \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we have $\omega_{\xi,\xi'} \boxtimes \omega_{\eta,\eta'} = \omega_{\theta,\theta'}$ and $\omega_{\xi',\xi} \boxtimes \omega_{\eta',\eta} = \omega_{\theta',\theta}$ where

$$\theta_{i(m,n)} = |\eta_n\rangle_2 \xi_m \in \alpha \lhd \gamma, \quad \theta_{i(m,n)} = |\xi'_m\rangle_1 \eta'_n \in \beta \rhd \delta \quad \textit{for all } m,n \in \mathbb{N}.$$

Proof. The proof of assertion i) is straightforward; we only prove the existence of the first contraction in ii). Let $\xi, \xi', \eta, \eta', i, \theta, \theta'$ be as above. Then $\theta \in (\alpha \lhd \gamma)^\infty$ and $\|\theta\| \leq \|\xi\| \|\eta\|$ because

$$\sum_k \theta_k^* \theta_k = \sum_{m,n} \xi_m^* \langle \eta_n |_2 |\eta_n\rangle_2 \xi_m = \sum_{m,n} \xi_m^* \rho_\beta(\eta_n^* \eta_n) \xi_m \leq \|\eta\|^2 \sum_m \xi_m^* \xi_m \leq \|\eta\|^2 \|\xi\|^2,$$

and similarly $\theta' \in (\beta \rhd \delta)^\infty$ and $\|\theta'\| \leq \|\xi'\| \|\eta'\|$. Next, we show that $\omega_{\theta,\theta'}$ does not depend on ξ and ξ' but only on $\omega_{\xi,\xi'} \in \Omega_{\alpha,\beta}(A)$. Let $\zeta' \in \mathfrak{K}$ and $x \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$. Then

$$\omega_{\theta,\theta'}(x)\zeta' = \sum_{m,n \in \mathbb{N}} \xi_m^* \langle \eta_n |_2 x |\xi'_m\rangle_1 \eta'_n \zeta',$$

where the sum converges in norm. Fix any $n \in \mathbb{N}$. Then we find a sequence $(k_r)_r$ in $\mathbb{N}$ and $\eta''_{r,1}, \dots, \eta''_{r,k_r} \in \gamma$, $\zeta''_{r,1}, \dots, \zeta''_{r,k_r} \in \mathfrak{K}$ such that the sum $\sum_{l=1}^{k_r} \eta''_{r,l}\zeta''_{r,l}$ converges in norm to $\eta'_n\zeta'$ as r tends to infinity. But then

$$\begin{aligned}
\sum_m \xi^*_m \langle \eta_n|_2 x|\xi'_m\rangle_1 \eta'_n\zeta' &= \lim_{r\to\infty} \sum_m \sum_{l=1}^{k_r} \xi^*_m \langle \eta_n|_2 x|\xi'_m\rangle_1 \eta''_{r,l}\zeta''_{r,l} \\
&= \lim_{r\to\infty} \sum_{l=1}^{k_r} \sum_m \xi^*_m \langle \eta_n|_2 x|\eta''_{r,l}\rangle_2 \xi'_m\zeta''_{r,l} \\
&= \lim_{r\to\infty} \sum_{l=1}^{k_r} \omega_{\xi,\xi'}(\langle \eta_n|_2 x|\eta''_{r,l}\rangle_2)\zeta''_{r,l}.
\end{aligned}$$

Note here that $\langle \eta_n|x|\eta''_{r,l}\rangle_2 \in A$. Therefore, the sum on the left hand side only depends on $\omega_{\xi,\xi'} \in \Omega_{\alpha,\beta}(A)$ but not on ξ,ξ', and since $n \in \mathbb{N}$ was arbitrary, the same is true for $\omega_{\theta,\theta'}(x)\zeta'$. A similar argument shows that $\omega_{\theta,\theta'}(x)^*\zeta$ depends on $\omega_{\eta,\eta'} \in \Omega_{\gamma,\delta}(B)$ but not on η,η' for each $\zeta \in \mathfrak{K}$. □

3.3 Concrete Hopf C^*-bimodules and their convolution algebras

The fiber product construction leads to the following generalization of a Hopf C^*-algebra and of related concepts.

Definition 3.8. *Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. A* comultiplication *on a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-algebra $A^{\beta,\alpha}_H$ is a jointly semi-normal morphism Δ from $A^{\beta,\alpha}_H$ to $A^{\beta,\alpha}_H \underset{\mathfrak{b}}{*} A^{\beta,\alpha}_H$ that is coassociative in the sense that the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[rr, "\Delta"] \arrow[d, "\Delta"] & & A\,{}_{\alpha}\underset{\mathfrak{b}}{*}{}_{\beta}A \arrow[d, "\mathrm{id}\underset{\mathfrak{b}}{*}\Delta"] \\
& & A\,{}_{\alpha}\underset{\mathfrak{b}}{*}{}_{(\beta\lhd\beta)}(A\,{}_{\alpha}\underset{\mathfrak{b}}{*}{}_{\beta}A) \arrow[d, hook] \\
A\,{}_{\alpha}\underset{\mathfrak{b}}{*}{}_{\beta}A \arrow[r, "\Delta\underset{\mathfrak{b}}{*}\mathrm{id}"] & (A\,{}_{\alpha}\underset{\mathfrak{b}}{*}{}_{\beta}A)\,{}_{(\alpha\rhd\alpha)}\underset{\mathfrak{b}}{*}{}_{\beta}A \arrow[r, hook] & \mathcal{L}(H\,{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H\,{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H).
\end{tikzcd}$$

A (semi-)normal Hopf C^*-bimodule over $\mathfrak{b}$ *is a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-algebra with a jointly (semi-)normal comultiplication. When we speak of a Hopf C^*-bimodule, we always mean a semi-normal one. A morphism of (semi-)normal Hopf C^*-bimodules $(A^{\beta,\alpha}_H, \Delta_A)$, $(B^{\delta,\gamma}_K, \Delta_B)$ over $\mathfrak{b}$ is a jointly (semi-)normal morphism π from $A^{\beta,\alpha}_H$ to $B^{\delta,\gamma}_K$ satisfying $\Delta_B \circ \pi = (\pi \underset{\mathfrak{b}}{*} \pi) \circ \Delta_A$.*

Let $(A^{\beta,\alpha}_H, \Delta)$ be a Hopf C^*-bimodule over $\mathfrak{b}$. A bounded left Haar weight *for $(A^{\beta,\alpha}_H, \Delta)$ is a completely positive contraction $\phi\colon A \to \mathfrak{B}$ satisfying*

$$\phi(a\rho_\beta(b)) = \phi(a)b, \qquad \phi(\langle \xi|_1\Delta(a)|\xi'\rangle_1) = \xi^*\rho_\beta(\phi(a))\xi'$$

for all $a \in A$, $b \in \mathfrak{B}$, $\xi, \xi' \in \alpha$. We call ϕ normal *if* $\phi \in \Omega_{M(\beta)}(A)$*, where* $M(\beta) = \{T \in \mathcal{L}(\mathfrak{K}, H) \mid T\mathfrak{B}^\dagger \subseteq \beta, T^*\beta \subseteq \mathfrak{B}^\dagger\}$. *Similarly, a* bounded right Haar weight *for* $(A_H^{\beta,\alpha}, \Delta)$ *is a completely positive contraction* $\psi \colon A \to \mathfrak{B}^\dagger$ *satisfying*

$$\psi(a\rho_\alpha(b^\dagger)) = \psi(a)b^\dagger, \qquad \psi(\langle\eta|_2\Delta(a)|\eta'\rangle_2) = \eta^*\rho_\alpha(\psi(a))\eta'$$

for all $a \in A$, $b^\dagger \in \mathfrak{B}^\dagger$, $\eta, \eta' \in \beta$. We call ψ normal *if* $\psi \in \Omega_{M(\alpha)}(A)$*, where* $M(\alpha) = \{S \in \mathcal{L}(\mathfrak{K}, H) \mid S\mathfrak{B} \subseteq \alpha, S^*\alpha \subseteq \mathfrak{B}\}$.

A bounded (left/right) counit *for* $(A_H^{\beta,\alpha}, \Delta)$ *is a jointly semi-normal morphism of* C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$*-algebras* $\varepsilon \colon A_H^{\beta,\alpha} \to \mathcal{L}(\mathfrak{K})_{\mathfrak{K}}^{\mathfrak{B}^\dagger, \mathfrak{B}}$ *that makes the (left/right one of the) following two diagrams commute:*

$$\begin{tikzcd}
A_\alpha \underset{\mathfrak{b}}{*}{}_\beta A \arrow[d, "\varepsilon \underset{\mathfrak{b}}{*} \mathrm{id}"'] & & A \arrow[ll, "\Delta"'] \arrow[d] \\
\mathcal{L}(\mathfrak{K})_{\mathfrak{B}} \underset{\mathfrak{b}}{*}{}_\beta A \arrow[r] & \mathcal{L}(\mathfrak{K}_{\mathfrak{B}} \underset{\mathfrak{b}}{\otimes}{}_\beta H) \arrow[r, "\cong"] & \mathcal{L}(H),
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "\Delta"] \arrow[d] & & A_\alpha \underset{\mathfrak{b}}{*}{}_\beta A \arrow[d, "\mathrm{id} \underset{\mathfrak{b}}{*} \varepsilon"] \\
\mathcal{L}(H) & \mathcal{L}(H_\alpha \underset{\mathfrak{b}}{\otimes}{}_{\mathfrak{B}^\dagger} \mathfrak{K}) \arrow[l, "\cong"'] & A_\alpha \underset{\mathfrak{b}}{*}{}_{\mathfrak{B}^\dagger} \mathcal{L}(\mathfrak{K}). \arrow[l]
\end{tikzcd} \tag{14}$$

Remark 3.9. Let $(A_H^{\beta,\alpha}, \Delta)$ be a Hopf C^*-bimodule over $\mathfrak{b}$. Evidently, a completely positive contraction $\phi \colon A \to \mathfrak{B}$ is a normal bounded left Haar weight for $(A_H^{\beta,\alpha}, \Delta)$ if and only if $\phi \in \Omega_{M(\beta)}(A)$ and $(\mathrm{id} * \phi) \circ \Delta = \rho_\beta \circ \phi$. A similar remark applies to normal bounded right Haar weights.

Let $(A_H^{\beta,\alpha}, \Delta)$ be a normal Hopf C^*-bimodule over $\mathfrak{b}$. Combining Propositions 3.5 and 3.7, we obtain for each of the spaces $\Omega = \Omega_\alpha(A), \Omega_\beta(A), \Omega_{\alpha,\beta}(A), \Omega_{\beta,\alpha}(A)$ a map

$$\Omega \times \Omega \to \Omega, \quad (\omega, \omega') \mapsto \omega * \omega' := (\omega \boxtimes \omega') \circ \Delta. \tag{15}$$

Theorem 3.10. *Let $(A_H^{\beta,\alpha}, \Delta)$ be a normal Hopf C^*-bimodule over $\mathfrak{b}$. Then $\Omega_\alpha(A)$, $\Omega_\beta(A)$, $\Omega_{\alpha,\beta}(A)$, $\Omega_{\beta,\alpha}(A)$ are Banach algebras with respect to the multiplication* (15)*.*

Proof. It only remains to show that the multiplication is associative, but this follows from the coassociativity of Δ. $\square$

3.4 The legs of a C^*-pseudo-multiplicative unitary

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, $(H, \widehat{\beta}, \alpha, \beta)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module and $V \colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H$ a C^*-pseudo-multiplicative unitary. We associate to V two algebras and, if V is well behaved, two Hopf C^*-bimodules as follows. Let

$$\widehat{A}_V := [\langle\beta|_2 V|\alpha\rangle_2] \subseteq \mathcal{L}(H), \qquad A_V := [\langle\alpha|_1 V|\widehat{\beta}\rangle_1] \subseteq \mathcal{L}(H), \tag{16}$$

where $|\alpha\rangle_2, |\widehat{\beta}\rangle_1 \subseteq \mathcal{L}(H, H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H)$ and $\langle\beta|_2, \langle\alpha|_1 \subseteq \mathcal{L}(H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H, H)$ are defined as in Subsection 2.1.

Proposition 3.11. *The following relations hold:*

$$\widehat{A}_{V^{op}} = A_V^*, \quad [\widehat{A}_V\widehat{A}_V] = \widehat{A}_V, \quad [\widehat{A}_V H] = H = [\widehat{A}_V^* H], \quad [\widehat{A}_V \beta] = \beta = [\widehat{A}_V^* \beta],$$
$$[\widehat{A}_V \rho_{\widehat{\beta}}(\mathfrak{B})] = [\rho_{\widehat{\beta}}(\mathfrak{B})\widehat{A}_V] = \widehat{A}_V = [\widehat{A}_V \rho_\alpha(\mathfrak{B}^\dagger)] = [\rho_\alpha(\mathfrak{B}^\dagger)\widehat{A}_V],$$
$$A_{V^{op}} = \widehat{A}_V^*, \quad [A_V A_V] = A_V, \quad [A_V H] = H = [A_V^* H], \quad [A_V \widehat{\beta}] = \widehat{\beta} = [A_V^* \widehat{\beta}],$$
$$[A_V \rho_\beta(\mathfrak{B})] = [\rho_\beta(\mathfrak{B}) A_V] = A_V = [A_V \rho_\alpha(\mathfrak{B}^\dagger)] = [\rho_\alpha(\mathfrak{B}^\dagger) A_V].$$

Proof. First, we have $\widehat{A}_{V^{op}} = [\langle\widehat{\beta}|_2 \Sigma V^* \Sigma|\alpha\rangle_2] = [\langle\widehat{\beta}|_1 V^*|\alpha\rangle_1] = A_V^*$ and $[\widehat{A}_V\beta] = [\langle\beta|_2 V|\alpha\rangle_2\beta] = [\langle\beta|_2|\beta\rangle_2\beta] = [\rho_\alpha(\mathfrak{B}^\dagger)\beta] = \beta$ because $V(\beta \triangleleft \alpha) = \beta \triangleleft \beta$. Similarly, one shows that $[\widehat{A}_V^*\beta] = \beta$, $A_{V^{op}} = \widehat{A}_V^*$, and $[A_V\widehat{\beta}] = \widehat{\beta} = [A_V^*\widehat{\beta}]$. The remaining equations are a particular case of Proposition 4.10 in subsection 4.2. ☐

Consider the $*$-homomorphisms

$$\widehat{\Delta}_V \colon \rho_\beta(\mathfrak{B})' \to \mathcal{L}\big(H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H\big),\ y \mapsto V^*(\mathrm{id} \underset{\mathfrak{b}}{\otimes} y)V,$$
$$\Delta_V \colon \rho_{\widehat{\beta}}(\mathfrak{B})' \to \mathcal{L}\big(H {}_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H\big),\ z \mapsto V(z \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})V^*.$$

Proposition 3.12. *$\widehat{\Delta}_V$ is a jointly normal morphism of C^*-$(\mathfrak{b},\mathfrak{b}^\dagger)$-algebras $(\rho_\beta(\mathfrak{B})')_H^{\alpha,\widehat{\beta}}$ and $\big((\rho_\beta(\mathfrak{B}) {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha \rho_\beta(\mathfrak{B}))'\big)^{(\alpha\triangleleft\alpha),(\widehat{\beta}\triangleright\widehat{\beta})}_{H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H}$, and Δ_V is a jointly normal morphism of C^*-$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebras $(\rho_{\widehat{\beta}}(\mathfrak{B})')_H^{\beta,\alpha}$ and $\big((\rho_{\widehat{\beta}}(\mathfrak{B}) {}_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta \rho_{\widehat{\beta}}(\mathfrak{B}))'\big)^{(\beta\triangleleft\beta),(\alpha\triangleright\alpha)}_{H {}_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H}$. Moreover, $\widehat{\Delta}_{V^{op}} = \mathrm{Ad}_\Sigma \circ \Delta_V$ and $\Delta_{V^{op}} = \mathrm{Ad}_\Sigma \circ \widehat{\Delta}_V$.*

Proof. We only prove the assertions concerning $\widehat{\Delta}_V$. The relation $\Delta_{V^{op}} = \mathrm{Ad}_\Sigma \circ \widehat{\Delta}_V$ is easily verified. Next, $\widehat{\Delta}_V(\rho_\beta(\mathfrak{B})') \subseteq (\rho_\beta(\mathfrak{B}) \underset{\mathfrak{b}^\dagger}{\otimes} \rho_\beta(\mathfrak{B}))'$ because $V(\rho_\beta(\mathfrak{B}) \underset{\mathfrak{b}^\dagger}{\otimes} \rho_\beta(\mathfrak{B})) = \rho_\beta(\mathfrak{B}) \underset{\mathfrak{b}}{\otimes} \rho_{\widehat{\beta}}(\mathfrak{B}) \subseteq \mathrm{id} \underset{\mathfrak{b}}{\otimes} \rho_\beta(\mathfrak{B})'$ by (7). To see that $\widehat{\Delta}_V$ is a jointly normal morphism, note that $V^*|\alpha\rangle_1 \subseteq \mathcal{L}^{\widehat{\Delta}_V}\big(H, H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H\big)$ because $\widehat{\Delta}(y)V^*|\xi\rangle_1 = V^*(\mathrm{id} \underset{\mathfrak{b}}{\otimes} y)|\xi\rangle_1 = V^*|\xi\rangle_1 y$ for all $y \in \rho_{\widehat{\beta}}(\mathfrak{B})'$, $\xi \in \alpha$, and that $\alpha \triangleleft \alpha = [V^*|\alpha\rangle_1\alpha]$ and $\widehat{\beta} \triangleright \widehat{\beta} = [V^*|\alpha\rangle_1\widehat{\beta}]$ by (7). ☐

Under favorable circumstances, $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$ and $((A_V)_H^{\beta,\alpha}, \Delta_V)$ will be concrete Hopf C^*-bimodules. A sufficient condition, regularity, will be given in subsection 5.1. Coassociativity of $\widehat{\Delta}_V$ and Δ_V follows easily from the commutativity of diagram (8):

Lemma 3.13. *If $\widehat{B} \subseteq \rho_{\widehat{\beta}}(\mathfrak{B})'$ is a C^*-algebra, $\rho_\alpha(\mathfrak{B}^\dagger)\widehat{B} + \rho_{\widehat{\beta}}(\mathfrak{B})\widehat{B} \subseteq \widehat{B}$ and $\widehat{\Delta}_V(\widehat{B}) \subseteq \widehat{B} {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{*}{}_\alpha \widehat{B}$, then $(\widehat{B}_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$ is a normal Hopf C^*-bimodule over $\mathfrak{b}^\dagger$. Similarly, if $B \subseteq \rho_\beta(\mathfrak{B})'$ is a C^*-algebra, $\rho_\beta(\mathfrak{B})B + \rho_\alpha(\mathfrak{B}^\dagger)B \subseteq B$ and $\Delta_V(B) \subseteq B {}_\alpha\underset{\mathfrak{b}}{*}{}_\beta B$, then $(B_H^{\beta,\alpha}, \Delta_V)$ is a normal Hopf C^*-bimodule over $\mathfrak{b}$.*

Proof. We only prove the assertion concerning $\widehat{B}$; the assertion concerning B follows similarly. Let $\widehat{B} \subseteq \rho_{\widehat{\beta}}(\mathfrak{B})'$ be a C^*-algebra satisfying the assumptions and put $\widehat{\Delta} := \widehat{\Delta}_V$. By Proposition 3.12, we only need to show that $(\widehat{\Delta} \underset{\mathfrak{b}^{\dagger}}{*} \mathrm{id})(\widehat{\Delta}(\widehat{b})) = (\mathrm{id} \underset{\mathfrak{b}^{\dagger}}{*} \widehat{\Delta})(\widehat{\Delta}(\widehat{b}))$ for all $\widehat{b} \in \widehat{B}$. But this is shown by the following commutative diagram:

$$\begin{tikzcd}
H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[r, "(\widehat{\Delta} \underset{\mathfrak{b}^{\dagger}}{*} \mathrm{id})(\widehat{\Delta}(\widehat{a}))"] \arrow[d, "V_{12}"] \arrow[ddddd, bend right=60, "\mathrm{id}"'] & H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \\
H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[r, "\mathrm{id} \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\widehat{a})"] \arrow[d, "V_{23}"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[u, "V_{12}^{*}"'] \\
H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "\mathrm{id} \underset{\mathfrak{b}}{\otimes} \mathrm{id} \underset{\mathfrak{b}}{\otimes} \widehat{a}"] \arrow[d, "V_{12}^{*}"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "V_{23}^{*}"'] \\
(H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H)_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "\mathrm{id} \underset{\mathfrak{b}^{\dagger}}{\otimes} \mathrm{id} \underset{\mathfrak{b}}{\otimes} \widehat{a}"] \arrow[d, "V_{12}^{*}\Sigma_{23}"] & (H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H)_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "V_{12}"'] \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H_{\beta}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[r, "\widehat{\Delta}(\widehat{a}) \underset{\mathfrak{b}^{\dagger}}{\otimes} \mathrm{id}"] \arrow[d, "\Sigma_{23}V_{23}"] & H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H_{\beta}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[u, "\Sigma_{23}V_{12}"'] \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[r, "(\mathrm{id} \underset{\mathfrak{b}^{\dagger}}{*} \widehat{\Delta})(\widehat{\Delta}(\widehat{a}))"] & H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[u, "V_{23}^{*}\Sigma_{23}"'] \arrow[uuuuu, bend right=60, "\mathrm{id}"']
\end{tikzcd}$$
□

Using the maps introduced in subsection 3.2, we construct convolution algebras $\tilde{\Omega}_{\beta,\alpha}$ and $\tilde{\Omega}_{\alpha,\widehat{\beta}}$ with homomorphisms onto dense subalgebras $\widehat{A}_V^0 \subseteq \widehat{A}_V$ and $A_V^0 \subseteq A_V$, respectively, as follows. Let

$$\tilde{\Omega}_{\beta,\alpha} := \Omega_{\beta,\alpha}\big(\rho_{\widehat{\beta}}(\mathfrak{B})'\big), \qquad \tilde{\Omega}_{\alpha,\widehat{\beta}} := \Omega_{\alpha,\widehat{\beta}}(\rho_{\beta}(\mathfrak{B})').$$

Theorem 3.14. *i) There exist linear contractions*

$$\tilde{\Omega}_{\beta,\alpha}\hat{\otimes}\tilde{\Omega}_{\beta,\alpha} \to \Omega_{(\beta\triangleleft\beta),(\alpha\triangleright\alpha)}\big((\rho_{\widehat{\beta}}(\mathfrak{B})_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}\rho_{\widehat{\beta}}(\mathfrak{B}))'\big), \quad \omega\otimes\omega' \mapsto \omega\boxtimes\omega',$$

$$\tilde{\Omega}_{\alpha,\widehat{\beta}}\hat{\otimes}\tilde{\Omega}_{\alpha,\widehat{\beta}} \to \Omega_{(\alpha\triangleleft\alpha),(\widehat{\beta}\triangleright\widehat{\beta})}\big((\rho_{\beta}(\mathfrak{B})_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}\rho_{\beta}(\mathfrak{B}))'\big), \quad \omega\otimes\omega' \mapsto \omega\boxtimes\omega',$$

such that for all $\xi,\xi' \in \beta^{\infty}$, $\eta,\eta' \in \alpha^{\infty}$, $\zeta,\zeta' \in \widehat{\beta}^{\infty}$ and each bijection $i\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, we have $\omega_{\xi,\eta}\boxtimes\omega_{\xi',\eta'} = \omega_{\theta,\theta'}$ and $\omega_{\eta,\zeta}\boxtimes\omega_{\eta',\zeta'} = \omega_{\kappa,\kappa'}$, where for all $m,n \in \mathbb{N}$,

$$\theta_{i(m,n)} = |\xi'_n\rangle_2\xi_m \in \beta\triangleleft\beta, \qquad \theta'_{i(m,n)} = |\eta_m\rangle_1\eta'_n \in \alpha\triangleright\alpha,$$

$$\kappa_{i(m,n)} = |\eta'_n\rangle_2\eta_m \in \alpha\triangleleft\alpha, \qquad \kappa'_{i(m,n)} = |\zeta_m\rangle_1\zeta'_n \in \widehat{\beta}\triangleright\widehat{\beta}.$$

ii) The Banach spaces $\tilde{\Omega}_{\beta,\alpha}$ and $\tilde{\Omega}_{\alpha,\widehat{\beta}}$ carry the structure of Banach algebras, where the multiplication is given by $\omega*\omega' = (\omega\boxtimes\omega')\circ\Delta_V$ and $\omega*\omega' = (\omega\boxtimes\omega')\circ\widehat{\Delta}_V$, respectively.

iii) There exist contractive algebra homomorphisms $\widehat{\pi}_V\colon \tilde{\Omega}_{\beta,\alpha} \to \widehat{A}_V$ and $\pi_V\colon \tilde{\Omega}_{\alpha,\widehat{\beta}} \to A_V$ such that for all $\xi \in \beta^{\infty}$, $\eta \in \alpha^{\infty}$, $\zeta \in \widehat{\beta}^{\infty}$,

$$\widehat{\pi}_V(\omega_{\xi,\eta}) = \sum_n \langle\xi_n|_2V|\eta_n\rangle_2, \qquad \pi_V(\omega_{\eta,\zeta}) = \sum_n \langle\eta_n|_1V|\zeta_n\rangle_1.$$

Proof. i) This is a slight modification of Proposition 3.7 and follows from similar arguments.

ii) The existence of the multiplication in ii) follows from i) and Propositions 3.5 and 3.12, and associativity from coassociativity of Δ_V and $\widehat{\Delta}_V$ (see the proof of Lemma 3.13).

iii) This is a special case of the more general Proposition 4.13 which is proven in subsection 4.3. $\square$

If $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$ and $((A_V)_H^{\beta,\alpha}, \Delta_V)$ are Hopf C^*-bimodules, they should be thought of as standing in a generalized Pontrjagin duality. This duality is captured by a pairing on the dense subalgebras

$$\widehat{A}_V^0 := \widehat{\pi}_V(\tilde{\Omega}_{\beta,\alpha}) \subseteq \widehat{A}_V, \qquad A_V^0 := \pi_V(\tilde{\Omega}_{\alpha,\widehat{\beta}}) \subseteq A_V.$$

Definition 3.15. *We call the algebra $\widehat{A}_V^0 \subseteq \widehat{A}_V$, equipped with the quotient norm from the surjection $\widehat{\pi}_V$, the* Fourier algebra *of V. Similarly, we call the algebra $A_V^0 \subseteq A_V$, equipped with the quotient norm from the surjection π_V, the* dual Fourier algebra *of V.*

Proposition 3.16. *i) There exists a bilinear map $(\cdot|\cdot)\colon \widehat{A}_V^0 \times A_V^0 \to \mathcal{L}(\mathfrak{K})$ such that $\omega(\pi_V(\upsilon)) = \big(\widehat{\pi}_V(\omega)\big|\pi_V(\upsilon)\big) = \upsilon(\widehat{\pi}_V(\omega))$ for all $\omega \in \tilde{\Omega}_{\beta,\alpha}, \upsilon \in \tilde{\Omega}_{\alpha,\widehat{\beta}}$.*

ii) This map is nondegenerate in the sense that for each $\widehat{a} \in \widehat{A}_V^0$ and $a \in A_V^0$, there exist $\widehat{a}' \in \widehat{A}_V^0$ and $a' \in A_V^0$ such that $(\widehat{a}|a') \neq 0$ and $(\widehat{a}'|a) \neq 0$.

iii) $(\widehat{\pi}_V(\omega)\widehat{\pi}_V(\omega')|a) = (\omega \boxtimes \omega')(\Delta_V(a))$ and $(\widehat{a}|\pi_V(\upsilon)\pi_V(\upsilon')) = (\upsilon \boxtimes \upsilon')(\widehat{\Delta}_V(\widehat{a}))$ for all $\omega, \omega' \in \tilde{\Omega}_{\beta,\alpha}$, $a \in A_V^0$, $\upsilon, \upsilon' \in \tilde{\Omega}_{\alpha,\widehat{\beta}}$, $\widehat{a} \in \widehat{A}_V^0$.

Proof. i) If $\omega = \omega_{\xi,\xi'}$ and $\upsilon = \omega_{\eta,\eta'}$, where $\xi \in \beta^\infty$, $\xi', \eta \in \alpha^\infty$, $\eta' \in \widehat{\beta}^\infty$, then

$$\omega(\pi_V(\upsilon)) = \sum_{m,n} \xi_m^* \langle \eta_n|_1 V|\eta_n'\rangle_1 \xi_m' = \sum_{m,n} \eta_n^* \langle \xi_m|_2 V|\xi_m'\rangle_2 \eta_n' = \upsilon(\widehat{\pi}_V(\omega)).$$

ii) Evident.

iii) For all ω, ω', a as above, $(\widehat{\pi}_V(\omega)\widehat{\pi}_V(\omega')|a) = (\widehat{\pi}_V(\omega * \omega')|a) = (\omega * \omega')(a) = (\omega \boxtimes \omega)(\Delta_V(a))$. The second equation follows similarly. $\square$

As a consequence of part ii) of the preceding result, we obtain the following simple relation between the Fourier algebra $\widehat{A}_V^0$ and the convolution algebra constructed in Theorem 3.10.

Proposition 3.17. *If $((A_V)_H^{\beta,\alpha}, \widehat{\Delta}_V)$ or $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \Delta_V)$ is a normal Hopf C^*-bimodule, then we have a commutative diagram of Banach algebras and homomorphisms*

$$\begin{tikzcd} \tilde{\Omega}_{\beta,\alpha} \arrow[r, "\widehat{\pi}_V"] \arrow[d, equal] & \widehat{A}_V^0 \\ \Omega_{\beta,\alpha}(\rho_{\widehat{\beta}}(\mathfrak{B})') \arrow[r, "q"] & \Omega_{\beta,\alpha}(A_V) \arrow[u, "\widehat{\pi}"'] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \tilde{\Omega}_{\alpha,\widehat{\beta}} \arrow[r, "\pi_V"] \arrow[d, equal] & A_V^0 \\ \Omega_{\alpha,\widehat{\beta}}(\rho_{\beta}(\mathfrak{B})') \arrow[r, "q"] & \Omega_{\alpha,\widehat{\beta}}(\widehat{A}_V), \arrow[u, "\pi"'] \end{tikzcd}$$

respectively, where q is the quotient map and $\widehat{\pi}$ or π an isometric isomorphism. $\square$

3.5 The legs of the unitary of a groupoid

The general preceding constructions are now applied to the C^*-pseudo-multiplicative unitary of a locally compact, Hausdorff, second countable groupoid G that was constructed in subsection 2.3. The algebras A_V and $\widehat{A}_V$ turn out to be the reduced groupoid C^*-algebra $C_r^*(G)$ and the function algebra $C_0(G)$, respectively, but unfortunately, we can not determine the Fourier algebras $\widehat{A}_V^0$ and A_V^0.

We use the same notation as in subsection 2.3 and let

$$\begin{gathered}\mathfrak{K} := L^2(G^0,\mu), \quad \mathfrak{B} = \mathfrak{B}^\dagger := C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K}), \quad \mathfrak{b} := (\mathfrak{K},\mathfrak{B},\mathfrak{B}^\dagger),\\ H := L^2(G,\nu), \quad \alpha = \beta := j(L^2(G,\lambda)), \quad \widehat{\beta} := \hat{j}(L^2(G,\lambda^{-1})),\\ V\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \cong L^2(G_s {\times}_r G, \nu^2_{s,r}) \to L^2(G_r {\times}_r G, \nu^2_{r,r}) \cong H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H,\\ (V\omega)(x,y) = \omega(x,x^{-1}y) \text{ for all } \omega \in C_c(G_s {\times}_r G),\, (x,y) \in G_r {\times}_r G.\end{gathered}$$

Denote by $m\colon C_0(G) \to \mathcal{L}(H)$ the representation given by multiplication operators, and by $L^1(G,\lambda)$ the completion of $C_c(G)$ with respect to the norm given by

$$\|f\| := \sup_{u\in G^0} \int_{G^u} |f(u)| \,\mathrm{d}\lambda^u(x) \quad \text{for all } f \in C_c(G).$$

Then $L^1(G,\lambda)$ is a Banach algebra with respect to the convolution product

$$(f*g)(y) = \int_{G^{r(y)}} g(x) f(x^{-1}y)\,\mathrm{d}\lambda^{r(y)}(x) \quad \text{for all } f,g \in L^1(G,\lambda), y \in G,$$

and there exists a norm-decreasing algebra homomorphism $L\colon L^1(G,\lambda) \to \mathcal{L}(H)$ such that

$$\big(L(f)\xi\big)(y) = \int_{G^{r(y)}} f(x) D^{-1/2}(x) \xi(x^{-1}y)\,\mathrm{d}\lambda^{r(y)}(x) \quad \text{for all } f,\xi \in C_c(G), y \in G.$$

For all $\xi,\xi' \in L^2(G,\lambda)$ and $\eta \in L^2(G,\lambda), \eta' \in L^2(G,\lambda^{-1})$, let

$$\widehat{a}_{\xi,\xi'} = \langle j(\xi)|_2 V | j(\xi')\rangle_2 \in \widehat{A}_V^0 \qquad \text{and} \qquad a_{\eta,\eta'} = \langle j(\eta)|_1 V | \hat{j}(\eta')\rangle_1 \in A_V^0.$$

Routine arguments show that there exists a unique continuous map

$$L^2(G,\lambda) \times L^2(G,\lambda) \to C_0(G),\ (\xi,\xi') \mapsto \overline{\xi} * \xi'^*,$$

such that

$$(\overline{\xi} * \xi'^*)(x) = \int_{G^{r(x)}} \overline{\xi(y)} \xi'(x^{-1}y)\,\mathrm{d}\lambda^{r(x)}(y) \quad \text{for all } \xi,\xi' \in C_c(G), x \in G.$$

Lemma 3.18. *Let $\xi,\xi' \in L^2(G,\lambda)$ and $\eta,\eta' \in C_c(G)$. Then $\widehat{a}_{\xi,\xi'} = m(\overline{\xi} * \xi'^*)$ and $a_{\eta,\eta'} = L(\overline{\eta}\eta')$.*

Proof. By continuity, we may assume $\xi, \xi' \in C_c(G)$. Then for all $\zeta, \zeta' \in C_c(G)$,

$$
\begin{aligned}
\langle \zeta | \widehat{a}_{\xi,\xi'} \zeta' \rangle &= \langle \zeta \oslash j(\xi) | V(\zeta' \oslash j(\xi')) \rangle \\
&= \int_G \int_{G^{r(x)}} \overline{\zeta(x)\xi(y)} \zeta'(x) \xi'(x^{-1}y) \, \mathrm{d}\lambda^{r(x)}(y) \, \mathrm{d}\nu(x) = \langle \zeta | m(\overline{\xi} * \xi'^*) \zeta' \rangle, \\
\langle \zeta | a_{\eta,\eta'} \zeta' \rangle &= \langle j(\eta) \oslash \zeta | V(\hat{j}(\eta') \oslash \zeta') \rangle \\
&= \int_G \int_{G^{r(y)}} \overline{\eta(x)\zeta(y)} \eta'(x) D^{-1/2}(x) \zeta'(x^{-1}y) \, \mathrm{d}\lambda^{r(y)}(x) \, \mathrm{d}\nu(y) \\
&= \langle \zeta | L(\overline{\eta}\eta') \zeta' \rangle. \qquad \square
\end{aligned}
$$

Remark 3.19. To extend the formula $a_{\eta,\eta'} = L(\overline{\eta}\eta')$ to all $\eta \in L^2(G,\lambda)$, $\eta' \in L^2(G,\lambda^{-1})$, we would have to extend the representation $L\colon C_c(G) \to \mathcal{L}(H)$ to some algebra X and the pointwise multiplication $(\eta, \eta') \mapsto \overline{\eta}\eta'$ to a map $L^2(G,\lambda) \times L^2(G,\lambda^{-1}) \to X$. Note that pointwise multiplication extends to a continuous map $L^2(G,\lambda) \times L^2(G,\lambda) \to L^1(G,\lambda)$, but in general this is not what we need. We expect that the map $L\colon C_c(G) \to A_V^0$ does not extend to an isometric isomorphism of Banach algebras $L^1(G,\lambda) \to A_V^0$.

The algebra $\widehat{A}_V^0$ can be considered as a continuous Fourier algebra of the locally compact groupoid G. Another Fourier algebra for locally compact groupoids was defined by Paterson in [25] as follows. He constructs a Fourier-Stieltjes algebra $B(G) \subseteq C(G)$ and defines the Fourier algebra $A(G)$ to be the norm-closed subalgebra of $B(G)$ generated by the set $A_{cf}(G) := \{\widehat{a}_{\xi,\xi'} \mid \xi \in L^2(G,\lambda)\}$. The definition of $B(G)$ in [25] immediately implies that $\|\widehat{\pi}_V(\omega_{\xi,\xi'})\|_{B(G)} \leq \|\xi\|\|\xi'\|$ for all $\xi \in \alpha^\infty, \xi' \in \beta^\infty$ with finitely many non-zero components, whence the following relation holds:

Proposition 3.20. *The identity on $A_{cf}(G)$ extends to a norm-decreasing homomorphism of Banach algebras $\widehat{A}_V^0 \to A(G)$.* $\square$

Another Fourier space $\tilde{\mathcal{A}}(G)$ considered in [25, Note after Proposition 13] is defined as follows. For each $\eta \in L^2(G,\lambda)$ and $u \in G^0$, write $\|\xi_n(u)\| := \langle \xi_n | \xi_n \rangle (u)^{1/2}$. Denote by M the set of all pairs (ξ, ξ') of sequences in $L^2(G,\lambda)$ such that the supremum $|(\xi,\xi')|_M := \sup_{u,v \in G^0} \sum_n \|\xi_n(u)\| \|\xi'_n(v)\|$ is finite, and denote by $\tilde{\mathcal{A}}(G)$ the completion of the linear span of $A_{cf}(G)$ with respect to the norm defined by

$$
\|\widehat{a}\|_{\tilde{\mathcal{A}}(G)} = \inf \left\{ |(\xi,\xi')|_M \,\middle|\, \widehat{a} = \sum_n \widehat{a}_{\xi_n,\xi'_n} \right\}.
$$

Proposition 3.21. *The identity on $A_{cf}(G)$ extends to a linear contraction $\widehat{A}_V^0 \to \tilde{\mathcal{A}}(G)$.*

Proof. For all $\xi, \xi' \in L^2(G,\lambda)^\infty$, we have

$$
\|\xi\|^2 = \sup_{u \in G^0} \sum_n \langle \xi_n | \xi_n \rangle (u) = \sup_{u \in G^0} \sum_n \|\xi_n(u)\|^2, \qquad \|\xi'\|^2 = \sup_{v \in G^0} \sum_n \|\xi_n(v)\|^2,
$$

and therefore $|(\xi,\xi')|_M = \sup_{u,v \in G^0} \sum_n \|\xi_n(u)\| \|\xi'_n(v)\| \leq \|\xi\| \|\xi'\|$. $\square$

Let us add that a Fourier algebra for measured groupoids was defined and studied by Renault [27], and for measured quantum groupoids by Vallin [37].

Finally, we consider the C^*-algebras associated to V. Recall that the reduced groupoid C^*-algebra $C_r^*(G)$ is the closed linear span of all operators of the $L(g)$, where $g \in L^1(G,\lambda)$ [26].

Theorem 3.22. *Let V be the C^*-pseudo-multiplicative unitary of a locally compact groupoid G. Then $((\widehat{A}_V)_H^{\alpha,\beta}, \widehat{\Delta}_V)$ and $((A_V)_H^{\widehat{\beta},\alpha}, \Delta_V)$ are Hopf C^*-bimodule and*

$$\begin{aligned} \widehat{A}_V &= m(C_0(G)), & \big(\widehat{\Delta}_V(m(f))\omega\big)(x,y) &= f(xy)\omega(x,y), \\ A_V &= C_r^*(G), & \big(\Delta_V(L(g))\omega'\big)(x',y') &= \int_{G^{u'}} g(z)D^{-1/2}(z)\omega'(z^{-1}x',z^{-1}y')\,\mathrm{d}\lambda^{u'}(z) \end{aligned}$$

for all $f \in C_0(G)$, $\omega \in H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H$, $(x,y) \in G_s \times_r G$ and $g \in C_c(G)$, $\omega' \in H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H$, $(x',y') \in G_r \times_r G$, where $u' = r(x') = r(y')$.

Proof. The first assertion will follow from Example 5.3 and Theorem 5.7 in subsection 5.1. The equations concerning $\widehat{A}_V$ and A_V follow directly from Lemma 3.18. Let us prove the formulas for $\widehat{\Delta}_V$ and Δ_V. For all $f, \omega, (x,y)$ as above,

$$\begin{aligned} \big(\widehat{\Delta}_V(m(f))\omega\big)(x,y) &= \big(V^*(\mathrm{id} \ogreaterthan m(f))V\omega\big)(x,y) \\ &= \big((\mathrm{id} \ogreaterthan m(f))V\omega\big)(x,xy) = f(xy)(V\omega)(x,xy) = f(xy)\omega(x,y), \end{aligned}$$

and for all $g, (x',y'), \omega', u'$ as above,

$$\begin{aligned} \big(\Delta_V(L(g))\omega'\big)(x',y') &= \big(V(L(g) \olessthan \mathrm{id})V^*\omega'\big)(x',y') \\ &= \big((L(g) \olessthan \mathrm{id})V^*\omega'\big)(x',x'^{-1}y') \\ &= \int_{G^{u'}} g(z)D^{-1/2}(z)(V^*\omega')(z^{-1}x',x'^{-1}y')\,\mathrm{d}\lambda^{u'}(z) \\ &= \int_{G^{u'}} g(z)D^{-1/2}(z)\omega'(z^{-1}x',z^{-1}y)\,\mathrm{d}\lambda^{u'}(z). \qquad \square \end{aligned}$$

4 Representations of a C^*-pseudo-multiplicative unitary

Let G be a locally compact group and let V be the multiplicative unitary on the Hilbert space $L^2(G,\lambda)$ given by formula (3). Then one can associate to every unitary representation π of G on a Hilbert space K a unitary operator X on $L^2(G,\lambda) \otimes K \cong L^2(G,\lambda;K)$ such that $(Xf)(x) = \pi(x)f(x)$ for all $x \in G$ and $f \in L^2(G,\lambda;K)$, and this operator satisfies the modified pentagon equation

$$V_{12}X_{13}X_{23} = V_{23}X_{12}. \tag{17}$$

For a general multiplicative unitary V on a Hilbert space H, Baaj and Skandalis defined a representation on a Hilbert space K to be a unitary X on $H \otimes K$ satisfying equation (17), equipped the class of all such representations with the structure of a C^*-tensor category and showed that under the assumption of regularity, this C^*-tensor category is the category of representations of a Hopf C^*-algebra $(A_{(u)}, \Delta_{(u)})$ (see [3]). In the case where V is the unitary associated to a group G as above, this category is isomorphic to the category of unitary representations of G, and $A_{(u)}$ is the full group C^*-algebra $C^*(G)$.

We carry over these definitions and constructions to C^*-pseudo-multiplicative unitaries and relate them to representations of groupoids. Throughout this section, let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, $(H, \widehat{\beta}, \alpha, \beta)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module and $V\colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \to H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H$ a C^*-pseudo-multiplicative unitary.

4.1 The C^*-tensor category of representations

Let ${}_\gamma K_{\widehat{\delta}}$ be a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-module and $X\colon K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \to K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H$ an operator satisfying

$$X(\gamma \triangleleft \alpha) = \gamma \triangleright \alpha, \quad X(\widehat{\delta} \triangleright \beta) = \widehat{\delta} \triangleleft \beta, \quad X(\widehat{\delta} \triangleright \widehat{\beta}) = \gamma \triangleright \widehat{\beta} \tag{18}$$

as subsets of $\mathcal{L}(\mathfrak{K}, K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H)$. Then all operators in the following diagram are well defined,

$$\begin{tikzcd}
K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[r, "{X \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id}}"] \arrow[d, "{\mathrm{id} \underset{\mathfrak{b}^\dagger}{\otimes} V}"] & K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[r, "{\mathrm{id} \underset{\mathfrak{b}}{\otimes} V}"] & K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H, \\
K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{(\alpha \triangleright \alpha)} (H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H) \arrow[d, "{\mathrm{id} \underset{\mathfrak{b}^\dagger}{\otimes} \Sigma}"] & & (K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H)_{(\gamma \triangleleft \alpha)} \underset{\mathfrak{b}}{\otimes} {}_\beta H \arrow[u, "{X \underset{\mathfrak{b}}{\otimes} \mathrm{id}}"'] \\
K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H_\beta \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[rr, "{X \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id}}"] & & \big(K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H\big)_{(\widehat{\delta} \triangleleft \beta)} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[u, "\Sigma_{23}"']
\end{tikzcd} \tag{19}$$

where the canonical isomorphism $\Sigma_{23}\colon \big(K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H\big)_{(\widehat{\delta} \triangleleft \beta)} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \cong (K_{\rho_\gamma} \otimes \beta)_{\rho_{(\widehat{\delta} \triangleleft \beta)}} \otimes \alpha \xrightarrow{\cong} (K_{\rho_{\widehat{\delta}}} \otimes \alpha)_{\rho_{(\gamma \triangleleft \alpha)}} \otimes \beta \cong (K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H)_{(\gamma \triangleleft \alpha)} \underset{\mathfrak{b}}{\otimes} {}_\beta H$ is given by $(\zeta \otimes \xi) \otimes \eta \mapsto (\zeta \otimes \eta) \otimes \xi$. We again adopt the leg notation [3] and write

$$X_{12} \text{ for } X \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id} \text{ and } X \underset{\mathfrak{b}}{\otimes} \mathrm{id}; \qquad X_{13} \text{ for } \Sigma_{23}(X \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})(\mathrm{id} \underset{\mathfrak{b}^\dagger}{\otimes} \Sigma).$$

Definition 4.1. *A* representation *of V consists of a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-module ${}_\gamma K_{\widehat{\delta}}$ and a unitary $X\colon K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \to K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H$ such that equation* (18) *holds and diagram* (19) *commutes. We also call X a representation of V (on ${}_\gamma K_{\widehat{\delta}}$). A* (semi-)morphism *of representations $({}_\gamma K_{\widehat{\delta}}, X)$ and $({}_\varepsilon L_{\widehat{\phi}}, Y)$ is an operator $T \in \mathcal{L}_{(s)}({}_\gamma K_{\widehat{\delta}}, {}_\varepsilon L_{\widehat{\phi}})$ satisfying $Y(T \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id}) = (T \underset{\mathfrak{b}}{\otimes} \mathrm{id})X$. Evidently, the class of all representations and (semi-)morphisms forms a category; we denote it by $\mathbf{C}^*\text{-}\mathbf{rep}_V^{(s)}$.*

Examples 4.2. i) Consider the canonical isomorphisms

$$\Phi\colon \mathfrak{K}_{\mathfrak{B}^\dagger} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \to H,\ b^\dagger \otimes \zeta \mapsto \rho_\alpha(b^\dagger)\zeta, \quad \Psi\colon \mathfrak{K}_{\mathfrak{B}} \underset{\mathfrak{b}}{\otimes} {}_\beta H \to H,\ b \otimes \zeta \mapsto \rho_\beta(b)\zeta. \tag{20}$$

The composition $\mathbf{1}_V := \Psi^*\Phi$ is a representation on ${}_{\mathfrak{B}}\mathfrak{K}_{\mathfrak{B}^\dagger}$ which we call the *trivial* representation of V.

ii) The pair $({}_\alpha H_{\widehat{\beta}}, V)$ is a representation which we call the *regular* representation.

iii) Let $(({}_{\gamma_i}K^i_{\widehat{\delta}_i}, X_i))_i$ be a family of representations. Then the operator

$$\boxplus_i X_i \colon \left(\boxplus_i K^i_{\widehat{\delta}_i}\right) \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \to \left(\boxplus_i K^i_{\gamma_i}\right) \underset{\mathfrak{b}}{\otimes} {}_\beta H$$

corresponding to $\bigoplus_i X_i$ with respect to the identifications $\left(\boxplus_i K^i_{\widehat{\delta}_i}\right) \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \cong \bigoplus_i \left({K^i}_{\widehat{\delta}^i} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H\right)$ and $\left(\boxplus_i K^i_{\gamma_i}\right) \underset{\mathfrak{b}}{\otimes} {}_\beta H \cong \bigoplus_i \left({K^i}_{\gamma^i} \underset{\mathfrak{b}}{\otimes} {}_\beta H\right)$ is a representation on $\boxplus_i {}_{\gamma_i}K^i_{\widehat{\delta}_i}$. We call it the *direct sum* of $(X_i)_i$.

iv) Let $\mathfrak{c}$ be a C^*-base, L_λ a C^*-$\mathfrak{c}$-module, $(K, \gamma, \widehat{\delta}, \kappa)$ a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger, \mathfrak{c}^\dagger)$-module, and X a representation on ${}_\gamma K_{\widehat{\delta}}$. If $X(\kappa \triangleleft \alpha) = \kappa \triangleleft \beta$, then the operator

$$\mathrm{id} \underset{\mathfrak{c}}{\otimes} X \colon L_\lambda \underset{\mathfrak{c}}{\otimes} {}_\kappa K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \to L_\lambda \underset{\mathfrak{c}}{\otimes} {}_\kappa K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H$$

is a representation on ${}_{\lambda \triangleright \gamma}(L_\lambda \underset{\mathfrak{c}}{\otimes} {}_\kappa K)_{\lambda \triangleright \widehat{\delta}}$, as one can easily check.

The category of representations admits a tensor product:

Lemma 4.3. *Let $({}_\gamma K_{\widehat{\delta}}, X)$ and $({}_\varepsilon L_{\widehat{\phi}}, Y)$ be representations of V. Then the operator*

$$X \boxtimes Y \colon K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\varepsilon L_{\widehat{\phi}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \xrightarrow{Y_{23}} K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{\varepsilon \triangleright \alpha}(L_\varepsilon \underset{\mathfrak{b}}{\otimes} {}_\beta H) \xrightarrow{X_{13}} (K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\varepsilon L)_{\gamma \triangleleft \varepsilon} \underset{\mathfrak{b}}{\otimes} {}_\beta H,$$

where $Y_{23} = \mathrm{id} \underset{\mathfrak{b}^\dagger}{\otimes} Y$ and where X_{13} acts like X on the first and last factor of the relative tensor product, is a representation of V on ${}_\gamma K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\varepsilon L_{\widehat{\phi}}$.

Proof. First, the relations (18) for X and Y imply

$$X_{13}Y_{23}(\gamma \triangleleft \varepsilon \triangleleft \alpha) = X_{13}(\gamma \triangleleft (\varepsilon \triangleright \alpha)) = (\gamma \triangleleft \varepsilon) \triangleright \alpha,$$
$$X_{13}Y_{23}(\widehat{\delta} \triangleright \widehat{\phi} \triangleright \beta) = X_{13}(\widehat{\delta} \triangleright (\widehat{\phi} \triangleleft \beta)) = (\widehat{\delta} \triangleright \widehat{\phi}) \triangleleft \beta,$$
$$X_{13}Y_{23}(\widehat{\delta} \triangleright \widehat{\phi} \triangleright \widehat{\beta}) = X_{13}(\widehat{\delta} \triangleright (\varepsilon \triangleright \widehat{\beta})) = (\gamma \triangleleft \varepsilon) \triangleleft \beta.$$

If V is an ordinary multiplicative unitary, then $Z := X \boxtimes Y$ satisfies $Z_{12}Z_{13}V_{23} = V_{23}Z_{12}$ because the equations $Y_{12}Y_{13}V_{23} = V_{23}Y_{12}$, $X_{12}X_{13}V_{23} = V_{23}X_{12}$ imply $X_{13}Y_{23}X_{14}Y_{24}V_{34} = X_{13}X_{14}Y_{23}Y_{24}V_{34} = X_{13}X_{14}V_{34}Y_{23} = V_{34}X_{13}Y_{23}$; here, we used the leg notation [3]. A similar calculation applies to the general case. $\square$

The tensor product turns $\mathbf{C}^*\text{-}\mathbf{rep}_V$ and $\mathbf{C}^*\text{-}\mathbf{rep}^s_V$ into (C^*-)tensor categories, which are frequently also called (C^*-)monoidal categories [8, 14, 20]:

Theorem 4.4. *The category $\mathbf{C}^*\text{-}\mathbf{rep}_V$ carries the structure of a C^*-tensor category and the category $\mathbf{C}^*\text{-}\mathbf{rep}^s_V$ carries the structure of a tensor category, where both times*

- *the tensor product is given by* $(X,Y) \mapsto X \boxtimes Y$ *for representations and* $(S,T) \mapsto S \underset{\mathfrak{b}^\dagger}{\otimes} T$ *for morphisms;*
- *the unit is the trivial representation* $\mathbf{1}_V$*;*
- *the associativity isomorphism* $a_{X,Y,Z} \colon (X \boxtimes Y) \boxtimes Z \to X \boxtimes (Y \boxtimes Z)$ *is the isomorphism* $a_{\mathcal{K},\mathcal{L},\mathcal{M}}$ *of equation* (6) *for all representations* $(\mathcal{K},X),(\mathcal{L},Y),(\mathcal{M},Z)$*;*
- *the unit isomorphisms* $l_X \colon \mathbf{1}_V \boxtimes X \to X$ *and* $r_X \colon X \boxtimes \mathbf{1}_V \to X$ *are the isomorphisms* $l_{\mathcal{K}}$ *and* $r_{\mathcal{K}}$*, respectively, of equation* (5) *for each representation* $(\mathcal{K},X)$.

Proof. Tedious but straightforward. □

The regular representation tensorially absorbs every other representation:

Proposition 4.5. *Let* $({}_\gamma K_{\widehat{\delta}}, X)$ *be a representation of* V*. Then* X *is an isomorphism between the representation* $X \boxtimes V$ *and the amplification* $\mathrm{id} \underset{\mathfrak{b}}{\otimes} V$ *on* ${}_{\gamma \triangleright \alpha}(K {}_\gamma\underset{\mathfrak{b}}{\otimes}{}_\beta H)_{\gamma \triangleright \widehat{\beta}}$.

Proof. This follows from commutativity of (19). □

We denote by $\mathrm{End}(\mathbf{1}_V)$ the algebra of endomorphisms of the trivial representation $\mathbf{1}_V$. This is a commutative C^*-algebra, and the category $\mathbf{C}^*\textbf{-rep}_V$ can be considered as a bundle of C^*-categories over the spectrum of $\mathrm{End_V}(\mathbf{1}_V)$ [39].

Proposition 4.6. $\mathrm{End}(\mathbf{1}_V) = \{b \in M(\mathfrak{B}) \cap M(\mathfrak{B}^\dagger) \subseteq \mathcal{L}(\mathfrak{K}) \mid \rho_\alpha(b) = \rho_\beta(b)\}$.

Proof. First, note that $\mathcal{L}({}_{\mathfrak{B}}\mathfrak{K}_{\mathfrak{B}^\dagger})$ is equal to $M(\mathfrak{B}) \cap M(\mathfrak{B}^\dagger) \subseteq \mathcal{L}(\mathfrak{K})$. Let Φ and Ψ be as in (20). Then for each $x \in \mathcal{L}({}_{\mathfrak{B}}\mathfrak{K}_{\mathfrak{B}^\dagger})$,

$$\begin{aligned} x \in \mathrm{End}(\mathbf{1}_V) &\Leftrightarrow \Psi^*\Phi(x \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id}) = (x \underset{\mathfrak{b}}{\otimes} \mathrm{id})\Psi^*\Phi \\ &\Leftrightarrow \Phi(x \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})\Phi^* = \Psi(x \underset{\mathfrak{b}}{\otimes} \mathrm{id})\Psi^* \Leftrightarrow \rho_\alpha(x) = \rho_\beta(x). \end{aligned}$$ □

4.2 The legs of representation operators

To every representation, we associate an algebra and a space of generalized matrix elements as follows. Given a representation X on a C^*-$(\mathfrak{b},\mathfrak{b}^\dagger)$-module ${}_\gamma K_{\widehat{\delta}}$, we put

$$\widehat{A}_X := [\langle\beta|_2 X|\alpha\rangle_2] \subseteq \mathcal{L}(K) \qquad \text{and} \qquad A_X := [\langle\gamma|_1 X|\widehat{\delta}\rangle_1] \subseteq \mathcal{L}(H), \tag{21}$$

where $|\alpha\rangle_2, |\widehat{\delta}\rangle_1, \langle\beta|_2, \langle\gamma|_1$ are defined as in subsection 2.1.

Examples 4.7. i) For the trivial representation $({}_{\mathfrak{B}}\mathfrak{K}_{\mathfrak{B}^\dagger}, \mathbf{1}_V)$, we have $\widehat{A}_{\mathbf{1}_V} = [\beta^*\alpha]$ and $A_{\mathbf{1}_V} = [\rho_\beta(\mathfrak{B})\rho_\alpha(\mathfrak{B}^\dagger)]$. The space $\widehat{A}_{\mathbf{1}_V}$ is related to the C^*-algebra $\mathrm{End}(\mathbf{1}_V)$ (see Proposition 4.6) as follows: $\mathrm{End}(\mathbf{1}_V) = \mathcal{L}({}_{\mathfrak{B}}\mathfrak{K}_{\mathfrak{B}^\dagger}) \cap (\widehat{A}_{\mathbf{1}_V})'$. This relation follows from Proposition 4.6 and the fact that an element $x \in \mathcal{L}({}_{\mathfrak{B}}\mathfrak{K}_{\mathfrak{B}^\dagger}) = M(\mathfrak{B}) \cap M(\mathfrak{B}^\dagger)$ satisfies $\rho_\alpha(x) = \rho_\beta(x)$ if and only if for all $\eta \in \beta$ and $\xi \in \alpha$, the elements $\eta^*\xi x = \eta^*\rho_\alpha(x)\xi$ and $x\eta^*\xi = \eta^*\rho_\beta(x)\xi$ coincide.

ii) For the regular representation $({}_{\alpha}H_{\widehat{\beta}}, V)$, the definition above is consistent with definition (16).

iii) If $(X_i)_i$ is a family of representations and $X = \boxplus_i X_i$, then $\widehat{A}_X \subseteq \prod_i \widehat{A}_{X_i}$ and $A_X = [\bigcup_i A_{X_i}]$.

iv) If $({}_{\lambda\triangleright\gamma}(L_{\lambda} \underset{\mathfrak{c}}{\otimes} {}_{\kappa}K)_{\lambda\triangleright\widehat{\delta}}, \mathrm{id} \underset{\mathfrak{c}}{\otimes} X)$ is the amplification of a representation $({}_{\gamma}K_{\widehat{\delta}}, X)$ as in Example 4.2 iv), then $\widehat{A}_{(\mathrm{id} \underset{\mathfrak{c}}{\otimes} X)} = \mathrm{id}_{\lambda} \underset{\mathfrak{c}}{\otimes} {}_{\kappa}\widehat{A}_X$ and $A_{(\mathrm{id} \underset{\mathfrak{c}}{\otimes} X)} = A_X$.

With respect to tensor products, the definition of $\widehat{A}_X$ and A_X behaves as follows:

Lemma 4.8. *Let $({}_{\gamma}K_{\widehat{\delta}}, X)$ and $({}_{\varepsilon}L_{\widehat{\phi}}, Y)$ be representations of V. Then*

$$A_{(X \boxtimes Y)} = [A_X A_Y], \qquad [\widehat{A}_{(X \boxtimes Y)} |\varepsilon\rangle_2] = [|\varepsilon\rangle_2 \widehat{A}_X], \qquad [\widehat{A}^*_{(X \boxtimes Y)} |\widehat{\delta}\rangle_1] = [|\widehat{\delta}\rangle_1 A^*_X].$$

The proof involves commutative diagrams of a special kind:

Notation 4.9. *We shall frequently prove equations for certain spaces of operators using commutative diagrams. In these diagrams, the vertexes are labelled by Hilbert spaces, the arrows are labelled by single operators or closed spaces of operators, and the composition is given by the closed linear span of all possible compositions of operators.*

Proof of Lemma 4.8. The following commutative diagrams shows that $A_{(X \boxtimes Y)} = [A_X A_Y]$:

$$\begin{tikzcd}
K_{\widehat{\delta}} \underset{\mathfrak{b}^{\dagger}}{\otimes} {}_{\varepsilon}L_{\widehat{\phi}} \underset{\mathfrak{b}^{\dagger}}{\otimes} {}_{\alpha}H \arrow[rrr, "Y_{23}"] & & & K_{\widehat{\delta}} \underset{\mathfrak{b}^{\dagger}}{\otimes} {}_{(\varepsilon\triangleright\alpha)} (L_{\varepsilon} \underset{\mathfrak{b}}{\otimes} {}_{\beta}H) \arrow[rrr, "X_{13}"] \arrow[drr, "\langle\varepsilon|_2"'] & & & (K_{\widehat{\delta}} \underset{\mathfrak{b}^{\dagger}}{\otimes} {}_{\varepsilon}L)_{(\gamma\triangleleft\varepsilon)} \underset{\mathfrak{b}}{\otimes} {}_{\beta}H \arrow[d, "\langle\varepsilon|_2"] \\
L_{\widehat{\phi}} \underset{\mathfrak{b}^{\dagger}}{\otimes} {}_{\alpha}H \arrow[u, "|\widehat{\delta}\rangle_1"] \arrow[r, "Y"] & L_{\varepsilon} \underset{\mathfrak{b}}{\otimes} {}_{\beta}H \arrow[urr, "|\widehat{\delta}\rangle_1"'] \arrow[drr, "\langle\varepsilon|_1"] & & & & K_{\widehat{\delta}} \underset{\mathfrak{b}^{\dagger}}{\otimes} {}_{\alpha}H \arrow[r, "X"] & K_{\gamma} \underset{\mathfrak{b}}{\otimes} {}_{\beta}H \arrow[d, "\langle\gamma|_1"] \\
H \arrow[u, "|\widehat{\phi}\rangle_1"] \arrow[rrr, "A_Y"] & & & H \arrow[rrr, "A_X"] \arrow[urr, "|\widehat{\delta}\rangle_1"] & & & H.
\end{tikzcd}$$

The relations concerning $\widehat{A}_{(X \boxtimes Y)}$ follow similarly. $\square$

We now collect some general properties of the spaces introduced above.

Proposition 4.10. *Let $({}_{\gamma}K_{\widehat{\delta}}, X)$ be a representation of V.*

i) The space $\widehat{A}_X \subseteq \mathcal{L}(K)$ satisfies

$$\begin{gathered}
[\widehat{A}_X \widehat{A}_X] = \widehat{A}_X, \quad [\widehat{A}_X K] = K, \quad [\widehat{A}_X \gamma] = [\gamma \widehat{A}_{\mathbf{1}_V}], \quad [\widehat{A}^*_X \widehat{\delta}] = [\widehat{\delta} \widehat{A}^*_{\mathbf{1}_V}], \\
[\widehat{A}_X \rho_{\widehat{\delta}}(\mathfrak{B})] = [\rho_{\widehat{\delta}}(\mathfrak{B}) \widehat{A}_X] = \widehat{A}_X = [\widehat{A}_X \rho_{\gamma}(\mathfrak{B}^{\dagger})] = [\rho_{\gamma}(\mathfrak{B}^{\dagger}) \widehat{A}_X],
\end{gathered} \tag{22}$$

and if $\widehat{A}_X = \widehat{A}^*_X$, then $(\widehat{A}_X)^{\gamma,\widehat{\delta}}_K$ is a C^*-$(\mathfrak{b}, \mathfrak{b}^{\dagger})$-algebra.

ii) The space $A_X \subseteq \mathcal{L}(H)$ satisfies

$$\begin{gathered}
[A_X \widehat{\beta}] = \widehat{\beta}, \quad [A_X \beta] = [\beta \gamma^* \widehat{\delta}], \quad [A^*_X \alpha] = [\alpha \widehat{\delta}^* \gamma], \quad [A_X A_V] = A_V, \\
[\Delta_V(A_X) |\beta\rangle_2] \subseteq [|\beta\rangle_2 A_X], \quad [\Delta_V(A^*_X) |\alpha\rangle_1] \subseteq [|\alpha\rangle_1 A^*_X], \\
[A_X \rho_{\beta}(\mathfrak{B})] = [\rho_{\beta}(\mathfrak{B}) A_X] = A_X = [A_X \rho_{\alpha}(\mathfrak{B}^{\dagger})] = [\rho_{\alpha}(\mathfrak{B}^{\dagger}) A_X].
\end{gathered} \tag{23}$$

Proof. i) First, $[\widehat{A}_X\widehat{A}_X] = [\langle\beta|_2\langle\alpha|_3 X_{12}|\alpha\rangle_3|\alpha\rangle_2]$ because the diagram below commutes:

$$
\begin{tikzcd}
& K \arrow[dl, "|\alpha\rangle_2"'] \arrow[d, "|\alpha\rangle_2"] \arrow[rr, "\widehat{A}_X"'] & & K \arrow[dr, "|\alpha\rangle_2"] \arrow[rr, "\widehat{A}_X"'] & & K & \\
K_{\widehat{\delta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[ddr, "|\alpha\rangle_3"'] & K_{\widehat{\delta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[r, "X"] \arrow[d, "|\alpha\rangle_2"] & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[ur, "\langle\beta|_2"] \arrow[dr, "|\alpha\rangle_2"'] & (\mathrm{C}) & K_{\widehat{\delta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[r, "X"] & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\beta|_2"] & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[ul, "\langle\beta|_2"'] \\
& K_{\widehat{\delta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{(\alpha\triangleright\alpha)}(H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H) \arrow[rr, "X_{13}"'] \arrow[d, "V_{23}^{*}"] & & (K_{\widehat{\delta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H)_{(\gamma\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[ur, "\langle\beta|_3"'] \arrow[rr, "X_{12}"'] \arrow[d, phantom, "(\mathrm{P})"] & & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\beta|_3"'] & \\
& K_{\widehat{\delta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[rrrr, "X_{12}"] & & {} & & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[u, "V_{23}"'] \arrow[uur, "\langle\alpha|_3"'] &
\end{tikzcd}
$$

Indeed, cell (C) commutes because for all $\xi \in \alpha$, $\eta, \eta' \in \beta$, $\zeta \in K$,

$$|\xi\rangle_2\langle\eta'|_2(\zeta \oslash \eta) = \rho_\gamma\big(\eta'^*\eta\big)\zeta \oslash \xi = \rho_{(\gamma\triangleleft\alpha)}\big(\eta'^*\eta\big)(\zeta \oslash \xi) = \langle\eta'|_3|\xi\rangle_2(\zeta \oslash \eta), \qquad (24)$$

cell (P) is diagram (8), and the other cells commute by definition of $\widehat{A}_X$ and because of (7). Next, $[\langle\beta|_2\langle\alpha|_3 X_{12}|\alpha\rangle_3|\alpha\rangle_2] = \widehat{A}_X$ because the following diagram commutes:

$$
\begin{tikzcd}
& K \arrow[dl, "|\alpha\rangle_2"'] \arrow[d, "|\alpha\rangle_2"] \arrow[r, "\widehat{A}_X"'] & K \\
K_{\widehat{\delta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[r, "\rho_{(\widehat{\delta}\triangleright\widehat{\beta})}(\mathfrak{B})"] \arrow[dr, "|\alpha\rangle_3"'] & K_{\widehat{\delta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[r, "V"] & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\beta|_2"] \\
& K_{\widehat{\delta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[u, "\langle\alpha|_3"] \arrow[r, "V_{12}"] & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[u, "\langle\alpha|_3"]
\end{tikzcd}
$$

We prove some of the other equations in (22); the remaining ones follow similarly.

$$
\begin{aligned}
[\widehat{A}_X K] &= [\langle\beta|_2 X|\alpha\rangle_2 K] = [\langle\beta|_2 X(K_{\widehat{\delta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H)] = [\langle\beta|_2(K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H)] = K, \\
[\widehat{A}_X\gamma] &= [\langle\beta|_2 X|\alpha\rangle_2\gamma] = [\langle\beta|_2|\gamma\rangle_1\alpha] = [\gamma\beta^*\alpha] = [\gamma\widehat{A}_{\mathbf{1}_V}], \\
[\widehat{A}_X\rho_{\widehat{\delta}}(\mathfrak{B})] &= [\langle\beta|_2 X|\alpha\rangle_2\rho_{\widehat{\delta}}(\mathfrak{B})] = [\langle\beta|_2 X|\alpha\mathfrak{B}\rangle_2] = \widehat{A}_X, \\
[\rho_{\widehat{\delta}}(\mathfrak{B})\widehat{A}_X] &= [\rho_{\widehat{\delta}}(\mathfrak{B})\langle\beta|_2 X|\alpha\rangle_2] = [\langle\beta|_2(\rho_{\widehat{\delta}}(\mathfrak{B})\underset{\mathfrak{b}^{\dagger}}{\otimes}\mathrm{id})X|\alpha\rangle_2] \\
&= [\langle\beta|_2 X(\mathrm{id}\underset{\mathfrak{b}^{\dagger}}{\otimes}\rho_\beta(\mathfrak{B}))|\alpha\rangle_2] = [\langle\beta|_2 X|\rho_\beta(\mathfrak{B})\alpha\rangle_2] = \widehat{A}_X.
\end{aligned}
$$

ii) First, $[A_X\widehat{\beta}] = [\langle\gamma|_1 X|\widehat{\delta}\rangle_1\widehat{\beta}] = [\langle\gamma|_1|\gamma\rangle_1\widehat{\beta}] = [\rho_\beta(\mathfrak{B})\widehat{\beta}] = \widehat{\beta}$ by (18), and similar calculations show that $[A_X\beta] = [\beta\gamma^*\widehat{\delta}]$ and $[A_X^*\alpha] = [\alpha\widehat{\delta}^*\gamma]$. Next, $[A_X A_V] = A_{X\boxtimes V} = A_V$ by Lemma 4.8, Lemma 4.5, and Example 4.7 iv). The equations in the last line of (23) follow from similar calculations as for $\widehat{A}_X$.

Finally, let us prove the equations in the middle line of (23). Since $A_X \subseteq \mathcal{L}(H_{\widehat{\beta}}) \subseteq$

$\rho_{\hat\beta}(\mathfrak{B})'$, $\Delta_V(A_X)$ is well defined. Consider the commutative diagram

$$\begin{tikzcd}
H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[r, "V^*"'] \arrow[d, "|\hat\delta\rangle_1"] & H_{\hat\beta} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[r, "A_X \underset{\mathfrak{b}^\dagger}{\otimes} 1"'] \arrow[d, "|\hat\delta\rangle_1"] & H_{\hat\beta} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[r, "V"'] & H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H. \\
K_{\hat\delta} \underset{\mathfrak{b}^\dagger}{\otimes}{}_{(\alpha\rhd\alpha)} (H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H) \arrow[r, "V_{23}^*"] & K_{\hat\delta} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H_{\hat\beta} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[r, "X_{12}"] & K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_{\hat\beta} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[r, "V_{23}"] \arrow[u, "\langle\gamma|_1"] & K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[u, "\langle\gamma|_1"]
\end{tikzcd}$$

Since the composition on top is $\Delta_V(A_X)$ and the composition on the bottom is $X_{12}X_{13}$, the following diagram commutes and shows that $[\Delta_V(A_X)|\beta\rangle_2] = [A_X|\beta\rangle_2]$:

$$\begin{tikzcd}
H \arrow[r, "|\hat\delta\rangle_1"] \arrow[rrr, bend left, "A_X"] \arrow[d, "|\beta\rangle_2"] & K_{\hat\delta} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[r, "X"] \arrow[d, "|\beta\rangle_3"] & K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[r, "\langle\gamma|_1"] \arrow[d, "|\beta\rangle_3"] & H \arrow[d, "|\beta\rangle_2"] \\
H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[r, "X_{13}|\hat\delta\rangle_1"] \arrow[rrr, bend right, "\Delta_V(A_X)"'] & (K_{\hat\delta} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H)_{(\gamma\lhd\alpha)} \underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[r, "X_{12}"] & K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[r, "\langle\gamma|_1"] & H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H
\end{tikzcd}$$

A similar argument shows that $[\Delta_V(A_X^*)|\alpha\rangle_1] = [|\alpha\rangle_1 A_X^*]$. □

Definition 4.11. *We call the C^*-pseudo-multiplicative unitary V* well-behaved *if $\widehat{A}_X = \widehat{A}_X^*$ for every representation X of V and $\widehat{A}_Y = \widehat{A}_Y^*$ for every representation Y of V^{op}.*

Proposition 4.12. *If V is well-behaved, then $((\widehat{A}_V)_H^{\alpha,\hat\beta}, \widehat{\Delta}_V)$ and $((A_V)_H^{\beta,\alpha}, \Delta_V)$ are concrete Hopf C^*-bimodules.*

Proof. By Proposition 3.11, the assumption implies that $(\widehat{A}_V)_H^{\alpha,\hat\beta}$ and $(A_V)_H^{\beta,\alpha} = (\widehat{A}_{V^{op}})_H^{\beta,\alpha}$ are C^*-algebras, that $\Delta_V(A_V) \subseteq A_V {}_\alpha\underset{\mathfrak{b}}{*}{}_\beta A_V$ and similarly that $\widehat{\Delta}_V(\widehat{A}_V) \subseteq \widehat{A}_V {}_{\hat\beta}\underset{\mathfrak{b}^\dagger}{*}{}_\alpha \widehat{A}_V$. Now, the assertion follows from Lemma 3.13. □

4.3 The universal Banach algebra of representations

Every representation of V induces a representation of the convolution algebra $\tilde{\Omega}_{\beta,\alpha}$ introduced in subsection 3.4 as follows.

Proposition 4.13. *i) Let $({}_\gamma K_{\hat\delta}, X)$ be a representation of V. Then there exists a contractive algebra homomorphism $\hat\pi_X : \tilde{\Omega}_{\beta,\alpha} \to \widehat{A}_X$ such that*

$$\hat\pi_X(\omega_{\xi,\xi'}) = \sum_n \langle \xi_n|_2 X|\xi'_n\rangle_2 \quad \textit{for all } \xi \in \beta^\infty, \xi' \in \alpha^\infty. \tag{25}$$

ii) Let $({}_\gamma K_{\hat\delta}, X)$ and $({}_\varepsilon L_{\hat\phi}, Y)$ be representations of V and let $T \in \mathcal{L}_{(s)}({}_\gamma K_{\hat\delta}, {}_\varepsilon L_{\hat\phi})$. Then T is a (semi-)morphism from $({}_\gamma K_{\hat\delta}, X)$ to $({}_\varepsilon L_{\hat\phi}, Y)$ if and only if T intertwines $\hat\pi_X$ and $\hat\pi_Y$ in the sense that $T\hat\pi_X(\omega) = \hat\pi_Y(\omega)T$ for all $\omega \in \tilde{\Omega}_{\beta,\alpha}$.

Proof. i) The sum on the right hand side of (25) depends on $\omega_{\xi,\xi'}$ but not on ξ,ξ' because $\eta^*\left(\sum_n \langle\xi_n|_2 X|\xi'_n\rangle_2\right)\eta' = \omega_{\xi,\xi'}(\langle\eta|_1 X|\eta'\rangle_1)$ for all $\eta\in\gamma,\eta'\in\widehat{\delta}$. Thus, $\widehat{\pi}_X$ is well defined by (25). It is a homomorphism because for all $\omega,\omega'\in\tilde{\Omega}_{\beta,\alpha}$, $\eta\in\gamma,\eta'\in\widehat{\delta}$,

$$\begin{aligned}
\eta^*\widehat{\pi}_X(\omega)\widehat{\pi}_X(\omega')\eta' &= (\omega\boxtimes\omega')(\langle\eta|_1 X_{12}X_{13}|\eta'\rangle_1) \\
&= (\omega\boxtimes\omega')(\langle\eta|_1 V_{23}X_{12}V_{23}^*|\eta'\rangle_1) \\
&= (\omega\boxtimes\omega')\big(V(\langle\eta|_1 X|\eta'\rangle_1 {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}\,\mathrm{id})V^*\big) \\
&= (\omega*\omega')(\langle\eta|_1 X|\eta'\rangle_1) = \eta^*\widehat{\pi}_X(\omega*\omega')\eta'.
\end{aligned}$$

ii) Straightforward. □

For later use, we note the following formula:

Lemma 4.14. $\widehat{\Delta}_V(\widehat{\pi}_V(\omega)) = \widehat{\pi}_{V\boxtimes V}(\omega)$ *for each* $\omega\in\tilde{\Omega}_{\beta,\alpha}$.

Proof. For all $\xi\in\alpha^\infty$ and $\xi'\in\beta^\infty$, we have $\widehat{\Delta}_V(\widehat{\pi}_V(\omega_{\xi',\xi})) = \sum_n\langle\xi'|_3 V_{12}V_{23}V_{12}^*|\xi\rangle_3 = \sum_n\langle\xi'|_3 V_{13}V_{23}|\xi\rangle_3 = \widehat{\pi}_{V\boxtimes V}(\omega_{\xi',\xi})$. □

Denote by $\widehat{A}_{(u)}$ the separated completion of $\tilde{\Omega}_{\beta,\alpha}$ with respect to the seminorm

$$|\omega| := \sup\{\|\widehat{\pi}_X(\omega)\| \mid (X \text{ is a representation of } V\} \text{ for each } \omega\in\tilde{\Omega}_{\beta,\alpha}$$

and by $\widehat{\pi}_{(u)}\colon\tilde{\Omega}_{\beta,\alpha}\to\widehat{A}_{(u)}$ the natural map.

Proposition 4.15. *i) There exists a unique algebra structure on $\widehat{A}_{(u)}$ such that $\widehat{A}_{(u)}$ is a Banach algebra and $\widehat{\pi}_{(u)}$ an algebra homomorphism.*

ii) For every representation X of V, there exists a unique algebra homomorphism $\widehat{\pi}^{(u)}_X\colon\widehat{A}_{(u)}\to\widehat{A}_X$ such that $\widehat{\pi}^{(u)}_X\circ\widehat{\pi}_{(u)} = \widehat{\pi}_X$.

iii) If V is well-behaved, then the Banach algebra $\widehat{A}_{(u)}$ carries a unique involution turning it into a C^*-algebra such that $\pi^{(u)}_X$ is a $*$-homomorphism for every representation X of V.

Proof. Assertions i) and ii) follow from routine arguments. Let us prove iii). For each $\omega\in\tilde{\Omega}_{\beta,\alpha}$ and $\varepsilon>0$, choose a representation $X_{(\omega,\varepsilon)}$ such that $\|\widehat{\pi}_{X_{(\omega,\varepsilon)}}(\omega)\| > |\omega|-\varepsilon$. Let $X := \boxplus_{\omega,\varepsilon}X_{(\omega,\varepsilon)}$, where the sum is taken over all $\omega\in\tilde{\Omega}_{\beta,\alpha}$ and $\varepsilon>0$. Then evidently $\widehat{\pi}^{(u)}_X\colon\widehat{A}_{(u)}\to\widehat{A}_X$ is an isometric isomorphism of Banach algebras. We can therefore define an involution on $\widehat{A}_{(u)}$ such that $\widehat{\pi}^{(u)}_X$ becomes a $*$-isomorphism. Now, let Y be a representation V. Then $X\boxplus Y$ is a representation again, and we have a commutative diagram

$$\begin{tikzcd}
 & \widehat{A}_{(u)} \arrow[dl, "\widehat{\pi}^{(u)}_X"'] \arrow[d, "\widehat{\pi}^{(u)}_{X\boxplus Y}"] \arrow[dr, "\widehat{\pi}^{(u)}_Y"] & \\
\widehat{A}_X & \widehat{A}_{X\boxplus Y} \arrow[l, "p_X"] \arrow[r, "p_Y"'] & \widehat{A}_Y,
\end{tikzcd}$$

where p_X and p_Y are the natural maps. Since $\widehat{\pi}_X^{(u)}$ is isometric, so is $\widehat{\pi}_{X\boxplus Y}^{(u)}$. But $\widehat{\pi}_{X\boxplus Y}^{(u)}$ also has dense image and therefore is surjective, whence p_X is injective. Since $\widehat{\pi}_X^{(u)}$, p_X, p_Y are $*$-homomorphisms, so is $\widehat{\pi}_{X\boxplus Y}^{(u)}$ and hence also $\widehat{\pi}_Y^{(u)}$. □

4.4 Universal representations and the universal Hopf C^*-bimodule

If the unitary V is well-behaved, then the universal Banach algebra $\widehat{A}_{(u)}$ constructed above can be equipped with the structure of a semi-normal Hopf C^*-bimodule, where the comultiplication corresponds to the tensor product of representations of V. The key idea is to identify $\widehat{A}_{(u)}$ with the C^*-algebra associated to a representation that is universal in the following sense.

Definition 4.16. *A representation $({}_\gamma K_{\widehat{\delta}}, X)$ of V is* universal *if for every representation $({}_\varepsilon L_{\widehat{\phi}}, Y)$ and every $\xi \in \varepsilon$, $\zeta \in L$, $\eta \in \widehat{\phi}$, there exists a semi-morphism T from $({}_\gamma K_{\widehat{\delta}}, X)$ to $({}_\varepsilon L_{\widehat{\phi}}, Y)$ that is a partial isometry and satisfies $\xi \in T\gamma$, $\zeta \in TK$, $\eta \in T\hat{\delta}$.*

Remark 4.17. Evidently, every universal representation is a generator [20] of $\mathbf{C}^*\text{-}\mathbf{rep}_V^s$ in the categorical sense.

We shall use a cardinality argument to show that $\mathbf{C}^*\text{-}\mathbf{rep}_V^s$ has a universal representation. Given a topological space X and a cardinal number c, let us say that X is *c-separable* if X has a dense subset of cardinality less than or equal to c. Let $\omega := |\mathbb{N}|$. Let us also say that a *subrepresentation* of a representation $({}_\gamma K_{\widehat{\delta}}, X)$ of V is a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-module ${}_\varepsilon L_{\widehat{\phi}}$ such that $L \subseteq K$, $\varepsilon \subseteq \gamma$, $\widehat{\phi} \subseteq \widehat{\delta}$, and $X(L_{\widehat{\phi}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H) = L_\varepsilon \underset{\mathfrak{b}^\dagger}{\otimes}{}_\beta H$.

Lemma 4.18. *Let $({}_\gamma K_{\widehat{\delta}}, X)$ be a representation, c, d cardinal numbers, $K_0 \subseteq K$, $\gamma_0 \subseteq \gamma$, $\widehat{\delta}_0 \subseteq \widehat{\delta}$ c-separable subsets, and assume that the spaces $\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger, \alpha, \beta, \widehat{\beta}$ are d-separable. Put $e := \omega \sum_{n=0}^{\infty} d^n$. Then there exists a subrepresentation ${}_\varepsilon L_{\widehat{\phi}}$ of $({}_\gamma K_{\widehat{\delta}}, X)$ such that $\gamma_0 \subseteq \varepsilon$, $\widehat{\delta}_0 \subseteq \widehat{\phi}$, $K_0 \subseteq L$ and such that $L, \gamma, \widehat{\phi}$ are $e(c+1)$-separable.*

Proof. Replacing K_0, γ_0, $\widehat{\delta}_0$ by dense subsets, we may assume that each of these sets has cardinality less than or equal to c. Moreover, replacing γ_0 and $\widehat{\delta}_0$ by larger sets, we may assume that $\mathfrak{B} = [\gamma_0^* \gamma_0]$, $\mathfrak{B}^\dagger = [\widehat{\delta}_0^* \widehat{\delta}_0]$, and $|\gamma_0| \leq c + \omega d$, $|\widehat{\delta}_0| \leq c + \omega d$.

Now, we can choose inductively $K_n \subseteq K$, $\widehat{\delta}_n \subseteq \widehat{\delta}$, $\gamma_n \subseteq \gamma$ for $n = 1, 2, \ldots$ such that for $n = 0, 1, 2, \ldots$, the following conditions hold:

i) K_{n+1} is large enough so that $K_n + \widehat{\delta}_n \mathfrak{K} + \gamma_n \mathfrak{K} \subseteq [K_{n+1}]$, but small enough so that

$$|K_{n+1}| \leq |K_n| + \omega d(|\gamma_n| + |\widehat{\delta}_n|) \leq \omega d(|K_n| + |\gamma_n| + |\widehat{\delta}_n|);$$

ii) γ_{n+1} is large enough so that

$$\gamma_n + \gamma_n \mathfrak{B} + \rho_{\widehat{\delta}}(\mathfrak{B}^\dagger)\gamma_n \subseteq [\gamma_{n+1}], \quad K_n \subseteq [\gamma_{n+1} \mathfrak{K}],$$
$$X|\alpha\rangle_2 \gamma_n \subseteq [|\gamma_{n+1}\rangle_1 \alpha], \quad X|\widehat{\delta}_n\rangle_1 \widehat{\beta} \subseteq [|\gamma_{n+1}\rangle_1 \widehat{\beta}],$$

but small enough so that

$$|\gamma_{n+1}| \leq |\gamma_n|(1+\omega d) + \omega|K_n| + \omega d|\gamma_n| + \omega d|\widehat{\delta}_n| \leq \omega d(|K_n| + |\gamma_n| + |\widehat{\delta}_n|);$$

iii) $\widehat{\delta}_{n+1}$ is large enough so that

$$\widehat{\delta}_n + \widehat{\delta}_n \mathfrak{B}^\dagger + \rho_\gamma(\mathfrak{B}^\dagger)\widehat{\delta}_n \subseteq [\widehat{\delta}_{n+1}], \quad K_n \subseteq [\widehat{\delta}_{n+1}\mathfrak{K}], \quad X|\widehat{\delta}_n\rangle_1\beta \subseteq [|\beta\rangle_2\widehat{\delta}_{n+1}],$$

but small enough so that

$$|\widehat{\delta}_{n+1}| \leq |\widehat{\delta}_n|(1+\omega d) + \omega|K_n| + \omega d|\widehat{\delta}_n| \leq \omega d(|K_n| + |\widehat{\delta}_n|).$$

Since $|K_0| + |\gamma_0| + |\widehat{\delta}_0| = 3c + 2\omega d$, we can conclude inductively that for all $n = 0, 1, 2, \ldots$,

$$|K_{n+1}| + |\gamma_{n+1}| + |\widehat{\delta}_{n+1}| \leq \omega d(|K_n| + |\gamma_n| + |\widehat{\delta}_n|) \leq (\omega d)^{n+1}(c + \omega d).$$

Therefore, the spaces $L := [\bigcup_n K_n]$, $\varepsilon := [\bigcup_n \gamma_n]$, $\widehat{\phi} := \left[\bigcup_n \widehat{\delta}_n\right]$ are $e(c+1)$-separable. By construction, ${}_\varepsilon L_{\widehat{\phi}}$ is a subrepresentation of $({}_\gamma K_{\widehat{\delta}}, X)$. $\square$

Proposition 4.19. *There exists a universal representation of V.*

Proof. Let d and e be as in Lemma 4.18. Then there exists a set $\mathcal{X}$ of representations of V such that every representation $({}_\varepsilon L_{\widehat{\phi}}, Y)$ of V, where the underlying Hilbert space L is e-separable, is isomorphic to some representation in $\mathcal{X}$. Using Lemma 4.18, one easily verifies that the direct sum $\boxplus_{X \in \mathcal{X}} X$ is a universal representation. $\square$

Theorem 4.20. *Let V be a well-behaved C^*-pseudo-multiplicative unitary and let $({}_\gamma K_{\widehat{\delta}}, X)$ be a universal representation of V.*

i) The $*$-homomorphism $\widehat{\pi}_X^{(u)} : \widehat{A}_{(u)} \to \widehat{A}_X$ is an isometric isomorphism.

ii) If $({}_\varepsilon L_{\widehat{\phi}}, Y)$ is a representation of V, then there exists a jointly semi-normal morphism $\widehat{\pi}_{X,Y}$ of C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-algebras $(\widehat{A}_X)_K^{\gamma,\widehat{\delta}}$, $(\widehat{A}_Y)_L^{\varepsilon,\widehat{\phi}}$ such that $\widehat{\pi}_Y^{(u)} = \widehat{\pi}_{X,Y} \circ \widehat{\pi}_X^{(u)}$.

iii) Let $\widehat{\Delta}_X := \widehat{\pi}_{X, X\boxtimes X}$. Then $((\widehat{A}_X)_K^{\gamma,\widehat{\delta}}, \widehat{\Delta}_X)$ is a semi-normal Hopf C^*-bimodule.

iv) $\widehat{\pi}_{X,V}$ is a morphism of the semi-normal Hopf C^*-bimodules $((\widehat{A}_X)_K^{\gamma,\widehat{\delta}}, \widehat{\Delta}_X)$ and $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$.

v) Let $({}_\varepsilon L_{\widehat{\phi}}, Y)$ be a universal representation of V and define $\widehat{\Delta}_Y$ similarly as $\widehat{\Delta}_X$. Then $\widehat{\pi}_{X,Y}$ is an isomorphism of the semi-normal Hopf C^*-bimodules $((\widehat{A}_X)_K^{\gamma,\widehat{\delta}}, \widehat{\Delta}_X)$ and $((\widehat{A}_Y)_L^{\varepsilon,\widehat{\phi}}, \widehat{\Delta}_Y)$.

Proof. i) Let $\omega \in \tilde{\Omega}_{\beta,\alpha}$, let $({}_{\varepsilon}L_{\widehat{\phi}}, Y)$ be a representation of V, and let $\zeta \in L$. Since X is universal, there exists a semi-morphism T from X to Y that is a partial isometry and satisfies $\zeta \in TL$. Then by Proposition 4.13, $\|\widehat{\pi}_Y(\omega)\zeta\| = \|\widehat{\pi}_Y(\omega)TT^*\zeta\| = \|T\widehat{\pi}_X(\omega)T^*\zeta\| \leq \|\widehat{\pi}_X(\omega)\|\|\zeta\|$. Since Y and ζ were arbitrary, we can conclude that $\|\widehat{\pi}_{(u)}(\omega)\| \leq \|\widehat{\pi}_X(\omega)\|$ and hence that $\widehat{\pi}_X^{(u)}$ is isometric.

ii) We have to show that $\widehat{\pi}_{X,Y} := \widehat{\pi}_Y^{(u)} \circ \widehat{\pi}_X^{(u)-1}$ is a jointly semi-normal morphism of C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-algebras. Let $\xi \in \varepsilon$ and $\eta \in \widehat{\phi}$. Since X is universal, there exists a semi-morphism T from X to Y such that $\xi \in T\gamma$ and $\eta \in T\widehat{\delta}$. By Proposition 4.13, $\widehat{\pi}_Y(\omega)T = \widehat{\pi}_X(\omega)T$ for all $\omega \in \tilde{\Omega}_{\beta,\alpha}$, and hence $T \in \mathcal{L}_s^{\widehat{\pi}_{X,Y}}({}_{\gamma}K_{\widehat{\delta}}, {}_{\varepsilon}L_{\widehat{\phi}})$. The claim follows.

iii) We need to show that $\widehat{\Delta}_X$ is coassociative. We shall prove that $(\widehat{\Delta}_X \underset{\mathfrak{b}^\dagger}{*} \mathrm{id}) \circ \widehat{\Delta}_X = \widehat{\pi}_{X,X \boxtimes X \boxtimes X}$, and a similar argument shows that $(\mathrm{id} \underset{\mathfrak{b}^\dagger}{*} \widehat{\Delta}_X) \circ \widehat{\Delta}_X = \widehat{\pi}_{X,X \boxtimes X \boxtimes X}$. Let S, T be semi-morphisms from X to $X \boxtimes X$. Then $R := (S \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id}) \circ T$ is a semi-morphism from X to $X \boxtimes X \boxtimes X$, and a generalization of Proposition 4.13 ii) shows that for each $\omega \in \tilde{\Omega}_{\beta,\alpha}$,

$$\begin{aligned}(\widehat{\Delta}_X \underset{\mathfrak{b}^\dagger}{*} \mathrm{id})(\widehat{\Delta}_X(\widehat{\pi}_X(\omega))) \cdot R &= (S \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id}) \cdot \widehat{\Delta}_X(\widehat{\pi}_X(\omega)) \cdot T \\ &= R \cdot \widehat{\pi}_X(\omega) = \widehat{\pi}_{X,X \boxtimes X \boxtimes X}(\omega) \cdot R.\end{aligned}$$

Since S and T were arbitrary and X is universal, we can conclude that $(\widehat{\Delta}_X \underset{\mathfrak{b}^\dagger}{*} \mathrm{id}) \circ \widehat{\Delta}_X(\widehat{\pi}_X(\omega)) = \widehat{\pi}_{X,X \boxtimes X \boxtimes X}(\omega)$.

iv) We have to show that $(\widehat{\pi}_{X,V} \underset{\mathfrak{b}^\dagger}{*} \widehat{\pi}_{X,V}) \circ \widehat{\Delta}_X = \widehat{\Delta}_V \circ \widehat{\pi}_{X,V}$. Let $\omega \in \tilde{\Omega}_{\beta,\alpha}$ and $S \in \mathcal{L}^{\widehat{\pi}_{X,V}}(K_{\widehat{\delta}}, H_{\widehat{\beta}})$, $T \in \mathcal{L}^{\widehat{\pi}_{X,V}}(K_\gamma, H_\alpha)$. Then $R := (S \underset{\mathfrak{b}^\dagger}{\otimes} T)$ satisfies $R(X \boxtimes X) = (V \boxtimes V)R$, and using Lemma 4.14, we find

$$\begin{aligned}(\widehat{\pi}_{X,V} \underset{\mathfrak{b}^\dagger}{*} \widehat{\pi}_{X,V})(\widehat{\Delta}_X(\widehat{\pi}_X(\omega)) \cdot R &= R \cdot \widehat{\Delta}_X(\widehat{\pi}_X(\omega)) \\ &= R \cdot \widehat{\pi}_{X \boxtimes X}(\omega) = \widehat{\pi}_{V \boxtimes V}(\omega) \cdot R = \widehat{\Delta}_V(\widehat{\pi}_V(\omega)) \cdot R.\end{aligned}$$

Since S and T were arbitrary, we can conclude $(\widehat{\pi}_{X,V} \underset{\mathfrak{b}^\dagger}{*} \widehat{\pi}_{X,V})(\widehat{\Delta}_X(\widehat{\pi}_X(\omega)) = \widehat{\Delta}_V(\widehat{\pi}_V(\omega))$.

v) We have to show that $(\widehat{\pi}_{X,Y} \underset{\mathfrak{b}^\dagger}{*} \widehat{\pi}_{X,Y}) \circ \widehat{\Delta}_X = \widehat{\Delta}_Y \circ \widehat{\pi}_{X,Y}$. Let $\omega \in \tilde{\Omega}_{\beta,\alpha}$ and $S \in \mathcal{L}^{\widehat{\pi}_{X,Y}}(K_{\widehat{\delta}}, H_{\widehat{\phi}})$, $T \in \mathcal{L}^{\widehat{\pi}_{X,Y}}(K_\gamma, H_\varepsilon)$. Then $R := (S \underset{\mathfrak{b}^\dagger}{\otimes} T)$ satisfies $R(X \boxtimes X) = (Y \boxtimes Y)R$, and using Proposition 4.13, we find

$$\begin{aligned}(\widehat{\pi}_{X,Y} * \widehat{\pi}_{X,Y})(\widehat{\Delta}_X(\widehat{\pi}_X(\omega)) \cdot R &= R \cdot \widehat{\Delta}_X(\widehat{\pi}_X(\omega)) \\ &= R \cdot \widehat{\pi}_{X \boxtimes X}(\omega) = \widehat{\pi}_{Y \boxtimes Y}(\omega) \cdot R = \widehat{\Delta}_Y(\widehat{\pi}_Y(\omega)) \cdot R.\end{aligned}$$

Since S and T were arbitrary, we can conclude $(\widehat{\pi}_{X,Y} * \widehat{\pi}_{X,Y})(\widehat{\Delta}_X(\widehat{\pi}_X(\omega)) = (\widehat{\Delta}_Y(\widehat{\pi}_Y(\omega)))$. $\square$

4.5 Corepresentations and W^*-representations

The notion of a representation of a C^*-pseudo-multiplicative unitary can be dualized so that one obtains the notion of a corepresentation, and adapted to W^*-modules instead of C^*-modules so that one obtains the notion of a W^*-representation. We briefly summarize the main definitions and properties of these concepts.

A *corepresentation* of V consists of a C^*-$(\mathfrak{b},\mathfrak{b}^\dagger)$-module ${}_\gamma K_\delta$ and of a unitary $X\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\gamma K \to H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\delta K$ that satisfies $X(\alpha\triangleleft\gamma)=\alpha\triangleright\gamma$, $X(\beta\triangleleft\gamma)=\beta\triangleleft\delta$, $X(\widehat{\beta}\triangleright\delta)=\widehat{\beta}\triangleleft\delta$ and makes the following diagram commute:

$$\begin{tikzcd}
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\gamma K \arrow[r, "V_{12}"] \arrow[d, "X_{23}"] & H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\gamma K \arrow[r, "X_{23}"] & H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\delta K, \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{(\alpha\triangleright\gamma)}(H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\delta K) \arrow[rr, "X_{13}"] & & (H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H)_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_\delta K \arrow[u, "V_{12}"']
\end{tikzcd} \tag{26}$$

where V_{12}, X_{13}, X_{23} are defined similarly as in subsection 4.1. A *(semi-)morphism* of corepresentations $({}_\gamma K_\delta, X)$ and $({}_\varepsilon L_\phi, Y)$ is an operator $T\in\mathcal{L}_{(s)}({}_\gamma K_\delta, {}_\varepsilon L_\phi)$ satisfying $Y(\mathrm{id}\underset{\mathfrak{b}^\dagger}{\otimes}T)=(\mathrm{id}\underset{\mathfrak{b}}{\otimes}T)X$. Evidently, the class of all corepresentations V with all (semi-)morphisms forms a category $\mathbf{C}^*\text{-}\mathbf{corep}_V^{(s)}$. One easily verifies that there exists an isomorphism of categories $\mathbf{C}^*\text{-}\mathbf{corep}_V^{(s)} \to \mathbf{C}^*\text{-}\mathbf{rep}_{V^{op}}^{(s)}$ given by $({}_\gamma K_\delta, X)\mapsto({}_\gamma K_\delta, \Sigma Y^*\Sigma)$ and $T\mapsto T$. Thus, all constructions and results on representations carry over to corepresentations. In particular, we can equip $\mathbf{C}^*\text{-}\mathbf{corep}_V$ with the structure of C^*-tensor category and $\mathbf{C}^*\text{-}\mathbf{corep}_V^s$ with the structure of a tensor category.

Replacing $\mathfrak{b}$ by the W^*-base $[\![\mathfrak{b}]\!]$ and C^*-modules by W^*-modules (see [31]) in definition 4.1, we obtain the notion of a W^*-representation. If we reformulate this notion using correspondences instead of W^*-modules, the definition reads as follows. A *W^*-representation* of V consists of a Hilbert space K with two commuting nondegenerate and normal representations $\widehat{\sigma}\colon \mathfrak{B}\to\mathcal{L}(K)$, $\sigma\colon\mathfrak{B}^\dagger\to\mathcal{L}(K)$ and a unitary $X\in\mathcal{L}(K_{\widehat{\sigma}}\olessthan\alpha, K_\sigma\olessthan\beta)$ that satisfies $X(\sigma(b^\dagger)\olessthan\mathrm{id})=(\mathrm{id}\olessthan\rho_\alpha(b^\dagger))X$, $X(\mathrm{id}\olessthan\rho_\beta(b))=(\widehat{\sigma}(b)\olessthan\mathrm{id})X$, $X(\mathrm{id}\olessthan\rho_{\widehat{\beta}}(b))=(\mathrm{id}\olessthan\rho_{\widehat{\beta}}(b))X$ for all $b^\dagger\in\mathfrak{B}^\dagger$, $b\in\mathfrak{B}$ and that makes the following diagram commute,

$$\begin{tikzcd}
K_{\widehat{\sigma}}\olessthan\alpha_{\rho_{\widehat{\beta}}}\olessthan\alpha \arrow[r, "X\olessthan\mathrm{id}"] \arrow[d, "\mathrm{id}\olessthan V"] & K_\sigma\olessthan\beta_{\rho_{\widehat{\beta}}}\olessthan\alpha \arrow[r, "\mathrm{id}\olessthan V"] & K_\sigma\olessthan\beta_{\rho_\alpha}\olessthan\beta, \\
K_{\widehat{\sigma}}\olessthan(\alpha\ogreaterthan_{\rho_\beta}\alpha) \arrow[d, "\Sigma_{23}"] & & (K_{\widehat{\sigma}}\olessthan\alpha)_{\sigma\olessthan\mathrm{id}}\olessthan\beta \arrow[u, "X\olessthan\mathrm{id}"'] \\
K_{\widehat{\sigma}}\olessthan\alpha_{\rho_\beta}\olessthan\alpha \arrow[rr, "X\olessthan\mathrm{id}"] & & (K_\sigma\olessthan\beta)_{\widehat{\sigma}\olessthan\mathrm{id}}\olessthan\alpha \arrow[u, "\Sigma_{23}"']
\end{tikzcd}$$

where Σ_{23} denotes the isomorphisms that exchange the second and the third factor in the iterated internal tensor products. Here, normality of $\sigma, \hat{\sigma}$ means that they extend to the von Neumann algebras generated by $\mathfrak{B}$ and $\mathfrak{B}^\dagger$, respectively, in $\mathcal{L}(\mathfrak{K})$. A *morphism* of W^*-representations $(K,\sigma,\hat{\sigma},X)$ and $(L,\tau,\hat{\tau},Y)$ is an operator $T\in\mathcal{L}(K,L)$ that intertwines σ and τ on one side and $\hat{\sigma}$ and $\hat{\tau}$ on the other side, and satisfies

$Y(T \otimes \mathrm{id}) = (T \otimes \mathrm{id})X$. Evidently, the class of all W^*-representations of V forms a category $\mathbf{W^*\text{-}rep}_V$. One easily verifies that there exists a functor $\mathbf{C^*\text{-}rep}_V^{(s)} \to \mathbf{W^*\text{-}rep}_V$ given by $({}_\gamma K_{\widehat{\delta}}, X) \mapsto (K, \rho_\gamma, \rho_{\widehat{\delta}}, X)$ and $T \mapsto T$. Using a relative tensor product of W^*-modules (see [31]), one can equip $\mathbf{W^*\text{-}rep}_V$ with the structure of a C^*-tensor category similarly like $\mathbf{C^*\text{-}rep}_V$ and finds that the functors above preserve the tensor product. Finally, one can consider W^*-corepresentations of V which are defined in a straightforward manner.

4.6 Representations of groupoids and of the associated unitaries

Let G be a locally compact, Hausdorff, second countable groupoid with a left Haar system. Then the C^*-tensor category of representations of G is equivalent to the C^*-tensor category of corepresentations of the C^*-pseudo-multiplicative unitary associated to G, as will be explained now. We use the notation and results of subsections 2.3 and 3.5,

$$\begin{gathered}
\mathfrak{K} := L^2(G^0,\mu), \quad \mathfrak{B} = \mathfrak{B}^\dagger := C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K}), \quad \mathfrak{b} := (\mathfrak{K},\mathfrak{B},\mathfrak{B}^\dagger), \\
H := L^2(G,\nu), \quad \alpha = \beta := j(L^2(G,\lambda)), \quad \widehat{\beta} := \hat{j}(L^2(G,\lambda^{-1})), \\
V\colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \cong L^2(G_s {\times_r} G, \nu^2_{s,r}) \to L^2(G_r {\times_r} G, \nu^2_{r,r}) \cong H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H, \\
(V\omega)(x,y) = \omega(x,x^{-1}y) \text{ for all } \omega \in C_c(G_s {\times_r} G),\, (x,y) \in G_r {\times_r} G, \\
C_0(G) \cong \widehat{A}_V \subseteq \mathcal{L}(H), \quad C^*_r(G) = A_V \subseteq \mathcal{L}(H), \\
\rho_{\widehat{\beta}} = s^*\colon C_0(G^0) \to C_b(G) \to \mathcal{L}(H), \quad \rho_\alpha = r^*\colon C_0(G^0) \to C_b(G) \to \mathcal{L}(H),
\end{gathered}$$

and fix further notation. Let X be a locally compact Hausdorff space, E a Hilbert C^*-module over $C_0(X)$ and $x \in X$. We denote by $\chi_x\colon C(X) \to \mathbb{C}$ the evaluation at x and by $E_x := E \otimes_{\chi_x} \mathbb{C}$ the fiber of E at x; this is the Hilbert space associated to the sesquilinear form $(\eta,\eta') \mapsto \langle\eta|\eta'\rangle(x)$ on E. Given an element $\xi \in E$ and an operator $T \in \mathcal{L}_{C_0(X)}(E)$, we denote by $\xi_x := \xi \otimes_{\chi_x} 1 \in E_x$ and $T_x := T \otimes_{\chi_x} \mathrm{id}_{\mathbb{C}} \in \mathcal{L}(E_x)$ the values of ξ and T, respectively, at x. Given a locally compact Hausdorff space Y and a continuous map $p\colon Y \to X$, the pull-back of E along p is the Hilbert C^*-module $p^*E := E \otimes_{p^*} C_0(Y)$ over $C_0(Y)$, where $p^*\colon C_0(X) \to M(C_0(Y))$ denotes the pull-back on functions. This pull-back is functorial, that is, if Z is a locally compact Hausdorff space and $q\colon Z \to Y$ is a continuous map, then $(p \circ q)^*E$ is naturally isomorphic to q^*p^*E. For ξ, T as above and all $y \in Y$, we have $(p^*\xi)_y = \xi_{p(y)}$ and $(p^*T)_y = T_y$.

The first part of the following definition is a special case of [18, Définition 4.4]:

Definition 4.21. *A* continuous representation of G *consists of a Hilbert C^*-module E over $C_0(G^0)$ and a unitary $U \in \mathcal{L}_{C_0(G)}(s^*E, r^*E)$ such that $U_xU_y = U_{xy}$ for all $(x,y) \in G_s {\times_r} G$. We denote by $\mathbf{C^*\text{-}rep}_G$ the category of continuous representations of G, where the morphisms between representations (E,U_E) and (F,U_F) are all operator $T \in \mathcal{L}_{C_0(G^0)}(E,F)$ satisfying $U_F \circ s^*T = r^*T \circ U_E$ in $\mathcal{L}_{C_0(G)}(s^*E, r^*F)$.*

The verification of the following result is straightforward:

Proposition 4.22. *i) Let (E,U_E) and (F,U_F) be continuous representations of G and represent $C_0(G^0)$ on F by right multiplication operators. Then $(E \otimes F)_x =$*

$E_x \otimes F_x$ *for all* $x \in G$*, and there exists a continuous representation* $U_E \boxtimes U_F$ *of* G *on* $E \ogreaterthan F$ *such that* $(U_E \boxtimes U_F)_x = (U_E)_x \otimes (U_F)_x$ *for all* $x \in G$.

ii) *If* S_i *is a morphism of continuous representations* $(E_i, U_{i,E})$ *and* $(F_i, U_{i,F})$ *for* $i = 1, 2$*, then* $S_1 \ogreaterthan S_2$ *is a morphism between* $(E_1 \ogreaterthan E_2, U_{1,E} \boxtimes U_{2,E})$ *and* $(F_1 \ogreaterthan F_2, U_{1,F} \boxtimes U_{2,F})$.

iii) *The category* $\mathbf{C^*\text{-}rep}_G$ *carries the structure of a* C^**-tensor category such that*
   - *the tensor product is given by the constructions in i) and ii);*
   - *the associativity isomorphism* $a_{(E_1,U_1),(E_2,U_2),(E_3,U_3)}$ *is the canonical isomorphism* $(E_1 \ogreaterthan E_2) \ogreaterthan E_3 \to E_1 \ogreaterthan (E_2 \ogreaterthan E_3)$ *for all* $(E_1, U_1), (E_2, U_2), (E_3, U_3)$*;*
   - *the unit consists of the Hilbert* C^**-module* $C_0(G^0)$ *and the canonical isomorphism* $s^*C_0(G^0) \cong C_0(G) \cong r^*C_0(G^0)$*;*
   - *the isomorphisms* $l_{(E,U)}$ *and* $r_{(E,U)}$ *are the canonical isomorphisms* $C_0(G^0) \ogreaterthan E \cong E \cong E \ogreaterthan C_0(G^0)$ *for each* (E, U). □

Define $p_1, p_2, m \colon G_s{\times}_rG \to G$ by $p_1(x,y) = x$, $p_2(x,y) = y$, $m(x,y) = xy$ and $r_1, t, s_2 \colon G_s{\times}_rG \to G^0$ by $r_1(x,y) = r(x), t(x,y) = s(x), s_2(x,y) = s(y)$. Then we have a commutative diagram

$$\begin{tikzcd}
 & G^0 & \\
G \arrow[ur, "s"] \arrow[d, "r"'] & G_s{\times}_rG \arrow[l, "p_1"] \arrow[r, "p_2"'] \arrow[u, "t"] \arrow[d, "m"] \arrow[dl, "r_1"] \arrow[dr, "s_2"'] & G \arrow[ul, "r"'] \arrow[d, "s"] \\
G^0 & G \arrow[l, "r"] \arrow[r, "s"'] & G^0.
\end{tikzcd} \tag{27}$$

Lemma 4.23. *Let* E *be a Hilbert* C^**-module over* $C_0(G^0)$ *and* $U \in \mathcal{L}_{C_0(G)}(s^*E, r^*E)$*. Then* $U_xU_y = U_{xy}$ *for all* $x, y \in G_s{\times}_rG$ *if and only if* m^*U *is equal to the composition*

$$s_2^*E \xrightarrow{p_2^*U} t^*E \xrightarrow{p_1^*U} r_1^*E.$$

Proof. $((p_1^*U)(p_2^*U))_{(x,y)} = U_xU_y$, $(m^*U)_{(x,y)} = U_{xy}$ for all $(x,y) \in G_s{\times}_rG$. □

We need the following straightforward result which involves the operators defined in (2):

Lemma 4.24. *Let* $p \colon Y \to X$ *be a continuous map of locally compact Hausdorff spaces, let* L *be a Hilbert space with a nondegenerate injective* $*$*-homomorphism* $C_0(Y) \hookrightarrow \mathcal{L}(L)$*, and let* γ *be a Hilbert* C^**-module over* $C_0(X)$.

i) *There exists an isomorphism of* $\Phi^f_{L,\gamma} \colon L \olessthan f^*\gamma \to L_{f^*} \olessthan \gamma$ *of Hilbert spaces given by* $\zeta \olessthan (\xi \ogreaterthan_{f^*} g) \mapsto g\zeta_{f^*} \olessthan \xi$.

ii) *There exists an isomorphism* $\Psi^f_{L,\gamma} \colon f^*\gamma \to [r_L^{f^*}(\gamma)C_0(Y)] \subseteq \mathcal{L}(L, L_{f^*} \olessthan \gamma)$ *of Hilbert* C^**–modules over* $C_0(Y)$ *given by* $\xi \olessthan_{f^*} g \mapsto r_L^{f^*}(\xi)g$.

iii) *For all* $U \in \mathcal{L}_{C_0(Y)}(f^*\gamma)$ *and* $\omega \in f^*\gamma$*, we have* $\Phi^f_{L,\gamma}(\mathrm{id}_L \olessthan U)(\Phi^f_{L,\gamma})^*\Psi^f_{L,\gamma}(\omega) = \Psi^f_{L,\gamma}(U\omega)$ *in* $\mathcal{L}(L, L_{f^*} \olessthan \gamma)$. □

To each representation of G, we functorially associate a corepresentation of V:

Proposition 4.25. *i) Let (E,U) be a continuous representation of G. Put $K := \mathfrak{K} \otimes E$ and identify E with the subspace $\gamma = \delta := \{r_{\mathfrak{K}}(\xi) \mid \xi \in E\} \subseteq \mathcal{L}(\mathfrak{K},K)$ via $\xi \mapsto r_{\mathfrak{K}}(\xi)$. Then ${}_{\gamma}K_{\delta}$ is a C^*-$(\mathfrak{b},\mathfrak{b}^\dagger)$-module, we have canonical identifications*

$$H_{s^*} \otimes E \cong H_{\rho_{\widehat{\beta}}} \otimes \gamma \cong H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{\gamma}K, \qquad H_{r^*} \otimes E \cong H_{\rho_{\alpha}} \otimes \delta \cong H_{\alpha} \underset{\mathfrak{b}}{\otimes} {}_{\delta}K,$$

and ${}_{\gamma}K_{\delta}$ together with the unitary $X := \Phi^{r}_{H,\gamma}(\mathrm{id}_H \otimes U)\Phi^{s}_{H,\gamma}$ form a corepresentation $\mathbf{F}(E,U) := ({}_{\gamma}K_{\delta}, X)$ of V.

ii) Let T be a morphism of continuous representations (E,U_E), (F,U_F) of G. Then $\mathbf{F}T := \mathrm{id}_{\mathfrak{K}} \otimes T$ is a morphism of the corepresentations $\mathbf{F}(E,U_E)$, $\mathbf{F}(F,U_F)$.

iii) The assignments $(E,U) \mapsto \mathbf{F}(E,U)$ and $T \mapsto \mathbf{F}T$ form a functor $\mathbf{F}\colon \mathbf{C}^*\text{-}\mathbf{rep}_G \to \mathbf{C}^*\text{-}\mathbf{corep}_V$.

Proof. i) The assertion on ${}_{\gamma}K_{\delta}$ is easily checked. In $\mathcal{L}(H, H_{\alpha} \underset{\mathfrak{b}}{\otimes} {}_{\delta}K)$, we have

$$\begin{aligned}\Phi^{r}_{H,\gamma}(\mathrm{id}_H \otimes U)(\Phi^{s}_{H,\gamma})^*[|\gamma\rangle_2 C_0(G)] &= \Phi^{r}_{H,\gamma}(\mathrm{id}_H \otimes U) r_H(s^*\gamma) \\ &= \Phi^{r}_{H,\gamma} r_H(Us^*\gamma) = \Phi^{r}_{H,\gamma} r_H(r^*\gamma) = [|\gamma\rangle_2 C_0(G)].\end{aligned}$$

Note here that $r_H(\cdot)$ denotes the operator defined in (2), while r denotes the range map of G. Using the relations $\gamma = \delta$ and $\alpha = \beta = [C_0(G)\alpha]$, we can conclude

$$X[|\gamma\rangle_2\alpha] = [|\gamma\rangle_2\alpha] = [|\alpha\rangle_1\gamma], \quad X[|\gamma\rangle_2\beta] = [|\delta\rangle_2\beta], \quad X[|\widehat{\beta}\rangle_1\delta] = X[|\delta\rangle_2\widehat{\beta}] = [|\delta\rangle_2\widehat{\beta}].$$

To finish the proof, we have to show that diagram (26) commutes. We apply Lemma 4.24 to the maps $p = r_1, t, s_2\colon G_s \times_r G \to G^0$ in diagram (27), the space $L = H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{\alpha}H$ and the representation $C_0(G_s \times_r G) \to \mathcal{L}(L)$ given by multiplication operators, use the relations $r_1^* = \rho_{(\alpha \lhd \alpha)}$ and $s_2^* = \rho_{(\widehat{\beta} \rhd \widehat{\beta})}$, and find that $X_{13}X_{23}$ is equal to the composition

$$L_{s_2^*} \otimes \gamma \xrightarrow{(\Phi^{s_2}_{L,\gamma})^*} L \otimes s_2^*\gamma \xrightarrow{\mathrm{id}_L \otimes p_2^* U} L \otimes t^*\gamma \xrightarrow{\mathrm{id}_L \otimes p_1^* U} L \otimes r_1^*\gamma \xrightarrow{\Phi^{r_1}_{L,\gamma}} L_{r_1^*} \otimes \gamma,$$

which coincides by Lemma 4.23 with $\Phi^{r_1}_{L,\gamma}(\mathrm{id}_L \otimes m^*U)(\Phi^{s_2}_{L,\gamma})^*$. Since $V^*_{12}p_2^*(f)V_{12} = \widehat{\Delta}_V(f) = m^*f$ for all $f \in C_0(G)$, this composition is equal to $V^*_{12}X_{23}V_{12}$.

ii), iii) Straightforward. ☐

Conversely, we functorially associate to every corepresentation of V a representation of G. In the formulation of this construction, we apply Lemma 4.24 to $Y = G$ and $L = H$.

Proposition 4.26. *i) Let X be a corepresentation of V on a C^*-$(\mathfrak{b},\mathfrak{b}^\dagger)$-module ${}_{\gamma}K_{\delta}$. Then $\gamma = \delta$, $X[|\gamma\rangle_2 C_0(G)] = [|\gamma\rangle_2 C_0(G)]$, and γ together with the unitary $U := \Psi^{r}_{H,\gamma}X(\Psi^{s}_{H,\gamma})^*\colon s^*\gamma \to r^*\gamma$ form a continuous representation $\mathbf{G}({}_{\gamma}K_{\delta}, X) := (\gamma, U)$ of G.*

ii) If T is a morphism of corepresentations $({}_{\gamma}K_{\delta}, X)$ and $({}_{\varepsilon}L_{\phi}, Y)$, then the map $\mathbf{G}T\colon \gamma \to \varepsilon$, $\xi \mapsto T\xi$, is a morphism of the continuous representations $\mathbf{G}({}_{\gamma}K_{\delta}, X)$, $\mathbf{G}({}_{\varepsilon}L_{\phi}, Y)$.

iii) The assignments $({}_{\gamma}K_{\delta}, X) \mapsto \mathbf{G}({}_{\gamma}K_{\delta}, X)$, $T \mapsto \mathbf{G}T$ form a functor $\mathbf{G}\colon \mathbf{C^*\text{-}corep}_V \to \mathbf{C^*\text{-}rep}_G$.

Proof. i) Since $\alpha = \beta$ and $[|\alpha\rangle_1\gamma] = X[|\gamma\rangle_2\alpha] = X[|\gamma\rangle_2\beta] = [|\delta\rangle_2\beta] = [|\beta\rangle_1\delta]$ as subsets of $\mathcal{L}(\mathfrak{K}, H_{\alpha} \underset{\mathfrak{b}}{\otimes} {}_{\delta}K)$, we can conclude $\gamma = [\rho_{\delta}(\mathfrak{B})\gamma] = [\langle\alpha|_1|\alpha\rangle_1\gamma] = [\langle\alpha|_1|\beta\rangle_1\delta] = [\rho_{\delta}(\mathfrak{B})\delta] = \delta$. The relation $X[|\gamma\rangle_2 C_0(G)] = [|\gamma\rangle_2 C_0(G)]$ will follow from Example 5.3 ii) and Proposition 5.8. Finally, $X = \Phi^r_{H,\gamma}(\mathrm{id}_H \otimes U)\Phi^s_{H,\gamma}$, and reversing the arguments in the proof of Proposition 4.25, we conclude from $X_{12}X_{13} = V^*_{12}X_{23}V_{12}$ that $p_1^*U \circ p_2^*U = m^*U$. By Lemma 4.23, U is a representation on γ.

ii), iii) Straightforward. □

We define an equivalence between C^*-tensor categories to be an equivalence of the underlying C^*-categories and tensor categories [20].

Theorem 4.27. *The functors $\mathbf{C^*\text{-}rep}_G \underset{\mathbf{F}}{\overset{\mathbf{G}}{\leftrightarrows}} \mathbf{C^*\text{-}corep}_V$ extend to an equivalence of C^*-tensor categories.*

Proof. Lemma 4.24 iii) implies that the functors $\mathbf{F}, \mathbf{G}$ form equivalences of categories. The verification of the fact that they preserve the monoidal structure is tedious but straightforward. □

Remark 4.28. Let us note that a similar equivalence holds between the categories of measurable representations of G and W^*-corepresentations of V.

5 Regular, proper and étale C^*-pseudo-multiplicative unitaries

In this section, we study particular classes of C^*-pseudo-multiplicative unitaries. As before, let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, let $(H, \widehat{\beta}, \alpha, \beta)$ be a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module, and let $V\colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{\alpha}H \to H_{\alpha} \underset{\mathfrak{b}}{\otimes} {}_{\beta}H$ be a C^*-pseudo-multiplicative unitary.

5.1 Regularity

In [3], Baaj and Skandalis showed that the pairs $(\widehat{A}_V, \widehat{\Delta}_V)$ and (A_V, Δ_V) associated to a multiplicative unitary V on a Hilbert space H form Hopf C^*-algebras if the unitary satisfies the regularity condition $[\langle H|_2 V|H\rangle_1] = \mathcal{K}(H)$. This condition was generalized by Baaj in [1, 2] and extended to pseudo-multiplicative unitaries by Enock [10].

We now formulate a generalized regularity condition for C^*-pseudo-multiplicative unitaries and show that the pairs $((\widehat{A}_V)^{\alpha,\widehat{\beta}}_H, \widehat{\Delta}_V)$ and $((A_V)^{\beta,\alpha}_H, \Delta_V)$ associated to such a unitary V in subsection 3.4 are concrete Hopf C^*-bimodules if V is regular. This regularity condition involves the space

$$C_V := [\langle\alpha|_1 V|\alpha\rangle_2] \subseteq \mathcal{L}(H).$$

Proposition 5.1. *We have*

$$[C_V C_V] = C_V, \quad C_{V^{op}} = C_V^*, \quad [C_V\alpha] = \alpha,$$
$$[C_V\rho_\beta(\mathfrak{B})] = [\rho_\beta(\mathfrak{B})C_V] = C_V = [C_V\rho_{\widehat{\beta}}(\mathfrak{B})] = [\rho_{\widehat{\beta}}(\mathfrak{B})C_V].$$

Proof. The proof is completely analogous to the proof of Proposition 3.11; for example, the first equation follows from the commutativity of the following two diagrams:

$$\begin{tikzcd}
 & H \arrow[rr, "C_V"] \arrow[dl, bend right, "|\alpha\rangle_2"'] \arrow[d, "|\alpha\rangle_2"] & & H \arrow[rr, "C_V"] \arrow[dr, "|\alpha\rangle_2"] & & H & \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[ddr, bend right, "|\alpha\rangle_2"'] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V"] \arrow[d, "|\alpha\rangle_3"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[ur, "\langle\alpha|_1"] \arrow[dr, "|\alpha\rangle_3"] & & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\alpha|_1"'] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[ul, bend right, "\langle\alpha|_1"'] \\
 & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[rr, "V_{12}"'] \arrow[d, "V_{23}"] & & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[ur, "\langle\alpha|_1"] \arrow[rr, "V_{23}"'] & & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\alpha|_1"'] & \\
 & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{(\alpha\triangleright\alpha)}(H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H) \arrow[rrrr, "V_{13}"] & & & & (H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H)_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "V_{12}"'] \arrow[uur, bend right, "\langle\alpha|_2"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & H \arrow[r, "C_V"] \arrow[dl, bend right, "|\alpha\rangle_2"'] \arrow[d, "|\alpha\rangle_2"] & H \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "\rho_{(\widehat{\beta}\triangleright\beta)}(\mathfrak{B})"] \arrow[dr, bend right, "|\alpha\rangle_2"'] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\alpha|_1"'] \\
 & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{(\alpha\triangleright\alpha)}(H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H) \arrow[u, "\langle\alpha|_2"'] \arrow[r, "V_{13}"] & (H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H)_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\alpha|_2"']
\end{tikzcd}$$

□

Definition 5.2. *A C^*-pseudo-multiplicative unitary $(\mathfrak{b}, H, \widehat{\beta}, \alpha, \beta, V)$ is* semi-regular *if $C_V \supseteq [\alpha\alpha^*]$, and* regular *if $C_V = [\alpha\alpha^*]$.*

Examples 5.3. i) By Proposition 5.1, V is (semi-)regular if and only if V^{op} is (semi-)regular.

ii) The C^*-pseudo-multiplicative unitary associated to a locally compact Hausdorff groupoid G as in Theorem 2.7 is regular. To prove this assertion, we use the notation introduced in subsection 2.3 and calculate that for each $\xi, \xi' \in C_c(G)$, $\zeta \in C_c(G) \subseteq L^2(G, \nu)$, $y \in G$,

$$\big(\langle j(\xi')|_1 V|j(\xi)\rangle_2\zeta\big)(y) = \int_{G^{r(y)}} \overline{\xi'(x)}\zeta(x)\xi(x^{-1}y)\,\mathrm{d}\lambda^{r(y)}(x),$$
$$\big(j(\xi')j(\xi)^*\zeta\big)(y) = \xi'(y)\int_{G^{r(y)}} \overline{\xi(x)}\zeta(x)\,\mathrm{d}\lambda^{r(y)}(x).$$

Using standard approximation arguments, we find $[\langle\alpha|_1 V|\alpha\rangle_2] = [S(C_c(G_r{\times}_r G))] = [\alpha\alpha^*]$, where for each $\omega \in C_c(G_r{\times}_r G)$, the operator $S(\omega)$ is given by

$$(S(\omega)\zeta)(y) = \int_{G^{r(y)}} \omega(x,y)\zeta(x)d\lambda^{r(y)}(x) \quad \text{for all } \zeta \in C_c(G), y \in G.$$

iii) In [35], we introduce compact C^*-quantum groupoids and construct for each such quantum groupoid a C^*-pseudo-multiplicative unitary that turns out to be regular.

We shall now deduce several properties of semi-regular and regular C^*-pseudo-multiplicative unitaries, using commutative diagrams as explained in Notation 4.9.

Proposition 5.4. *If V is semi-regular, then C_V is a C^*-algebra.*

Proof. Assume that V is regular. Then the following two diagrams commute, whence $[C_V C_V^*] = [\langle\alpha|_1\langle\alpha|_1 V_{23}|\alpha\rangle_1|\alpha\rangle_2] = C_V$:

$$\begin{tikzcd}
 & H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[dr, bend left, "|\alpha\rangle_1"] & & \\
H \arrow[ur, bend left, "|\alpha\rangle_2"] \arrow[r, "|\alpha\rangle_1"'] \arrow[d, "C_V^*"'] & H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "|\alpha\rangle_3"'] \arrow[d, "V^*"] & H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H{}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[d, "V_{12}^*"] \arrow[dddr, bend left, "V_{23}"] & \\
H \arrow[d, equal] & H{}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[l, "\langle\alpha|_2"] \arrow[r, "|\alpha\rangle_3"] \arrow[dr, phantom, "(R)"] & H{}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H{}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[d, "V_{23}"] & \\
H \arrow[r, "|\alpha\rangle_2"] \arrow[d, "C_V"'] & H{}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[d, "V"] & H{}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{(\alpha\rhd\alpha)}(H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H) \arrow[l, "\langle\alpha|_2"'] \arrow[d, "V_{13}"] & \\
H & H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[l, "\langle\alpha|_1"'] & (H{}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H){}_{(\alpha\lhd\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[l, "\langle\alpha|_2"'] & H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[l, "V_{12}^*"'] \arrow[dll, bend left, "\langle\alpha|_1"] \\
 & H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[ul, bend left, "\langle\alpha|_1"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
H \arrow[r, "C_V"'] \arrow[d, "|\alpha\rangle_2"] & H & \\
H{}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[r, "V"] \arrow[d, "|\alpha\rangle_1"] & H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\alpha|_1"] \arrow[r, "\rho_{(\beta\lhd\beta)}(\mathfrak{B})"] \arrow[d, "\langle\alpha|_1"] & H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[ul, bend right, "|\alpha\rangle_1"'] \\
H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H{}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha}H \arrow[r, "V_{23}"] & H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H{}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[ur, bend right, "|\alpha\rangle_1"'] &
\end{tikzcd}$$

Now, assume that V is semi-regular. Then cell (R) in the first diagram need not commute, but still $[|\alpha\rangle_2\langle\alpha|_2] \subseteq [\langle\alpha|_2 V_{23}|\alpha\rangle_3]$ and hence $[C_V C_V^*] \subseteq [\langle\alpha|_1\langle\alpha|_1 V_{23}|\alpha\rangle_1|\alpha\rangle_2] = C_V$. A similar argument shows that also $[C_V^* C_V] \subseteq C_V$, and from Proposition 3.11 and [1, Lemme 3.3], it follows that C_V is a C^*-algebra. ☐

Proposition 5.5. *Assume that $C_V = C_V^*$.*

i) Let $({}_{\gamma}K_{\widehat{\delta}}, X)$ be a representation of V. Then $(\widehat{A}_X)_K^{\gamma,\widehat{\delta}}$ is a C^*-$(\mathfrak{b},\mathfrak{b}^{\dagger})$-algebra.

ii) $(\widehat{A}_V)_H^{\alpha,\widehat{\beta}}$ is a C^*-$(\mathfrak{b},\mathfrak{b}^{\dagger})$-algebra and $(A_V)_H^{\beta,\alpha}$ a C^*-$(\mathfrak{b}^{\dagger},\mathfrak{b})$-algebra.

The proof uses the following central lemma:

Lemma 5.6. *Let $({}_\gamma K_{\widehat{\delta}}, X)$ be a representation of V. Then $[X(1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V)X^*|\beta\rangle_2] = [|\beta\rangle_2 \widehat{A}_X^*]$.*

Proof. The following diagram commutes and shows that we have $[X(1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V)X^*|\beta\rangle_2] = [|\beta\rangle_2 \langle\alpha|_2 X^*|\beta\rangle_2] = [|\beta\rangle_2 \widehat{A}_X^*]$:

$$\begin{tikzcd}
K \arrow[rrr, "|\beta\rangle_2"'] \arrow[d, "|\beta\rangle_2"] & & & K {}_\gamma\underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[dl, "|\beta\rangle_3"] \arrow[ddd, "X^*"] \\
K {}_\gamma\underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[r, "|\alpha\rangle_3"] \arrow[d, "X^*"] & K {}_\gamma\underset{\mathfrak{b}}{\otimes}{}_\beta H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[r, "V_{23}"] \arrow[d, "X_{12}^*"] & K {}_\gamma\underset{\mathfrak{b}}{\otimes}{}_\beta H {}_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[dd, "X_{12}^*"] & \\
K {}_{\widehat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[r, "|\alpha\rangle_3"] \arrow[d, "1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V"] & K {}_{\widehat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[d, "V_{23}"] \arrow[r, phantom, "(\mathrm{P})"] & & \\
K {}_{\widehat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[d, "X"] & K {}_{\widehat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{(\alpha \triangleright \alpha)} (H {}_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H) \arrow[l, "\langle\alpha|_2"] \arrow[r, "X_{13}"] & (K {}_{\widehat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H) {}_{(\gamma \triangleleft \alpha)}\underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[dll, "\langle\alpha|_2"] & K {}_{\widehat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[l, "|\beta\rangle_3"'] \arrow[d, "\langle\alpha|_2"'] \\
K {}_\gamma\underset{\mathfrak{b}}{\otimes}{}_\beta H & & & K \arrow[lll, "|\beta\rangle_2"]
\end{tikzcd}$$

Indeed, cell (P) commutes by (19), and the remaining cells because of (18) or by inspection. ☐

Proof of Proposition 5.5. i) By Proposition 4.10, it suffices to show that $\widehat{A}_X = \widehat{A}_X^*$. But by Proposition 4.10 and Lemma 5.6, $\widehat{A}_X^* = [\rho_\alpha(\mathfrak{B}^\dagger)\widehat{A}_X^*] = [\langle\beta|_2|\beta\rangle_2 \widehat{A}_X^*] = [\langle\beta|_2 X(1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V)X^*|\beta\rangle_2]$.

ii) Statement i) applied to $({}_\gamma K_{\widehat{\delta}}, X) = ({}_\alpha H_{\widehat{\beta}}, V)$ yields the first assertion. The second one follows after replacing V by V^{op}, where we use Propositions 3.11 and 5.1. ☐

The main result of this subsection is the following:

Theorem 5.7. *If V is semi-regular, then $\big((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V\big)$ and $\big((A_V)_H^{\beta,\alpha}, \Delta_V\big)$ are normal Hopf C^*-bimodules.*

Proof. We prove the assertion concerning $\big((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V\big)$; for $\big((A_V)_H^{\beta,\alpha}, \Delta_V\big)$, the arguments are similar. By Proposition 5.5, $(\widehat{A}_V)_H^{\alpha,\widehat{\beta}}$ is a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-algebra, and by Proposition 4.10, applied to $A_{V^{op}} = \widehat{A}_V^* = \widehat{A}_V$, we have $\Delta_{V^{op}}(\widehat{A}_V) \subseteq (\widehat{A}_V) {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{*}{}_\alpha (\widehat{A}_V)$. Now, the claim follows from Lemma 3.13. ☐

We collect several auxiliary results on regular C^*-pseudo-multiplicative unitaries.

Proposition 5.8. *Assume that V is regular.*

i) Let $({}_\gamma K_{\widehat{\delta}}, X)$ be a representation of V. Then $[X|\alpha\rangle_2 \widehat{A}_X] = [|\beta\rangle_2 \widehat{A}_X]$ and $[X|\widehat{\delta}\rangle_1 A_V] = [|\gamma\rangle_1 A_V]$.

ii) $[V|\alpha\rangle_2 \widehat{A}_V] = [|\beta\rangle_2 \widehat{A}_V]$ and $[V|\widehat{\beta}\rangle_1 A_V] = [|\alpha\rangle_1 A_V]$.

Proof. Using Lemma 5.6 and the relation $\widehat{A}_X = \widehat{A}_X^*$ (Proposition 5.5), we find that $[X|\alpha\rangle_2\widehat{A}_X] = [X|\alpha\rangle_2\langle\alpha|_2 X^*|\beta\rangle_2] = [X(1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V)X^*|\beta\rangle_2] = [|\beta\rangle_2\widehat{A}_X]$.

Replacing $({}_{\gamma}K_{\widehat{\delta}}, X)$ by $({}_{\alpha}H_{\widehat{\beta}}, V)$, we obtain the first equation in ii), and replacing V by V^{op} and using Proposition 3.11, we obtain $[\Sigma V^*\Sigma|\alpha\rangle_2 A_V^*] = [|\widehat{\beta}\rangle_2 A_V^*]$, which yields $[V|\widehat{\beta}\rangle_1 A_V] = [|\alpha\rangle_1 A_V]$.

Finally, let us prove the equation $[X|\widehat{\delta}\rangle_1 A_V] = [|\gamma\rangle_1 A_V]$. The following commutative diagram shows that $[X_{13}|\widehat{\delta}\rangle_1|\alpha\rangle_1 A_V] = [|\alpha\rangle_2|\gamma\rangle_1 A_V]$:

$$\begin{tikzcd}
H \arrow[r, "A_V"] \arrow[d, equal] & H \arrow[r, "|\alpha\rangle_1"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "|\widehat{\delta}\rangle_1"] \arrow[d, "V^*"] & K_{\widehat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{(\alpha\triangleright\alpha)}(H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H) \arrow[d, "V_{23}^*"] \arrow[dddd, bend left=60, "X_{13}"] \\
H \arrow[r, "A_V"] \arrow[d, equal] & H \arrow[r, "|\widehat{\beta}\rangle_1"] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "|\widehat{\delta}\rangle_1"] & K_{\widehat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[d, "X_{12}"] \\
H \arrow[r, "A_V"] \arrow[d, equal] & H \arrow[r, "|\widehat{\beta}\rangle_1"] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "|\gamma\rangle_1"] \arrow[d, "V"] & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[d, "V_{23}"] \\
H \arrow[r, "A_V"] \arrow[d, equal] & H \arrow[r, "|\alpha\rangle_1"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "|\gamma\rangle_1"] & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[d, "X_{12}"] \\
H \arrow[r, "A_V"] & H \arrow[r, "|\gamma\rangle_1"] & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "|\alpha\rangle_2"] & (K_{\widehat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H)_{(\gamma\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H
\end{tikzcd}$$

Moreover, also the following diagram commutes,

$$\begin{tikzcd}
H \arrow[r, "|\alpha\rangle_1"] \arrow[dr, "\rho_\beta(\mathfrak{B})"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "X_{13}|\widehat{\delta}\rangle_1"] \arrow[d, "\langle\alpha|_1"] & (K_{\widehat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H)_{(\gamma\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[d, "\langle\alpha|_2"] & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[l, "|\alpha\rangle_2"'] \arrow[dl, "\rho_{(\widehat{\delta}\triangleleft\beta)}(\mathfrak{B})"'] & \\
H \arrow[u, "A_V"] \arrow[r, "A_V"] & H \arrow[r, "X|\widehat{\delta}\rangle_1"] & K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H & H \arrow[l, "|\gamma\rangle_1"'] \arrow[u, "|\gamma\rangle_1"'] & H, \arrow[l, "A_V"']
\end{tikzcd}$$

and hence $[X|\widehat{\delta}\rangle_1 A_V] = [\langle\alpha|_2 X_{13}|\widehat{\delta}\rangle_1|\alpha\rangle_1] = [\langle\alpha|_2|\alpha\rangle_2|\gamma\rangle_1 A_V] = [|\gamma\rangle_1 A_V]$. $\square$

The last result in this subsection involves the algebras $\widehat{A}_{\mathbf{1}_V} = [\beta^*\alpha]$ and $\widehat{A}_{\mathbf{1}_{V^{op}}} = [\widehat{\beta}^*\alpha]$ associated to the trivial representations of V and V^{op}, respectively.

Proposition 5.9. *If V is regular, then $[\beta\widehat{A}_{\mathbf{1}_V}] = [\alpha\widehat{A}_{\mathbf{1}_V}]$ and $[\widehat{\beta}A_{\mathbf{1}_{V^{op}}}] = [\alpha A_{\mathbf{1}_{V^{op}}}]$.*

Proof. The following diagram commutes

$$\begin{tikzcd}
\mathfrak{K} \arrow[rr, "\beta"'] \arrow[d, "\beta"] & & H \arrow[r, "\alpha^*"'] \arrow[d, "|\beta\rangle_2"] & \mathfrak{K} \arrow[d, "\beta"] \\
H \arrow[r, "|\alpha\rangle_2"] \arrow[rrr, bend right, "\alpha\alpha^*"'] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "\langle\alpha|_1"] & H
\end{tikzcd}$$

and shows that $[\beta\widehat{A}_{\mathbf{1}_V}] = [\beta\widehat{A}_{\mathbf{1}_V}^*] = [\beta\alpha^*\beta] = [\alpha\alpha^*\beta] = [\alpha\widehat{A}_{\mathbf{1}_V}^*] = [\alpha\widehat{A}_{\mathbf{1}_V}]$. The second equation follows by replacing V with V^{op}. $\square$

5.2 Proper and étale C^*-pseudo-multiplicative unitaries

In [3], Baaj and Skandalis characterized multiplicative unitaries that correspond to compact or discrete quantum groups by the existence of fixed or cofixed vectors, respectively, and showed that from such vectors, one can construct a Haar state and a counit on the associated legs. We adapt some of their constructions to C^*-pseudo-multiplicative unitaries as follows.

Given a C^*-$\mathfrak{b}^{(\dagger)}$-module K_γ, let $M(\gamma) = \{T \in \mathcal{L}(\mathfrak{K}, K) \mid T\mathfrak{B}^{(\dagger)} \subseteq \gamma, T^*\gamma \subseteq \mathfrak{B}^{(\dagger)}\}$.

Definition 5.10. *A* fixed element *for V is an operator $\eta \in M(\widehat{\beta}) \cap M(\alpha) \subseteq \mathcal{L}(\mathfrak{K}, H)$ satisfying $V|\eta\rangle_1 = |\eta\rangle_1$. A* cofixed element *for V is an operator $\xi \in M(\alpha) \cap M(\beta) \subseteq \mathcal{L}(\mathfrak{K}, H)$ satisfying $V|\xi\rangle_2 = |\xi\rangle_2$. We denote the set of all fixed/cofixed elements for V by* $\mathrm{Fix}(V)$/$\mathrm{Cofix}(V)$.

Example 5.11. Let us consider the C^*-pseudo-multiplicative unitary associated to a locally compact, Hausdorff, second countable groupoid G in subsection 2.3. We identify $M(L^2(G,\lambda))$ in the natural way with the completion of the space

$$\Big\{ f \in C(G) \,\Big|\, r\colon \operatorname{supp} f \to G \text{ is proper, } \sup_{u \in G^0} \textstyle\int_{G^u} |f(x)|^2 \,\mathrm{d}\lambda^u(x) \text{ is finite} \Big\}$$

with respect to the norm $f \mapsto \sup_{u \in G^0} \big(\int_{G^u} |f(x)|^2 \,\mathrm{d}\lambda^u(x)\big)^{1/2}$. Similarly as in [34, Lemma 7.11], one easily verifies that

i) $\eta_0 \in M(L^2(G,\lambda))$ is a fixed element for V if and only if for each $u \in G^0$, $\eta_0|_{G^u \setminus \{u\}} = 0$ almost everywhere with respect to λ^u;

ii) $\xi_0 \in M(L^2(G,\lambda))$ is a cofixed element for V if and only if $\xi_0(x) = \xi_0(s(x))$ for all $x \in G$.

Let us collect some easy properties of fixed and cofixed elements.

Remarks 5.12. i) $\mathrm{Fix}(V) = \mathrm{Cofix}(V^{op})$ and $\mathrm{Cofix}(V) = \mathrm{Fix}(V^{op})$.

ii) $\mathrm{Fix}(V)^*\mathrm{Fix}(V)$ and $\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)$ are contained in $M(\mathfrak{B}) \cap M(\mathfrak{B}^\dagger)$.

iii) The relations $\mathrm{Fix}(V) \subseteq M(\widehat{\beta}) \cap M(\alpha)$ imply $\rho_\alpha(\mathfrak{B}^\dagger)\mathrm{Fix}(V) = \mathrm{Fix}(V)\mathfrak{B}^\dagger \subseteq \widehat{\beta}$ and $\rho_{\widehat{\beta}}(\mathfrak{B})\mathrm{Fix}(V) = \mathrm{Fix}(V)\mathfrak{B} \subseteq \alpha$. Likewise, we have $\rho_\beta(\mathfrak{B})\mathrm{Cofix}(V) \subseteq \alpha$ and $\rho_\alpha(\mathfrak{B}^\dagger)\mathrm{Cofix}(V) \subseteq \beta$.

Lemma 5.13. *Let $\xi, \xi' \in \mathrm{Cofix}(V)$ and $\eta, \eta' \in \mathrm{Fix}(V)$. Then*

$$\langle \xi|_2 V|\xi'\rangle_2 = \rho_\alpha(\xi^*\xi') = \rho_{\widehat{\beta}}(\xi^*\xi'), \qquad \langle \eta|_1 V|\eta'\rangle_1 = \rho_\beta(\eta^*\eta') = \rho_\alpha(\eta^*\eta').$$

Proof. Let $\zeta \in H$. Then $\langle \xi|_2 V|\xi'\rangle_2\zeta = \langle \xi|_2|\xi'\rangle_2\zeta = \rho_\alpha(\xi^*\xi')\zeta$ and $(\langle \xi|_2 V|\xi'\rangle_2)^*\zeta = \langle \xi'|_2|\xi\rangle_2\zeta = \rho_\beta((\xi')^*\xi)\zeta$. The second equation follows similarly. □

Proposition 5.14. *i)* $\rho_{\widehat{\beta}}(M(\mathfrak{B}))\mathrm{Cofix}(V) \subseteq \mathrm{Cofix}(V)$ *and* $\rho_\beta(\mathfrak{B})\mathrm{Fix}(V) \subseteq \mathrm{Fix}(V)$.

ii) $[\mathrm{Cofix}(V)\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)] = \mathrm{Cofix}(V)$ *and* $[\mathrm{Fix}(V)\mathrm{Fix}(V)^*\mathrm{Fix}(V)] = \mathrm{Fix}(V)$.

iii) $[\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)]$ *and* $[\mathrm{Fix}(V)^*\mathrm{Fix}(V)]$ *are C^*-subalgebras of $M(\mathfrak{B}) \cap M(\mathfrak{B}^\dagger)$; in particular, they are commutative.*

Proof. We only prove the assertions concerning $\mathrm{Cofix}(V)$; the other assertions follow similarly.

i) Let $T \in M(\mathfrak{B})$ and $\xi \in \mathrm{Cofix}(V)$. Then $\rho_{\widehat{\beta}}(T)\xi \subseteq M(\beta) \cap M(\alpha)$ because $\rho_{\widehat{\beta}}(\mathfrak{B})\beta \subseteq \beta$ and $\rho_{\widehat{\beta}}(\mathfrak{B})\alpha \subseteq \alpha$. The relation $V(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}$ furthermore implies

$$V|\rho_{\widehat{\beta}}(T)\xi\rangle_2 = V\rho_{(\widehat{\beta}\triangleright\widehat{\beta})}(T)|\xi\rangle_2 = \rho_{(\alpha\triangleright\widehat{\beta})}(T)V|\xi\rangle_2 = \rho_{(\alpha\triangleright\widehat{\beta})}(T)|\xi\rangle_2 = |\rho_{\widehat{\beta}}(T)\xi\rangle_2.$$

ii) Using i) and the relation $\mathrm{Cofix}(V) \subseteq M(\widehat{\beta})$, we find that

$$\begin{aligned}[]
[\mathrm{Cofix}(V)\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)] &\subseteq [\mathrm{Cofix}(V)M(\mathfrak{B}^\dagger)] \\
&= [\rho_{\widehat{\beta}}(M(\mathfrak{B}^\dagger))\mathrm{Cofix}(V)] \subseteq \mathrm{Cofix}(V).
\end{aligned}$$

Therefore, $[\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)]$ is a C^*-algebra and $\mathrm{Cofix}(V)$ is a Hilbert C^*-module over $[\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)]$. Now, [17, p. 5] implies that the inclusion above is an equality.

iii) This follows from ii) and Remark 5.12 ii). ☐

Definition 5.15. *We call the C^*-pseudo-multiplicative unitary V* étale *if $\eta^*\eta = \mathrm{id}_{\mathfrak{K}}$ for some $\eta \in \mathrm{Fix}(V)$,* proper *if $\xi^*\xi = \mathrm{id}_{\mathfrak{K}}$ for some $\xi \in \mathrm{Cofix}(V)$, and* compact *if it is proper and $\mathfrak{B}, \mathfrak{B}^\dagger$ are unital.*

Example 5.16. The C^*-pseudo-multiplicative unitary associated to a locally compact, Hausdorff, second countable groupoid G (Theorem 2.7) is étale/proper/compact if and only if G is étale/proper/compact. This follows from similar arguments as in [34, Theorem 7.12].

Remarks 5.17. i) By Remark 5.12, V is étale/proper if and only if V^{op} is proper/étale.

ii) If V is proper, then $\mathrm{id}_H \in \widehat{A}_V$; if V is étale, then $\mathrm{id}_H \in A_V$. This follows directly from Lemma 5.13.

The first main result of this subsection shows how one can construct a counit on $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$ from a fixed element for V.

Theorem 5.18. *Let V be an étale C^*-pseudo-multiplicative unitary.*

i) There exists a unique contractive homomorphism $\widehat{\varepsilon}\colon \widehat{A}_V \to \widehat{A}_{\mathbf{1}}$ such that $\widehat{\pi}_{\mathbf{1}} = \widehat{\varepsilon} \circ \widehat{\pi}_V : \tilde{\Omega}_{\beta,\alpha} \to \widehat{A}_{\mathbf{1}}$.

ii) Assume that V is regular. Then $\widehat{\varepsilon}$ is a jointly normal morphism from $(\widehat{A}_V)_H^{\alpha,\widehat{\beta}}$ to $(\widehat{A}_{\mathbf{1}})_{\mathfrak{K}}^{\mathfrak{B},\mathfrak{B}^\dagger}$ and a bounded counit for $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$.

Proof. Choose an $\eta_0 \in \mathrm{Fix}(V)$ with $\eta_0^*\eta_0 = \mathrm{id}_{\mathfrak{K}}$ and define $\widehat{\varepsilon}\colon \widehat{A}_V \to \mathcal{L}(\mathfrak{K})$ by $\hat{a} \mapsto \eta_0^*\hat{a}\eta_0$. Then $\widehat{\varepsilon}$ is contractive. For all $\xi \in \alpha, \eta \in \beta, \zeta \in \mathfrak{K}$,

$$\langle\eta|_2 V|\xi\rangle_2\eta_0\zeta = \langle\eta|_2 V(\eta_0 \otimes \xi\zeta) = \langle\eta|_2(\eta_0 \otimes \xi\zeta) = \eta_0(\eta^*\xi)\zeta,$$

and hence $\widehat{\pi}_V(\omega)\eta_0 = \eta_0\widehat{\pi}_{\mathbf{1}}(\omega)$ for all $\omega \in \tilde{\Omega}_{\beta,\alpha}$. In particular, $\widehat{\varepsilon}(\widehat{\pi}_V(\omega)) = \eta_0^*\widehat{\pi}_V(\omega)\eta_0 = \eta_0^*\eta_0\widehat{\pi}_{\mathbf{1}}(\omega) = \widehat{\pi}_{\mathbf{1}}(\omega)$.

Assume that V is regular. Then $\widehat{\varepsilon}$ is a morphism as claimed because by construction, $\widehat{\varepsilon}$ is a $*$-homomorphism, $\eta_0^* \in \mathcal{L}^{\widehat{\varepsilon}}({}_\alpha H_{\widehat{\beta}}, {}_{\mathfrak{B}}\mathfrak{K}_{\mathfrak{B}^\dagger})$, and $[\eta_0^*\alpha] \supseteq [\eta_0^*\eta_0\mathfrak{B}] = \mathfrak{B}$ and $[\eta_0^*\widehat{\beta}] \supseteq [\eta_0^*\eta_0\mathfrak{B}^\dagger] = \mathfrak{B}^\dagger$. It remains to show that diagram (14) commutes. Clearly, $(\widehat{\varepsilon} \underset{\mathfrak{b}^\dagger}{*} \mathrm{id})(x) = \langle\eta_0|_1 x|\eta_0\rangle_1$ and $(\mathrm{id} \underset{\mathfrak{b}^\dagger}{*} \widehat{\varepsilon})(x) = \langle\eta_0|_2 x|\eta_0\rangle_2$ for all $x \in (\widehat{A}_V)_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{*} {}_\alpha(\widehat{A}_V)$. Now, the left square in diagram (14) commutes because for all $\hat{a} \in \widehat{A}_V$,

$$\langle\eta_0|_1\widehat{\Delta}_V(\hat{a})|\eta_0\rangle_1 = \langle\eta_0|_1 V^*(1 \underset{\mathfrak{b}}{\otimes} \hat{a})V|\eta_0\rangle_1 = \langle\eta_0|_1(1 \underset{\mathfrak{b}}{\otimes} \hat{a})|\eta_0\rangle_1 = \rho_\beta(\eta_0^*\eta_0)\hat{a} = \hat{a}.$$

To see that the left square in diagram (14) commutes, let $\eta \in \beta, \xi \in \alpha$ and consider the following diagram:

$$\begin{tikzcd}
H \arrow[r, "|\xi\rangle_2"] \arrow[d, "|\eta_0\rangle_2"] \arrow[dr, phantom, ""] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[r, "V"] \arrow[d, "|\eta_0\rangle_2"] \arrow[dr, phantom, "(*)"] & H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[rr, "\mathrm{id}"] \arrow[dr, "|\eta_0\rangle_2"'] & & H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[r, "\langle\eta|_2"] & H \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[r, "|\xi\rangle_3"] \arrow[rrrrr, bend right, "\widehat{\Delta}(\langle\eta|_2V|\xi\rangle_2)"'] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[rr, "V_{13}V_{23}"] & & (H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H)_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[ur, "\langle\eta_0|_2"'] \arrow[rr, "\langle\eta|_3"] & & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[u, "\langle\eta_0|_2"']
\end{tikzcd}$$

The lower cell commutes by Lemma 4.14, cell (*) commutes because $V_{23}|\eta_0\rangle_2 = |\eta_0\rangle_2$, and the other cells commute as well. Since $\eta \in \beta$ and $\xi \in \alpha$ were arbitrary, the claim follows. ☐

As an example, we consider the unitary associated to a groupoid (subsection 2.3) .

Proposition 5.19. *Let G be a locally compact, Hausdorff, second countable groupoid and let $V\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H$ be the associated C^*-pseudo-multiplicative unitary.*

i) Let G be étale. Then V is étale, $\widehat{A}_V \cong C_0(G)$, $\widehat{A}_{\mathbf{1}} \cong C_0(G^0)$, and $\widehat{\varepsilon}\colon \widehat{A}_V \to \widehat{A}_{\mathbf{1}}$ is given by the restriction of functions on G to functions on G^0.

ii) Let G be proper. Then V^{op} is étale, $A_V = \widehat{A}_{V^{op}} = C_r^*(G)$, and for each $f \in C_c(G)$, the operator $\widehat{\varepsilon}(L(f)) \in \mathcal{L}(L^2(G^0,\mu))$ is given by

$$(\widehat{\varepsilon}(L(f))\zeta)(u) = \int_{G^u} f(x)D^{-1/2}(x)\zeta(s(x))\,\mathrm{d}\lambda^u(x) \quad \textit{for all } \zeta \in L^2(G^0,\mu),\ x \in G.$$

Proof. For all $\xi,\xi' \in C_c(G)$, $\zeta \in L^2(G^0,\mu)$ and $u \in G^0$, we have by Lemma 3.18

$$\begin{aligned}
(\widehat{\varepsilon}(m(\overline{\xi}*\xi'^*))\zeta)(u) = (\widehat{\varepsilon}(\hat{a}_{\xi,\xi'})\zeta)(u) &= (j(\xi)^* j(\xi')\zeta)(u) \\
&= \int_{G^u} \overline{\xi(x)}\xi'(x)\zeta(u)\,\mathrm{d}\lambda^u(x) \\
&= (\overline{\xi}*\xi'^*)(u)\zeta(u), \\
(\widehat{\varepsilon}(L(\overline{\xi}\xi'))\zeta)(u) = (\widehat{\varepsilon}(a_{\xi,\xi'})\zeta)(u) &= (j(\xi)^*\hat{j}(\xi')\zeta)(u) \\
&= \int_{G^u} \overline{\xi(x)}\xi'(x)D^{-1/2}(x)\zeta(s(x))\,\mathrm{d}\lambda^u(x).
\end{aligned}$$ ☐

The second main result of this subsection shows how one can construct a Haar weight on $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$ from a cofixed element for V.

Theorem 5.20. *Let V be a proper regular C^*-pseudo-multiplicative unitary. Then there exists a normal bounded left Haar weight ϕ for $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$.*

Proof. Choose $\xi_0 \in \mathrm{Cofix}(V)$ with $\xi_0^*\xi_0 = \mathrm{id}_{\mathfrak{K}}$. By Proposition 3.11 and Remark 5.12 i), $[\xi_0^*\widehat{A}_V\xi_0] = [\xi_0^*\rho_\alpha(\mathfrak{B}^\dagger)\widehat{A}_V\rho_\alpha(\mathfrak{B}^\dagger)\xi_0] \subseteq [\beta^*\widehat{A}_V\beta] \subseteq \mathfrak{B}^\dagger$. Hence, we can define a completely positive map $\phi\colon \widehat{A}_V \to \mathfrak{B}^\dagger$ by $\hat{a} \mapsto \xi_0^*\hat{a}\xi_0$, and $\phi \in \Omega_{M(\alpha)}(\widehat{A}_V)$. For all $\hat{a} \in \widehat{A}_V$, $(\mathrm{id} * \phi)(\widehat{\Delta}_V(\hat{a})) = \langle \xi_0|_2 V^*(\mathrm{id} \underset{\mathfrak{b}}{\otimes} \hat{a})V|\xi_0\rangle_2 = \langle \xi_0|_2(\mathrm{id} \underset{\mathfrak{b}}{\otimes} \hat{a})|\xi_0\rangle_2 = \rho_\alpha(\xi_0^*\hat{a}\xi_0)$. ☐

As an example, we again consider the unitary associated to a groupoid.

Proposition 5.21. *Let G be a locally compact, Hausdorff, second countable groupoid and let $V\colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H$ be the associated C^*-pseudo-multiplicative unitary.*

- *i) Let G be proper. Then V is proper, $\widehat{A}_V \cong C_0(G)$, and the map $\phi\colon \widehat{A}_V \to C_0(G^0)$ given by $(\phi(f))(u) = \int_{G^u} f(x)\, d\lambda^u(x)$ is a normal bounded left Haar weight for $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$.*
- *ii) Let G be étale. Then V^{op} is proper and there exists a normal bounded left and right Haar weight ϕ for $((A_V)_H^{\beta,\alpha}, \Delta_V)$ given by $L(f) \mapsto f|_{G^0}$ for all $f \in C_c(G)$.*

Proof. This follows from Theorem 5.20 and similar calculations as in 5.19. ☐

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