Coactions of Hopf C^*-bimodules

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Abstract

Coactions of Hopf C^*-bimodules simultaneously generalize coactions of Hopf C^*-algebras and actions of groupoids. Following an approach of Baaj and Skandalis, we construct reduced crossed products and establish a duality for fine coactions. Examples of coactions arise from Fell bundles on groupoids and actions of a groupoid on bundles of C^*-algebras. Continuous Fell bundles on an étale groupoid correspond to coactions of the reduced groupoid algebra, and actions of a groupoid on a continuous bundle of C^*-algebras correspond to coactions of the function algebra.

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1 Introduction and preliminaries

Actions of quantum groupoids that simultaneously generalize actions of quantum groups and actions of groupoids have been studied in various settings, including that of weak Hopf algebras or finite quantum groupoids [24, 25], Hopf algebroids or algebraic quantum groupoids [7, 13], and Hopf-von Neumann bimodules or measured quantum groupoids [10, 11, 29]. In this article, we introduce and investigate coactions of Hopf C^*-bimodules or reduced locally compact quantum groupoids within the framework developed in [27, 28].

In the first part of this article, we construct reduced crossed products and dual coactions, and show that the bidual of a fine coaction is Morita equivalent to the initial coaction. These constructions apply to pairs of Hopf C^*-bimodules that appear as the left and the right leg of a (weak) C^*-pseudo-Kac system, which consists of a C^*-pseudo-multiplicative unitary [28] and an additional symmetry. We associate such a C^*-pseudo-Kac system to every groupoid and to every compact C^*-quantum groupoid and expect that the same can be done for every reduced locally compact quantum groupoid once this concept has been defined properly. The constructions in this part generalize corresponding constructions of Baaj and Skandalis [3] for coactions of Hopf C^*-algebras.

Coactions of the two Hopf C^*-bimodules associated to a locally compact Hausdorff groupoid — the function algebra on one side and the reduced groupoid algebra on the other — are studied in detail in the second part of this article. We show that actions of the groupoid on continuous bundles of C^*-algebras correspond to coactions of the first Hopf C^*-bimodule, and that continuous Fell bundles on G naturally yield coactions of the second Hopf C^*-bimodule. Generalizing results of Quigg [21] and Baaj and Skandalis [2] from groups to groupoids, we show that if the groupoid is étale, every coaction of the reduced groupoid algebra arises from a Fell bundle.

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This article is organized as follows. The first part is concerned with coactions of Hopf C^*-bimodules and associated reduced crossed products. Section 2 summarizes the relative tensor product of C^*-modules and the fiber product of C^*-algebras over C^*-bases [27] which are fundamental to everything that follows, and introduces coactions of Hopf C^*-bimodules. Section 3 is concerned with C^*-pseudo-Kac systems. Every C^*-pseudo-Kac system gives rise to two Hopf C^*-bimodules, called the legs of the system, which are dual to each other in a suitable sense. Coactions of these legs on C^*-algebras, associated reduced crossed products, dual coactions and a duality theorem concerning iterated crossed products are discussed in Section 4. Section 5 gives the construction of the C^*-pseudo-Kac system of a locally compact Hausdorff groupoid G. The associated Hopf C^*-bimodules are the function algebra on one side and the reduced groupoid C^*-algebra of G on the other side. The second part of the article relates coactions of these Hopf C^*-bimodules to well-known notions. Section 6 shows that actions of a groupoid G on continuous bundles of C^*-algebras correspond to certain fine coactions of the function algebra of G. Section 7 contains preliminaries on Fell bundles, their morphisms and multipliers. Section 8 shows that continuous Fell bundles on G give rise to coactions of the reduced groupoid C^*-algebra of G, and section 9 gives a reverse construction that associates to every sufficiently nice coaction of the groupoid algebra a Fell bundle provided that the groupoid G is étale.

Preliminaries We use the following notation. Given a subset Y of a normed space X, we denote by $[Y] \subset X$ the closed linear span of Y. All sesquilinear maps like inner products of Hilbert spaces are assumed to be conjugate-linear in the first component and linear in the second one. Given a Hilbert space H, we use the ket-bra notation and define for each $\xi \in H$ operators $|\xi\rangle\colon \mathbb{C} \to H$, $\lambda \mapsto \lambda\xi$, and $\langle\xi| = |\xi\rangle^*\colon H \to \mathbb{C}$, $\xi' \mapsto \langle\xi|\xi'\rangle$. Given a C^*-algebra A and a subspace $B \subset A$, we denote by $A \cap B'$ the relative commutant $\{a \in A \mid [a, B] = 0\}$.

We shall make extensive use of (right) Hilbert C^*-modules; see [16]. In particular, we use the internal tensor product and the KSGNS-construction. Let E be a Hilbert C^*-module over a C^*-algebra A, let F be a Hilbert C^*-module over a

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C^*-algebra B, and let $\phi\colon A \to \mathcal{L}(F)$ be a completely positive map. We denote by $E \ominus_\phi F$ the Hilbert C^*-module over B which is the closed linear span of elements $\eta \ominus_\phi \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \eta \ominus_\phi \xi | \eta' \ominus_\phi \xi' \rangle = \langle \xi | \phi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\eta \ominus_\phi \xi)b = \eta \ominus_\phi \xi b$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. If ϕ is a $*$-homomorphism, this is the usual internal tensor product; if $F = B$, this is the KSGNS-construction. If $S \in \mathcal{L}(E)$ and $T \in \mathcal{L}(F) \cap \phi(A)'$, then there exists a unique operator $S \ominus_\phi T \in \mathcal{L}(E \ominus_\phi E)$ such that $(S \ominus_\phi T)(\eta \ominus_\phi \xi) = S\eta \ominus_\phi T\xi$ for all $\eta \in E$, $\xi \in F$; see [9, Proposition 1.34]. We sloppily write "$\ominus_A$" or "$\ominus$" instead of "$\ominus_\phi$" if no confusion may arise. We also define a flipped product $F {}_\phi{\oslash} E$ as follows. We equip the algebraic tensor product $F \odot E$ with the structure maps $\langle \xi \odot \eta | \xi' \odot \eta' \rangle := \langle \xi | \phi(\langle \eta | \eta' \rangle) \xi' \rangle$, $(\xi \odot \eta)b := \xi b \odot \eta$, form the separated completion, and obtain a Hilbert C^*-module $F {}_\phi{\oslash} E$ over B which is the closed linear span of elements $\xi {}_\phi{\oslash} \eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \xi {}_\phi{\oslash} \eta | \xi' {}_\phi{\oslash} \eta' \rangle = \langle \xi | \phi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\xi {}_\phi{\oslash} \eta)b = \xi b {}_\phi{\oslash} \eta$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. Again, we sloppily write "${}_A{\oslash}$" or "$\oslash$" instead of "${}_\phi{\oslash}$" if no confusion may arise. Evidently, there exists a unitary $\Sigma\colon F \ominus E \xrightarrow{\cong} E \oslash F$, $\eta \ominus \xi \mapsto \xi \oslash \eta$.

2 Hopf C^*-bimodules and coactions

A groupoid differs from a group in the non-triviality of its unit space. In almost every approach to quantum groupoids, the unit space is replaced by a nontrivial algebra, and a relative tensor product of modules and a fiber product of algebras over that algebra become fundamentally important. We shall use the corresponding constructions for C^*-algebras introduced in [27] and briefly summarize the main definitions and results below. For additional details and motivation, see [27, 28].

The relative tensor product A C^*-*base* is a triple $(\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ consisting of a Hilbert space $\mathfrak{K}$ and two commuting nondegenerate C^*-algebras $\mathfrak{B}, \mathfrak{B}^\dagger \subseteq \mathcal{L}(\mathfrak{K})$. It should be thought of as a C^*-algebraic counterpart to pairs consisting of a von Neumann algebra and its commutant. Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. Its *opposite* is the C^*-base $\mathfrak{b}^\dagger := (\mathfrak{K}, \mathfrak{B}^\dagger, \mathfrak{B})$.

A C^*-$\mathfrak{b}$-*module* is a pair $H_\alpha = (H, \alpha)$, where H is a Hilbert space and $\alpha \subseteq \mathcal{L}(\mathfrak{K}, H)$ is a closed subspace satisfying $[\alpha\mathfrak{K}] = H$, $[\alpha\mathfrak{B}] = \alpha$, and $[\alpha^*\alpha] = \mathfrak{B} \subseteq \mathcal{L}(\mathfrak{K})$. If H_α is a C^*-$\mathfrak{b}$-module, then α is a Hilbert C^*-module over B with inner product $(\xi, \xi') \mapsto \xi^*\xi'$ and there exist isomorphisms

$$\alpha \ominus \mathfrak{K} \to H,\ \xi \ominus \zeta \mapsto \xi\zeta, \qquad \mathfrak{K} \oslash \alpha \to H,\ \zeta \oslash \xi \mapsto \xi\zeta, \tag{1}$$

and a nondegenerate representation

$$\rho_\alpha\colon \mathfrak{B}^\dagger \to \mathcal{L}(H), \quad \rho_\alpha(b^\dagger)(\xi\zeta) = \xi b^\dagger \zeta \quad \text{for all } b^\dagger \in \mathfrak{B}^\dagger, \xi \in \alpha, \zeta \in \mathfrak{K}.$$

A *semi-morphism* between C^*-$\mathfrak{b}$-modules H_α and K_β is an operator $T \in \mathcal{L}(H, K)$ satisfying $T\alpha \subseteq \beta$. If additionally $T^*\beta \subseteq \alpha$, we call T a *morphism.* We denote the set of all (semi-)morphisms by $\mathcal{L}_{(s)}(H_\alpha, K_\beta)$. If $T \in \mathcal{L}_s(H_\alpha, K_\beta)$, then $T\rho_\alpha(b^\dagger) = \rho_\beta(b^\dagger)T$ for all $b^\dagger \in \mathfrak{B}^\dagger$, and if additionally $T \in \mathcal{L}(H_\alpha, K_\beta)$, then left multiplication by T defines an operator in $\mathcal{L}(\alpha, \beta)$ which we again denote by T.

We shall use the following notion of C^*-bi- and C^*-n-modules. Let $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ be C^*-bases, where $\mathfrak{b}_i = (\mathfrak{K}_i, \mathfrak{B}_i, \mathfrak{B}_i^\dagger)$ for each i. A C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$*-module* is a tuple $(H, \alpha_1, \ldots, \alpha_n)$, where H is a Hilbert space and (H, α_i) is a C^*-$\mathfrak{b}_i$-module for each i such that $[\rho_{\alpha_i}(\mathfrak{B}_i^\dagger)\alpha_j] = \alpha_j$ whenever $i \neq j$. In the case $n = 2$, we abbreviate ${}_\alpha H_\beta := (H, \alpha, \beta)$. If $(H, \alpha_1, \ldots, \alpha_n)$ is a C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-module, then $[\rho_{\alpha_i}(\mathfrak{B}_i^\dagger), \rho_{\alpha_j}(\mathfrak{B}_j^\dagger)] = 0$ whenever $i \neq j$. The set of *(semi-)morphisms* between C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-modules $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_n)$ and $\mathcal{K} = (K, \beta_1, \ldots, \beta_n)$ is $\mathcal{L}_{(s)}(\mathcal{H}, \mathcal{K}) := \bigcap_{i=1}^n \mathcal{L}_{(s)}(H_{\alpha_i}, K_{\beta_i}) \subseteq \mathcal{L}(H, K)$.

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, H_β C^*-$\mathfrak{b}$-module, and K_γ a C^*-$\mathfrak{b}^\dagger$-module. The *relative tensor product* of H_β and K_γ is the Hilbert space

$$H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K := \beta \otimes \mathfrak{K} \otimes \gamma.$$

It is spanned by elements $\xi \otimes \zeta \otimes \eta$, where $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, and the inner product is given by $\langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta' \rangle = \langle \zeta | \xi^* \xi' \eta^* \eta' \zeta' \rangle = \langle \zeta | \eta^* \eta' \xi^* \xi' \zeta' \rangle$ for all $\xi, \xi' \in \beta$, $\zeta, \zeta' \in \mathfrak{K}$, $\eta, \eta' \in \gamma$. Obviously, there exists a unitary *flip*

$$\Sigma \colon H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K \to K_\gamma \underset{\mathfrak{b}^\dagger}{\otimes}_\beta H, \quad \xi \otimes \zeta \otimes \eta \mapsto \eta \otimes \zeta \otimes \xi.$$

Using the unitaries in (1) on H_β and K_γ, respectively, we shall make the following identifications without further notice:

$$H_{\rho_\beta} \otimes \gamma \cong H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K \cong \beta \otimes_{\rho_\gamma} K, \quad \xi\zeta \otimes \eta \equiv \xi \otimes \zeta \otimes \eta \equiv \xi \otimes \eta\zeta.$$

For all $S \in \rho_\beta(\mathfrak{B}^\dagger)'$ and $T \in \rho_\gamma(\mathfrak{B})'$, we have operators

$$S \otimes \mathrm{id} \in \mathcal{L}(H_{\rho_\beta} \otimes \gamma) = \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K), \qquad \mathrm{id} \otimes T \in \mathcal{L}(\beta \otimes_{\rho_\gamma} K) = \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K).$$

If $S \in \mathcal{L}_s(H_\beta)$ or $T \in \mathcal{L}_s(K_\gamma)$, then $(S \otimes \mathrm{id})(\xi \otimes \eta\zeta) = S\xi \otimes \eta\zeta$ or $(\mathrm{id} \otimes T)(\xi\zeta \otimes \eta) = \xi\zeta \otimes T\eta$, respectively, for all $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, so that we can define

$$S \underset{\mathfrak{b}}{\otimes} T := (S \otimes \mathrm{id})(\mathrm{id} \otimes T) = (\mathrm{id} \otimes T)(S \otimes \mathrm{id}) \in \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K)$$
$$\text{for all } (S, T) \in \big(\mathcal{L}_s(H_\beta) \times \rho_\gamma(\mathfrak{B})'\big) \cup \big(\rho_\beta(\mathfrak{B}^\dagger)' \times \mathcal{L}_s(K_\gamma)\big).$$

For each $\xi \in \beta$ and $\eta \in \gamma$, there exist bounded linear operators

$$|\xi\rangle_1 \colon K \to H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K,\ \omega \mapsto \xi \otimes \omega, \qquad |\eta\rangle_2 \colon H \to H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K,\ \omega \mapsto \omega \otimes \eta,$$

whose adjoints $\langle \xi|_1 := |\xi\rangle_1^*$ and $\langle \eta|_2 := |\eta\rangle_2^*$ are given by

$$\langle \xi|_1 \colon \xi' \otimes \omega \mapsto \rho_\gamma(\xi^* \xi')\omega, \qquad \langle \eta|_2 \colon \omega \otimes \eta' \mapsto \rho_\beta(\eta^* \eta')\omega.$$

We write $|\beta\rangle_1 := \{|\xi\rangle_1 \,|\, \xi \in \beta\} \subseteq \mathcal{L}(K, H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K)$ and similarly define $\langle \beta|_1$, $|\gamma\rangle_2$, $\langle \gamma|_2$.

Let $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_m, \beta)$ be a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{b})$-module and let $\mathcal{K} = (K, \gamma, \delta_1, \ldots, \delta_n)$ be a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, where $\mathfrak{a}_i = (\mathfrak{H}_i, \mathfrak{A}_i, \mathfrak{A}_i^\dagger)$ and $\mathfrak{c}_j = (\mathfrak{L}_j, \mathfrak{C}_j, \mathfrak{C}_j^\dagger)$ are C^*-bases for all i, j. We define

$$\alpha_i \triangleleft \gamma := [|\gamma\rangle_2 \alpha_i] \subseteq \mathcal{L}(\mathfrak{H}_i, H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K), \qquad \beta \triangleright \delta_j := [|\beta\rangle_1 \delta_j] \subseteq \mathcal{L}(\mathfrak{L}_j, H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K)$$

for all i, j. Then $(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K, \alpha_1 \triangleleft \gamma, \ldots, \alpha_m \triangleleft \gamma, \beta \triangleright \delta_1, \ldots, \beta \triangleright \delta_n)$ is a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, called the *relative tensor product* of $\mathcal{H}$ and $\mathcal{K}$ and denoted by $\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}$. For all i, j and $a^\dagger \in \mathfrak{A}_i^\dagger$, $c^\dagger \in \mathfrak{C}_j^\dagger$,

$$\rho_{(\alpha_i \triangleleft \gamma)}(a^\dagger) = \rho_{\alpha_i}(a^\dagger) \underset{\mathfrak{b}}{\otimes} \mathrm{id}, \qquad \rho_{(\beta \triangleright \delta_j)}(c^\dagger) = \mathrm{id} \underset{\mathfrak{b}}{\otimes} \rho_{\delta_j}(c^\dagger).$$

The relative tensor product is functorial in the following sense. Let $\tilde{\mathcal{H}} = (\tilde{H}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, \tilde{\beta})$ be a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{b})$-module, $\tilde{\mathcal{K}} = (\tilde{K}, \tilde{\gamma}, \tilde{\delta}_1, \ldots, \tilde{\delta}_n)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, and $S \in \mathcal{L}_{(s)}(\mathcal{H}, \tilde{\mathcal{H}})$, $T \in \mathcal{L}_{(s)}(\mathcal{K}, \tilde{\mathcal{K}})$. Then there exists a unique operator $S \underset{\mathfrak{b}}{\otimes} T \in \mathcal{L}_{(s)}(\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}, \tilde{\mathcal{H}} \underset{\mathfrak{b}}{\otimes} \tilde{\mathcal{K}})$ satisfying $(S \underset{\mathfrak{b}}{\otimes} T)(\xi \otimes \zeta \otimes \eta) = S\xi \otimes \zeta \otimes T\eta$ for all $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$.

Finally, the relative tensor product is associative in the following sense. Let $\mathfrak{d}, \mathfrak{e}_1, \ldots, \mathfrak{e}_l$ be C^*-bases, $\hat{\mathcal{K}} = (K, \gamma, \delta_1, \ldots, \delta_n, \epsilon)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n, \mathfrak{d})$-module and $\mathcal{L} = (L, \phi, \psi_1, \ldots, \psi_l)$ a C^*-$(\mathfrak{d}^\dagger, \mathfrak{e}_1, \ldots, \mathfrak{e}_l)$-module. Then there exists a canonical isomorphism

$$a_{\mathcal{H},\mathcal{K},\mathcal{L}} \colon (H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K) {}_{\beta \triangleright \epsilon}\underset{\mathfrak{d}}{\otimes}{}_\phi L \to \beta \otimes_{\rho_\gamma} K_{\rho_\epsilon} \otimes \phi \to H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_{\gamma \triangleleft \phi} (K {}_\epsilon\underset{\mathfrak{d}}{\otimes}{}_\phi L) \tag{2}$$

which is an isomorphism of C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{c}_1, \ldots, \mathfrak{c}_n, \mathfrak{e}_1, \ldots, \mathfrak{e}_l)$-modules $(\mathcal{H} \underset{\mathfrak{b}}{\otimes} \hat{\mathcal{K}}) \underset{\mathfrak{d}}{\otimes} \mathcal{L} \to \mathcal{H} \underset{\mathfrak{b}}{\otimes} (\hat{\mathcal{K}} \underset{\mathfrak{d}}{\otimes} \mathcal{L})$. From now on, we identify the Hilbert spaces in (2) and denote them by $H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K {}_\epsilon\underset{\mathfrak{d}}{\otimes}{}_\phi L$.

The fiber product of C^*-algebras Let $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ be C^*-bases, where $\mathfrak{b}_i = (\mathfrak{K}_i, \mathfrak{B}_i, \mathfrak{B}_i^\dagger)$ for each i. A *(nondegenerate) C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-algebra* consists of a C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-module $(H, \alpha_1, \ldots, \alpha_n)$ and a (nondegenerate) C^*-algebra $A \subseteq \mathcal{L}(H)$ such that $\rho_{\alpha_i}(\mathfrak{B}_i^\dagger)A$ is contained in A for each i. We shall only be interested in the cases $n = 1, 2$, where we abbreviate $A_H^\alpha := (H_\alpha, A)$, $A_H^{\alpha,\beta} := ({}_\alpha H_\beta, A)$. Given a C^*-$(b_1, \ldots, b_n)$-algebra $\mathcal{A} = ((H, \alpha_1, \ldots, \alpha_n), A)$, we identify $M(A)$ with a C^*-subalgebra of $\mathcal{L}([AH]) \subseteq \mathcal{L}(H)$ and obtain C^*-$(b_1, \ldots, b_n)$-algebra $M(\mathcal{A}) = ((H, \alpha_1, \ldots, \alpha_n), M(A))$.

We need several natural notions of a morphism. Let $\mathcal{A} = (\mathcal{H}, A)$ and $\mathcal{C} = (\mathcal{K}, C)$ be C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-algebras, where $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_n)$ and $\mathcal{K} = (K, \gamma_1, \ldots, \gamma_n)$. A $*$-homomorphism $\pi \colon A \to C$ is called a *jointly (semi-)normal morphism* or briefly *(semi-)morphism* from $\mathcal{A}$ to $\mathcal{C}$ if $[\mathcal{L}_{(s)}^\pi(\mathcal{H}, \mathcal{K})\alpha_i] = \gamma_i$ for each i, where

$$\mathcal{L}_{(s)}^\pi(\mathcal{H}, \mathcal{K}) = \{T \in \mathcal{L}_{(s)}(\mathcal{H}, \mathcal{K}) \mid Ta = \pi(a)T \text{ for all } a \in A\}.$$

One easily verifies that every (semi-)morphism π between C^*-$\mathfrak{b}$-algebras A_H^α and C_K^γ satisfies $\pi(\rho_\alpha(b^\dagger)) = \rho_\gamma(b^\dagger)$ for all $b^\dagger \in \mathfrak{B}^\dagger$.

We construct a fiber product of C^*-algebras over C^*-bases as follows. Given Hilbert spaces H, K, a closed subspace $E \subseteq \mathcal{L}(H, K)$, and a C^*-algebra $A \subseteq \mathcal{L}(H)$, we define a C^*-algebra

$$\mathrm{Ind}_E(A) := \{T \in \mathcal{L}(K) \mid TE \subseteq [EA] \text{ and } T^*E \subseteq [EA]\} \subseteq \mathcal{L}(K).$$

Let $\mathfrak{b}$ be a C^*-base, A^β_H a C^*-$\mathfrak{b}$-algebra, and B^γ_K a C^*-$\mathfrak{b}^\dagger$-algebra. The *fiber product* of A^β_H and B^γ_K is the C^*-algebra

$$A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B := \mathrm{Ind}_{|\beta\rangle_1}(B) \cap \mathrm{Ind}_{|\gamma\rangle_2}(A) \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K).$$

To define coactions, we also need to consider the C^*-algebra

$$A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B := \mathrm{Ind}_{[|\beta\rangle_1 B]}(B) \cap \mathrm{Ind}_{|\gamma\rangle_2}(A) \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K),$$

which evidently contains $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$. If A and B are unital, so is $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$, but otherwise, $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$ and $A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B$ may be degenerate. Clearly, conjugation by the flip $\Sigma\colon H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K \to K_\gamma \underset{\mathfrak{b}^\dagger}{\otimes}{}_\beta H$ yields an isomorphism

$$\mathrm{Ad}_\Sigma\colon A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \to B_\gamma \underset{\mathfrak{b}^\dagger}{*}{}_\beta A.$$

If $\mathfrak{a}, \mathfrak{c}$ are C^*-bases, $A^{\alpha,\beta}_H$ is a C^*-$(\mathfrak{a},\mathfrak{b})$-algebra and $B^{\gamma,\delta}_K$ a C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-algebra, then

$$A^{\alpha,\beta}_H \underset{\mathfrak{b}}{*} B^{\gamma,\delta}_K := ({}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta,\ A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)$$

is a C^*-$(\mathfrak{a},\mathfrak{c})$-algebra, called the *fiber product* of $A^{\alpha,\beta}_H$ and $B^{\gamma,\delta}_K$ [27, Proposition 3.18], and likewise $({}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta,\ A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B)$ is a C^*-$(\mathfrak{a},\mathfrak{c})$-algebra.

The construction of the fiber product and of the algebra above is functorial with respect to (semi-)morphisms [28, Theorem 3.2] in the following sense.

Lemma 2.1. *Let $\mathfrak{c}$ be a C^*-base, π a semi-morphism of C^*-$\mathfrak{b}$-algebras A^β_H, C^λ_L, and ${}_\gamma K_\delta$ a C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-module. Let $I := \mathcal{L}^\pi_s(H_\beta, L_\lambda) \underset{\mathfrak{b}}{\otimes} \mathrm{id} \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K, L_\lambda \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$ and*

$$X := (I^*I)' \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K), \qquad Y := (II^*)' \subseteq \mathcal{L}(L_\lambda \underset{\mathfrak{b}}{\otimes}{}_\gamma K).$$

i) $\mathcal{X} := (H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K_\delta, X)$ and $\mathcal{Y} := (L_\lambda \underset{\mathfrak{b}}{\otimes}{}_\gamma K_\delta, Y)$ are C^*-$\mathfrak{c}$-algebras.

ii) There exists a semi-morphism $\mathrm{Ind}_{|\gamma\rangle_2}(\pi)\colon \mathcal{X} \to \mathcal{Y}$ such that $(\mathrm{Ind}_{|\gamma\rangle_2}(\pi))(x)z = zx$ for all $x \in X$ and $z \in I$.

iii) Assume that B^γ_K is a C^*-$\mathfrak{b}^\dagger$-algebra. Then we have $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \subseteq A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B \subseteq X$, $(\mathrm{Ind}_{|\gamma\rangle_2}(\pi))(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B) \subseteq C_\lambda \underset{\mathfrak{b}}{*}{}_\gamma B$ and $(\mathrm{Ind}_{|\gamma\rangle_2}(\pi))(A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B) \subseteq C_\lambda \underset{\mathfrak{b}}{*}{}^\gamma B$.

iv) $[|\gamma\rangle_2 A\langle\gamma|_2] \subseteq X$ and $(\mathrm{Ind}_{|\gamma\rangle_2}(\pi))([|\gamma\rangle_2 A\langle\gamma|_2]) = [|\gamma\rangle_2 \pi(A)\langle\gamma|_2]$.

Proof. The existence of the $*$-homomorphism $\mathrm{Ind}_{|\gamma\rangle_2}(\pi)$ follows as in [28, Lemma 3.1]. The proof of the remaining assertions is straightforward; compare [27, §3.4] and [28, §3.1]. □

Theorem 2.2. *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be C^*-bases, ϕ a (semi-)morphism of C^*-$(\mathfrak{a},\mathfrak{b})$-algebras $\mathcal{A} = A^{\alpha,\beta}_H$ and $\mathcal{C} = C^{\kappa,\lambda}_L$, and ψ a (semi-)morphism of C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-algebras $\mathcal{B} = B^{\gamma,\delta}_K$*

and $\mathcal{D} = D_M^{\mu,\nu}$. *Then there exists a unique (semi-)morphism of* C^*-$(\mathfrak{a},\mathfrak{c})$-*algebras* $\phi * \psi$ *from* $({}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta,\ A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B)$ *to* $({}_\kappa L_\lambda \underset{\mathfrak{b}}{\otimes} {}_\mu M_\nu,\ C_\lambda \underset{\mathfrak{b}}{*}{}^\mu D)$ *such that*

$$(\phi * \psi)(x)R = Rx \quad \text{for all } x \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \text{ and } R \in I_M J_H + J_L I_K,$$

where $I_X = \mathcal{L}^\phi(H, L) \underset{\mathfrak{b}}{\otimes} \mathrm{id}_X$, $J_Y = \mathrm{id}_Y \underset{\mathfrak{b}}{\otimes} \mathcal{L}^\psi(K, M)$ *for* $X \in \{K, M\}, Y \in \{H, L\}$, *and* $\phi * \psi$ *restricts to a (semi-)morphism from* $A_H^{\alpha,\beta} \underset{\mathfrak{b}}{*} B_K^{\gamma,\delta}$ *to* $C_L^{\kappa,\lambda} \underset{\mathfrak{b}}{*} D_M^{\mu,\nu}$.

Proof. This follows from Lemma 2.1 and a similar argument as in the proof of [27, Theorem 3.13]. □

Unfortunately, the fiber product need not be associative, but in our applications, it will only appear as the target of a comultiplication or coaction whose coassociativity will compensate the non-associativity of the fiber product.

Hopf C^*-bimodules and coactions The notion of a Hopf C^*-bimodule was introduced in [28].

Definition 2.3. *Let* $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ *be a* C^**-base. A* Hopf C^*-bimodule *over* $\mathfrak{b}$ *is a* C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-*algebra* $A_H^{\beta,\alpha}$ *together with a morphism* Δ *from* $A_H^{\beta,\alpha}$ *to* $A_H^{\beta,\alpha} \underset{\mathfrak{b}}{*} A_H^{\beta,\alpha}$ *that is coassociative in the sense that* $(\delta * \mathrm{id}) \circ \delta = (\mathrm{id} * \Delta) \circ \delta$ *as maps from* A *to* $\mathcal{L}(H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H)$.

Let $(\mathcal{A}, \Delta)$ *be a Hopf* C^**-bimodule, where* $\mathcal{A} = A_H^{\beta,\alpha}$. *A* coaction *of* $(\mathcal{A}, \Delta)$ *consists of a* C^*-$\mathfrak{b}$-*algebra* C_K^γ *and a semi-morphism* δ *from* (K_γ, C) *to* $(K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H_\alpha, C_\gamma \underset{\mathfrak{b}}{*}{}^\beta A)$ *such that* $(\delta * \mathrm{id}) \circ \delta = (\mathrm{id} * \Delta) \circ \delta$ *as maps from* C *to* $\mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H)$. *We call such a coaction* (C_K^γ, δ)

- left-full *if* $[\delta(C)|\gamma\rangle_1 A] = [|\gamma\rangle_1 A]$, *and* right-full *if* $[\delta(C)|\beta\rangle_2] = [|\beta\rangle_2 C]$;
- fine *if* δ *is injective, a morphism, and right-full, and if* $[\rho_\gamma(\mathfrak{B}^\dagger)C] = C$;
- very fine *if it is fine and if* $\delta^{-1}\colon \delta(C) \to C$ *is a morphism of* C^*-$\mathfrak{b}$-*algebras from* $(K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H_\alpha, \delta(C))$ *to* (K_γ, C).

A morphism *between coactions* (C_K^γ, δ_C) *and* (D_L^ϵ, δ_D) *is a semi-morphism* ρ *of* C^*-$\mathfrak{b}$-*algebras from* C_K^γ *to* $M(D)_L^\epsilon$ *that is nondegenerate in the sense that* $[\rho(C)D] = D$ *and equivariant in the sense that* $\delta_D(d) \cdot (\rho * \mathrm{id})(\delta_C(c)) = \delta_D(d\rho(c))$ *for all* $d \in D$, $c \in C$. *We denote the category of all coactions of* $(\mathcal{A}, \Delta)$ *by* $\mathbf{Coact}_{(\mathcal{A},\Delta)}$.

Examples of Hopf C^*-bimodules and coactions will be discussed in detail in Sections 5, 6, and 8.

3 Weak C^*-pseudo-Kac systems

To form a reduced crossed product for a coaction of a Hopf C^*-bimodule $(\mathcal{A}, \Delta)$ and to equip this reduced crossed product with a dual coaction, one needs a second Hopf C^*-bimodule $(\widehat{\mathcal{A}}, \widehat{\Delta})$ that is dual to $(\mathcal{A}, \Delta)$ in a suitable sense. We shall see that a good notion of duality is that $(\mathcal{A},\Delta)$ and $(\widehat{\mathcal{A}},\widehat{\Delta})$ are the legs of a weak C^*-pseudo-Kac system, which is a generalization of the balanced multiplicative unitaries and Kac systems introduced by Baaj and Skandalis [1, 3].

C^*-pseudo-multiplicative unitaries A weak C^*-pseudo-Kac system consists of a well-behaved C^*-pseudo-multiplicative unitary V and a symmetry U satisfying a number of axioms. Before we state these axioms, we recall from [28] the notion of a C^*-pseudo-multiplicative unitary and the construction of the associated Hopf C^*-bimodules.

Let $\mathfrak{b}$ be a C^*-base. A *C^*-pseudo-multiplicative unitary* over $\mathfrak{b}$ consists of a C^*-$(\mathfrak{b}^\dagger,\mathfrak{b},\mathfrak{b}^\dagger)$-module $(H,\widehat{\beta},\alpha,\beta)$ and a unitary $V\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H$ such that

$$V(\alpha\triangleleft\alpha)=\alpha\triangleright\alpha,\quad V(\widehat{\beta}\triangleright\beta)=\widehat{\beta}\triangleleft\beta,\quad V(\widehat{\beta}\triangleright\widehat{\beta})=\alpha\triangleright\widehat{\beta},\quad V(\beta\triangleleft\alpha)=\beta\triangleleft\beta \tag{3}$$

in $\mathcal{L}(\mathfrak{K},H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H)$ and $V_{12}V_{13}V_{23}=V_{23}V_{12}$ in the sense that the following diagram commutes,

$$\begin{array}{ccc} H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \xrightarrow{V_{12}} H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H & \xrightarrow{V_{23}} & H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H, \\ \downarrow V_{23} & & V_{12}\uparrow \\ H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha\triangleright\alpha}(H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H) & \xrightarrow{V_{13}} & (H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H)_{\alpha\triangleleft\alpha}\underset{\mathfrak{b}}{\otimes}{}_\beta H \end{array} \tag{4}$$

where V_{ij} is the leg notation for the operator that acts like V on the ith and jth factor in the relative tensor product; see [28].

Let V be a C^*-pseudo-multiplicative unitary as above, let

$$\widehat{A}=\widehat{A}_V=[\langle\beta|_2V|\alpha\rangle_2]\subseteq\mathcal{L}(H),\quad \widehat{\Delta}=\widehat{\Delta}_V\colon\widehat{A}\to\mathcal{L}(H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H),\ \widehat{a}\mapsto V^*(1\underset{\mathfrak{b}}{\otimes}\widehat{a})V,$$
$$A=A_V=[\langle\alpha|_1V|\widehat{\beta}\rangle_1]\subseteq\mathcal{L}(H),\quad \Delta=\Delta_V\colon A\to\mathcal{L}(H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H),\ a\mapsto V(a\underset{\mathfrak{b}^\dagger}{\otimes}1)V^*,$$

and let $\widehat{\mathcal{A}}=\widehat{A}_H^{\alpha,\widehat{\beta}}$, $\mathcal{A}=A_H^{\beta,\alpha}$. We say that V is *well-behaved* if $(\widehat{\mathcal{A}},\widehat{\Delta})$ and $(\mathcal{A},\Delta)$ are Hopf C^*-bimodules. This happens for example if V is *regular* in the sense that $[\langle\alpha|_1V|\alpha\rangle_2]=[\alpha\alpha^*]\subseteq\mathcal{L}(H)$ [28, Theorem 5.7].

The *opposite* of V is the C^*-pseudo-multiplicative unitary

$$V^{op}:=\Sigma V^*\Sigma\colon H_\beta\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H\xrightarrow{\Sigma}H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H\xrightarrow{V^*}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H\xrightarrow{\Sigma}H_\alpha\underset{\mathfrak{b}}{\otimes}{}_{\widehat{\beta}}H.$$

If V is well-behaved or regular, then the same is true for V^{op}, and then

$$\widehat{A}_{V^{op}}=A_V,\qquad\widehat{\Delta}_{V^{op}}=\mathrm{Ad}_\Sigma\circ\Delta_V,\qquad A_{V^{op}}=\widehat{A}_V,\qquad\Delta_{V^{op}}=\mathrm{Ad}_\Sigma\circ\widehat{\Delta}_V. \tag{5}$$

Balanced C^*-pseudo-multiplicative unitaries Let $(H,\widehat{\alpha},\widehat{\beta},\alpha,\beta)$ be a C^*-$(\mathfrak{b},\mathfrak{b}^\dagger,\mathfrak{b},\mathfrak{b}^\dagger)$-module and $U\in\mathcal{L}({}_{\widehat{\alpha}}H_{\widehat{\beta}},{}_\alpha H_\beta)$ a symmetry, that is, $U=U^*=U^{-1}$.

Then $U\widehat{\alpha} = \alpha$, $U\widehat{\beta} = \beta$, $U\alpha = \widehat{\alpha}$, $U\beta = \widehat{\beta}$, and the following diagram commutes,

$$\begin{array}{ccc}
H_{\widehat{\alpha}}\underset{\mathfrak{b}}{\otimes}_{\widehat{\beta}}H & \overset{(1 \underset{\mathfrak{b}^{(\dagger)}}{\otimes} U)\Sigma}{\longleftrightarrow} & H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}_{\alpha}H \\
{\scriptstyle (U \underset{\mathfrak{b}^{(\dagger)}}{\otimes} 1)\Sigma}\updownarrow & \times & \updownarrow{\scriptstyle (1 \underset{\mathfrak{b}^{(\dagger)}}{\otimes} U)\Sigma} \\
H_{\beta}\underset{\mathfrak{b}^{\dagger}}{\otimes}_{\widehat{\alpha}}H & \underset{(U \underset{\mathfrak{b}^{(\dagger)}}{\otimes} 1)\Sigma}{\longleftrightarrow} & H_{\alpha}\underset{\mathfrak{b}}{\otimes}_{\beta}H
\end{array}$$

where each arrow can be read in both directions and the diagonal maps are $U \underset{\mathfrak{b}^{(\dagger)}}{\otimes} U$. We adopt the leg notation and write U_1 for $U \underset{\mathfrak{b}^{(\dagger)}}{\otimes} 1$ and U_2 for $1 \underset{\mathfrak{b}^{(\dagger)}}{\otimes} U$. For each $T \in \mathcal{L}(H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}_{\alpha}H, H_{\alpha}\underset{\mathfrak{b}}{\otimes}_{\beta}H)$, let

$$\begin{aligned}
\check{T} &:= \Sigma(1 \underset{\mathfrak{b}}{\otimes} U)T(1 \underset{\mathfrak{b}^{\dagger}}{\otimes} U)\Sigma \colon H_{\widehat{\alpha}}\underset{\mathfrak{b}}{\otimes}_{\widehat{\beta}}H \to H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}_{\alpha}H, \\
\widehat{T} &:= \Sigma(U \underset{\mathfrak{b}}{\otimes} 1)T(U \underset{\mathfrak{b}^{\dagger}}{\otimes} 1)\Sigma \colon H_{\alpha}\underset{\mathfrak{b}}{\otimes}_{\beta}H \to H_{\beta}\underset{\mathfrak{b}^{\dagger}}{\otimes}_{\widehat{\alpha}}H.
\end{aligned}$$

Switching from $(\mathfrak{b}, H, \widehat{\alpha}, \widehat{\beta}, \alpha, \beta)$ to $(\mathfrak{b}^{\dagger}, H, \beta, \widehat{\alpha}, \widehat{\beta}, \alpha)$ or $(\mathfrak{b}^{\dagger}, H, \widehat{\beta}, \alpha, \beta, \widehat{\alpha})$, respectively, we can iterate the assignments $T \mapsto \check{T}$ and $T \mapsto \widehat{T}$, and obtain

$$\check{\check{\check{T}}} = \widehat{T}, \qquad \check{\check{T}} = (U \underset{\mathfrak{b}}{\otimes} U)T(U \underset{\mathfrak{b}^{\dagger}}{\otimes} U) = \widehat{\widehat{T}}, \qquad \check{T} = \widehat{\widehat{\widehat{T}}}. \tag{6}$$

Definition 3.1. *A* balanced C^*-pseudo-multiplicative unitary (V, U) *on a* C^*-$(\mathfrak{b}, \mathfrak{b}^{\dagger}, \mathfrak{b}, \mathfrak{b}^{\dagger})$*-module* $(H, \widehat{\alpha}, \widehat{\beta}, \alpha, \beta)$ *consists of a symmetry* $U \in \mathcal{L}({}_{\widehat{\alpha}}H_{\widehat{\beta}}, {}_{\alpha}H_{\beta})$ *and a* C^**-pseudo-multiplicative unitary* $V\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}_{\alpha}H \to H_{\alpha}\underset{\mathfrak{b}}{\otimes}_{\beta}H$ *such that* $\check{V}$ *and* $\widehat{V}$ *are* C^**-pseudo-multiplicative unitaries.*

Remark 3.2. In the definition above, $(\check{V}, U)$ is a C^*-pseudo-multiplicative unitary if and only if $(\widehat{V}, U)$ is one because $\check{V} = (U \underset{\mathfrak{b}^{\dagger}}{\otimes} U)\widehat{V}(U \underset{\mathfrak{b}}{\otimes} U)$.

Let (V, U) be a balanced C^*-pseudo-multiplicative unitary as above.

Remarks 3.3. i) One easily verifies that $(\check{V}, U)$, $(\widehat{V}, U)$, (V^{op}, U) are balanced C^*-pseudo-multiplicative unitaries again. We call them the *predual*, *dual*, and *opposite* of (V, U), respectively.

ii) The relations (3) for the unitaries $\check{V}$, $\widehat{V}$ read as follows:

$$\begin{array}{llll}
\widehat{\beta} \triangleleft \widehat{\beta} \xrightarrow{\check{V}} \widehat{\beta} \triangleright \widehat{\beta}, & \widehat{\alpha} \triangleright \alpha \xrightarrow{\check{V}} \widehat{\alpha} \triangleleft \alpha, & \widehat{\alpha} \triangleright \widehat{\alpha} \xrightarrow{\check{V}} \widehat{\beta} \triangleright \widehat{\alpha}, & \alpha \triangleleft \widehat{\beta} \xrightarrow{\check{V}} \alpha \triangleleft \alpha, \\
\beta \triangleleft \beta \xrightarrow{\widehat{V}} \beta \triangleright \beta, & \alpha \triangleright \widehat{\alpha} \xrightarrow{\widehat{V}} \alpha \triangleleft \widehat{\alpha}, & \alpha \triangleright \alpha \xrightarrow{\widehat{V}} \beta \triangleright \alpha, & \widehat{\alpha} \triangleleft \beta \xrightarrow{\widehat{V}} \widehat{\alpha} \triangleleft \widehat{\alpha},
\end{array}$$

where $X \xrightarrow{W} Y$ means $WX = Y$. They furthermore imply

$$\begin{array}{lll}
\widehat{\beta} \triangleright \widehat{\alpha} \xrightarrow{V} \alpha \triangleright \widehat{\alpha}, & \widehat{\alpha} \triangleright \beta \xrightarrow{\check{V}} \widehat{\beta} \triangleright \beta, & \alpha \triangleright \widehat{\beta} \xrightarrow{\widehat{V}} \beta \triangleright \widehat{\beta}, \\
\widehat{\alpha} \triangleleft \alpha \xrightarrow{V} \widehat{\alpha} \triangleleft \beta, & \beta \triangleleft \widehat{\beta} \xrightarrow{\check{V}} \beta \triangleleft \alpha, & \widehat{\beta} \triangleleft \beta \xrightarrow{\widehat{V}} \widehat{\beta} \triangleleft \widehat{\alpha}.
\end{array}$$

iii) The spaces $\widehat{A}$ and A are contained in $\mathcal{L}(H_{\widehat{\alpha}})$ since $[\widehat{A}\widehat{\alpha}] = [\langle\beta|_2 V|\alpha\rangle_2\widehat{\alpha}] = [\langle\beta|_2|\beta\rangle_2\widehat{\alpha}] = [\rho_\alpha(\mathfrak{B}^\dagger)\widehat{\alpha}] = \widehat{\alpha}$ and similarly $[A\widehat{\alpha}] = [\langle\alpha|_1 V|\widehat{\beta}\rangle_1\widehat{\alpha}] = \widehat{\alpha}$.

Lemma 3.4. *$V_{13}V_{23}\check{V}_{12} = \check{V}_{12}V_{13}$ and $\widehat{V}_{23}V_{12}V_{13} = V_{13}\widehat{V}_{23}$, that is, the following diagrams commute:*

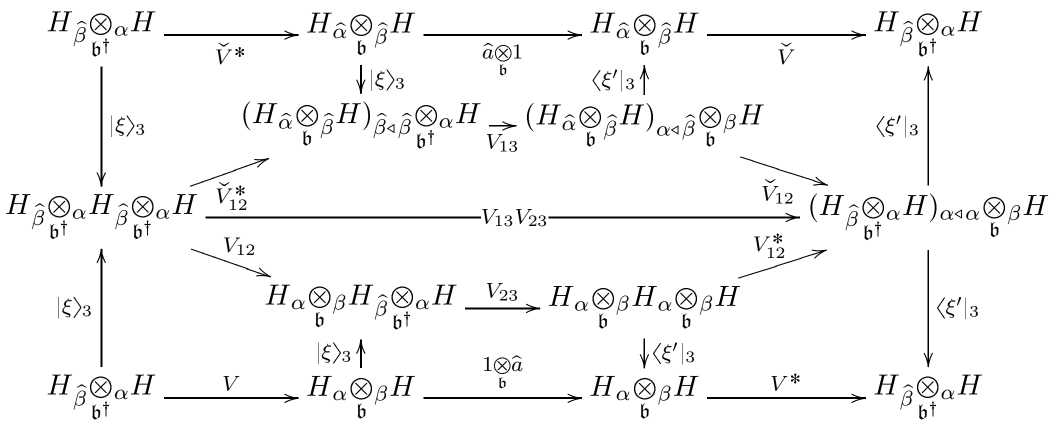
$$\begin{array}{ccccc} (H {}_{\widehat{\alpha}}\underset{\mathfrak{b}}{\otimes}{}_{\widehat{\beta}} H)_{\widehat{\beta}\triangleleft\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H & \xrightarrow{V_{13}} & (H {}_{\widehat{\alpha}}\underset{\mathfrak{b}}{\otimes}{}_{\widehat{\beta}} H)_{\alpha\triangleleft\widehat{\beta}}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H & \xrightarrow{\check{V}_{12}} & (H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H)_{\alpha\triangleleft\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H, \\ \downarrow{\scriptstyle \check{V}_{12}} & & & & \uparrow{\scriptstyle V_{13}} \\ H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H & & \xrightarrow{\qquad V_{23} \qquad} & & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha\triangleright\alpha} (H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H) \end{array} \tag{7}$$

$$\begin{array}{ccccc} H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha\triangleright\alpha} (H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H) & \xrightarrow{\widehat{V}_{23}} & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\beta\triangleright\alpha} (H {}_{\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\widehat{\alpha}} H) & \xrightarrow{V_{13}} & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta\triangleright\beta} (H {}_{\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\widehat{\alpha}} H). \\ \downarrow{\scriptstyle V_{13}} & & & & \uparrow{\scriptstyle \widehat{V}_{23}} \\ (H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H)_{\alpha\triangleleft\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H & & \xrightarrow{\qquad V_{12} \qquad} & & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \end{array} \tag{8}$$

Proof. Let $W := \Sigma V \Sigma$. We insert the relation $\check{V} = U_1 W U_1$ into the pentagon equation $\check{V}_{12}\check{V}_{13}\check{V}_{23} = \check{V}_{23}\check{V}_{12}$ and obtain $U_1W_{12}U_1 \cdot U_1W_{13}U_1 \cdot \check{V}_{23} = \check{V}_{23} \cdot U_1W_{12}U_1$ and hence $W_{12}W_{13}\check{V}_{23} = \check{V}_{23}W_{12}$. We conjugate both sides of this equation by the automorphism $\Sigma_{23}\Sigma_{12}$, which amounts to renumbering the legs of the operators according to the permutation $(1,2,3) \mapsto (2,3,1)$, and obtain $V_{13}V_{23}\check{V}_{12} = \check{V}_{12}V_{13}$. A similar calculation shows that $\widehat{V}_{23}V_{12}V_{13} = V_{13}\widehat{V}_{23}$. □

Proposition 3.5. *$\widehat{A}_{\check{V}} = UA_VU$, $\widehat{\Delta}_{\check{V}} = \mathrm{Ad}_{(U\underset{\mathfrak{b}}{\otimes}U)} \circ \Delta_V \circ \mathrm{Ad}_U$, $A_{\check{V}} = \widehat{A}_V$, $\Delta_{\check{V}} = \widehat{\Delta}_V$ and $A_{\widehat{V}} = U\widehat{A}_VU$, $\Delta_{\widehat{V}} = \mathrm{Ad}_{(U\underset{\mathfrak{b}^\dagger}{\otimes}U)} \circ \widehat{\Delta}_V \circ \mathrm{Ad}_U$, $\widehat{A}_{\widehat{V}} = A_V$, $\widehat{\Delta}_{\widehat{V}} = \Delta_V$.*

Proof. By definition, $A_{\check{V}} = [\langle\widehat{\beta}|_1 \Sigma U_2 V U_2 \Sigma|\widehat{\alpha}\rangle_1] = [\langle U\widehat{\beta}|_2 V|U\widehat{\alpha}\rangle_2] = [\langle\beta|_2 V|\alpha\rangle_2] = \widehat{A}_V$. Next, let $\widehat{a} = \langle\xi'|_2 V|\xi\rangle_2 \in \widehat{A}_V$, where $\xi' \in \beta$, $\xi \in \alpha$. The following diagram



commutes because diagrams (7) and (4) commute. Therefore, $\Delta_{\check{V}}(\widehat{a}) = \check{V}(\widehat{a} \underset{\mathfrak{b}}{\otimes} 1)\check{V}^* = V^*(1 \underset{\mathfrak{b}}{\otimes} \widehat{a})V = \widehat{\Delta}_V(\widehat{a})$. Since elements of the form like $\widehat{a}$ are dense in $\widehat{A}_V$, we can conclude $\Delta_{\check{V}} = \widehat{\Delta}_{\check{V}}$. The proof of the remaining assertions is similar. □

Corollary 3.6. *If V is well-behaved, then also $\check{V}$ and $\widehat{V}$ are well-behaved.*

Weak C^*-pseudo-Kac systems Let (V, U) as above.

Lemma 3.7. *For each $\hat{a} \in \hat{A}$ and $a \in A$, we have equivalences*

$$(1 \underset{\mathfrak{b}^\dagger}{\otimes} \hat{a})\hat{V} = \hat{V}(1 \underset{\mathfrak{b}}{\otimes} \hat{a}) \quad \Leftrightarrow \quad (U\hat{a}U \underset{\mathfrak{b}}{\otimes} 1)V = V(U\hat{a}U \underset{\mathfrak{b}^\dagger}{\otimes} 1) \quad \Leftrightarrow \quad [U\hat{a}U, \hat{A}] = 0,$$

$$(a \underset{\mathfrak{b}^\dagger}{\otimes} 1)\check{V} = \check{V}(a \underset{\mathfrak{b}}{\otimes} 1) \quad \Leftrightarrow \quad (1 \underset{\mathfrak{b}}{\otimes} UaU)V = V(1 \underset{\mathfrak{b}^\dagger}{\otimes} UaU) \quad \Leftrightarrow \quad [UaU, A] = 0.$$

These equivalent conditions hold for all $\hat{a} \in \hat{A}$ and $a \in A$ if and only if $V_{23}\hat{V}_{12} = \hat{V}_{12}V_{23}$ and $\check{V}_{23}V_{12} = V_{12}\check{V}_{23}$ in the sense that the following diagrams commute:

$$\begin{array}{ccc}
H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H & \xrightarrow{\hat{V}_{12}} & H {}_{\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\hat{\alpha}} H {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H \\
\downarrow{\scriptstyle V_{23}} & & \downarrow{\scriptstyle V_{23}} \\
H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H & \xrightarrow{\hat{V}_{12}} & H {}_{\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\hat{\alpha}} H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H,
\end{array}
\qquad
\begin{array}{ccc}
H {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H {}_{\hat{\alpha}}\underset{\mathfrak{b}}{\otimes}{}_{\hat{\beta}} H & \xrightarrow{V_{12}} & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H {}_{\hat{\alpha}}\underset{\mathfrak{b}}{\otimes}{}_{\hat{\beta}} H \\
\downarrow{\scriptstyle \check{V}_{23}} & & \downarrow{\scriptstyle \check{V}_{23}} \\
H {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H & \xrightarrow{V_{12}} & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H.
\end{array}$$

Proof. This is straightforward, for example, $V_{23}\hat{V}_{12} = \hat{V}_{12}V_{23}$ holds if and only if $\langle \xi'|_3 V_{23}\hat{V}_{12}|\xi\rangle_3 = \langle \xi'|_3 \hat{V}_{12}V_{23}|\xi\rangle_3$ for all $\xi \in \alpha, \xi' \in \beta$ and hence if and only if $(1 \underset{\mathfrak{b}^\dagger}{\otimes} \hat{a})\hat{V} = \hat{V}(1 \underset{\mathfrak{b}}{\otimes} \hat{a})$ for all $\hat{a} \in \hat{A}$. ☐

Definition 3.8. *We call (V, U) a* weak C^*-pseudo-Kac system *if V is well-behaved and if the equivalent conditions in Lemma 3.7 hold, and a* C^*-pseudo-Kac-system *if $V, \check{V}, \hat{V}$ are regular and $(\Sigma(1 \underset{\mathfrak{b}}{\otimes} U)V)^3 = \mathrm{id}$, where $\Sigma(1 \underset{\mathfrak{b}}{\otimes} U)V \colon H {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H \to H {}_{\hat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H$.*

Remark 3.9. In leg notation, the equation $\big(\Sigma(1 \underset{\mathfrak{b}}{\otimes} U)V\big)^3 = 1$ reads $(\Sigma U_2 V)^3 = 1$. Conjugating by Σ or V, we see that this condition is equivalent to the relation $(U_2 V \Sigma)^3 = 1$ and to the relation $(V\Sigma U_2)^3 = 1$.

Lemma 3.10. $(\Sigma U_2 V)^3 = 1$ *if and only if* $\hat{V} V \check{V} = U_1 \Sigma$.

Proof. Use the relation $U_1 U_2 (\Sigma U_2 V)^3 U_2 \Sigma = \Sigma U_1 V U_1 \Sigma \cdot V \cdot \Sigma U_2 V U_2 \Sigma = \hat{V} \cdot V \cdot \check{V}$. ☐

Proposition 3.11. *Every C^*-pseudo-Kac system is a weak C^*-pseudo-Kac system.*

Proof. Let (V, U) be a C^*-pseudo-Kac system. Then $V, \check{V}, \hat{V}$ are regular and therefore well-behaved. Using diagrams (4) and (7), we find

$$V_{12}\check{V}_{12}\Sigma_{12}V_{23} = V_{12}\check{V}_{12}V_{13}\Sigma_{12} = V_{12}V_{13}V_{23}\check{V}_{12}\Sigma_{12} = V_{23}V_{12}\check{V}_{12}\Sigma_{12}.$$

By Lemma 3.10, $V_{12}\check{V}_{12}\Sigma_{12} = \hat{V}^*_{12}U_1$ and hence $\hat{V}^*_{12}U_1V_{23} = V_{23}\hat{V}^*_{12}U_1$. Since $\hat{V}_{12}$ is unitary and $U_1V_{23} = V_{23}U_1$, we can conclude $\hat{V}_{12}V_{23} = V_{23}\hat{V}_{12}$. A similar argument shows that $\check{V}_{23}V_{12} = V_{12}\check{V}_{23}$. ☐

The following result is crucial for the duality presented in the next section.

Proposition 3.12. *Let (V, U) be a C^*-pseudo-Kac system. Then $[A\hat{A}] = [\hat{\alpha}\hat{\alpha}^*]$.*

Proof. The relation $[\widehat{\alpha}^*\widehat{A}] = \widehat{\alpha}^*$ (Remark 3.3 iii)), regularity of V, and the relations $V^* = \Sigma U_2 V \Sigma U_2 V \Sigma U_2$ and $[V|\alpha\rangle_2\widehat{A}] = [|\beta\rangle_2\widehat{A}]$ [28, Proposition 5.8] imply

$$\begin{aligned}
[\widehat{\alpha}\widehat{\alpha}^*] = [U\alpha\alpha^* U\widehat{A}] &= [U\langle\alpha|_2 V^*|\alpha\rangle_1 U\widehat{A}] \\
&= [U\langle\alpha|_2 \Sigma U_2 V \Sigma U_2 V \Sigma U_2|\alpha\rangle_1 U\widehat{A}] \\
&= [\langle\alpha|_1 V\Sigma U_2 V|\alpha\rangle_2\widehat{A}] \\
&= [\langle\alpha|_1 V\Sigma U_2|\beta\rangle_2\widehat{A}] = [\langle\alpha|_1 V|\widehat{\beta}\rangle_1\widehat{A}] = [A\widehat{A}].
\end{aligned}$$ □

Lemma 3.13. *Let (V,U) be a (weak) C^*-pseudo-Kac system. Then also $(\check{V},U)$, $(\widehat{V},U)$, and (V^{op},U) are (weak) C^*-pseudo-Kac systems.*

Proof. If (V,U) is a weak C^*-pseudo-Kac system, then the tuples above are balanced C^*-pseudo-multiplicative unitaries by Remark 3.3 i), and the remaining necessary conditions follow easily from Proposition 3.5 and equation (5).

If (V,U) is a C^*-pseudo-Kac system, then equation (6), the relation $\widecheck{(V^{op})} = U_1V^*U_1 = (\widehat{V})^{op}$, and the fact that V^{op} is regular, imply that the tuples above satisfy the regularity condition in Definition 3.8. To check that they also satisfy the second condition, we use Remark 3.9 and calculate $(\Sigma U_2\widehat{V})^3 = (V\Sigma U_2)^3 = 1$, $(\check{V}\Sigma U_2)^3 = (\Sigma U_2 V)^3 = 1$, $(U_2V^{op}\Sigma)^3 = (U_2\Sigma V^*)^3 = ((V\Sigma U_2)^3)^* = 1$. □

The C^*-pseudo-Kac system of a compact C^*-quantum groupoid In [26], we introduced compact C^*-quantum groupoids and associated to each such object a regular C^*-pseudo-multiplicative unitary V. We now recall this construction and define a symmetry U such that (V,U) is a C^*-pseudo-Kac system.

A *compact C^*-quantum graph* consists of a unital C^*-algebra B with a faithful KMS-state μ, a unital C^*-algebra A with unital embeddings $r\colon B \to A$ and $s\colon B^{op} \to A$ such that $[r(B), s(B^{op})] = 0$, and faithful conditional expectations $\phi\colon A \to r(B) \cong B$ and $\psi\colon A \to s(B^{op}) \cong B^{op}$ such that the compositions $\nu := \mu\circ\phi$ and $\nu^{-1} := \mu^{op}\circ\psi$ are KMS-states related by some positive invertible element $\delta \in A \cap r(B)' \cap s(B^{op})'$ via the formula $\nu^{-1}(a) = \nu(\delta^{1/2}a\delta^{1/2})$, valid for all $a \in A$. An *involution* for such a compact C^*-quantum graph is a $*$-antiisomorphism $R\colon A \to A$ such that $R\circ R = \mathrm{id}_A$, $R(r(b)) = s(b^{op})$ and $\phi(R(a)) = \psi(a)^{op}$ for all $b \in B$, $a \in A$.

Let (B,μ,A,r,s,ϕ,ψ) be a compact C^*-quantum graph with involution R. We denote by (H_μ,ζ_μ,J_μ) and (H_ν,ζ_ν,J_ν) the GNS-spaces, canonical cyclic vectors, and modular conjugations for the KMS-states μ and ν, respectively, and let $\zeta_{\nu^{-1}} = \delta^{1/2}\zeta_\nu$. As usual, we have representations $B^{op} \to \mathcal{L}(H_\mu)$, $b^{op} \mapsto J_\mu b^* J_\mu$, and $A^{op} \to \mathcal{L}(H_\nu)$, $a^{op} \mapsto J_\nu a^* J_\nu$. Using the isometries

$$\zeta_\phi\colon H_\mu \to H_\nu,\ b\zeta_\mu \mapsto r(b)\zeta_\nu, \qquad \zeta_\psi\colon H_{\mu^{op}} \to H_\nu,\ b^{op}\zeta_{\mu^{op}} \mapsto s(b^{op})\zeta_{\nu^{-1}},$$

we define subspaces $\widehat{\alpha},\widehat{\beta},\alpha,\beta \subseteq \mathcal{L}(H_\mu,H_\nu)$ by $\widehat{\alpha} := [A\zeta_\phi]$, $\widehat{\beta} := [A\zeta_\psi]$, $\beta := [A^{op}\zeta_\phi]$, $\beta := [A^{op}\zeta_\psi]$. Let $H = H_\nu$ and $\mathfrak{b} = (\mathfrak{K},\mathfrak{B},\mathfrak{B}^\dagger)$, where $\mathfrak{K} = H_\mu$, $\mathfrak{B} = B \subseteq \mathcal{L}(H_\mu)$, $\mathfrak{B}^\dagger = B^{op} \subseteq \mathcal{L}(H_\mu)$. Then $(H,\widehat{\alpha},\widehat{\beta},\alpha,\beta)$ is a C^*-$(\mathfrak{b},\mathfrak{b}^\dagger,\mathfrak{b},\mathfrak{b}^\dagger)$-module and $\mathcal{A} := A_H^{\beta,\alpha}$ a C^*-$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebra [26].

A *compact C^*-quantum groupoid* consists of a compact C^*-quantum graph with involution as above and a morphism $\mathcal{A} \to \mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{A}$ of C^*-$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebras such that

i) $(\Delta * \mathrm{id}) \circ \Delta = (\mathrm{id} * \Delta) \circ \Delta$ as maps from A to $\mathcal{L}(H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H)$;

ii) $\langle \zeta_\phi|_2 \Delta(a)|\zeta_\phi\rangle_2 = \rho_\beta(\phi(a))$ and $\langle \zeta_\psi|_1 \Delta(a)|\zeta_\psi\rangle_1 = \rho_\alpha(\psi(a))$ for all $a \in A$;

iii) $[\Delta(A)|\alpha\rangle_1] = [|\alpha\rangle_1 A] = [\Delta(A)|\zeta_\psi\rangle_1 A]$, $[\Delta(A)|\beta\rangle_2] = [|\beta\rangle_2 A] = [\Delta(A)|\zeta_\phi\rangle_2 A]$;

iv) $R(\langle\zeta_\psi|_1\Delta(a)(d^{op} \underset{\mathfrak{b}}{\otimes} 1)|\zeta_\psi\rangle_1) = \langle\zeta_\psi|_1(a^{op} \underset{\mathfrak{b}}{\otimes} 1)\Delta(d)|\zeta_\psi\rangle_1$ for all $a, d \in A$.

By [26, Theorem 5.4], there exists a unique regular C^*-pseudo-multiplicative unitary $V \colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \to H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H$ such that $V|a\zeta_\psi\rangle_1 = \Delta(a)|\zeta_\psi\rangle_1$ for all $a \in A$. Denote by $J = J_\nu$ the modular conjugation for ν as above, by $I \colon H \to H$ the antiunitary given by $Ia\zeta_{\nu^{-1}} = R(a)^*\zeta_\nu$ for all $a \in A$, and let $U = IJ \in \mathcal{L}(H)$.

Proposition 3.14. (V, U) *is a C^*-pseudo-Kac system.*

Proof. First, $U^2 = IJIJ = IJJI = II = \mathrm{id}_H$ because $IJ = JI$, and $U\zeta_\phi = \zeta_\psi$, $U\zeta_\nu = \zeta_{\nu^{-1}}$, $U\widehat{\alpha} = I\beta = \alpha$, $U\widehat{\beta} = I\alpha = \beta$ by [26, Lemma 2.7, Proposition 3.8]. The relation $(J_{\alpha}\underset{J_\mu}{\otimes}{}_{\beta}I)V(J_{\alpha}\underset{J_\mu}{\otimes}{}_{\beta}I) = V^*$ [26, Theorem 5.6] implies

$$\begin{aligned}\check{V} = \Sigma(1 \underset{\mathfrak{b}}{\otimes} JI)V(1 \underset{\mathfrak{b}^\dagger}{\otimes} JI)\Sigma &= (J_{\alpha}\underset{J_\mu}{\otimes}{}_{\widehat{\beta}}J)\Sigma(J_{\alpha}\underset{J_\mu}{\otimes}{}_{\beta}I)V(J_{\alpha}\underset{J_\mu}{\otimes}{}_{\beta}I)\Sigma(J_{\widehat{\alpha}}\underset{J_\mu}{\otimes}{}_{\widehat{\beta}}J) \\ &= (J_{\alpha}\underset{J_\mu}{\otimes}{}_{\widehat{\beta}}J)\Sigma V^*\Sigma(J_{\widehat{\alpha}}\underset{J_\mu}{\otimes}{}_{\widehat{\beta}}J) = (J_{\alpha}\underset{J_\mu}{\otimes}{}_{\widehat{\beta}}J)V^{op}(J_{\widehat{\alpha}}\underset{J_\mu}{\otimes}{}_{\widehat{\beta}}J).\end{aligned}$$

Since V^{op} is a regular C^*-pseudo-multiplicative unitary, so is $\check{V}$. In particular, (V, U) is a balanced C^*-pseudo-multiplicative unitary. We show that $\widehat{V}V = U_1\Sigma\check{V}^*$, and then the claim follows from Lemma 3.10. Let $a, b \in A$ and $\omega = \widehat{V}V(a\zeta_\psi \ogreaterthan Ub\zeta_{\nu^{-1}})$. Since $\Delta(a) = \widehat{V}^*(1 \underset{\mathfrak{b}^\dagger}{\otimes} a)\widehat{V}$ by Proposition 3.5,

$$\begin{aligned}\omega = \widehat{V}\Delta(a)(\zeta_\psi \ogreaterthan Ub\zeta_{\nu^{-1}}) &= (1 \underset{\mathfrak{b}^\dagger}{\otimes} a)\widehat{V}(\zeta_\psi \ogreaterthan Ub\zeta_{\nu^{-1}}) \\ &= \Sigma(U \underset{\mathfrak{b}}{\otimes} 1)(UaU \underset{\mathfrak{b}}{\otimes} 1)V(b\zeta_{\nu^{-1}} \olessthan \zeta_\psi) \\ &= \Sigma(U \underset{\mathfrak{b}}{\otimes} 1)(UaU \underset{\mathfrak{b}}{\otimes} 1)\Delta(b)(\zeta_\psi \ogreaterthan \zeta_{\nu^{-1}}).\end{aligned}$$

Using the relations $UaU = JIaIJ = R(a)^{op}$ and $[UaU\underset{\mathfrak{b}}{\otimes}1, \Delta(b)] \in [A^{op}\underset{\mathfrak{b}}{\otimes}1, A_{\alpha}\underset{\mathfrak{b}}{*}{}_{\beta}A] = 0$, we find

$$\omega = \Sigma(U \underset{\mathfrak{b}}{\otimes} 1)\Delta(b)(UaU\zeta_\psi \ogreaterthan \zeta_{\nu^{-1}}) = (U \underset{\mathfrak{b}^\dagger}{\otimes} 1)\Sigma(U \underset{\mathfrak{b}}{\otimes} U)\Delta(b)(U \underset{\mathfrak{b}}{\otimes} U)(aU\zeta_\psi \ogreaterthan \zeta_\nu).$$

Since $\check{V}^*(1 \underset{\mathfrak{b}^\dagger}{\otimes} UbU)\check{V} = (U \underset{\mathfrak{b}}{\otimes} U)\Delta(b)(U \underset{\mathfrak{b}}{\otimes} U)$ by Proposition 3.5, we obtain

$$\begin{aligned}\omega &= (U \underset{\mathfrak{b}^\dagger}{\otimes} 1)\Sigma\check{V}^*(1 \underset{\mathfrak{b}^\dagger}{\otimes} UbU)\check{V}(aU\zeta_\psi \ogreaterthan \zeta_\nu) \\ &= (U \underset{\mathfrak{b}^\dagger}{\otimes} 1)\Sigma\check{V}^*(1 \underset{\mathfrak{b}^\dagger}{\otimes} UbU)\Sigma(1 \underset{\mathfrak{b}}{\otimes} U)V(\zeta_\nu \olessthan UaU\zeta_\psi).\end{aligned}$$

By Proposition [26, Proposition 5.5], $V(\zeta_\nu \olessthan UaU\zeta_\psi) = \zeta_\nu \olessthan UaU\zeta_\phi$, whence

$$\omega = (U \underset{\mathfrak{b}^\dagger}{\otimes} 1)\Sigma\check{V}^*(1 \underset{\mathfrak{b}^\dagger}{\otimes} UbU)(aU\zeta_\phi \ogreaterthan \zeta_\nu) = (U \underset{\mathfrak{b}^\dagger}{\otimes} 1)\Sigma\check{V}^*(a\zeta_\psi \ogreaterthan Ub\zeta_{\nu^{-1}}). \qquad \square$$

4 Reduced crossed products and duality

Let (V, U) be a weak C^*-pseudo-Kac system and let $(\mathcal{A}, \Delta)$, $(\widehat{\mathcal{A}}, \widehat{\Delta})$ be the Hopf C^*-bimodules associated to V as in the preceding section. Generalizing the corresponding constructions and results for coactions of Hopf C^*-algebras [3], we now associate to every coaction of one of these Hopf C^*-bimodules a reduced crossed product that carries a dual coaction of the other Hopf C^*-bimodule, and prove a duality theorem concerning the iteration of this construction.

Reduced crossed products for coactions of $(\mathcal{A}, \Delta)$ Let δ be a coaction of the Hopf C^*-bimodule $(\mathcal{A}, \Delta)$ on a C^*-$\mathfrak{b}$-algebra $\mathcal{C} = C_K^\gamma$ and let[2]

$$C \rtimes_r \widehat{A} := [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})] \subseteq \mathcal{L}(K {}_\gamma{\underset{\mathfrak{b}}{\otimes}}_\beta H), \qquad \mathcal{C} \rtimes_r \widehat{\mathcal{A}} := (K {}_\gamma{\underset{\mathfrak{b}}{\otimes}}_\beta H_{\widehat{\beta}}, C \rtimes_r \widehat{A}).$$

Proposition 4.1. *i)* $[\delta(C)(\gamma \triangleright \widehat{\beta})] \subseteq \gamma \triangleright \widehat{\beta}$ *with equality if* δ *is left-full.*

ii) $C \rtimes_r \widehat{A}$ *is a* C^**-algebra and* $\mathcal{C} \rtimes_r \widehat{\mathcal{A}}$ *is a* C^*-$\mathfrak{b}^\dagger$*-algebra.*

iii) There exist nondegenerate $*$*-homomorphisms* $C \to M(C \rtimes_r \widehat{A})$ *and* $\widehat{A} \to M(C \rtimes_r \widehat{A})$*, given by* $c \mapsto \delta(c)$ *and* $\widehat{a} \mapsto 1 \underset{\mathfrak{b}}{\otimes} \widehat{a}$*, respectively.*

Proof. i) The relation $\widehat{\beta} = [A\widehat{\beta}]$ [28, Proposition 3.11] implies that $[\delta(C)|\gamma\rangle_1\widehat{\beta}] = [\delta(C)|\gamma\rangle_1 A\widehat{\beta}] \subseteq [|\gamma\rangle_1 A\widehat{\beta}] = [|\gamma\rangle_1\widehat{\beta}]$.

ii) We first show that $[(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})\delta(C)] \subseteq [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})]$. Let $\delta^{(2)} := (\mathrm{id} * \Delta) \circ \delta = (\delta * \mathrm{id}) \circ \delta \colon C \to \mathcal{L}(K {}_\gamma{\underset{\mathfrak{b}}{\otimes}}_\beta H {}_\gamma{\underset{\mathfrak{b}}{\otimes}}_\beta H)$. By definition of $\widehat{A}$ and Δ,

$$\begin{aligned}
[(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})\delta(C)] = [\langle\beta|_3(1 \underset{\mathfrak{b}}{\otimes} V)|\alpha\rangle_3\delta(C)] &= [\langle\beta|_3(1 \underset{\mathfrak{b}}{\otimes} V)(\delta(C) \underset{\mathfrak{b}^\dagger}{\otimes} 1)|\alpha\rangle_3] \\
&= [\langle\beta|_3\delta^{(2)}(C)(1 \underset{\mathfrak{b}}{\otimes} V)|\alpha\rangle_3] \\
&\subseteq [\delta(C)\langle\beta|_3(1 \underset{\mathfrak{b}}{\otimes} V)|\alpha\rangle_3] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})].
\end{aligned}$$

Consequently, $C \rtimes_r \widehat{A}$ is a C^*-algebra. By [28, Proposition 3.11], $[\widehat{A}\rho_{\widehat{\beta}}(\mathfrak{B})] = \widehat{A}$, and hence $[(C \rtimes_r \widehat{A})\rho_{(\gamma \triangleright \widehat{\beta})}(\mathfrak{B})] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A}\rho_{\widehat{\beta}}(\mathfrak{B}))] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})] = C \rtimes_r \widehat{A}$.

iii) Immediate. □

Theorem 4.2. *There exists a unique coaction* $\widehat{\delta}$ *of* $(\widehat{\mathcal{A}}, \widehat{\Delta})$ *on* $\mathcal{C} \rtimes_r \widehat{\mathcal{A}}$ *such that* $\widehat{\delta}(\delta(c)(1 \underset{\mathfrak{b}}{\otimes} \widehat{a})) = (\delta(c) \underset{\mathfrak{b}^\dagger}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\widehat{a}))$ *for all* $c \in C$, $\widehat{a} \in \widehat{A}$*. If* $\widehat{\Delta}$ *is a fine coaction, then* $\widehat{\delta}$ *is a very fine coaction. If* δ *is left-full, then* $\widehat{\delta}$ *is left-full.*

Proof. Define $\widehat{\delta} \colon C \rtimes_r \widehat{A} \to \mathcal{L}(K {}_\gamma{\underset{\mathfrak{b}}{\otimes}}_\beta H_{\widehat{\beta}} {\underset{\mathfrak{b}^\dagger}{\otimes}}_\alpha H)$ by $x \mapsto (1 \underset{\mathfrak{b}}{\otimes} \check{V})(x \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \check{V}^*)$. Then $\widehat{\delta}$ is injective and satisfies $\widehat{\delta}(\delta(c)(1 \underset{\mathfrak{b}}{\otimes} \widehat{a})) = (\delta(c) \underset{\mathfrak{b}^\dagger}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\widehat{a}))$ for all $c \in C$, $\widehat{a} \in \widehat{A}$ because $\check{V}(\widehat{a} \underset{\mathfrak{b}}{\otimes} 1)\check{V}^* = \widehat{\Delta}(\widehat{a})$ by Proposition 3.5 and $(1 \underset{\mathfrak{b}}{\otimes} \check{V})\delta(c)(1 \underset{\mathfrak{b}}{\otimes} \check{V}^*) = \delta(c)$ as

[2] The notation $C \rtimes_r \widehat{A}$ is consistent with [3] but not with [10], where $C \rtimes_r A$ is used instead.

a consequence of the relation $\check{V}(a \underset{\mathfrak{b}}{\otimes} 1)\check{V}^* = a \underset{\mathfrak{b}^\dagger}{\otimes} 1$. We show that $\hat{\delta}$ is a coaction of $(\hat{\mathcal{A}}, \hat{\Delta})$. First, $[\hat{\delta}(C \rtimes_r \hat{A})|\alpha\rangle_3] \subseteq [|\alpha\rangle_3(C \rtimes_r \hat{A})]$ because

$$[(\delta(C) \underset{\mathfrak{b}^\dagger}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \hat{\Delta}(\hat{A}))|\alpha\rangle_3] \subseteq [(\delta(C) \underset{\mathfrak{b}^\dagger}{\otimes} 1)|\alpha\rangle_3(1 \underset{\mathfrak{b}}{\otimes} \hat{A})] = [|\alpha\rangle_3\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \hat{A})]. \quad (9)$$

Next, $[\hat{\delta}(C \rtimes_r \hat{A})|\gamma \triangleright \hat{\beta}\rangle_1\hat{A}] \subseteq [|\gamma \triangleright \hat{\beta}\rangle_1\hat{A}]$ because by Proposition 4.1 i),

$$\begin{aligned}[(1 \underset{\mathfrak{b}}{\otimes} \hat{\Delta}(\hat{A}))(\delta(C) \underset{\mathfrak{b}^\dagger}{\otimes} 1)|\gamma \triangleright \hat{\beta}\rangle_1\hat{A}] &\subseteq [(1 \underset{\mathfrak{b}}{\otimes} \hat{\Delta}(\hat{A}))|\gamma \triangleright \hat{\beta}\rangle_1\hat{A}] \\ &= [|\gamma\rangle_1\hat{\Delta}(\hat{A})|\hat{\beta}\rangle_1\hat{A}] \subseteq [|\gamma\rangle_1|\hat{\beta}\rangle_1\hat{A}]. \quad (10)\end{aligned}$$

Furthermore, $\hat{\delta}(x)(1 \underset{\mathfrak{b}}{\otimes} \check{V})|\xi\rangle_3 = (1 \underset{\mathfrak{b}}{\otimes} \check{V})|\xi\rangle_3 x$ for each $x \in C \rtimes_r \hat{A}$, $\xi \in \hat{\beta}$, and by Remark 3.3 ii), $[(1 \underset{\mathfrak{b}}{\otimes} \check{V})|\hat{\beta}\rangle_3(\gamma \triangleright \hat{\beta})] = \gamma \triangleright \hat{\beta} \triangleright \hat{\beta}$ and $[\langle\hat{\beta}|_3(1 \underset{\mathfrak{b}}{\otimes} \check{V})^*(\gamma \triangleright \hat{\beta} \triangleright \hat{\beta})] = \gamma \triangleright \hat{\beta}$. The maps $(\hat{\delta} * \mathrm{id}) \circ \hat{\delta}$ and $(\mathrm{id} * \hat{\Delta}) \circ \hat{\delta}$ from $C \rtimes_r \hat{A}$ to $\mathcal{L}\big(K_\gamma \underset{\mathfrak{b}}{\otimes}_\beta H_{\hat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}_\alpha H_{\hat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}_\alpha H\big)$ are given by $\delta(c)(1 \underset{\mathfrak{b}}{\otimes} \hat{a}) \mapsto \big(\delta(c) \underset{\mathfrak{b}^\dagger}{\otimes} 1 \underset{\mathfrak{b}^\dagger}{\otimes} 1\big)\big(1 \underset{\mathfrak{b}}{\otimes} \hat{\Delta}^{(2)}(\hat{a})\big)$ for all $c \in C, \hat{a} \in \hat{A}$, where $\hat{\Delta}^{(2)} := (\hat{\Delta} * \mathrm{id}) \circ \hat{\Delta} = (\mathrm{id} * \hat{\Delta}) \circ \hat{\Delta}$. Thus, $(\mathcal{C} \rtimes_r \hat{\mathcal{A}}, \hat{\delta})$ is a coaction of $(\hat{\mathcal{A}}, \hat{\Delta})$. If the coactions $\hat{\Delta}$ is fine, then the inclusion (9) is an equality and in any case $[\langle\hat{\beta}|_3(1 \underset{\mathfrak{b}}{\otimes} \check{V})^*(\gamma \triangleright \hat{\beta} \triangleright \hat{\beta})] = \gamma \triangleright \hat{\beta}$, whence $\hat{\delta}$ will be very fine. If δ is left-full, then the inclusion (10) is an equality by Proposition 4.1 i) and hence $\hat{\delta}$ is left-full. ☐

Definition 4.3. *We call $\mathcal{C} \rtimes_r \hat{\mathcal{A}}$ the* reduced crossed product *and $(\mathcal{C} \rtimes_r \hat{\mathcal{A}}, \hat{\delta})$ the* reduced dual coaction *of $(\mathcal{C}, \delta)$.*

The construction of reduced dual coactions is functorial in the following sense:

Proposition 4.4. *Let ρ be a morphism between coactions $(\mathcal{C}, \delta_{\mathcal{C}})$ and $(\mathcal{D}, \delta_{\mathcal{D}})$ of $(\mathcal{A}, \Delta)$. Then there exists a unique morphism $\rho \rtimes_r \mathrm{id}$ from $(\mathcal{C} \rtimes_r \hat{\mathcal{A}}, \hat{\delta}_{\mathcal{C}})$ to $(\mathcal{D} \rtimes_r \hat{\mathcal{A}}, \hat{\delta}_{\mathcal{D}})$ such that $(\rho \rtimes_r \mathrm{id})\big((1 \underset{\mathfrak{b}}{\otimes} \hat{a})\delta_{\mathcal{C}}(c)\big) \cdot \delta_{\mathcal{D}}(d)(1 \underset{\mathfrak{b}}{\otimes} \hat{a}') = (1 \underset{\mathfrak{b}}{\otimes} \hat{a})\delta_{\mathcal{D}}(\rho(c)d)(1 \underset{\mathfrak{b}}{\otimes} \hat{a}')$ for all $c \in C$, $d \in D$, $\hat{a}, \hat{a}' \in \hat{A}$.*

Proof. The semi-morphism $\mathrm{Ind}_{|\hat{\beta}\rangle_2}(\rho)$ of Lemma 2.1 restricts to a semi-morphism $\rho \rtimes_r \mathrm{id}$ from $\mathcal{C} \rtimes_r \hat{\mathcal{A}}$ to $M(\mathcal{D} \rtimes_r \hat{\mathcal{A}})$ which satisfies the formula given above, and this formula implies that $\rho \rtimes_r \mathrm{id}$ is a morphism of coactions as claimed. ☐

Corollary 4.5. *There exists a functor $- \rtimes_r \hat{\mathcal{A}}$: $\mathbf{Coact}_{(\mathcal{A},\Delta)} \to \mathbf{Coact}_{(\hat{\mathcal{A}},\hat{\Delta})}$ given by $(\mathcal{C}, \delta) \mapsto (\mathcal{C} \rtimes_r \hat{\mathcal{A}}, \hat{\delta})$ and $\rho \mapsto \rho \rtimes_r \mathrm{id}$.* ☐

Reduced crossed products for coactions of $(\hat{\mathcal{A}}, \hat{\Delta})$ The construction in the preceding paragraph carries over to coactions of the Hopf C^*-bimodule $(\hat{\mathcal{A}}, \hat{\Delta})$ as follows. Let δ be a coaction of $(\hat{\mathcal{A}}, \hat{\Delta})$ on a C^*-$\mathfrak{b}^\dagger$-algebra $\mathcal{C} = C_K^\gamma$ and let

$$C \rtimes_r A := [\delta(C)(1 \underset{\mathfrak{b}^\dagger}{\otimes} UAU)] \subseteq \mathcal{L}(K_\gamma \underset{\mathfrak{b}^\dagger}{\otimes}_\alpha H), \quad \mathcal{C} \rtimes_r \mathcal{A} = (K_\gamma \underset{\mathfrak{b}^\dagger}{\otimes}_\alpha H_{\hat{\alpha}}, C \rtimes_r A).$$

Using straightforward modifications of the preceding proofs, one shows:

Proposition 4.6. *i)* $[\delta(C)(\gamma \triangleright \widehat{\alpha})] \subseteq \gamma \triangleright \widehat{\alpha}$ *with equality if* δ *is fine.*

ii) $C \rtimes_r A$ *is a* C^**-algebra and* $\mathcal{C} \rtimes_r \mathcal{A}$ *is a* C^*-$\mathfrak{b}$*-algebra.*

iii) There exist nondegenerate $*$*-homomorphisms* $C \to M(C \rtimes_r A)$ *and* $A \to M(C \rtimes_r A)$*, given by* $c \mapsto \delta(c)$ *and* $a \mapsto 1 \underset{\mathfrak{b}^\dagger}{\otimes} a$*, respectively.* ☐

Theorem 4.7. *There exists a unique coaction* $(\mathcal{C} \rtimes_r \mathcal{A}, \widehat{\delta})$ *of* $(\mathcal{A}, \Delta)$ *such that* $\widehat{\delta}(\delta(c)(1 \underset{\mathfrak{b}^\dagger}{\otimes} UaU)) = (\delta(c) \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{Ad}_{(U \underset{\mathfrak{b}}{\otimes} 1)} \Delta(a))$ *for all* $c \in C$*,* $a \in A$*. If* Δ *is a fine coaction, then* $\widehat{\delta}$ *is a very fine coaction. If* δ *is left-full, then* $\widehat{\delta}$ *is left-full.* ☐

Definition 4.8. *Let* $(\mathcal{C}, \delta)$ *be a coaction of* $(\widehat{\mathcal{A}}, \widehat{\Delta})$*. Then we call* $\mathcal{C} \rtimes_r \mathcal{A}$ *the* reduced crossed product *and* $(\mathcal{C} \rtimes_r \mathcal{A}, \widehat{\delta})$ *the* reduced dual coaction *of* $(\mathcal{C}, \delta)$*.*

Proposition 4.9. *Let* ρ *be a morphism between coactions* $(\mathcal{C}, \delta_{\mathcal{C}})$ *and* $(\mathcal{D}, \delta_{\mathcal{D}})$ *of* $(\widehat{\mathcal{A}}, \widehat{\Delta})$*. Then there exists a unique morphism* $\rho \rtimes_r \mathrm{id}$ *from* $(\mathcal{C} \rtimes_r \mathcal{A}, \widehat{\delta_{\mathcal{C}}})$ *to* $(\mathcal{D} \rtimes_r \mathcal{A}, \widehat{\delta_{\mathcal{D}}})$ *such that* $(\rho \rtimes_r \mathrm{id})\big((1 \underset{\mathfrak{b}^\dagger}{\otimes} UaU)\delta_{\mathcal{C}}(c)\big) \cdot \delta_{\mathcal{D}}(d)(1 \underset{\mathfrak{b}^\dagger}{\otimes} Ua'U) = (1 \underset{\mathfrak{b}^\dagger}{\otimes} UaU)\delta_{\mathcal{D}}(\rho(c)d)(1 \underset{\mathfrak{b}^\dagger}{\otimes} Ua'U)$ *for all* $c \in C$*,* $d \in D$*,* $a, a' \in A$*.* ☐

Corollary 4.10. *There exists a functor* $- \rtimes_r \mathcal{A}$: $\mathbf{Coact}_{(\widehat{\mathcal{A}}, \widehat{\Delta})} \to \mathbf{Coact}_{(\mathcal{A}, \Delta)}$ *given by* $(\mathcal{C}, \delta) \mapsto (\mathcal{C} \rtimes_r \mathcal{A}, \widehat{\delta})$ *and* $\rho \mapsto \rho \rtimes_r \mathrm{id}$*.* ☐

The duality theorem The preceding constructions yield for each coaction $(\mathcal{C}, \delta_{\mathcal{C}})$ of $(\mathcal{A}, \Delta)$ and each coaction $(\mathcal{D}, \delta_{\mathcal{D}})$ of $(\widehat{\mathcal{A}}, \widehat{\Delta})$ a bidual $(\mathcal{C} \rtimes_r \widehat{\mathcal{A}} \rtimes_r \mathcal{A}, \widehat{\widehat{\delta}}_{\mathcal{C}})$ and $(\mathcal{D} \rtimes_r \mathcal{A} \rtimes_r \widehat{\mathcal{A}}, \widehat{\widehat{\delta}}_{\mathcal{D}})$, respectively. The following generalization of the Baaj-Skandalis duality theorem [3] identifies these biduals in the case where (V, U) is a C^*-pseudo-Kac system and the initial coactions are fine. Morally, it says that up to Morita equivalence, the functors $- \rtimes_r \widehat{\mathcal{A}}$ and $- \rtimes_r \mathcal{A}$ implement an equivalence of the categories $\mathbf{Coact}^f_{(\mathcal{A}, \Delta)}$ and $\mathbf{Coact}^f_{(\widehat{\mathcal{A}}, \widehat{\Delta})}$.

Theorem 4.11. *Assume that* (V, U) *is a* C^**-pseudo-Kac system.*

i) Let $(\mathcal{C}, \delta)$ *be a (very) fine coaction of* $(\mathcal{A}, \Delta)$*, where* $\mathcal{C} = C^\gamma_K$*. Then there exists an isomorphism* $\Phi\colon C \rtimes_r \widehat{A} \rtimes_r A \to [|\beta\rangle_2 C\langle\beta|_2] \subseteq \mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes}_\beta H)$ *such that* Φ^{-1} *is an (iso)morphism from* $(K_\gamma \underset{\mathfrak{b}}{\otimes}_\beta H_{\widehat{\alpha}}, [|\beta\rangle_2 C\langle\beta|_2])$ *to* $\mathcal{C} \rtimes_r \widehat{\mathcal{A}} \rtimes_r \mathcal{A}$ *and* $\widehat{\widehat{\delta}} \circ \Phi^{-1} = (\Phi^{-1} * \mathrm{id}) \circ \mathrm{Ad}_{(1 \underset{\mathfrak{b}}{\otimes} \Sigma\widehat{V})} \circ \mathrm{Ind}_{|\beta\rangle_2}(\delta)$*.*

ii) Let $(\mathcal{D}, \delta)$ *be a (very) fine coaction of* $(\widehat{\mathcal{A}}, \widehat{\Delta})$*, where* $\mathcal{D} = D^\epsilon_L$*. Then there exists an isomorphism* $\Phi\colon D \rtimes_r A \rtimes_r \widehat{A} \cong [|\alpha\rangle_2 D\langle\alpha|_2] \subseteq \mathcal{L}(L_\epsilon \underset{\mathfrak{b}^\dagger}{\otimes}_\alpha H)$ *such that* Φ^{-1} *is an (iso)morphism from* $(L_\epsilon \underset{\mathfrak{b}^\dagger}{\otimes}_\alpha H_{\widehat{\beta}}, [|\alpha\rangle_2 D\langle\alpha|_2])$ *to* $\mathcal{D} \rtimes_r \mathcal{A} \rtimes_r \widehat{\mathcal{A}}$ *and* $\widehat{\widehat{\delta}} \circ \Phi^{-1} = (\Phi^{-1} * \mathrm{id}) \circ \mathrm{Ad}_{(1 \underset{\mathfrak{b}^\dagger}{\otimes} \Sigma V)} \circ \mathrm{Ind}_{|\alpha\rangle_2}(\delta)$*.*

Proof. We only prove i); assertion ii) follows similarly after replacing (V, U) by $(\check{V}, U)$. By Proposition 3.5 and Proposition 3.12, applied to the C^*-pseudo-Kac system $(\check{V}, U)$, we have $[\widehat{A}\,\mathrm{Ad}_U(A)] = [A_{\check{V}}\widehat{A}_{\check{V}}] = [\beta\beta^*]$, and since δ is fine,

$$[|\beta\rangle_2 C\langle\beta|_2] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \beta\beta^*)] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A}\,\mathrm{Ad}_U(A))].$$

One easily verifies that the $*$-homomorphism $\mathrm{Ind}_{|\beta\rangle_2}(\delta)$ (see Lemma 2.1) yields an (iso)morphism of C^*-$\mathfrak{b}$-algebras

$$\mathrm{Ind}_{|\beta\rangle_2}(\delta)\colon \big(K_\gamma \underset{\mathfrak{b}}{\otimes}_\beta H_{\widehat{\alpha}}, [|\beta\rangle_2 C\langle\beta|_2]\big) \to \big(K_\gamma \underset{\mathfrak{b}}{\otimes}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes}_\beta H_{\widehat{\alpha}}, [|\beta\rangle_2 \delta(C)\langle\beta|_2]\big).$$

Denote by Ψ the composition of this (iso)morphism with the isomorphism $\mathrm{Ad}_{(1 \underset{\mathfrak{b}}{\otimes} V^*)}$ and let $\delta^{(2)} = (\delta * \mathrm{id}) \circ \delta = (\mathrm{id} * \Delta) \circ \delta$. Let $x = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} \widehat{a}UaU) \in [|\beta\rangle_2 C\langle\beta|_2]$, where $c \in C, \widehat{a} \in \widehat{A}, a \in A$. By Lemma 3.7,

$$\Psi(x) = \mathrm{Ad}_{(1 \underset{\mathfrak{b}}{\otimes} V^*)}(\delta^{(2)}(c)(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}}{\otimes} \widehat{a}UaU)) = (\delta(c) \underset{\mathfrak{b}^\dagger}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\widehat{a}))(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}^\dagger}{\otimes} UaU).$$

Consequently, $\Psi([|\beta\rangle_2 C\langle\beta|_2]) = C \rtimes_r \widehat{A} \rtimes_r A$. The relations $\mathcal{C} \rtimes_r \widehat{\mathcal{A}} \rtimes_r \mathcal{A} = (K_\gamma \underset{\mathfrak{b}}{\otimes}_\beta H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}_\alpha H_{\widehat{\alpha}}, C \rtimes_r \widehat{A} \rtimes_r A)$ and $(1 \underset{\mathfrak{b}}{\otimes} V^*)(\gamma \triangleright \alpha \triangleright \widehat{\alpha}) = \gamma \triangleright \widehat{\beta} \triangleright \widehat{\alpha}$ imply that Ψ is a morphism of C^*-$\mathfrak{b}$-algebras as claimed. Using the definition of $\widehat{\widehat{\delta}}$, Proposition 3.5, and Lemma 3.7, we find

$$\begin{aligned}
\widehat{\widehat{\delta}}(\Psi(x)) &= (\delta(c) \underset{\mathfrak{b}^\dagger}{\otimes} 1 \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\hat{a}) \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{Ad}_{(U \underset{\mathfrak{b}}{\otimes} 1)}(\Delta(a))) \\
&= \mathrm{Ad}_{(1 \underset{\mathfrak{b}}{\otimes} V^* \underset{\mathfrak{b}}{\otimes} 1)}\big((\delta^{(2)}(c) \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}}{\otimes} \widehat{a} \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}}{\otimes} \mathrm{Ad}_{(U \underset{\mathfrak{b}}{\otimes} 1)}(\Delta(a)))\big) \\
&= (\Psi * \mathrm{id})\big((\delta(c) \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \widehat{a} \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \mathrm{Ad}_{(U \underset{\mathfrak{b}}{\otimes} 1)}(\Delta(a)))\big) \\
&= (\Psi * \mathrm{id})\big((1 \underset{\mathfrak{b}}{\otimes} \Sigma\widehat{V})\delta^{(2)}(c)(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}}{\otimes} \widehat{a}UaU)(1 \underset{\mathfrak{b}}{\otimes} \widehat{V}^*\Sigma)\big) \\
&= (\Psi * \mathrm{id})\big((1 \underset{\mathfrak{b}}{\otimes} \Sigma\widehat{V})(\mathrm{Ind}_{|\beta\rangle_2}(\delta)(x))(1 \underset{\mathfrak{b}}{\otimes} \widehat{V}^*\Sigma)\big).
\end{aligned}$$

□

5 The C^*-pseudo-Kac system of a groupoid

For the remainder of this article, we fix a locally compact, Hausdorff, second countable groupoid G with a left Haar system λ. In [28], we associated to such a groupoid a regular C^*-pseudo-multiplicative unitary V and identified the underlying C^*-algebras of the Hopf C^*-bimodules $(\widehat{\mathcal{A}}, \widehat{\Delta})$ and $(\mathcal{A}, \Delta)$ of V with the function algebra $C_0(G)$ and the reduced groupoid C^*-algebra $C_r^*(G)$, respectively. We now recall this construction and define a symmetry U such that (V, U) becomes a C^*-pseudo-Kac system. For background on groupoids, see [19, 22].

Denote by λ^{-1} the right Haar system associated to λ and let μ be a measure on the unit space G^0 with full support. We denote the range and the source map of G by r and s, respectively, let $G^u := r^{-1}(u)$ and $G_u := s^{-1}(u)$ for each $u \in G^0$, and define measures ν, ν^{-1} on G such that

$$\int_G f\,\mathrm{d}\nu = \int_{G^0}\int_{G^u} f(x)\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u), \quad \int_G f d\nu^{-1} = \int_{G^0}\int_{G_u} f(x)\,\mathrm{d}\lambda_u^{-1}(x)\,\mathrm{d}\mu(u)$$

for all $f \in C_c(G)$. We assume that μ is quasi-invariant in the sense that ν and ν^{-1} are equivalent, and denote by $D := \mathrm{d}\nu/\mathrm{d}\nu^{-1}$ the Radon-Nikodym derivative. One can choose D such that it is a Borel homomorphism, see [19, p. 89], and we do so.

We identify functions in $C_b(G^0)$ and $C_b(G)$ with multiplication operators on the Hilbert spaces $L^2(G^0,\mu)$ and $L^2(G,\nu)$, respectively, and let $\mathfrak{K} = L^2(G^0,\mu)$, $\mathfrak{B} = \mathfrak{B}^\dagger = C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K})$, $\mathfrak{b} = (\mathfrak{K},\mathfrak{B},\mathfrak{B}^\dagger) = \mathfrak{b}$, $H = L^2(G,\nu)$.

Pulling functions on G^0 back to G along r or s, we obtain representations $r^*\colon C_0(G^0) \to C_b(G) \hookrightarrow \mathcal{L}(H)$ and $s^*\colon C_0(G^0) \to C_b(G) \hookrightarrow \mathcal{L}(H)$. We define Hilbert C^*-$C_0(G^0)$-modules $L^2(G,\lambda)$ and $L^2(G,\lambda^{-1})$ as the respective completions of the pre-C^*-module $C_c(G)$, the structure maps being given by

$$\begin{aligned}
\langle\xi'|\xi\rangle(u) &= \int_{G^u} \overline{\xi'(x)}\xi(x)\,\mathrm{d}\lambda^u(x), \qquad \xi f = r^*(f)\xi \quad \text{in the case of } L^2(G,\lambda),\\
\langle\xi'|\xi\rangle(u) &= \int_{G_u} \overline{\xi'(x)}\xi(x)\,\mathrm{d}\lambda^{-1}_u(x), \quad\ \xi f = s^*(f)\xi \quad \text{in the case of } L^2(G,\lambda^{-1})
\end{aligned}$$

respectively, for all $\xi,\xi' \in C_c(G)$, $u \in G^0$, $f \in C_0(G^0)$. Then there exist isometric embeddings $j\colon L^2(G,\lambda) \to \mathcal{L}(\mathfrak{K},H)$ and $\hat{j}\colon L^2(G,\lambda^{-1}) \to \mathcal{L}\big(\mathfrak{K},H\big)$ such that

$$\big(j(\xi)\zeta\big)(x) = \xi(x)\zeta(r(x)), \qquad \big(\hat{j}(\xi)\zeta\big)(x) = \xi(x)D^{-1/2}(x)\zeta(s(x))$$

for all $\xi \in C_c(G)$, $\zeta \in C_c(G^0)$. Let $\alpha = \beta := j(L^2(G,\lambda))$ and $\widehat{\alpha} = \widehat{\beta} := \hat{j}(L^2(G,\lambda^{-1}))$. Then $(H,\widehat{\alpha},\widehat{\beta},\alpha,\beta)$ is a C^*-$(\mathfrak{b},\mathfrak{b}^\dagger,\mathfrak{b},\mathfrak{b}^\dagger)$-module, $\rho_\alpha = \rho_\beta = r^*$ and $\rho_{\widehat{\alpha}} = \rho_{\widehat{\beta}} = s^*$, and $j(\xi)^*j(\xi') = \langle\xi|\xi'\rangle$ and $\hat{j}(\eta)^*\hat{j}(\eta') = \langle\eta|\eta'\rangle$ for all $\xi,\xi' \in L^2(G,\lambda)$, $\eta,\eta' \in L^2(G,\lambda^{-1})$ [28].

The Hilbert spaces $H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H$ and $H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H$ can be described as follows. Define measures $\nu^2_{s,r}$ on $G_s{\times}_rG$ and $\nu^2_{r,r}$ on $G_r{\times}_rG$ such that

$$\begin{aligned}
\int_{G_s\times_rG} f\,\mathrm{d}\nu^2_{s,r} &= \int_{G^0}\int_{G^u}\int_{G^{s(x)}} f(x,y)\,\mathrm{d}\lambda^{s(x)}(y)\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u),\\
\int_{G_r\times_rG} g\,\mathrm{d}\nu^2_{r,r} &= \int_{G^0}\int_{G^u}\int_{G^u} g(x,y)\,\mathrm{d}\lambda^u(y)\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u)
\end{aligned} \tag{11}$$

for all $f \in C_c(G_s{\times}_rG)$, $g \in C_c(G_r{\times}_rG)$. Then there exist unitaries

$$\Phi\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to L^2(G_s{\times}_rG,\nu^2_{s,r}) \qquad \text{and} \qquad \Psi\colon H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H \to L^2(G_r{\times}_rG,\nu^2_{r,r})$$

such that for all $\eta,\xi \in C_c(G)$, $\zeta \in C_c(G^0)$,

$$\begin{aligned}
\Phi\big(\hat{j}(\eta) \otimes \zeta \otimes j(\xi)\big)(x,y) &= \eta(x)D^{-1/2}(x)\zeta(s(x))\xi(y),\\
\Psi\big(j(\eta) \otimes \zeta \otimes j(\xi)\big)(x,y) &= \eta(x)\zeta(r(x))\xi(y).
\end{aligned}$$

From now on, we use these isomorphisms without further notice.

Theorem 5.1. *There exists a C^*-pseudo-Kac system (V,U) on $(H,\widehat{\alpha},\widehat{\beta},\alpha,\beta)$ such that for all $\omega \in C_c(G_s{\times}_rG)$, $(x,y) \in G_r{\times}_rG$, $\xi \in C_c(G)$, $z \in G$,*

$$(V\omega)(x,y) = \omega(x,x^{-1}y) \quad \textit{and} \quad (U\xi)(x) = \xi(x^{-1})D(x)^{-1/2}. \tag{12}$$

Proof. By [28, Theorem 2.7, Example 5.3 ii)], there exists a regular C^*-pseudo-multiplicative unitary V as claimed. The second formula in (12) defines a unitary $U \in \mathcal{L}(H)$ by definition of the Radon-Nikodym derivative $D = d\nu/d\nu^{-1}$, and $U^2 = \mathrm{id}$ because $(U^2\xi)(x) = (U\xi)(x^{-1})D(x)^{-1/2} = \xi(x)D(x)^{1/2}D(x)^{-1/2} = \xi(x)$ for all $\xi \in C_c(G)$ and $x \in G$. The unitary $\widehat{V} = \Sigma U_1 V U_1 \Sigma$ is equal to $V^{op} = \Sigma V^* \Sigma$ because

$$\begin{aligned}(U_1 V U_1 \omega)(x, y) &= (V U_1 \omega)(x^{-1}, y)D(x)^{-1/2} \\ &= (U_1\omega)(x^{-1}, xy)D(x)^{-1/2} \\ &= \omega(x, xy)D(x^{-1})^{-1/2}D(x)^{-1/2} = \omega(x, xy)\end{aligned}$$

for all $\omega \in C_c(G_r{\times_r}G)$, $(x, y) \in G_s{\times_r}G$. In particular, $\widehat{V}$ is a regular C^*-pseudo-multiplicative unitary. It remains to show that the map $Z := \Sigma U_2 V \colon H {}_{\widehat{\beta}}{\otimes}_{\mathfrak{b}^\dagger} {}_\alpha H \to H {}_{\widehat{\beta}}{\otimes}_{\mathfrak{b}^\dagger} {}_\alpha H$ satisfies $Z^3 = 1$. But for all $\omega \in C_c(G_s{\times_r}G)$ and $(x, y) \in G_s{\times_r}G$,

$$\begin{aligned}(Z\omega)(x, y) &= (V\omega)(y, x^{-1})D(x)^{-1/2} = \omega(y, y^{-1}x^{-1})D(x)^{-1/2}, \\ (Z^3\omega)(x, y) &= (Z^2\omega)(y, y^{-1}x^{-1})D(x)^{-1/2} \\ &= (Z\omega)(y^{-1}x^{-1}, xyy^{-1})\big(D(x)D(y)\big)^{-1/2} \\ &= \omega(x, x^{-1}xy)\big(D(x)D(y)D(y^{-1}x^{-1})\big)^{-1/2} = \omega(x, y).\end{aligned}$$ □

The Hopf C^*-bimodules $(\widehat{\mathcal{A}}, \widehat{\Delta})$ and $(\mathcal{A}, \Delta)$ associated to V can be described as follows [28, Theorem 3.22]. Given $g \in C_c(G)$, define $L(g) \in C^*_r(G) \subseteq \mathcal{L}(H)$ by

$$(L(g)f)(x) = \int_{G^{r(x)}} g(z)f(z^{-1}x)D^{-1/2}(z)\, \mathrm{d}\lambda^{r(x)}(z)$$

for all $x \in G$, $f \in C_c(G) \subseteq L^2(G, \nu) = H$. Then

$$\widehat{A} = C_0(G) \subseteq \mathcal{L}(H), \quad \big(\widehat{\Delta}(f)\omega\big)(x, y) = f(xy)\omega(x, y), \tag{13}$$

$$A = C^*_r(G), \quad \big(\Delta(L(g))\omega'\big)(x', y') = \int_{G^{u'}} g(z)D^{-1/2}(z)\omega'(z^{-1}x', z^{-1}y')\, \mathrm{d}\lambda^{u'}(z)$$

for all $f \in C_0(G)$, $\omega \in C_c(G_s{\times_r}G)$, $(x, y) \in G_s{\times_r}G$ and $g \in C_c(G)$, $\omega' \in C_c(G_r{\times_r}G)$, $(x', y') \in G_r{\times_r}G$, where $u' = r(x') = r(y')$. We shall loosely refer to $C_0(G)$ and $C^*_r(G)$ as Hopf C^*-bimodules, having in mind $(\widehat{\mathcal{A}}, \widehat{\Delta})$ and $(\mathcal{A}, \Delta)$, respectively.

6 Actions of G and coactions of $C_0(G)$

Let G be a groupoid and consider $C_0(G)$ as a Hopf C^*-bimodule as in the preceding section. Then coactions of $C_0(G)$ can be related to actions of G as follows. Let us say that a tuple $(\mathbf{F}, \mathbf{G}, \eta, \epsilon)$ is *an embedding of a category* $\mathbf{C}$ *into a category* $\mathbf{D}$ *as a full and coreflective subcategory* if $\mathbf{F} \colon \mathbf{C} \to \mathbf{D}$ is a full and faithful functor and $\mathbf{G} \colon \mathbf{D} \to \mathbf{C}$ is a faithful right adjoint to $\mathbf{F}$, where $\eta \colon \mathrm{id}_{\mathbf{C}} \to \mathbf{GF}$ is the unit and $\epsilon \colon \mathbf{FG} \to \mathrm{id}_{\mathbf{D}}$ is the counit of the adjunction; see also [18, §IV.3]. In this section, we construct such an embedding of the category of actions of G on continuous $C_0(G^0)$-algebras into the category of certain admissible coactions of $C_0(G)$. We keep the notation introduced in the preceding section.

$C_0(G^0)$-algebras and C^*-$\mathfrak{b}$-algebras We shall embed the category of admissible $C_0(G^0)$-algebras into the category of admissible C^*-$\mathfrak{b}$-algebras as a full and coreflective subcategory.

Recall that a *$C_0(X)$-algebra*, where X is some locally compact Hausdorff space, is a C^*-algebra C with a fixed nondegenerate $*$-homomorphism of $C_0(X)$ into the center of the multiplier algebra $M(C)$ [6, 14]. We denote the fiber of a $C_0(X)$-algebra C at a point $x \in X$ by C_x and write the quotient map $p_x \colon C \to C_x$ as $c \mapsto c_x$. Recall that C is a *continuous $C_0(X)$-algebra* if the map $X \to \mathbb{R}$ given by $x \mapsto \|c_x\|$ is continuous for each $c \in C$. A *morphism* of $C_0(X)$-algebras C, D is a nondegenerate $*$-homomorphism $\pi \colon C \to M(D)$ such that $\pi(fc) = f\pi(c)$ for all $f \in C_0(X)$, $c \in C$.

Definition 6.1. *We call a $C_0(G^0)$-algebra C* admissible *if it is continuous and if $C_u \neq 0$ for each $u \in G^0$, and we call a C^*-$\mathfrak{b}$-algebra C^γ_K* admissible *if $[\rho_\gamma(C_0(G^0))C] = C$ and $[C\gamma] = \gamma$. A* morphism *between admissible C^*-$\mathfrak{b}$-algebras C^γ_K, D^ϵ_L is a semi-morphism π from C^γ_K to $M(D)^\epsilon_L$ that is nondegenerate in the sense that $[\pi(C)D] = D$. Denote by $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$ the category of all admissible $C_0(G^0)$-algebras, and by $\mathbf{C^*}$-$\mathfrak{b}$-$\mathbf{alg}^a$ the category of all admissible C^*-$\mathfrak{b}$-algebras.*

Lemma 6.2. *i) Let C^γ_K be an admissible C^*-$\mathfrak{b}$-algebra. Then C is an admissible $C_0(G^0)$-algebra with respect to ρ_γ.*

ii) Let π be a morphism between admissible C^*-$\mathfrak{b}$-algebras C^γ_K and D^ϵ_L. Then π is a morphism of $C_0(G^0)$-algebras from (C, ρ_γ) to (D, ρ_ϵ).

Proof. i) The subalgebra $\rho_\gamma(C_0(G^0)) \subseteq M(C)$ is central because $C \subseteq \mathcal{L}(K_\gamma) \subseteq \rho_\gamma(C_0(G^0))'$. The map $C \hookrightarrow \mathcal{L}(K_\gamma) \cong \mathcal{L}(\gamma)$ is a faithful field of representations in the sense of [6, Theorem 3.3], and therefore C is a continous $C_0(G^0)$-algebra. We have $C_u \neq 0$ for each $u \in G^0$ because otherwise $C = [CI_u]$, where $I_u = C_0(G^0 \backslash \{u\})$, and then $[\gamma^*\gamma] = [\gamma^*C\gamma] = [\gamma^*I_uC\gamma] = [\gamma^*\gamma I_u] = I_u \neq C_0(G^0)$, contradicting the fact that K_γ is a C^*-$\mathfrak{b}$-module.

ii) This follows from [27, Proposition 3.5]. ☐

We embed $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$ into $\mathbf{C^*}$-$\mathfrak{b}$-$\mathbf{alg}^a$ using a KSGNS-construction for the following kind of weights.

Definition 6.3. *A $C_0(G^0)$-*weight *on a $C_0(G^0)$-algebra C is a positive $C_0(G^0)$-linear map $\phi \colon C \to C_0(G^0)$. We denote the set of all such $C_0(G^0)$-weights by $\mathcal{W}(C)$.*

Let C be an admissible $C_0(G^0)$-algebra. The results in [4] imply:

Lemma 6.4. $\bigcap_{\phi \in \mathcal{W}(C)} \ker \phi = \{0\}$ *and* $[\bigcup_{\phi \in \mathcal{W}(C)} \phi(C)] = C_0(G^0)$. ☐

Let $\phi \in \mathcal{W}(C)$. Then ϕ is completely positive [20, Theorem 3.9] and bounded [16, Lemma 5.1]. Let $E_\phi = C \otimes_\phi \mathfrak{K}$ (see Section 1) and define $\eta_\phi \colon C \to \mathcal{L}(E_\phi)$ and $l_\phi \colon C \to \mathcal{L}(\mathfrak{K}, E_\phi)$ by $\eta_\phi(c)(d \otimes_\phi \zeta) = cd \otimes_\phi \zeta$ and $l_\phi(c)\zeta = c \otimes_\phi \zeta$ for all $c, d \in C$, $\zeta \in \mathfrak{K}$. One easily verifies that for all $c, d \in C$, $f \in C_0(G^0)$, $\zeta \in \mathfrak{K}$,

$$\begin{aligned} l_\phi(c)^* l_\phi(d) &= \phi(c^*d), \quad l_\phi(c)f = l_\phi(cf), \\ \eta_\phi(c)(d \otimes_\phi f\zeta) &= cdf \otimes_\phi \zeta = \eta_\phi(cf)(d \otimes_\phi \zeta). \end{aligned} \tag{14}$$

The *universal* $C_0(G^0)$*-representation* $\eta_C\colon C \to \mathcal{L}(E_C)$ of C is the direct sum of the representations $\eta_\phi\colon C \to \mathcal{L}(E_\phi)$, where $\phi \in \mathcal{W}(C)$. Denote by $l_C \subseteq \mathcal{L}(\mathfrak{K}, E_C)$ the closed linear span of all maps $l_\phi(c)\colon \mathfrak{K} \to E_\phi \hookrightarrow E_C$, where $c \in C$, $\phi \in \mathcal{W}(C)$.

Lemma 6.5. *$\eta_C(C)^{l_C}_{E_C}$ is an admissible C^*-$\mathfrak{b}$-algebra and η_C is an isomorphism of $C_0(G^0)$-algebras from C to $(\eta_C(C), \rho_{l_C})$.*

Proof. The definition of l_C, the equations (14) and Lemma 6.4 imply that $[l_C\mathfrak{K}] = \bigoplus_\phi E_\phi = E_C$, $[l_C^* l_C] = [\bigcup_\phi \phi(C)] = C_0(G^0)$ and $[l_C C_0(G^0)] = l_C$, whence (E_C, l_C) is a C^*-$\mathfrak{b}$-module, and that $[\eta_C(C)\rho_{l_C}(C_0(G^0))] = [\eta_C(CC_0(G^0))] = \eta_C(C)$ and $[\eta_C(C)l_C] = l_C$, whence $\eta_C(C)^{l_C}_{E_C}$ is an admissible C^*-$\mathfrak{b}$-algebra. Lemma 6.4 implies that η_C is injective and hence an isomorphism of C onto $\eta_C(C)$, and the last equation in (14) implies that $\eta_C(c)\rho_\gamma(f) = \eta_C(cf)$ for all $c \in C$, $f \in C_0(G^0)$. □

Theorem 6.6. *There exists an embedding* $(\mathbf{F}, \mathbf{G}, \eta, \epsilon)$ *of* $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$ *into* $\mathbf{C^*}$-$\mathfrak{b}$-$\mathbf{alg}^a$ *as a full and coreflective subcategory such that*

i) $\mathbf{F}$ *is given by* $C \mapsto \eta_C(C)^{l_C}_{E_C}$ *on objects and by* $\mathbf{F}\pi\colon \eta_C(c) \mapsto \eta_D(\pi(c))$ *for each morphism* π *between objects* C, D *in* $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$*;*

ii) $\mathbf{G}$ *is given by* $C^\gamma_K \mapsto (C, \rho_\gamma)$ *on objects and* $\pi \mapsto \pi$ *on morphisms;*

iii) η_C *is defined as above for each object* C *in* $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$*;*

iv) $\epsilon_{\mathcal{C}} = \eta^{-1}_{\mathbf{G}\mathcal{C}}$ *for each object* $\mathcal{C}$ *in* $\mathbf{C^*}$-$\mathfrak{b}$-$\mathbf{alg}^a$.

Proof of Theorem 6.6. The functor $\mathbf{G}\colon \mathbf{C^*}$-$\mathfrak{b}$-$\mathbf{alg}^a \to \mathbf{C_0(G^0)}$-$\mathbf{alg}^a$ is well defined by Lemma 6.2 and evidently faithful.

Let C be an admissible $C_0(G^0)$-algebra, $\mathcal{D} = D^\gamma_K$ an admissible C^*-$\mathfrak{b}$-algebra, and $\pi\colon C \to \mathbf{G}\mathcal{D}$ a morphism in $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$. We claim that $\pi \circ \eta_C^{-1}$ is a morphism from $\mathbf{F}C$ to $\mathcal{D}$ in $\mathbf{C^*}$-$\mathfrak{b}$-$\mathbf{alg}^a$. Let $\xi \in \gamma$. Then the map $\phi\colon C \to C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K})$ given by $c \mapsto \xi^*\pi(c)\xi$ is a $C_0(G^0)$-weight, and there exists an isometry $S\colon E_\phi \to K$ such that $S(c \otimes_\phi \zeta) = \pi(c)\xi\zeta$ for all $c \in C$, $\zeta \in \mathfrak{K}$. Denote by $P\colon E_C \to E_\phi$ the natural projection. Then $[SPl_C] = [Sl_\phi(C)] = [\pi(C)\xi]$ lies in γ and contains ξ, and $SP\eta_C(c) = S\eta_\phi(c) = \pi(c)$ for each $c \in C$. Since $\xi \in \gamma$ was arbitrary, the claim follows.

Using Lemma 6.5, we conclude that $\mathbf{F}$ is well defined and that η is a natural isomorphism from id to $\mathbf{GF}$. Indeed, if $\pi\colon C \to D$ is a morphism in $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$, then $\mathbf{F}\pi = \eta_D \circ \pi \circ \eta_C^{-1}$ is a morphism from $\mathbf{F}C$ to $\mathbf{F}D$ by the argument above.

Finally, let $\mathcal{D}$ be an admissible C^*-$\mathfrak{b}$-algebra. The argument above, applied to the identity on $\mathbf{G}\mathcal{D}$, yields a morphism $\epsilon_{\mathcal{D}}$ from $\mathbf{FG}\mathcal{D}$ to $\mathcal{D}$ in $\mathbf{C^*}$-$\mathfrak{b}$-$\mathbf{alg}^a$ such that the composition $\mathbf{G}\mathcal{D} \xrightarrow{\eta_{\mathbf{G}\mathcal{D}}} \mathbf{GFG}\mathcal{D} \xrightarrow{\mathbf{G}\epsilon_{\mathcal{D}}} \mathbf{G}\mathcal{D}$ is the identity. Since η is a natural transformation, also $\epsilon\colon \mathbf{FG} \to \mathrm{id}$ is one. For each admissible $C_0(G^0)$-algebra C, the composition $\mathbf{F}C \xrightarrow{\mathbf{F}\eta_C} \mathbf{FGF}C \xrightarrow{\epsilon_{\mathbf{F}C}} \mathbf{F}C$ is the identity by construction. From [18, §IV.2 Theorem 2], we can conclude that $\mathbf{F}$ is a left adjoint to $\mathbf{G}$ such that η and ϵ form the unit and counit, respectively, of the adjunction. Since η is a natural isomorphism, $\mathbf{F}$ is full and faithful [18, §IV.3 Theorem 1]. □

Actions of G and coactions of $C_0(G)$ We next embed the category of admissible actions of G as a full and coreflective subcategory into the category of all admissible coactions of $C_0(G)$.

The definition of an action of G requires the following preliminaries. Given $C_0(G^0)$-algebras (C,ρ) and (D,σ), where D is commutative, we denote by $C_\rho \boxtimes_\sigma D$ the $C_0(G^0)$-tensor product [5], and drop the subscript ρ or σ if this map is understood. Given a $C_0(G^0)$-algebra C and a continuous surjection $t\colon G \to G^0$, we consider $C_0(G)$ as a $C_0(G^0)$-algebra via $t^*\colon C_0(G^0) \to M(C_0(G))$ and let $t^*C := C \boxtimes_{t^*} C_0(G)$, which is a $C_0(G)$-algebra in a natural way. Each morphism π of $C_0(G^0)$-algebras C, D induces a morphism of $t^*\pi$ of $C_0(G)$-algebras from t^*C to t^*D via $c \boxtimes f \mapsto \pi(c) \boxtimes f$. An *action* of G on a $C_0(G^0)$-algebra C is an isomorphism $\sigma\colon s^*C \to r^*C$ of $C_0(G)$-algebras such that the restrictions of σ to the fibers satisfy $\sigma_x \circ \sigma_y = \sigma_{xy}$ for all $(x,y) \in G_s\times_r G$ [17]. A *morphism* between actions (C,σ^C) and (D,σ^D) of G is a morphism of $C_0(G^0)$-algebras π from C to D satisfying $\sigma^D \circ s^*\pi = r^*\pi \circ \sigma^C$.

Definition 6.7. *We call an action (C,σ) of G* admissible *if the $C_0(G^0)$-algebra C is admissible, and we call a coaction (C_K^γ, δ) of $C_0(G)$* admissible *if C_K^γ is an admissible C^*-$\mathfrak{b}$-algebra and $[\delta(C)(1 \underset{\mathfrak{b}}{\otimes} C_0(G))] = C \underset{\mathfrak{b}}{\otimes} C_0(G)$ in $\mathcal{L}(K_\gamma {\underset{\mathfrak{b}}{\otimes}}_\alpha H)$.*

Remark 6.8. If σ is an action of G on a continuous $C_0(G^0)$-algebra, then the subset $Y := \{u \in G^0 \mid C_u \neq 0\} \subseteq G^0$ is open, C is an admissible $C_0(Y)$-algebra, and σ restricts to an action of the subgroupoid $G|_Y := \{x \in G \mid r(x), s(x) \in Y\} \subseteq G$.

Lemma 6.9. *Let C_K^γ and D_L^ϵ be admissible C^*-$\mathfrak{b}$-algebras, where D is commutative. Then there exists an isomorphism $C_{\rho_\gamma} \boxtimes_{\rho_\epsilon} D \to C {}_\gamma{\underset{\mathfrak{b}}{\otimes}}_\epsilon D$, $c \boxtimes d \mapsto c \underset{\mathfrak{b}}{\otimes} d$.*

Proof. Use [5, Lemma 2.7] and apply [5, Proposition 4.1] to the field of representations $C \hookrightarrow \mathcal{L}(K_\gamma) \cong \mathcal{L}(\gamma)$, noting that $\gamma \circledcirc_{\rho_\epsilon} D \cong [|\gamma\rangle_1 D]$ as a Hilbert C^*-D-module via $\xi \circledcirc d \mapsto |\xi\rangle_1 d$ and that $(C {}_\gamma{\underset{\mathfrak{b}}{\otimes}}_\epsilon D)[|\gamma\rangle_1 D] \subseteq [|\gamma\rangle_1 D]$. ☐

We use the isomorphism above without further notice.

Proposition 6.10. *i) Let (C_K^γ, δ) be an admissible coaction of $C_0(G)$. There exists a unique action σ_δ of G on (C, ρ_γ) given by $c \boxtimes f \mapsto \delta(c)(1 \underset{\mathfrak{b}}{\otimes} f)$.*

ii) Let (C,σ) be an admissible action of G. There exists a unique admissible, injective coaction δ_σ of $C_0(G)$ on $\mathbf{F}C$ given by $\eta_C(c) \mapsto (r^*\eta_C)(\sigma(c \boxtimes 1))$.

Proof. i) Since $\delta(C)$ and $1 \underset{\mathfrak{b}}{\otimes} C_0(G)$ commute, there exists a unique $*$-homomorphism $\tilde\sigma$ from the algebraic tensor product $C \odot C_0(G)$ to r^*C such that $\tilde\sigma(c \odot f) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} f)$ for all $c \in C$, $f \in C_0(G)$. Since δ is a coaction, $\delta(c\rho_\gamma(g)) = \delta(c)\rho_{(\gamma \triangleright \widehat{\beta})}(g) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} s^*(g))$ for all $g \in C_0(G^0)$. From [5, Lemma 2.7], we can conclude that $\tilde\sigma$ factorizes to a $*$-homomorphism $\sigma = \sigma_\delta\colon s^*C \to r^*C$ satisfying the formula in i). This σ is surjective because $[\delta(C)(1 \underset{\mathfrak{b}}{\otimes} C_0(G))] = C \underset{\mathfrak{b}}{\otimes} C_0(G)$. In particular, σ_x is surjective for each $x \in G$. We claim that $\sigma_x \circ \sigma_y = \sigma_{xy}$ for all $(x,y) \in G_s\times_r G$. Define $r_1\colon G_s\times_r G \to G^0$ by $(x,y) \mapsto r(x)$. By Lemma 6.9, we have isomorphisms

$C_{\gamma}\underset{\mathfrak{b}}{\otimes}_{\alpha}C_0(G)_{\widehat{\beta}}\underset{\mathfrak{b}}{\otimes}_{\alpha}C_0(G) \cong C \boxtimes_{r^*} C_0(G)_{s^*} \boxtimes_{r^*} C_0(G) \cong C \boxtimes_{r_1^*} C_0(G_s \times_r G) \cong r_1^* C$. Using formula (13), we find

$$\begin{aligned} \sigma_x \circ \sigma_y \circ p_{s(y)} = \sigma_x \circ p_y \circ \delta = p_{(x,y)} \circ (\delta * \mathrm{id}) \circ \delta, \\ \sigma_{xy} \circ p_{s(y)} = p_{xy} \circ \delta = p_{(x,y)} \circ (\mathrm{id} * \widehat{\Delta}) \circ \delta, \end{aligned} \tag{15}$$

and the claim follows. Finally, $\sigma_u = \mathrm{id}_{C_u}$ for each $u \in G^0$ because σ_u is surjective and idempotent, and σ_x is injective for each $x \in G$ because $\sigma_{s(x)} = \sigma_{x^{-1}} \circ \sigma_x$ is injective. Therefore, σ is injective.

ii) Let $D := \eta_C(C)_{l_C}\underset{\mathfrak{b}}{\otimes}_{\alpha}C_0(G)$. Then $\mathcal{D} := ((E_C)_{l_C}\underset{\mathfrak{b}}{\otimes}_{\alpha}H_{\widehat{\beta}}, D)$ is an admissible C^*-$\mathfrak{b}$-algebra. Define $\delta\colon C \to D$ by $c \mapsto (r^*\eta_C)(\sigma(c \boxtimes 1))$. Let $c \in C$, $g \in C_0(G^0)$. Then $cg \boxtimes 1 = c \boxtimes s^*(g)$ in $M(C \boxtimes_{s^*} C_0(G))$ and therefore $\delta(cg) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} s^*(g)) = \delta(c)\rho_{(l_C \triangleright \widehat{\beta})}(g)$. Consequently, δ is a morphism of $C_0(G^0)$-algebras from C to $(D, \rho_{l_C \triangleright \widehat{\beta}}) = \mathbf{G}\mathcal{D}$. By definition of $\mathbf{F}$ and ϵ, the morphism $\delta_\sigma := \epsilon_{\mathcal{D}} \circ \mathbf{F}\delta\colon \mathbf{F}C \to \mathbf{F}\mathbf{G}\mathcal{D} \to \mathcal{D}$ satisfies $\delta_\sigma \circ \eta_C = \delta$, and a similar calculation as in (15) shows that $(\delta_\sigma * \mathrm{id}) \circ \delta_\sigma = (\mathrm{id} * \widehat{\Delta}) \circ \delta_\sigma$. Consequently, δ_σ is a coaction of $(\widehat{\mathcal{A}}, \widehat{\Delta})$. Since σ is injective, so are δ and δ_σ. Finally, δ_σ is admissible because $[\delta_\sigma(\eta_C(C))(1 \underset{\mathfrak{b}}{\otimes} C_0(G))] = (r^*\eta_C)(\sigma(s^*C)) = r^*\eta_C(C) = [\eta_C(C) \underset{\mathfrak{b}}{\otimes} C_0(G)]$. ☐

Corollary 6.11. *Every admissible coaction of $C_0(G)$ is injective, left-full, and right-full.*

Proof. If (C_K^γ, δ) is an admissible coaction, then $[\delta(C)|\alpha\rangle_2] = [\delta(C)(1\underset{\mathfrak{b}}{\otimes}C_0(G))|\alpha\rangle_2] = [(C \underset{\mathfrak{b}}{\otimes} C_0(G))|\alpha\rangle_2] = [|\alpha\rangle_2 C]$ and $[\delta(C)|\gamma\rangle_1 C_0(G)] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} C_0(G))|\gamma\rangle_1] = [(C \underset{\mathfrak{b}}{\otimes} C_0(G))|\gamma\rangle_1] = [|\gamma\rangle_1 C_0(G)]$ because $[C_0(G)\alpha] = \alpha$ and $[C\gamma] = \gamma$. Finally, δ is injective because σ_δ is injective and $\delta(c) = \sigma_\delta(c \boxtimes 1)$ for all $c \in C$. ☐

Proposition 6.12. *Let $(\mathcal{C}, \delta^{\mathcal{C}})$, $(\mathcal{D}, \delta^{\mathcal{D}})$ be admissible coactions with associated actions $\sigma^{\mathcal{C}} = \sigma_{\delta^{\mathcal{C}}}$, $\sigma^{\mathcal{D}} = \sigma_{\delta^{\mathcal{D}}}$, and let $\pi \in \mathbf{C^*}\text{-}\mathfrak{b}\text{-}\mathbf{alg}^a(\mathcal{C}, \mathcal{D}) = \mathbf{C_0(G^0)}\text{-}\mathbf{alg}^a(\mathbf{G}\mathcal{C}, \mathbf{G}\mathcal{D})$. Then $(\pi * \mathrm{id}) \circ \delta^{\mathcal{C}} = \delta^{\mathcal{D}} \circ \pi$ if and only if $r^*\pi \circ \sigma^{\mathcal{C}} = \sigma^{\mathcal{D}} \circ s^*\pi$.*

Proof. Write $\mathcal{C} = C_K^\gamma$. The assertion holds because for all $c \in C$ and $f \in C_0(G)$,

$$\begin{aligned} ((\pi * \mathrm{id})(\delta^{\mathcal{C}}(c)))(1 \underset{\mathfrak{b}}{\otimes} f) = (\pi * \mathrm{id})(\delta^{\mathcal{C}}(c)(1 \underset{\mathfrak{b}}{\otimes} f)) = (r^*\pi \circ \sigma^{\mathcal{C}})(c \boxtimes f), \\ \delta^{\mathcal{D}}(\pi(c))(1 \underset{\mathfrak{b}}{\otimes} f) = \sigma^{\mathcal{D}}(\pi(c) \boxtimes f) = (\sigma^{\mathcal{D}} \circ s^*\pi)(c \boxtimes f). \end{aligned}$$ ☐

We denote by $\mathbf{G}\text{-}\mathbf{act}^a$ and $\mathbf{Coact}^a_{C_0(G)}$ the categories of all admissible actions of G and all admissible coactions of $C_0(G)$, respectively.

Theorem 6.13. *There exists an embedding $(\hat{\mathbf{F}}, \hat{\mathbf{G}}, \hat{\eta}, \hat{\epsilon})$ of $\mathbf{G}\text{-}\mathbf{act}^a$ into $\mathbf{Coact}^a_{C_0(G)}$ as a full and coreflective subcategory such that*

i) $\hat{\mathbf{F}}$ is given by $(C, \sigma) \mapsto (\mathbf{F}C, \delta_\sigma)$ on objects and $\pi \mapsto \mathbf{F}\pi$ on morphisms;

ii) $\hat{\mathbf{G}}$ is given by $(\mathcal{C}, \delta) \mapsto (\mathbf{G}\mathcal{C}, \sigma_\delta)$ on objects and $\pi \mapsto \mathbf{G}\pi = \pi$ on morphisms;

iii) $\hat{\eta}_{(C,\sigma)} = \eta_C$ and $\hat{\epsilon}_{(\mathcal{C},\delta)} = \epsilon_{\mathcal{C}}$ for all objects (C, σ) and $(\mathcal{C}, \delta)$.

Proof. The assignments $\hat{\mathbf{G}}$ and $\hat{\mathbf{F}}$ are well defined on objects and morphisms by Proposition 6.10 and 6.12. For each admissible action (C,σ), we have that $\eta_C \in \mathbf{G}\text{-}\mathbf{act}^a((C,\sigma), \hat{\mathbf{G}}\hat{\mathbf{F}}(C,\sigma))$ because $\sigma_{\delta_\sigma}(\eta_C(c) \boxtimes f) = \delta_\sigma(\eta_C(c))(1 \underset{\mathfrak{b}}{\otimes} f) = r^*\eta_C(\sigma(c \boxtimes f))$ for all $c \in C$, $f \in C_0(G)$, and Proposition 6.12 implies that $\epsilon_{\mathcal{C}} = \eta^{-1}_{\mathbf{G}\mathcal{C}} \in \mathbf{Coact}^a_{C_0(G)}(\hat{\mathbf{F}}\hat{\mathbf{G}}(\mathcal{C},\delta), (\mathcal{C},\delta))$ for each admissible coaction $(\mathcal{C},\delta)$. Now, the assertion follows from Theorem 6.6. □

Comparison of the associated reduced crossed products The reduced crossed product for an action (C,σ) of G is defined as follows [17]. The subspace $C_c(G;C,\sigma) := C_c(G)r^*C \subseteq r^*C$ carries the structure of a $*$-algebra and the structure of a pre-Hilbert C^*-module over C such that

$$(ab)_x = \int_{G^{r(x)}} a_y \sigma_y(b_{y^{-1}x})\,\mathrm{d}\lambda^{r(x)}(y), \qquad (a^*)_x = \sigma_x(a^*_{x^{-1}}),$$
$$\langle a|b\rangle_u = \int_{G^u} \sigma_y((a_{y^{-1}})^* b_{y^{-1}})\,\mathrm{d}\lambda^u(y) = (a^*b)_u, \qquad (ac)_x = a_x\sigma_x(c_{s(x)})$$

for all $a,b \in C_c(G;C,\sigma)$, $u \in G^0$ and $c \in C$, $x \in G$. Denote the completion of this pre-Hilbert C^*-module by $L^2(G,\lambda^{-1};C,\sigma)$. Using the relation $\langle a|bd\rangle_u = (abd)_u = \langle b^*a|d\rangle_u$, which holds for all $a,b,d \in C_c(G;C,\sigma)$, $u \in G^0$, and a routine norm estimate, one verifies the existence of a $*$-homomorphism $\pi\colon C_c(G;C,\sigma) \to \mathcal{L}(L^2(G,\lambda^{-1};C,\sigma))$ such that $\pi(b)d = bd$ for all $b,d \in C_c(G;C,\sigma)$. Then the *reduced crossed product* of (C,σ) is the C^*-algebra $C \rtimes_{\sigma,r} G := [\pi(C_c(G;C,\sigma))] \subseteq \mathcal{L}(L^2(G,\lambda^{-1};C,\sigma))$.

Proposition 6.14. *Let (C^γ_K,δ) be an admissible coaction of $C_0(G)$, consider C as a $C_0(G^0)$-algebra via ρ_γ, and let $\sigma = \sigma_\delta$. Then there exists an isomorphism $C \rtimes_{\sigma_\delta,r} G \to C \rtimes_r C^*_r(G)$ given by $\pi(c \boxtimes f) \mapsto \delta(c)(\mathrm{id} \underset{\mathfrak{b}}{\otimes} UL(f)U)$ for all $c \in C$, $f \in C_c(G)$.*

Proof. Let $\delta_U := \mathrm{Ad}_{(\mathrm{id} \underset{\mathfrak{b}}{\otimes} U)} \circ \delta\colon C \to \mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes}_{\hat{\alpha}} H)$. We equip $C_c(G;C,\sigma)$ with the structure of a pre-Hilbert C^*-module over C such that $\langle a|b\rangle_u = \int_{G_u} (a_x)^* b_x\,\mathrm{d}\lambda^{-1}_u(x)$ and $(ac)_x = a_x c_{s(x)}$ for all $a,b \in C_c(G;C,\sigma)$, $c \in C$, $u \in G^0$, and denote by $L^2(G,\lambda^{-1};C)$ the completion. One easily checks that there exists a unique unitary $\Phi\colon L^2(G,\lambda^{-1};C) \to [|\hat{\alpha}\rangle_2 C] = [\delta_U(C)|\hat{\alpha}\rangle_2\rangle]$ given by $c \boxtimes f \mapsto |\hat{j}(f)\rangle_2 c$, and that for all $c \in C, f \in C_c(G), y \in G$,

$$\Phi^{-1}(\delta_U(c)|\hat{j}(f)\rangle_2)_y = \sigma_{y^{-1}}(c_{r(y)})f(y).$$

Hence, there exists a unitary $\Psi\colon L^2(G,\lambda^{-1};C,\sigma) \to [\delta_U(C)|\hat{\alpha}\rangle_2\rangle]$ given by $c \boxtimes f \mapsto \delta_U(c)|\hat{j}(f)\rangle_2$. Let $c,d \in C$, $f,g \in C_c(G)$ and $\omega = \Phi^{-1}(\Psi(c \boxtimes f))$. Then $\delta_U(d)(\mathrm{id} \underset{\mathfrak{b}}{\otimes} L(g))\Psi = \Psi\pi(d \boxtimes g)$ because for all $x \in G$,

$$\begin{aligned}
\Phi^{-1}(\delta_U(d)(\mathrm{id} \underset{\mathfrak{b}}{\otimes} L(g))\Phi(\omega))_x &= \int_{G_u} \sigma_{x^{-1}}(d_{r(x)})g(xy^{-1})\omega_y\,\mathrm{d}\lambda^{-1}_u(y) \\
&= \int_{G_u} \sigma_{x^{-1}}\big(d_{r(xy^{-1})}g(xy^{-1})\sigma_{xy^{-1}}(c_{r(y)})f(y)\big)\,\mathrm{d}\lambda^{-1}_u(y) \\
&= \Phi^{-1}(\Psi(\pi(d \boxtimes g)(c \boxtimes f)))_x.
\end{aligned}$$

Since $d \in C$ and $g \in C_c(G)$ were arbitrary, the assertion follows. □

7 Fell bundles on groupoids

We now gather preliminaries on Fell bundles that are needed in Sections 8 and 9. We use the notion of a Banach bundle and standard notation; a reference is [8].

Fell bundles on groupoids and their C^*-algebras We first recall the notion of a Fell bundle on G and the definition of the associated reduced C^*-algebra [15]. Given an upper semicontinuous Banach bundle $p\colon \mathcal{F} \to G$, denote by $\mathcal{F}^0$ the restriction of $\mathcal{F}$ to G^0, by $\mathcal{F}_{sp}\times_{rp}\mathcal{F}$ the restriction of $\mathcal{F} \times \mathcal{F}$ to $G_s\times_r G$, by $\mathcal{F}_x$ for each $x \in G$ the fiber at x, by $\Gamma_c(\mathcal{F})$ the space of continuous sections of $\mathcal{F}$ with compact support, and by $\Gamma_0(\mathcal{F}^0)$ the space of continuous sections of $\mathcal{F}^0$ that vanish at infinity in norm.

Definition 7.1. *A* Fell bundle *on G is an upper semicontinuous Banach bundle $p\colon \mathcal{F} \to G$ with a continuous multiplication $\mathcal{F}_{sp}\times_{rp}\mathcal{F} \to \mathcal{F}$ and a continuous involution $*\colon \mathcal{F} \to \mathcal{F}$ such that for all $e \in \mathcal{F}$, $(e_1, e_2) \in \mathcal{F}_{sp}\times_{rp}\mathcal{F}$, $(x, y) \in G_s\times_r G$,*

i) $p(e_1e_2) = p(e_1)p(e_2)$ and $p(e^*) = p(e)^{-1}$;

ii) the map $\mathcal{F}_x \times \mathcal{F}_y \to \mathcal{F}_{xy}, (e'_1, e'_2) \mapsto e'_1e'_2$, is bilinear and the map $\mathcal{F}_x \to \mathcal{F}_{x^{-1}}$, $e' \mapsto e'^*$, is conjugate linear;

iii) $(e_1e_2)e_3 = e_1(e_2e_3)$, $(e_1e_2)^* = e_2^*e_1^*$, and $(e^*)^* = e$;

iv) $\|e_1e_2\| \leqslant \|e_1\|\|e_2\|$, $\|e^*e\| = \|e\|^2$, and $e^*e \geqslant 0$ in the C^*-algebra $\mathcal{F}_{s(p(e))}$.

We call $\mathcal{F}$ saturated *if $[\mathcal{F}_x\mathcal{F}_y] = \mathcal{F}_{xy}$ for all $(x, y) \in G_s\times_r G$, and* admissible *if $\Gamma_0(\mathcal{F}^0)$ is an admissible $C_0(G^0)$-algebra with respect to the pointwise operations.*

Let $\mathcal{F}$ be a Fell bundle on G. The associated reduced C^*-algebra is defined as follows. The space $\Gamma_c(\mathcal{F})$ is a $*$-algebra with respect to the multiplication and involution given by

$$(cd)(x) = \int_{G^{r(x)}} c(y)d(y^{-1}x)\,\mathrm{d}\lambda^{r(x)}(y) = \int_{G_{s(x)}} c(xz^{-1})d(z)\,\mathrm{d}\lambda^{-1}_{s(x)}(z) \tag{16}$$

and $c^*(x) = c(x^{-1})^*$, respectively, and a pre-Hilbert C^*-module over $\Gamma_0(\mathcal{F}^0)$ with respect to the structure maps

$$\langle c|d\rangle(u) = \int_{G_u} c(x)^*d(x)\,\mathrm{d}\lambda^{-1}_u(x) = (c^*d)(u), \qquad (ce)(x) = c(x)e(s(x)),$$

where $c, d \in \Gamma_c(\mathcal{F})$, $e \in \Gamma_0(\mathcal{F}^0)$, $x \in G$. Denote by $\Gamma^2(\mathcal{F}, \lambda^{-1})$ the completion of this pre-Hilbert C^*-module. Then there exists a $*$-homomorphism

$$L_{\mathcal{F}}\colon \Gamma_c(\mathcal{F}) \to \mathcal{L}(\Gamma^2(\mathcal{F}, \lambda^{-1})), \quad L_{\mathcal{F}}(a)b = ab \text{ for all } a, b \in \Gamma_c(\mathcal{F}),$$

and $C^*_r(\mathcal{F}) := [L_{\mathcal{F}}(\Gamma_c(\mathcal{F}))] \subseteq \mathcal{L}(\Gamma^2(\mathcal{F}, \lambda^{-1}))$ is the *reduced C^*-algebra of $\mathcal{F}$*. We identify $\Gamma_c(\mathcal{F})$ with $L_{\mathcal{F}}(\Gamma_c(\mathcal{F})) \subseteq C^*_r(\mathcal{F})$ via $L_{\mathcal{F}}$.

We equip $\Gamma_c(\mathcal{F})$ with the *inductive limit topology*; thus, a net converges if it converges uniformly and if the supports of its members are contained in some compact set. We shall frequently use the following general result; see [8, Proposition 2.3].

Lemma 7.2. *Let $\mathcal{E}$ be an upper semicontinuous Banach bundle on a locally compact, second countable, Hausdorff space X and let $\Gamma' \subseteq \Gamma_c(\mathcal{E})$ be a subspace such that*

i) Γ' *is closed under pointwise multiplication with elements of* $C_c(X)$*;*

ii) $\{f(x) \mid f \in \Gamma'\} \subseteq \mathcal{E}_x$ *is dense for each* $x \in X$*.*

Then Γ' *is dense in* $\Gamma_c(\mathcal{E})$*.* □

Given $f \in \Gamma_c(\mathcal{F})$ and $g \in \Gamma_0(\mathcal{F}^0)$, define $fg, gf \in \Gamma_c(\mathcal{F})$ by $(fg)(x) = f(x)g(s(x))$, $(gf)(x) = g(r(x))f(x)$ for all $x \in G$. Using the relation $[\mathcal{F}_x] = [\mathcal{F}_x\mathcal{F}_x^*\mathcal{F}_x]$, where $x \in G$, and Lemma 7.2, we find:

Lemma 7.3. $\Gamma_c(\mathcal{F})\Gamma_0(\mathcal{F}^0)$ *and* $\Gamma_0(\mathcal{F}^0)\Gamma_c(\mathcal{F})$ *are linearly dense in* $\Gamma_c(\mathcal{F})$*.* □

The multiplier bundle of a Fell bundle Given a Fell bundle $\mathcal{F}$ on G, we define a multiplier bundle $\mathcal{M}(\mathcal{F})$ on G, extending the definition in [12, §VIII.2.14]. Given a subspace $C \subseteq G$, we denote by $\mathcal{F}|_C$ the restriction of $\mathcal{F}$ to C.

Definition 7.4. *Let $x \in G$. A* multiplier of $\mathcal{F}$ of order x *is a map $T\colon \mathcal{F}|_{G^{s(x)}} \to \mathcal{F}|_{G^{r(x)}}$ such that $T\mathcal{F}_y \subseteq \mathcal{F}_{xy}$ for all $y \in G^{s(x)}$ and such that there exists a map $T^*\colon \mathcal{F}|_{G^{r(x)}} \to \mathcal{F}|_{G^{s(x)}}$ such that $e^*Tf = (T^*e)^*f$ for all $e \in \mathcal{F}|_{G^{r(x)}}$, $f \in \mathcal{F}|_{G^{s(x)}}$. We denote by $\mathcal{M}(\mathcal{F})_x$ the set of all multipliers of $\mathcal{F}$ of order x.*

As for adjointable operators of Hilbert C^*-modules, one deduces from the definition the following simple properties. Let $x \in G$. Then for each $T \in \mathcal{M}(\mathcal{F})_x$, the map T^* is uniquely determined, $T^* \in \mathcal{M}(\mathcal{F})_{x^{-1}}$, and $T^{**} = T$. Moreover, each $T \in \mathcal{M}(\mathcal{F}_x)$ is fiberwise linear in the sense that $T(\kappa e + f) = \kappa Te + Tf$ for all $\kappa \in \mathbb{C}$, $e, f \in \mathcal{F}_y$, $y \in G^{s(x)}$. The restrictions $T_{s(x)}\colon \mathcal{F}_{s(x)} \to \mathcal{F}_x$ and $(T^*)_x\colon \mathcal{F}_x \to \mathcal{F}_{s(x)}$ are adjoint operators of Hilbert C^*-modules over $\mathcal{F}_{s(x)}$, and since $\mathcal{F}_y = [\mathcal{F}_{r(y)}\mathcal{F}_y]$ for each $y \in G^{s(x)}$, the map $\mathcal{M}(\mathcal{F})_x \to \mathcal{L}(\mathcal{F}_{s(x)}, \mathcal{F}_x)$, $T \mapsto T_{s(x)}$, is a bijection. Clearly, we have a natural embedding $\mathcal{F}_x \hookrightarrow \mathcal{M}(\mathcal{F})_x$, where each $f \in \mathcal{F}$ acts as a multiplier via left multiplication. For each $y \in G^{s(x)}$, we have $\mathcal{M}(\mathcal{F})_x\mathcal{M}(\mathcal{F})_y \subseteq \mathcal{M}(\mathcal{F})_{xy}$, and for each $f \in \mathcal{F}_z$, $z \in G_{r(x)}$, we let $fT := (T^*f^*)^*$.

Definition 7.5. *For each $x \in G$, consider $\mathcal{M}(\mathcal{F})_x$ as a Banach space via the identification with $\mathcal{L}(\mathcal{F}_{s(x)}, \mathcal{F}_x)$. Let $\mathcal{M}(\mathcal{F}) = \coprod_{x \in G} \mathcal{M}(\mathcal{F})_x$ and denote by $\tilde{p}\colon \mathcal{M}(\mathcal{F}) \to G$ the natural map. The* strict topology *on $\mathcal{M}(\mathcal{F})$ is the weakest topology that makes $\tilde{p}$ and the maps $\mathcal{M}(\mathcal{F}) \to \mathcal{F}$ of the form $c \mapsto c \cdot d(s(\tilde{p}(c)))$ and $c \mapsto d(r(\tilde{p}(c))) \cdot c$ continuous for each $d \in \Gamma_c(\mathcal{F}^0)$. Denote by $\Gamma_c(\mathcal{M}(\mathcal{F}))$ the space of all sections that are strictly continuous, norm-bounded, and compactly supported.*

Remark 7.6. The bundle $\mathcal{M}(\mathcal{F})$ satisfies all axioms of a Fell bundle except for the fact that it is no Banach bundle with respect to the strict topology unless $\mathcal{M}(\mathcal{F}) = \mathcal{F}$. Indeed, for each $u \in G^0$, the subspace topology on $\mathcal{M}(\mathcal{F})_u \cong \mathcal{L}(\mathcal{F}_u) \cong M(\mathcal{F}_u)$ is the strict topology and coincides with the norm topology only if $M(\mathcal{F}_u) = \mathcal{F}_u$.

Given $f \in \Gamma_c(\mathcal{M}(\mathcal{F}))$ and $g \in \Gamma_0(\mathcal{F}^0)$, define $fg, gf \in \Gamma_c(\mathcal{F})$ by $(fg)(x) = f(x)g(s(x))$, $(gf)(x) = g(r(x))f(x)$ for all $x \in G$ again.

Lemma 7.7. *i) Let $c \in \Gamma_c(\mathcal{M}(\mathcal{F}))$ and $d \in \Gamma_c(\mathcal{F})$. Then there exists a section $cd \in \Gamma_c(\mathcal{F})$ such that $(cd)(x) = \int_{G^{r(x)}} c(y)d(y^{-1}x)\,\mathrm{d}\lambda^{r(x)}(y)$ for all $x \in G$.*

ii) $\Gamma_c(\mathcal{M}(\mathcal{F}))$ *carries a structure of a $*$-algebra such that* $c^*(x) = c(x^{-1})^*$ *and* $(cd)(x)e = \int_{G^{r(x)}} c(y)d(y^{-1}x)e \,\mathrm{d}\lambda^{r(x)}$ *for all* $c, d \in \Gamma_c(\mathcal{M}(\mathcal{F}))$, $x \in G$, $e \in \mathcal{F}_{s(x)}$.

iii) There exists a $*$-homomorphism $L_{\mathcal{M}(\mathcal{F})}: \Gamma_c(\mathcal{M}(\mathcal{F})) \to M(C_r^*(\mathcal{F}))$ *such that* $L_{\mathcal{M}(\mathcal{F})}(c)L_{\mathcal{F}}(d) = L_{\mathcal{F}}(cd)$ *for all* $c \in \Gamma_c(\mathcal{M}(\mathcal{F}))$, $d \in \Gamma_c(\mathcal{F})$.

iv) $\Gamma_c(\mathcal{M}(\mathcal{F}))$ *is closed under pointwise multiplication with elements of* $C_c(G)$.

Proof. i) Define $cd: G \to \mathcal{F}$ as above, and let $\epsilon > 0$. Using Lemma 7.3, we find a sequence $(g_n)_n$ in the span of $\Gamma_0(\mathcal{F}^0)\Gamma_c(\mathcal{F})$ that converges to d in the inductive limit topology. Since $\Gamma_c(\mathcal{M}(\mathcal{F}))\Gamma_0(\mathcal{F}^0) \subseteq \Gamma_c(\mathcal{F})$, the map $h_n: x \mapsto \int_{G^{r(x)}} c(y)g_n(y^{-1}x) \,\mathrm{d}\lambda^{r(x)}(y)$ lies in $\Gamma_c(\mathcal{F})$ for each n. Using the fact that c has compact support and bounded norm, one easily concludes that $(h_n)_n$ converges in the inductive limit topology to cd which therefore is in $\Gamma_c(\mathcal{F})$.

ii) Note that $(cd)(x)$ is well defined because the map $y \mapsto d(y^{-1}x)e$ is in $\Gamma_c(\mathcal{F})$ and thus i) applies. Now, the assertion follows from standard arguments.

iii) One easily verifies that there exists a representation $L_{\mathcal{M}(\mathcal{F})}: \Gamma_c(\mathcal{M}(\mathcal{F})) \to \mathcal{L}(\Gamma^2(\mathcal{F}))$ such that $L_{\mathcal{M}(\mathcal{F})}(c)d = cd$ for all $c \in \Gamma_c(\mathcal{M}(\mathcal{F}))$, $d \in \Gamma_c(\mathcal{F})$, and that $L_{\mathcal{M}(\mathcal{F})}(c)L_{\mathcal{F}}(d)e = cde = L_{\mathcal{F}}(cd)e$ for all $c \in \Gamma_c(\mathcal{M}(\mathcal{F}))$, $d, e \in \Gamma_c(\mathcal{F})$.

iv) This follows immediately from the fact that $\Gamma_c(\mathcal{F})$ is closed under pointwise multiplication by elements of $C_c(G)$. $\square$

Morphisms between Fell bundles Let $\mathcal{F}$ and $\mathcal{G}$ be Fell bundles on G.

Definition 7.8. *A* (fibrewise nondegenerate) morphism *from* $\mathcal{F}$ *to* $\mathcal{G}$ *is a continuous map* $T: \mathcal{F} \to \mathcal{M}(\mathcal{G})$ *that satisfies the following conditions:*

i) for each $x \in G$, *the map* T *restricts to a linear map* $T_x: \mathcal{F}_x \to \mathcal{M}(\mathcal{G})_x$;

ii) $T(e_1)T(e_2) = T(e_1e_2)$ *and* $T(e)^* = T(e^*)$ *for all* $(e_1, e_2) \in \mathcal{F}_{sp}\times_{rp}\mathcal{F}$, $e \in \mathcal{F}$;

iii) $\mathcal{G}_x = [T(\mathcal{F}_x)\mathcal{G}_{s(x)}]$ *for each* $x \in G^0$.

Let T be a morphism from $\mathcal{F}$ to $\mathcal{G}$. Then $T_u: \mathcal{F}_u \to \mathcal{M}(\mathcal{G})_u$ is a nondegenerate $*$-homomorphism for each $u \in G^0$; in particular, $\|T_u\| \leqslant 1$. One easily concludes that $\|T_x\| \leqslant 1$ for each $x \in G$. Hence, the formula $f \mapsto T \circ f$ defines $*$-homomorphisms $T_*: \Gamma_c(\mathcal{F}) \to \Gamma_c(\mathcal{M}(\mathcal{G}))$ and $T_*^0: \Gamma_0(\mathcal{F}^0) \to M(\Gamma_0(\mathcal{G}^0))$.

Proposition 7.9. *i)* $T_*^0: \Gamma_0(\mathcal{F}^0) \to M(\Gamma_0(\mathcal{G}^0))$ *is nondegenerate.*

ii) $T_*(\Gamma_c(\mathcal{F}))\Gamma_c(\mathcal{G}^0)$ *is dense in* $\Gamma_c(\mathcal{F})$.

iii) T_* *extends to a nondegenerate $*$-homomorphism* $T_*: C_r^*(\mathcal{F}) \to M(C_r^*(\mathcal{G}))$.

Proof. i), ii) This follows immediately from Lemma 7.2 and 7.7.

iii) Part ii) and a straightforward calculation show that there exists a unitary $\Psi: \Gamma^2(\mathcal{F}, \lambda^{-1}) \otimes_{T_*^0} \Gamma_0(\mathcal{G}^0) \to \Gamma^2(\mathcal{G}, \lambda^{-1})$ such that $(\Psi(f \otimes g))(x) = T_*(f)g$ for all $f \in \Gamma_c(\mathcal{F})$, $g \in \Gamma_0(\mathcal{G}^0)$. The map $C_r^*(\mathcal{F}) \to \mathcal{L}(\Gamma^2(\mathcal{G}, \lambda^{-1}))$ given by $f \mapsto \Psi(f \otimes \mathrm{id})\Psi^*$ is the desired extension. Lemma 7.3 and part ii) imply that $[T_*(\Gamma_c(\mathcal{F}))\Gamma_c(\mathcal{G})] = [T_*(\Gamma_c(\mathcal{F}))\Gamma_0(\mathcal{G}^0)\Gamma_c(\mathcal{G})] = [\Gamma_c(\mathcal{G})\Gamma_c(\mathcal{G})] = C_r^*(\mathcal{G})$. $\square$

8 From Fell bundles on G to coactions of $C_r^*(G)$

Let G be a groupoid, V the associated C^*-pseudo-multiplicative unitary, and $C_r^*(G)$ or, more precisely, $(\mathcal{A}, \Delta)$ the associated Hopf C^*-bimodule as in Section 5. We relate Fell bundles on G to coactions of $C_r^*(G)$ as follows. Let $\mathcal{F}$ be an admissible Fell bundle $\mathcal{F}$ on G. We shall construct a coaction of $C_r^*(G)$ on $C_r^*(\mathcal{F})$ which is unitarily implemented by a representation of V, and identify the reduced crossed product of this coaction with the reduced C^*-algebra of another Fell bundle. Finally, we show that this construction is functorial.

Recall that a *representation of unitary* V is a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-module ${}_\gamma K_{\widehat{\delta}}$ together with a unitary $X\colon K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H$ that satisfies $X(\gamma \triangleleft \alpha) = \gamma \triangleright \alpha$, $X(\widehat{\delta} \triangleright \beta) = \widehat{\delta} \triangleleft \beta$, $X(\widehat{\delta} \triangleright \widehat{\beta}) = \gamma \triangleright \widehat{\beta}$, and $X_{12}X_{13}V_{23} = V_{23}X_{12}$ [28, Definition 4.1]. We construct a coaction out of such a representation as follows.

Lemma 8.1. *Let $({}_\gamma K_{\widehat{\delta}}, X)$ be a representation of V, let C_K^γ be a C^*-$\mathfrak{b}$-algebra such that $[C, \rho_{\widehat{\beta}}(\mathfrak{B})] = 0$, define $\delta\colon C \to \mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H)$ by $c \mapsto X(c \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})X^*$, and assume that $[\delta(C)|\gamma\rangle_1 A] \subseteq [|\gamma\rangle_1 A]$ and $[\delta(C)|\beta\rangle_2] \subseteq [|\beta\rangle_2 C]$. Then δ is injective, a morphism from (K_γ, C) to $(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_\alpha, C_\gamma \underset{\mathfrak{b}}{*}{}^\beta A)$, and a coaction of $(\mathcal{A}, \Delta)$ on C_K^γ. If the inclusions above are equalities, then δ is left- or right-full, respectively.*

Proof. Evidently, δ is injective. It is a morphism of C^*-$\mathfrak{b}$-algebras because $X|\xi\rangle_2 c = \delta(c)X|\xi\rangle_2$ for each $\xi \in \alpha$, $c \in C$ and because $[X|\alpha\rangle_2\gamma] = \gamma \triangleright \alpha$ and $[(X|\alpha\rangle_2)^*(\gamma \triangleright \alpha)] = [\langle\alpha|_2(\gamma \triangleleft \alpha)] = \gamma$. A similar diagram as in [28, Proof of Lemma 3.13] shows that $(\delta * \mathrm{id})(\delta(c)) = (\mathrm{id} * \Delta)(\delta(c))$ for each $c \in C$. □

The representation of V associated to $\mathcal{F}$ Denote by $\mathcal{W} = \mathcal{W}(\Gamma_0(\mathcal{F}^0))$ the set of all $C_0(G^0)$-weights on $\Gamma_0(\mathcal{F}^0)$ and let $\phi \in \mathcal{W}$.

Lemma 8.2. *Let $c, d \in \Gamma_c(\mathcal{F})$. Then the map $x \mapsto \phi_{s(x)}(c(x)^*d(x))$ lies in $C_c(G)$.*

Proof. The function $G \to s^*\mathcal{F}^0$ given by $x \mapsto c(x)^*d(x)$ is continuous and has compact support, and the composition $h\colon x \mapsto \phi_{s(x)}(c(x)^*d(x))$ is continuous because the map $\mathcal{F}^0 \to \mathbb{C}$ given by $f \mapsto \phi_{p(f)}(f)$ is continuous. □

Define Hilbert C^*-$C_0(G^0)$-modules $\Gamma^2(\mathcal{F}, \lambda; \phi)$, $\Gamma^2(\mathcal{F}, \lambda^{-1}; \phi)$ and a Hilbert space $K_\phi = \Gamma^2(\mathcal{F}, \nu; \phi)$ as the respective completions of $\Gamma_c(\mathcal{F})$, where for all $c, d \in \Gamma_c(\mathcal{F})$, $f \in C_0(G^0)$, the inner product $\langle c|d\rangle$ and the product cf are given by

$$u \mapsto \int_{G^u} \phi_{s(x)}(c(x)^*d(x))\,\mathrm{d}\lambda^u(x), \quad y \mapsto c(y)f(r(y)) \quad \text{in case of } \Gamma^2(\mathcal{F}, \lambda; \phi),$$
$$u \mapsto \int_{G_u} \phi_{s(x)}(c(x)^*d(x))\,\mathrm{d}\lambda_u^{-1}(x), \quad y \mapsto c(y)f(s(y)) \quad \text{in case of } \Gamma^2(\mathcal{F}, \lambda^{-1}; \phi),$$
$$\text{and} \int_G \phi_{s(x)}(c(x)^*d(x))\,\mathrm{d}\nu(x) \quad \text{in case of } \Gamma^2(\mathcal{F}, \nu; \phi).$$

Lemma 8.3. $[\langle E|E\rangle] = [\phi(\Gamma_0(\mathcal{F}^0))]$ *for* $E \in \{\Gamma^2(\mathcal{F}, \lambda, \phi), \Gamma^2(\mathcal{F}, \lambda^{-1}; \phi)\}$.

Proof. Assume that $(\phi(c^*c))(u) \neq 0$ for some $c \in \Gamma_c(\mathcal{F}^0)$, $u \in G^0$. Choose $d \in \Gamma_c(\mathcal{F})$ such that $d|_{G^0} = c$. Then the function on G given by $x \mapsto \phi_{s(x)}(d(x)^*d(x))$ is non-negative and nonzero at u, whence $\langle d|d\rangle_E(u) \neq 0$. Now, the assertion follows because $[\langle E|E\rangle]$ and $[\phi(\Gamma_0(\mathcal{F}^0))]$ are closed ideals in $C_0(G^0)$. □

Let $K = \bigoplus_{\phi \in \mathcal{W}} K_\phi$ and identify each K_ϕ with a subspace of K. Given $c \in \Gamma_c(\mathcal{F})$ and $f \in C_0(G^0)$, define $fc, cf, cD^{-1/2} \in \Gamma_c(\mathcal{F})$ by

$$fc\colon x \mapsto f(r(x))c(x), \qquad cf\colon x \mapsto c(x)f(s(x)), \qquad cD^{-1/2}\colon x \mapsto c(x)D^{-1/2}(x).$$

Let $\phi \in \mathcal{W}$. Straightforward calculations show that there exist maps

$$j_\phi\colon \Gamma^2(\mathcal{F}, \lambda; \phi) \to \mathcal{L}(\mathfrak{K}, K_\phi) \qquad \text{and} \qquad \hat{j}_\phi\colon \Gamma^2(\mathcal{F}, \lambda^{-1}; \phi) \to \mathcal{L}(\mathfrak{K}, K_\phi)$$

such that $j_\phi(c)f = fc$ and $\hat{j}_\phi(c)f = (cD^{-1/2})f$ for all $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G^0)$, and

$$j_\phi(c)^*j_\phi(d) = \langle c|d\rangle_{\Gamma^2(\mathcal{F},\lambda;\phi)}, \quad \hat{j}_\phi(c)^*\hat{j}_\phi(d) = \langle c|d\rangle_{\Gamma^2(\mathcal{F},\lambda^{-1};\phi)} \text{for all } c, d \in \Gamma_c(\mathcal{F}).$$

Denote by $\gamma \subseteq \mathcal{L}(\mathfrak{K}, K)$ and $\widehat{\delta} \subseteq \mathcal{L}(\mathfrak{K}, K)$ the closed linear span of all subspaces $j_\phi(\Gamma^2(\mathcal{F}, \lambda; \phi))$ and $\hat{j}_\phi(\Gamma^2(\mathcal{F}, \lambda^{-1}; \phi))$, respectively, where $\phi \in \mathcal{W}$. Lemmas 6.4 and 8.3 imply:

Lemma 8.4. *${}_\gamma K_{\widehat{\delta}}$ is a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-module and $\rho_\gamma(f)(c_\phi)_\phi = (fc_\phi)_\phi$ and $\rho_{\widehat{\delta}}(f)(c_\phi)_\phi = (c_\phi f)_\phi$ for all $f \in C_0(G^0)$, $(c_\phi)_\phi \in \bigoplus_\phi \Gamma_c(\mathcal{F}) \subseteq K$.* □

For $t = s, r$, denote by $p_1^{t,r}\colon G_t \times_r G \to G$ the projection onto the first component, by $\mathcal{F}^2_{t,r} = (p_1^{t,r})^*\mathcal{F}$ the corresponding pull-back of $\mathcal{F}$, and by $\Gamma^2(\mathcal{F}^2_{t,r}, \nu^2_{t,r}; \phi)$ the Hilbert space that is the completion of $\Gamma_c(\mathcal{F}^2_{t,r})$ with respect to the inner product

$$\langle c|d\rangle = \int_{G_t \times_r G} \phi_{s(x)}(c(x,y)^*d(x,y))\, \mathrm{d}\nu^2_{t,r}(x,y).$$

Straightforward calculations show that there exist unitaries

$$\Phi\colon K_{\widehat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to \bigoplus_{\phi \in \mathcal{W}} \Gamma^2(\mathcal{F}^2_{s,r}, \nu^2_{s,r}; \phi), \qquad \Psi\colon K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H \to \bigoplus_{\phi \in \mathcal{W}} \Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi),$$

such that for all $\phi \in \mathcal{W}$, $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G^0)$, $g \in C_c(G)$,

- $\Phi\big(\hat{j}_\phi(c) \otimes f \otimes j(g)\big)$ is in $\Gamma^2(\mathcal{F}^2_{s,r}; \nu^2_{s,r}; \phi)$ and given by $(x,y) \mapsto ((cD^{-1/2})f)(x)g(y)$,
- $\Psi\big(j_\phi(c) \otimes f \otimes j(g)\big)$ is in $\Gamma^2(\mathcal{F}^2_{r,r}; \nu^2_{r,r}; \phi)$ and given by $(x,y) \mapsto (fc)(x)g(y)$.

We use the isomorphisms above without further notice. If $(T_\phi)_\phi$ is a norm-bounded family of operators between Hilbert spaces $(H^1_\phi)_\phi$ and $(H^2_\phi)_\phi$, we denote by $\bigoplus_\phi T_\phi \in \mathcal{L}(\bigoplus_\phi H^1_\phi, \bigoplus_\phi H^2_\phi)$ the operator given by $(\xi_\phi)_\phi \mapsto (T_\phi \xi_\phi)_\phi$. Similar arguments as those used for the construction of V in [28, Theorem 2.7] show:

Proposition 8.5. *For each $\phi \in \mathcal{W}$, there exists a unitary $X_\phi\colon \Gamma^2(\mathcal{F}^2_{s,r}, \nu^2_{s,r}; \phi) \to \Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi)$ such that $(X_\phi f)(x,y) = f(x, x^{-1}y)$ for all $f \in \Gamma_c(\mathcal{F}^2_{s,r})$, $(x,y) \in G_r \times_r G$. The pair $({}_\gamma K_{\widehat{\delta}}, \bigoplus_\phi X_\phi)$ is a representation of V.* □

The coaction of $C_r^*(G)$ on $C_r^*(\mathcal{F})$ We apply Lemma 8.1 to the representation $({}_\gamma K_{\widehat{\delta}}, X)$ and obtain a coaction of $C_r^*(G)$ on $C_r^*(\mathcal{F})$ as follows.

Lemma 8.6. *Let $\phi \in \mathcal{W}$. There exists a representation $\pi_\phi \colon C_r^*(\mathcal{F}) \to \mathcal{L}(K_\phi)$ such that for all $c, d \in \Gamma_c(\mathcal{F})$, $x \in G$,*

$$(\pi_\phi(c)d)(x) = \int_{G^{r(x)}} c(z)d(z^{-1}x)D^{-1/2}(z)\,\mathrm{d}\lambda^{r(x)}(z)$$

and $\pi_\phi(c)\hat{j}_\phi(d) = \hat{j}_\phi(cd)$ and $\pi_\phi(c)\rho_\gamma(f) = \pi_\phi(cf)$ for all $c, d \in \Gamma_c(\mathcal{F})$, $f \in C_0(G^0)$.

Proof. Identify $\Gamma^2(\mathcal{F}, \lambda^{-1}) \circleddash_\phi L^2(G^0, \mu)$ with K_ϕ via $c \circleddash f \mapsto \hat{j}_\phi(c)f$ for all $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G^0)$, and define π_ϕ by $c \mapsto c \circleddash_\phi \mathrm{id}$. ☐

Define $\pi \colon C_r^*(\mathcal{F}) \to \mathcal{L}(K)$ by $c \mapsto \bigoplus_\phi \pi_\phi(c)$. Lemmas 6.4 and 8.6 imply:

Lemma 8.7. *The representation π is faithful, $\pi(C_r^*(\mathcal{F}))_K^\gamma$ is a C^*-$\mathfrak{b}$-algebra, and $[\pi(C_r^*(\mathcal{F}))\widehat{\delta}] = \widehat{\delta}$.* ☐

Define $\delta \colon \pi(C_r^*(\mathcal{F})) \to \mathcal{L}(K {}_\gamma\!\underset{\mathfrak{b}}{\otimes}{}_\beta H)$ by $\pi(c) \mapsto X(\pi(c) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})X^*$. Then $\delta(\pi(c)) = \bigoplus_\phi \delta(\pi(c))_\phi$ for each $c \in C_r^*(\mathcal{F})$, where $\delta(\pi(c))_\phi \in \mathcal{L}(\Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi))$ acts as follows.

Lemma 8.8. *For all $c \in \Gamma_c(\mathcal{F})$, $\phi \in \mathcal{W}$, $d \in \Gamma_c(\mathcal{F}^2_{r,r})$, $(x, y) \in G_r \times_r G$,*

$$\big(\delta(\pi(c))_\phi d\big)(x, y) = \int_{G^{r(x)}} c(z)d(z^{-1}x, z^{-1}y)D^{-1/2}(z)\,\mathrm{d}\lambda^{r(x)}(z).$$

Proof. The verification is straightforward and similar to the calculation of the comultiplication Δ on $C_r^*(G)$; see [28, Theorem 3.22]. ☐

Theorem 8.9. *$(\pi(C_r^*(\mathcal{F}))_K^\gamma, \delta)$ is a very fine and left-full coaction of $C_r^*(G)$.*

The proof involves the following two lemmas.

Lemma 8.10. *Let $\phi \in \mathcal{W}$. There exist maps $T_\phi \colon \Gamma_c(\mathcal{F}^2_{r,r}) \to \mathcal{L}(K_\phi, \Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi))$ and $S_\phi \colon \Gamma_c(\mathcal{F}^2_{r,r}) \to \mathcal{L}(H, \Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi))$ that are continuous with respect to the inductive topology on $\Gamma_c(\mathcal{F}^2_{r,r})$ and the operator norm, respectively, such that for all $c \in \Gamma_c(\mathcal{F}^2_{r,r})$, $d \in \Gamma_c(\mathcal{F})$, $f \in C_c(G)$, $(x, y) \in G_r \times_r G$,*

$$(T_\phi(c)d)(x, y) = \int_{G^{r(x)}} c(z, y)d(z^{-1}x)D^{-1/2}(z)\,\mathrm{d}\lambda^{r(x)}(z),$$
$$(S_\phi(c)f)(x, y) = \int_{G^{r(y)}} c(x, z)f(z^{-1}y)D^{-1/2}(z)\,\mathrm{d}\lambda^{r(y)}(z).$$

Proof. Let $c, d, T_\phi(c)d$ as above. Then

$$\|T_\phi(c)d\|^2 = \int_G \int_{G^{r(x)}} \int_{G^{r(x)}} \int_{G^{r(x)}} \phi_{s(x)}\left(d(z_1^{-1}x)^*c(z_1, y)^*c(z_2, y)d(z_2^{-1}x)\right) \cdot \\ \cdot D^{-1/2}(z_1)D^{-1/2}(z_2)\,\mathrm{d}\lambda^{r(x)}(y)\,\mathrm{d}\lambda^{r(x)}(z_1)\,\mathrm{d}\lambda^{r(x)}(z_2)\,\mathrm{d}\nu(x).$$

We substitute $x' = z_1^{-1}x$, $z = z_1^{-1}z_2$, use the relations $D(z_2) = D(z_1)D(z)$ and

$$\begin{aligned}D^{-1}(z_1)\,\mathrm{d}\lambda^{r(x)}(z_1)\,\mathrm{d}\nu(x) &= D^{-1}(z_1)\,\mathrm{d}\lambda^{r(z_1)}(x)\,\mathrm{d}\nu(z_1)\\ &= \mathrm{d}\lambda^{s(z_1)}(x')\,\mathrm{d}\nu^{-1}(z_1) = \mathrm{d}\lambda^{-1}_{r(x')}(z_1)\,\mathrm{d}\nu(x'),\end{aligned}$$

and find

$$\begin{aligned}\|T_\phi(c)d\|^2 &= \int_G\int_{G_{r(x')}}\int_{G^{r(x')}}\int_{G^{s(z_1)}} \phi_{s(x)}(d(x')^*c(z_1,y)^*c(z_1z,y)d(z^{-1}x'))\cdot\\ &\qquad\qquad \cdot D^{-1/2}(z)\,\mathrm{d}\lambda^{s(z_1)}(y)\,\mathrm{d}\lambda^{r(x')}(z)\,\mathrm{d}\lambda^{-1}_{r(x')}(z_1)\,\mathrm{d}\nu(x')\\ &= \int_G\int_{G^{r(x')}} \phi_{s(x')}(d(x')R_c(z)d(z^{-1}x'))\,\mathrm{d}\lambda^{r(x')}(z)\,\mathrm{d}\nu(x') = \langle d|\pi_\phi(R_c)d\rangle_{K_\phi},\end{aligned}$$

where $R_c \in \Gamma_c(\mathcal{F})$ is given by

$$R_c(z) = \int_{G_{r(z)}}\int_{G^{s(z_1)}} c(z_1,y)^*c(z_1z,y)\,\mathrm{d}\lambda^{s(z_1)}(y)\,\mathrm{d}\lambda^{-1}_{r(z)}(z_1) \quad \text{for all } z \in G.$$

Hence, $T_\phi(c)$ extends to a bounded linear operator of norm $\|T_\phi(c)\|^2 \leqslant \|\pi_\phi(R_c)\|$. If $(c_n)_n$ is a sequence in $\Gamma_c(\mathcal{F}^2_{r,r})$ converging to c in the inductive limit topology, then the functions $R_{(c-c_n)}$ defined similarly as R_c converge to 0 in the inductive limit topology and hence $\|T_\phi(c-c_n)\|^2 \leqslant \|\pi_\phi(R_{(c-c_n)})\|$ converges to 0.

The proof of the assertion concerning S_ϕ is very similar. □

Given $c, d \in \Gamma_c(\mathcal{F})$ and $f \in C_c(G)$, define $\omega_{c,d,f} \in \Gamma_c(\mathcal{F}^2_{r,r})$ by

$$(x,y) \mapsto \int_{G^{r(x)}} c(z)d(z^{-1}x)f(z^{-1}y)\,\mathrm{d}\lambda^{r(x)}(z).$$

Lemma 8.11. *The linear span of all elements $\omega_{c,d,f}$ as above is dense in $\Gamma_c(\mathcal{F}^2_{r,r})$ with respect to the inductive limit topology.*

Proof. Let $(x,y) \in G_r \times_r G$, $e \in \mathcal{F}_x$, let $C \subseteq G_r \times_r G$ be a compact neighbourhood of (x,y), and let $\epsilon > 0$. Since $[\mathcal{F}_{r(x)}\mathcal{F}_x] = \mathcal{F}_x$, we can choose $c', d' \in \Gamma_c(\mathcal{F})$ such that $\|c'(z)d'(z^{-1}x) - e\| < \epsilon$ for all z in some neighbourhood of $r(x)$ in $G^{r(x)}$. Next, we can choose $h_c, h_d, f \in C_c(G)$ such that the elements $c, d \in \Gamma_c(\mathcal{F})$ given by $c(z) = c'(z)h_c(z)$ and $d(z) = d'(z)h_d(z)$ for all $z \in G$ satisfy $\|\omega_{c,d,f}(x,y) - e\| < \epsilon$ and $\operatorname{supp}\omega_{c,d,f} \subseteq C$. A standard partition of unity argument concludes the proof. □

Proof of Theorem 8.9. We show that Lemma 8.1 applies. Let $\phi \in \mathcal{W}$, $c, d \in \Gamma_c(\mathcal{F})$, $f, g \in C_c(G)$. Define $e_1, e_2, e_3, e_4 \in \Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi)$ and $\omega_1, \omega_2, \omega_3, \omega_4 \in \Gamma_c(\mathcal{F}^2_{r,r})$ by

$$\begin{aligned}
e_1 &= \delta(\pi(c))_\phi|j(f)\rangle_2 d, & \omega_1(z,y) &= c(z)f(z^{-1}y) \text{ for all } (z,y) \in G_r\times_r G,\\
e_2 &= |j(f)\rangle_2\pi_\phi(c)d, & \omega_2(z,y) &= c(z)f(y) \text{ for all } (z,y) \in G_r\times_r G,\\
e_3 &= |j_\phi(c)\rangle_1 L(f)g, & \omega_3(x,z) &= c(x)f(z) \text{ for all } (x,z) \in G_r\times_r G,\\
e_4 &= \delta(\pi(c))_\phi|j_\phi(d)\rangle_1 L(f)g, & \omega_4 &= \omega_{c,d,f}.
\end{aligned}$$

Using Lemma 8.8, we find that for all $(x,y) \in G_r {}_r\!\times_r G$,

$$\begin{aligned}
e_1(x,y) &= \int_{G^{r(x)}} c(z)D^{-1/2}(z)d(z^{-1}x)f(z^{-1}y)\,\mathrm{d}\lambda^{r(x)}(z) = (T_\phi(\omega_1)d)(x,y),\\
e_2(x,y) &= \int_{G^{r(x)}} c(z)d(z^{-1}x)D^{-1/2}(z)\,\mathrm{d}\lambda^{r(x)}(z)f(y) = (T_\phi(\omega_2)d)(x,y),\\
e_3(x,y) &= c(x)\int_{G^{r(y)}} f(z)D^{-1/2}(z)g(z^{-1}y)\,\mathrm{d}\lambda^{r(y)}(z) = (S_\phi(\omega_3)g)(x,y),\\
e_4(x,y) &= \int_{G^{r(x)}} c(z_1)D^{-1/2}(z_1)d(z_1^{-1}x)(L(f)g)(z_1^{-1}y)\,\mathrm{d}\lambda^{r(x)}(z_1)\\
&= \int_{G^{r(x)}}\int_{G^{s(z_1)}} c(z_1)D^{-1/2}(z_1)d(z_1^{-1}x)f(z_2)\cdot\\
&\qquad\qquad \cdot D^{-1/2}(z_2)g(z_2^{-1}z_1^{-1}y)\,\mathrm{d}\lambda^{s(z_1)}(z_2)\,\mathrm{d}\lambda^{r(x)}(z_1)\\
&= \int_{G^{r(x)}}\int_{G^{r(x)}} c(z_1)d(z_1^{-1}x)f(z_1^{-1}z_2')D^{-1/2}(z_2')g(z_2'^{-1}y)\,\mathrm{d}\lambda^{r(x)}(z_2')\,\mathrm{d}\lambda^{r(x)}(z_1)\\
&= (S_\phi(\omega_{c,d,f})g)(x,y).
\end{aligned}$$

By Lemmas 7.2 and 8.11, sections of the form like $\omega_1, \omega_2, \omega_3$ or ω_4, respectively, are linearly dense in $\Gamma_c(\mathcal{F}^2_{r,r})$. Therefore, $[\delta(\pi(C^*_r(\mathcal{F})))_\phi|\alpha\rangle_2] = [T_\phi(\Gamma_c(\mathcal{F}^2_{r,r}))] = [|\alpha\rangle_2\pi_\phi(C^*_r(\mathcal{F}))]$ and similarly $[\delta(\pi(C^*_r(\mathcal{F})))|\gamma\rangle_1 C^*_r(G)] = [\bigcup_{\phi\in\mathcal{W}} S_\phi(\Gamma_c(\mathcal{F}^2_{r,r}))] = [|\gamma\rangle_1 C^*_r(G)]$. ☐

Given $g, g' \in C_c(G)$, define $h_{g,g'} \in C_c(G)$ by

$$h_{g,g'}(z) = \int_{G^{r(z)}} \overline{g(y)}g'(z^{-1}y)\,\mathrm{d}\lambda^{r(z)}(y) \quad \text{for all } z \in G. \tag{17}$$

Lemma 8.12. *Let $c \in \Gamma_c(\mathcal{F})$, $g, g' \in C_c(G)$. Then $\langle j(g)|_2\delta(\pi(c))_\phi|j(g')\rangle_2 = \pi_\phi(c')$, where $c'(x) = c(x)h_{g,g'}(x)$ for all $x \in G$.*

Proof. The operators on both sides map each $d \in \Gamma_c(\mathcal{F})$ to the section

$$x \mapsto \int_{G^{r(x)}}\int_{G^{r(x)}} \overline{g(y)}c(z)d(z^{-1}x)g'(z^{-1}y)D^{-1/2}(z)\,\mathrm{d}\lambda^{r(x)}(z)\,\mathrm{d}\lambda^{r(x)}(y). \qquad ☐$$

The reduced crossed product of the coaction The bundle $\mathcal{F}^2_{s,r}$ carries the structure of a Fell bundle, and the reduced crossed product $\pi(C^*_r(\mathcal{F})) \rtimes_r C_0(G)$ for the coaction δ constructed above can be identified with $C^*_r(\mathcal{F}^2_{s,r})$ as follows.

Denote by $G \ltimes G$ the transformation groupoid for the action of G on itself given by right multiplication. Thus, $G \ltimes G = G_s {}_s\!\times_r G$ as a set, $(G \ltimes G)^0 = \bigcup_{u\in G^0}\{u\} \times G^u$ can be identified with G via $(r(y), y) \equiv y$, the range map $\tilde{r}$, the source map $\tilde{s}$, and the multiplication are given by $(x,y) \overset{\tilde{r}}{\mapsto} xy$, $(x,y) \overset{\tilde{s}}{\mapsto} y$, and $((x,y),(x',y')) \mapsto (xx', y')$, respectively, and the topology on $G \ltimes G$ is the weakest topology that makes $\tilde{r}, \tilde{s}$ and the map $(x,y) \mapsto x$ continuous. We equip $G \ltimes G$ with the right Haar system $\tilde{\lambda}^{-1}$ given by $\tilde{\lambda}^{-1}_y(C \times \{y\}) = \lambda^{-1}_{r(y)}(C)$ for all $C \subseteq G_{r(y)}$, $y \in G$.

The bundle $\mathcal{F}^2_{s,r}$ is a Fell bundle on $G \ltimes G$ with respect to the multiplication and involution given by $((f,y),(f',y')) \mapsto (ff', y')$ and $(f,y) \mapsto (f^*, p(f)y)$. The convolution product in $\Gamma_c(\mathcal{F}^2_{s,r})$ is given by

$$(cd)(x,y) = \int_{G_{r(y)}} c(xz^{-1}, zy)d(z,y)\,\mathrm{d}\lambda^{-1}_{r(y)}(z) \tag{18}$$

for all $c, d \in \Gamma_c(\mathcal{F}^2_{s,r})$, $(x,y) \in G_s{\times_r}G$, because $(G \ltimes G)_{\tilde{s}(x,y)} = G_{r(y)} \times \{y\}$ and $(x,y)(z,y)^{-1} = (xz^{-1}, zy)$ for all $z \in G_{r(y)}$.

Proposition 8.13. *There exists an isomorphism $\pi(C^*_r(\mathcal{F})) \rtimes_r C_0(G) \to C^*_r(\mathcal{F}^2_{s,r})$ such that $\delta(\pi(c))(1 \underset{\mathfrak{b}}{\otimes} f) \mapsto L_{\mathcal{F}^2_{s,r}}(d)$ whenever $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G)$, and $d(x,y) = c(x)f(y)$ for all $(x,y) \in G_s{\times_r}G$.*

Let $\phi \in \mathcal{W}$. Then the map $r^*\phi\colon \Gamma_0((\mathcal{F}^2_{s,r})^0) \to C_0(G)$ given by $(r^*\phi(c))(y) = \phi_{r(y)}(c(r(y),y))$ for all $c \in \Gamma_0((\mathcal{F}^2_{s,r})^0)$ and $y \in G$ is a $C_0(G)$-weight. One easily verifies that there exists a representation $L_{r^*\phi}\colon C^*_r(\mathcal{F}^2_{s,r}) \to \mathcal{L}(\Gamma^2(\mathcal{F}^2_{s,r}, \tilde{\lambda}^{-1}; r^*\phi))$ such that $L_{r^*\phi}(c)d = cd$ for all $c, d \in \Gamma_c(\mathcal{F}^2_{s,r})$.

Lemma 8.14. *i) There exists a unique unitary $U_\phi\colon \Gamma^2(\mathcal{F}^2_{s,r}, \tilde{\lambda}^{-1}; r^*\phi) \ominus H \to \Gamma^2(\mathcal{F}^2_{s,r}, \nu^2_{s,r}; \phi) \subseteq K_{\hat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H$ such that $(U_\phi(e \ominus g))(x,y) = e(x,y)g(y)D^{-1/2}(x)$ for all $e \in \Gamma_c(\mathcal{F}^2_{s,r})$, $g \in C_c(G)$, $(x,y) \in G_s{\times_r}G$.*

ii) $\delta(\pi(c))(1 \underset{\mathfrak{b}}{\otimes} f)X_\phi U_\phi = X_\phi U_\phi(L_{r^*\phi}(d) \ominus \mathrm{id})$ for all c, d, f as in Proposition 8.13.

Proof. i) For all e, g as in above,

$$\|U_\phi(e \ominus g)\|^2 = \int_G \int_{G_{r(y)}} \phi_{s(x)}(e(x,y)^* e(x,y))|g(y)|^2 \,\mathrm{d}\lambda^{-1}_{r(y)}(x)\,\mathrm{d}\nu(y) = \|e \ominus g\|^2.$$

ii) Let $c, d, e, f, g, (x,y)$ as above and $\widehat{\Delta}(f)_\phi = X^*_\phi(1 \underset{\mathfrak{b}}{\otimes} f)X_\phi$. A short calculation shows that $\big(\widehat{\Delta}(f)_\phi U_\phi(e \ominus g)\big)(x,y) = f(xy)e(x,y)g(y)D^{-1/2}(x)$. Using (18), we find that $\big((\pi_\phi(c) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})\widehat{\Delta}(f)_\phi U_\phi(e \ominus g)\big)(x,y)$ is equal to

$$\begin{aligned}
&\int_{G^{r(x)}} c(z)f(z^{-1}xy)e(z^{-1}x,y)g(y)D^{-1/2}(z)D^{-1/2}(z^{-1}xy)\,\mathrm{d}\lambda^{r(x)}(z) \\
&\qquad = \int_{G_{s(x)}} c(xz^{-1})f(zy)e(z,y)g(y)D^{-1/2}(xy)\,\mathrm{d}\lambda^{-1}_{s(x)}(z) \\
&\qquad = \int_{G_{s(x)}} d(xz^{-1}, zy)e(z,y)g(y)D^{-1/2}(xy)\,\mathrm{d}\lambda^{-1}_{s(x)}(z) = (U_\phi(de \ominus g))(x,y).
\end{aligned}$$

Thus, $\delta(\pi(c))(1 \underset{\mathfrak{b}}{\otimes} f)X_\phi U_\phi = X_\phi(\pi_\phi(c) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})\widehat{\Delta}(f)_\phi X_\phi U_\phi = X_\phi U_\phi(L_{r^*\phi}(d) \ominus \mathrm{id})$. □

Proof of Proposition 8.13. Consider the $*$-homomorphism

$$\Phi\colon C^*_r(\mathcal{F}^2_{s,r}) \to \mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H), \quad L_{\mathcal{F}^2_{s,r}}(d) \mapsto \bigoplus_{\phi \in \mathcal{W}} X_\phi U_\phi(L_{r^*\phi}(d) \ominus \mathrm{id})U^*_\phi X^*_\phi.$$

By part ii) of the lemma above, $\Phi(C_r^*(\mathcal{F}_{s,r}^2))$ contains $[\delta(\pi(C_r^*(\mathcal{F})))(1 \underset{\mathfrak{b}}{\otimes} C_0(G))] = \pi(C_r^*(\mathcal{F})) \rtimes_r C_0(G)$. The same lemma implies that this inclusion is an equality because the map $a \to \bigoplus_\phi a \otimes \mathrm{id}$ is continuous with respect to the inductive limit topology on $\Gamma_c(\mathcal{F}_{s,r}^2)$ and sections of the form $(x,y) \mapsto c(x)f(y)$, where $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G)$, are dense in $\Gamma_c(\mathcal{F}_{s,r}^2)$ by Lemma 7.2. Lemma 6.4 implies that $[\bigcap_\phi \ker r^*\phi] = 0$, and therefore Φ is injective. □

Proposition 8.15. *If $\mathcal{F}$ is saturated, then $C_r^*(\mathcal{F}_{s,r}^2) \cong \mathcal{K}(\Gamma^2(\mathcal{F}, \lambda^{-1}))$.*

Proof. To simplify notation, let $\Gamma^2 = \Gamma^2(\mathcal{F}, \lambda^{-1})$, $\tilde{\Gamma}^2 = \Gamma^2(\mathcal{F}_{s,r}^2, \tilde{\lambda}^{-1})$, $\Gamma_0 = \Gamma_0(\mathcal{F}^0)$, $\tilde{\Gamma}_0 = \Gamma_0((\mathcal{F}_{s,r}^2)^0)$. There exists a unitary $\Psi \colon \Gamma^2 \otimes_{s^*} C_0(G) \to \tilde{\Gamma}^2$ such that $(\Psi(c \otimes f))(x,y) = c(xy)f(y)$ for all $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G)$, $(x,y) \in G_s {\times}_r G$, because

$$\begin{aligned}\langle \Psi(c \otimes f) | \Psi(c' \otimes f') \rangle((r(y), y)) &= \int_{G_{r(y)}} c(xy)^* c'(xy) \,\mathrm{d}\lambda_{r(y)}^{-1}(x) \overline{f(y)} f'(y) \\ &= \overline{f(y)} \langle c | c' \rangle_{\Gamma^2}(s(y)) f(y) = \langle c \otimes f | c' \otimes f' \rangle(y)\end{aligned}$$

for all $c, c' \in \Gamma_c(\mathcal{F}), f, f' \in C_c(G)$, $y \in G$ by right-invariance of λ^{-1}. The $*$-homomorphism $\Phi \colon \mathcal{K}(\Gamma^2) \to \mathcal{L}(\tilde{\Gamma}^2)$ given by $k \mapsto \Psi(k \otimes_{s^*} \mathrm{id})\Psi^*$ is injective because $s^* \colon C_0(G^0) \to \mathcal{L}(C_0(G))$ is injective, and the claim follows once we have shown that $\Phi(\mathcal{K}(\Gamma^2)) = C_r^*(\mathcal{F}_{s,r}^2)$. Let $d, d' \in \Gamma_c(\mathcal{F})$ and denote by $|d\rangle\langle d'| \in \mathcal{K}(\Gamma^2)$ the operator given by $e \mapsto d\langle d' | e \rangle$. Then for all $c, f, (x,y)$ as above,

$$\begin{aligned}\big(\Psi(|d\rangle\langle d'| c \otimes f)\big)(x,y) &= \int_{G_{s(y)}} d(xy) d'(z)^* c(z) f(y) \,\mathrm{d}\lambda_{s(y)}^{-1}(z) \\ &= \int_{G_{s(y)}} d(xy) d'(z)^* \big(\Psi(c \otimes f)\big)(zy^{-1}, y) \,\mathrm{d}\lambda_{s(y)}^{-1}(z) \\ &= \int_{G_{r(y)}} d(xy) d'(z'y)^* \big(\Psi(c \otimes f)\big)(z', y) \,\mathrm{d}\lambda_{r(y)}^{-1}(z').\end{aligned}$$

Comparing with equation (18), we find that $\Psi(|d\rangle\langle d'| \otimes \mathrm{id})\Psi^* = L_{\mathcal{F}_{s,r}^2}(e)$, where $e \in \Gamma_c(\mathcal{F}_{s,r}^2)$ is given by $e(xz^{-1}, zy) = d(xy)d'(zy)^*$, or equivalently, by $e(x', y') = d(x'y')d'(y')^*$ for all $(x', y') \in G_s {\times}_r G$. Since $\mathcal{F}$ is saturated, Lemma 7.2 implies that sections of this form are dense in $\Gamma_c(\mathcal{F}_{s,r}^2)$ with respect to the inductive limit topology, and since the map $e \mapsto L_{\mathcal{F}_{s,r}^2}(e)$ is continuous with respect to this topology, we can conclude that $\Phi(\mathcal{K}(\Gamma^2)) = \Psi(\mathcal{K}(\Gamma^2) \otimes \mathrm{id})\Psi^* = C_r^*(\mathcal{F}_{s,r}^2)$. □

Corollary 8.16. *If $\mathcal{F}$ is saturated, then $\pi(C_r^*(\mathcal{F})) \rtimes_r C_0(G)$ and $\Gamma_0(\mathcal{F}^0)$ are Morita equivalent.*

Proof. One easily verifies that $\Gamma^2(\mathcal{F}, \lambda^{-1})$ is full. □

Example 8.17. Let σ be an action of G on an admissible $C_0(G^0)$-algebra C and let δ_σ be the corresponding coaction of $C_0(G)$ on $\mathbf{F}C$ (Proposition 6.10). Then there exists an admissible Fell bundle $\mathcal{C}$ on G with fibre $\mathcal{C}_x = C_{r(x)}$ for each $x \in G$, continuous sections $\Gamma_0(\mathcal{C}) = r^*C$, and multiplication and involution given by $cd = c\sigma_x(d)$, $c^* = \sigma_{x^{-1}}(c^*)$ for all $c \in \mathcal{C}_x$, $d \in \mathcal{C}_y$, $(x,y) \in G_s {\times}_r G$ [15], and the identity on

$\Gamma_c(\mathcal{C}) = C_c(G)r^*C$ extends to an isomorphism $C_r^*(\mathcal{C}) \to C \rtimes_r G$. One easily verifies that with respect to the isomorphism $\pi(C_r^*(\mathcal{C})) \cong C_r^*(\mathcal{C}) \cong C \rtimes_r G \cong \mathbf{F}C \rtimes_r C_r^*(G)$ of Proposition 6.14, the coaction of Theorem 8.9 coincides with the dual coaction on $\mathbf{F}C \rtimes_r C_r^*(G)$. Moreover, the Fell bundle $\mathcal{C}$ is saturated and $C_r^*(\mathcal{C}) \rtimes_r C_0(G) \cong \mathbf{F}C \rtimes_r C_r^*(G) \rtimes_r C_0(G)$ is Morita equivalent to $\Gamma_0(\mathcal{C}^0) \cong C$, as we already know by Theorem 4.11.

Remark 8.18. The Fell bundle $\mathcal{F}$ can be equipped with the structure of an $\mathcal{F}^2_{s,r}$-$\mathcal{F}^0$-equivalence in the sense of [23] in a straightforward way.

Functoriality of the construction Let $\mathcal{G}$, $\mathcal{F}$ be admissible Fell bundles on G with associated representations $((K_{\mathcal{G}}, \gamma_{\mathcal{G}}, \widehat{\delta}_{\mathcal{G}}), X_{\mathcal{G}})$, $((K_{\mathcal{F}}, \gamma_{\mathcal{F}}, \widehat{\delta}_{\mathcal{F}}), X_{\mathcal{F}})$ and coactions $(\pi_{\mathcal{G}}(C_r^*(\mathcal{G}))^{\gamma_{\mathcal{G}}}_{K_{\mathcal{G}}}, \delta_{\mathcal{G}})$, $(\pi_{\mathcal{F}}(C_r^*(\mathcal{F}))^{\gamma_{\mathcal{F}}}_{K_{\mathcal{F}}}, \delta_{\mathcal{F}})$, and let T be a morphism from $\mathcal{G}$ to $\mathcal{F}$.

Proposition 8.19. *There exists a unique morphism $\tilde{T}_*$ from $\big(\pi_{\mathcal{G}}(C_r^*(\mathcal{G}))^{\gamma_{\mathcal{G}}}_{K_{\mathcal{G}}}, \delta_{\mathcal{G}}\big)$ to $\big(\pi_{\mathcal{F}}(C_r^*(\mathcal{F}))^{\gamma_{\mathcal{F}}}_{K_{\mathcal{F}}}, \delta_{\mathcal{F}}\big)$ that satisfies $\tilde{T}_*(\pi_{\mathcal{G}}(a)) = \pi_{\mathcal{F}}(T_*(a))$ for all $a \in \Gamma_c(\mathcal{G})$.*

The proof involves the following construction.

Lemma 8.20. *Let $\phi \in \mathcal{W}(\Gamma_0(\mathcal{F}^0))$, $f \in \Gamma_0(\mathcal{F}^0)$ and define $\psi \in \mathcal{W}(\Gamma_0(\mathcal{G}^0))$ by $g \mapsto \phi(f^*T^0_*(g)f)$.*

i) There exists a unique isometry $T^f_\phi \colon K_\psi \to K_\phi$ such that $T^\phi_f g = T_*(g)f$ for all $g \in \Gamma_c(\mathcal{G})$.

ii) $T^f_\phi j_\psi(g) = j_\phi(T_*(g)f)$, $T^f_\phi \hat{j}_\psi(g) = \hat{j}_\psi(T_*(g)f)$, $T^f_\phi \pi_\psi(g) = \pi_\phi(T_*(g))T^f_\phi$ for all $g \in \Gamma_c(\mathcal{G})$.

Denote also the map $K_{\mathcal{G}} \to K_\psi \hookrightarrow K_{\mathcal{F}}$ given by $(\xi_{\psi'})_{\psi'} \mapsto T^f_\phi \xi_\psi$ by T^f_ϕ.

iii) T^f_ϕ is a semi-morphism from $(K_{\mathcal{G}}, \widehat{\delta}_{\mathcal{G}}, \gamma_{\mathcal{G}})$ to $(K_{\mathcal{F}}, \widehat{\delta}_{\mathcal{F}}, \gamma_{\mathcal{F}})$ and $(T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})X_{\mathcal{G}} = X_{\mathcal{F}}(T^f_\phi \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})$.

iv) $\delta_{\mathcal{F}}(\pi_{\mathcal{F}}(h))(T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})\delta_{\mathcal{G}}(\pi_{\mathcal{G}}(g)) = \delta_{\mathcal{F}}(\pi_{\mathcal{F}}(hT_*(g)))(T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})$ for all $h \in \Gamma_c(\mathcal{F})$, $g \in \Gamma_c(\mathcal{G})$.

Proof. i) Uniqueness is clear. Existence follows from the fact that for all $g, g' \in \Gamma_c(\mathcal{G})$,

$$\begin{aligned}\langle T_*(g)f | T_*(g')f\rangle_{K_\phi} &= \int_G \phi_{s(x)}\big(f(s(x))^* T(g(x)^* g'(x)) f(s(x))\big)\, \mathrm{d}\nu(x) \\ &= \int_G \psi_{s(x)}(g(x)^* g'(x))\, \mathrm{d}\nu(x) = \langle g|g\rangle_{K_\psi}.\end{aligned}$$

ii) Straightforward.

iii) By ii), $T^f_\phi \gamma_{\mathcal{G}} \subseteq \gamma_{\mathcal{F}}$ and $T^f_\phi \widehat{\delta}_{\mathcal{G}} \subseteq \widehat{\delta}_{\mathcal{F}}$. For all $\omega \in \Gamma_c(\mathcal{G}^2_{s,r})$ and $(x,y) \in G_r \times_r G$,

$$((T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})X_{\mathcal{G}}\omega)(x,y) = \omega(x, x^{-1}y)f(s(x)) = (X_{\mathcal{F}}(T^f_\phi \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})\omega)(x,y).$$

iv) By ii) and iii), we have $X_{\mathcal{F}}(\pi_{\mathcal{F}}(h) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})X^*_{\mathcal{F}}(T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})X_{\mathcal{G}}(\pi_{\mathcal{G}}(g) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})X^*_{\mathcal{G}} = X_{\mathcal{F}}(\pi_{\mathcal{F}}(hT_*(g)) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})X^*_{\mathcal{F}}(T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})$ for all $g \in \Gamma_c(\mathcal{G})$ and $h \in \Gamma_c(\mathcal{F})$. ∎

Proof of Proposition 8.19. Denote by $\mathcal{T} \subseteq \mathcal{L}(K_{\mathcal{G}}, K_{\mathcal{F}})$ the closed linear span of all operators T_{ϕ}^{f}, where $\phi \in \mathcal{W}(\Gamma_0(\mathcal{F}^0))$ and $f \in \Gamma_0(\mathcal{F}^0)$. Then Lemma 8.20 and Proposition 7.9 imply that $S\pi_{\mathcal{G}}(g) = \pi_{\mathcal{F}}(T_*(g))S$ for all $S \in \mathcal{T}, g \in \Gamma_c(\mathcal{G})$ and that

$$[\mathcal{T}\gamma_{\mathcal{G}}] = \Big[\bigcup_{\phi} j_{\phi}\big(T_*(\Gamma_c(\mathcal{G}))\Gamma_0(\mathcal{F}^0)\big)\Big] = \Big[\bigcup_{\phi} j_{\phi}(\Gamma_c(\mathcal{F}))\Big] = \gamma_{\mathcal{F}}.$$

By Proposition 7.9, T_* extends to a nondegenerate $*$-homomorphism $C_r^*(\mathcal{G}) \to M(C_r^*(\mathcal{F}))$. Thus, there exists a semi-morphism $\tilde{T}_*$ from $\pi_{\mathcal{G}}(C_r^*(\mathcal{G}))_{K_{\mathcal{G}}}^{\gamma_{\mathcal{G}}}$ to $\pi_{\mathcal{F}}(C_r^*(\mathcal{F}))_{K_{\mathcal{F}}}^{\gamma_{\mathcal{F}}}$ such that $\tilde{T}_*(\pi_{\mathcal{G}}(g)) = \pi_{\mathcal{F}}(T_*(g))$ for all $g \in \Gamma_c(\mathcal{G})$. For all $h \in \pi_{\mathcal{F}}(\Gamma_c(\mathcal{F}))$, $g \in \pi_{\mathcal{G}}(\Gamma_c(\mathcal{G}))$, $S \in \mathcal{T}$,

$$\delta_{\mathcal{F}}(h) \cdot (\tilde{T}_* * \mathrm{id})(\delta_{\mathcal{G}}(g)) \cdot (S \underset{\mathfrak{b}}{\otimes} \mathrm{id}) = \delta_{\mathcal{F}}(h)(S \underset{\mathfrak{b}}{\otimes} \mathrm{id})\delta_{\mathcal{G}}(g) = \delta_{\mathcal{F}}(h\tilde{T}_*(g))(S \underset{\mathfrak{b}}{\otimes} \mathrm{id})$$

by Lemma 8.20, and therefore $\delta_{\mathcal{F}}(h) \cdot (\tilde{T}_* * \mathrm{id})(\delta_{\mathcal{G}}(g)) = \delta_{\mathcal{F}}(h\tilde{T}_*(g))$. □

Denote by $\mathbf{Fell}_G^a$ the category of all admissible Fell bundles on G, and by $\mathbf{Coact}^a_{C_r^*(G)}$ the category of very fine left-full coactions of $C_r^*(G)$.

Theorem 8.21. *The assignments $\mathcal{F} \mapsto (\pi_{\mathcal{F}}(C_r^*(\mathcal{F}))_{K_{\mathcal{F}}}^{\gamma_{\mathcal{F}}}, \delta_{\mathcal{F}})$ and $T \mapsto \tilde{T}_*$ form a faithful functor $\check{\mathbf{F}}\colon \mathbf{Fell}_G^a \to \mathbf{Coact}^a_{C_r^*(G)}$.*

Proof. Functoriality of the constructions is evident. Assume that $\check{\mathbf{F}}S = \check{\mathbf{F}}T$ for some morphisms S, T from $\mathcal{F}$ to $\mathcal{G}$ in $\mathbf{Fell}_G^a$. Then the maps $S_*, T_*\colon \Gamma_c(\mathcal{F}) \to \Gamma_c(\mathcal{M}(\mathcal{G}))$ coincide because $\pi_{\mathcal{G}}$ is injective. Since $\{a(x) \mid a \in \Gamma_c(\mathcal{F})\} = \mathcal{F}_x$ for each $x \in G$ and $S(a(x)) = (S_*a)(x) = (T_*a)(x) = T(a(x))$ for each $a \in \Gamma_c(\mathcal{F})$, $x \in G$, we can conclude that $S = T$. □

9 From coactions of $C_r^*(G)$ to Fell bundles for étale G

We now assume that the groupoid G is étale [22] in the sense that the set $\mathfrak{G}$ of all open subsets $U \subseteq G$ for which the restrictions $r_U = r|_U\colon U \to r(U)$ and $s_U = s|_U\colon U \to s(U)$ are homeomorphisms is a cover of G. Moreover, we assume that the Haar systems λ and λ^{-1} are the families of counting measures. Then the functor $\check{\mathbf{F}}$ has a right adjoint $\check{\mathbf{G}}$ and embeds the category of admissible Fell bundles into a category of very fine coactions of $C_r^*(G)$ as a full and coreflective subcategory. The construction of the functor $\check{\mathbf{G}}$ uses the correspondence between Banach bundles and convex Banach modules developed in [8].

The Fell bundle of a coaction of $C_r^*(G)$ Let δ be an injective coaction of $C_r^*(G)$ on a C^*-$\mathfrak{b}$-algebra $\mathcal{C} = C_K^{\gamma}$. Since G is étale, $\rho_{\beta}(\mathfrak{B}) \subseteq C_r^*(G)$ and $\delta(C)|\gamma\rangle_1 \subseteq [|\gamma\rangle_1 C_r^*(G)]$. For each $U \in \mathfrak{G}$, we define a closed subspace

$$C_U := \{c \in [C\rho_{\gamma}(C_0(s(U)))] \mid \delta(c)|\gamma\rangle_1 \subseteq [|\gamma\rangle_1 L(C_0(U))]\} \subseteq C,$$

denote by $s_{U*}\colon C_0(U) \to C_0(s(U))$ and $r_{U*}\colon C_0(U) \to C_0(r(U))$ the push-forward of functions along s_U and r_U, respectively, and consider C_U as a right Banach $C_0(U)$-module via the formula $c \cdot f := c\rho_\gamma(s_{U*}(f))$. Denote by $\Gamma_f(\mathcal{F})$ the space of all sections of $\mathcal{F}$ that can be written as finite sums of sections in $\Gamma_0(\mathcal{F}|_U)$, where $U \in \mathfrak{G}$. Then $\Gamma_f(\mathcal{F})$ is a $*$-algebra with respect to the operations defined in (16), and one has natural inclusions $\Gamma_c(\mathcal{F}) \subseteq \Gamma_f(\mathcal{F}) \subseteq C^*_r(\mathcal{F})$ of $*$-algebras.

Proposition 9.1. *There exist a continuous Fell bundle $\mathcal{F}$ on G and a $*$-homomorphism $\iota\colon \Gamma_f(\mathcal{F}) \to C$ such that for each $U \in \mathfrak{G}$, the map ι restricts to an isometric isomorphism $\iota_U\colon \Gamma_0(\mathcal{F}|_U) \to C_U$ of Banach $C_0(U)$-modules. If $(\mathcal{F}', \iota')$ is another such pair, then there exists an isomorphism $T\colon \mathcal{F} \to \mathcal{F}'$ such that $\iota' \circ T_* = \iota$.*

The proof requires some preliminaries. First, note that for all $c \in C$, $f \in C_0(G^0)$,

$$\delta(c\rho_\gamma(f)) = \delta(c)\rho_{(\gamma \triangleright \alpha)}(f) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} \rho_\alpha(f)) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} r^*(f)).$$

Lemma 9.2. *Let $U, V \in \mathfrak{G}$.*

i) $c \cdot f = \rho_\gamma(r_{U*}(f))c$ for each $c \in C_U$ and $f \in C_0(U)$.

ii) $C_V C_U \subseteq C_{VU}$, $(C_U)^* = C_{U^{-1}}$, and $C_U = [C_V C_0(U)] \subseteq C_V$ if $U \subseteq V$.

iii) $C_{s(U)}$ is a continuous $C_0(s(U))$-algebra.

iv) C_U is a convex and continuous Banach $C_0(U)$-module.

Proof. i) Let c, f as above. Since $L(g)r^*(s_{U*}(f)) = r^*(r_{U*}(f))L(g)$ for all $g \in C_0(U)$, we have $\delta(c \cdot f) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} r^*(s_{U*}(f))) = (1 \underset{\mathfrak{b}}{\otimes} r^*(r_{U*}(f)))\delta(c) = \delta(\rho_\gamma(r_{U*}(f))c)$ and by injectivity of δ also $c \cdot f = \rho_\gamma(r_{U*}(f))c$.

ii) Clearly, $\delta(C_V C_U)|\gamma\rangle_1 \subseteq |\gamma\rangle_1 L(C_0(VU))$. Using i) twice, we find

$$\begin{aligned} C_V C_U &\subseteq [C_V \rho_\gamma(C_0(s(V))C_0(r(U)))C_U] \\ &= [C_V \rho_\gamma(C_0(s(V) \cap r(U))C_U] \subseteq [C\rho_\gamma(C_0(s(VU)))]. \end{aligned}$$

Consequently, $C_V C_U \subseteq C_{VU}$. By i) again, we have $(C_U)^* = [\rho_\gamma(C_0(r(U)))C_U]^* \subseteq [C\rho_\gamma(C_0(s(U^{-1})))]$, and using the relation $\delta(C_U^*)|\gamma\rangle_1 \subseteq [|\gamma\rangle_1 C^*_r(G)]$, we obtain

$$\delta(C_U^*)|\gamma\rangle_1 \subseteq [|\gamma\rangle_1 \langle\gamma|_1 \delta(C_U)^*|\gamma\rangle_1] \subseteq [|\gamma\rangle_1 L(C_0(U))^* \langle\gamma|_1|\gamma\rangle_1] = [|\gamma\rangle_1 L(C_0(U^{-1}))].$$

If $U \subseteq V$, then $C_U \subseteq [C_V C_0(U)] \subseteq C_V$, and $C_V C_0(U) \subseteq C_U$ because

$$\begin{aligned} \delta(C_V C_0(U))|\gamma\rangle_1 &= \delta(C_V)|\gamma\rangle_1 r^*(C_0(s(U))) \\ &\subseteq [|\gamma\rangle_1 L(C_0(V))r^*(C_0(s(U)))] = [|\gamma\rangle_1 L(C_0(U))]. \end{aligned}$$

iii) By ii), $C_{s(U)}$ is a C^*-algebra. Consider $|\gamma\rangle_1$ as a Hilbert C^*-module over $r^*(C_0(G^0)) \cong C_0(G^0)$. Since $\delta(C_{G^0})|\gamma\rangle_1 \subseteq |\gamma\rangle_1$ and $\delta(c \cdot f)|\eta\rangle_1 = \delta(c)|\eta\rangle_1 r^*(f)$ for all $c \in C_{G^0}$, $f \in C_0(G^0)$, $\eta \in \gamma$, the formula $c \cdot |\eta\rangle_1 := \delta(c)|\eta\rangle_1$ defines a faithful field of representations $C_{G^0} \to \mathcal{L}(|\gamma\rangle_1)$ in the sense of [6, Theorem 3.3]. Consequently, C_{G^0} is a continuous $C_0(G^0)$-algebra and $C_{s(U)}$ a continuous $C_0(s(U))$-algebra.

iv) Let $c, c' \in C_U$ and $f, f' \in C_0(U)$ such that $0 \leqslant f, f'$ and $f + f' \leqslant 1$. Then $\|c \cdot f + c' \cdot f'\|^2 = \|c^*c \cdot g^2 + c^*c' \cdot gg' + c'^*c \cdot g'g + c'^*c' \cdot g'^2\|$, where $g = s_{U*}(f)$, $g' = s_{U*}(f')$. Since $g^2 + gg'' + g'g + g'^2 \leqslant 1$ and $c^*c, c'^*c', c^*c', c'^*c' \in C_{U^{-1}U}$, which is a continuous $C_0(s(U))$-algebra and hence a convex Banach $C_0(s(U))$-module, we get $\|cf + c'f'\|^2 \leqslant \max\{\|c\|, \|c'\|\}^2$. Finally, the norm $\|c_u\|^2 = \|(c^*c)_{u^{-1}u}\|$ depends continuously on $u \in U$ because $C_{U^{-1}U}$ is a continuous $C_0(s(U))$-algebra. ☐

Proof of Proposition 9.1. Using Lemma 9.2 and [8], one easily verifies that there exists a continuous Fell bundle $\mathcal{F}$ on G with an isometric isomorphism $\iota_U \colon \Gamma_0(\mathcal{F}|_U) \to C_U$ of Banach $C_0(U)$-modules for each $U \in \mathfrak{G}$ such that for all $U, V \in \mathfrak{G}$, the following properties hold. First, the map $\Gamma_0(\mathcal{F}|_U) \hookrightarrow \Gamma_0(\mathcal{F}|_V) \xrightarrow{\iota_V} C_V$ is equal to $\Gamma_0(\mathcal{F}|_U) \xrightarrow{\iota_U} C_U \hookrightarrow C_V$ if $U \subseteq V$, and second, $\iota_U(f)^* = \iota_{U^{-1}}(f^*)$, $\iota_{UV}(fg) = \iota_U(f)\iota_V(g)$ for all $f \in \Gamma_0(\mathcal{F}|_U)$, $g \in \Gamma_0(\mathcal{F}|_V)$. Define $\iota \colon \Gamma_f(\mathcal{F}) \to C$ as follows. Given $a = \sum_i a_i \in \Gamma_f(\mathcal{F})$, where $a_i \in \Gamma_0(\mathcal{F}|_{U_i})$ and $U_i \in \mathfrak{G}$, let $\iota(a) = \sum_i \iota_{U_i}(a_i)$. Using the preceding two properties of ι, one easily verifies that ι is well-defined and a $*$-homomorphism. ◻

Denote by $p_0 \colon \Gamma_f(\mathcal{F}) \to \Gamma_0(\mathcal{F}^0)$ the restriction.

Proposition 9.3. *There exists a faithful conditional expectation* $p \colon [\iota(\Gamma_f(\mathcal{F}))] \to C_{G^0}$ *such that* $p \circ \iota = \iota_{G^0} \circ p_0$.

In the following lemma, $fh_{\xi,\xi'}$ denotes the pointwise product of functions $f, h_{\xi,\xi'} \in C_c(G)$, where $h_{\xi,\xi'}$ was defined in (17).

Lemma 9.4. *Let* $\xi, \xi' \in C_c(G)$, $c \in C$, $f \in C_c(G)$.

i) $\langle \eta |_1 \delta(\langle j(\xi)|_2 \delta(c)|j(\xi')\rangle_2)|\eta'\rangle_1 = \langle j(\xi)|_2 \Delta(\langle \eta|_1 \delta(c)|\eta'\rangle_1)|j(\xi')\rangle_2$ *for all* $\eta, \eta' \in \gamma$.

ii) $\langle j(\xi)|_2 \Delta(L(f))|j(\xi')\rangle_2 = L(fh_{\xi,\xi'})$.

iii) $\langle j(\xi)|_2 \delta(c \cdot f)|j(\xi')\rangle_2 = c \cdot (fh_{\xi,\xi'})$ *if* $c \in C_U$ *and* $f \in C_0(U)$, *where* $U \in \mathfrak{G}$.

Proof. i) Let $d = \langle j(\xi)|_2 \delta(c)|j(\xi')\rangle_2$. Then $\delta(d) = \langle j(\xi)|_3 (\delta * \mathrm{id})(\delta(c))|j(\xi')\rangle_3 = \langle j(\xi)|_3 (\mathrm{id} * \Delta)(\delta(c))|j(\xi')\rangle_3$ and $\langle \eta|_1 \delta(d)|\eta'\rangle_1 = \langle j(\xi)|_2 \Delta(\langle \eta|_1 \delta(c)|\eta'\rangle_1)|j(\xi')\rangle_2$.

ii) This is a special case of Lemma 8.12.

iii) Let $\eta, \eta' \in \gamma$. Since $c \in C_U$, we have $\langle \eta|_1 \delta(c)|\eta'\rangle_1 = L(g)$ for some $g \in C_0(U)$. Let $\xi'' = r^*(s_{U*}(f))\xi$ and denote by $d_l, d_r \in C$ the left and the right hand side of the equation in iii), respectively. Then $d_l = \langle j(\xi)|_2 \delta(c)|j(\xi'')\rangle_2$, and by i) and ii),

$$\langle \eta|_1 \delta(d_l)|\eta'\rangle_1 = \langle j(\xi)|_2 \Delta(\langle \eta|_1 \delta(c)|\eta'\rangle_1)|j(\xi'')\rangle_2 = \langle j(\xi)|_2 L(g)|j(\xi'')\rangle_2 = L(gh_{\xi,\xi''}),$$
$$\langle \eta|_1 \delta(d_r)|\eta'\rangle_1 = \langle \eta|_1 \delta(c)|\eta'\rangle_1 r^*(s_{U*}(fh_{\xi,\xi'})) = L(g)L(s_{U*}(fh_{\xi,\xi'})).$$

We can conclude that $\langle \eta|_1 \delta(d_l)|\eta'\rangle_1 = \langle \eta|_1 \delta(d_r)|\eta'\rangle_1$ because for all $x \in G$,

$$(gh_{\xi,\xi''})(x) = g(x) \int_{G^{r(x)}} \overline{\xi(y)} f(x) \xi'(x^{-1}y) \, \mathrm{d}\lambda^{r(x)}(y) = g(x)(s_{U*}(fh_{\xi,\xi'}))(s(x)).$$

Since $\eta, \eta' \in \gamma$ were arbitrary and δ is injective, we must have $d_l = d_r$. ◻

Proof of Proposition 9.3. Given a subset $U \subseteq G$, denote by χ_U its characteristic function. Using the same formulas as for elements of $C_c(G)$, we can define a map $j(\xi) \colon \mathfrak{K} \to H$ and the function $h_{\xi,\xi'}$ for the characteristic function $\xi = \xi' = \chi_{G^0}$ of $G^0 \subseteq G$, and then Lemma 9.4 still holds. Define $p \colon C \to C$ by $c \mapsto \langle j(\chi_{G^0})|_2 \delta(c)|j(\chi_{G^0})\rangle_2$. Then $\|p\| \leqslant \|j(\chi_{G^0})\|^2 = 1$, and the relation $h_{\chi_{G^0},\chi_{G^0}} = \chi_{G^0}$ and Lemma 9.4 imply that $p|_{C_{G^0}} = \mathrm{id}$ and $p|_{C_U} = 0$ whenever $U \in \mathfrak{G}$ and $U \cap G^0 = \emptyset$. Using a partition of unity argument and the fact that $G^0 \subseteq G$ is open and closed, we can conclude that $p \circ \iota = \iota_{G^0} \circ p_0$.

It remains to show that p is faithful. Using the right-regular representation of G, one easily verifies that $[C_r^*(G)'j(\chi_{G^0})\mathfrak{K}] = H$. Therefore, the map $q\colon C_r^*(G) \to \mathcal{L}(\mathfrak{K})$, $a \mapsto j(\chi_{G^0})^* a j(\chi_{G^0})$, is faithful in the sense that $q(a^*a) \neq 0$ whenever $a \neq 0$. If $c \in [\iota(\Gamma_f(\mathcal{F}))]$ and $p(c^*c) = 0$, then $\eta^* p(c^*c)\eta = q(\langle \eta^*|_1 \delta(c^*c)|\eta\rangle_1) = 0$ and hence $\langle \eta^*|_1\delta(c^*c)|\eta\rangle_1 = 0$ and $\delta(c)|\eta\rangle_1 = 0$ for all $\eta \in \gamma$, whence $\delta(c) = 0$ and $c = 0$ by injectivity of δ. □

Proposition 9.3 and [15, Fact 3.11] imply:

Corollary 9.5. *ι extends to an embedding $C_r^*(\mathcal{F}) \to C$.* □

We denote the extension above by ι again.

Proposition 9.6. *If δ is fine, then $\iota\colon C_r^*(\mathcal{F}) \to C$ is a $*$-isomorphism.*

Proof. We only need to show that C is equal to the linear span of all C_U, where $U \in \mathfrak{G}$. Consider an element $d \in C$ of the form $d = \langle j(\xi)|_2\delta(c)|j(\xi')\rangle_2$, where $c \in C, \xi \in C_c(V), \xi' \in C_c(V')$ for some $V, V' \in \mathfrak{G}$. Since G is étale and δ is fine, the closed linear span of all elements of the form like d is equal to $[\langle \alpha|_2\delta(C)|\alpha\rangle_2] = [\langle \alpha|_2|\alpha\rangle_2 C] = C$. We show that $d \in C_U$, where $U = VV'^{-1} \in \mathfrak{G}$, and then the claim follows. Let $\eta, \eta' \in \gamma$. By Lemma 9.4,

$$\langle \eta|_1\delta(d)|\eta'\rangle_1 \in \langle j(\xi)|_2\Delta(C_r^*(G))|j(\xi')\rangle_2 \subseteq [L(C_c(G)h_{\xi,\xi'})] \subseteq L(C_0(U)).$$

Using the relation $\delta(d)|\gamma\rangle_1 \subseteq [|\gamma\rangle_1 C_r^*(G)]$, we find $\delta(d)|\gamma\rangle_1 \subseteq [|\gamma\rangle_1\langle\gamma|_1\delta(d)|\gamma\rangle_1] \subseteq [|\gamma\rangle_1 L(C_0(U))]$. Moreover, since $h_{\xi,\xi'} \in C_c(U)$, we can choose $g \in C_0(U)$ with $h_{\xi,\xi'}g = h_{\xi,\xi'}$. Then $L(fh_{\xi,\xi'})r^*(s_{U*}(g)) = L(fh_{\xi,\xi'})$ for each $f \in C_0(U)$, and hence $\langle \eta|_1\delta(d\rho_\gamma(s_{U*}(g)))|\eta'\rangle_1 = \langle\eta|_1\delta(d)|\eta'\rangle_1 r^*(s_{U*}(g)) = \langle \eta|_1\delta(d)|\eta'\rangle_1$. Since δ is injective, we can conclude $d = d\rho_\gamma(s_{U*}(g)) \in C\rho_\gamma(C_0(s(U)))$ and finally $d \in C_U$. □

Proposition 9.7. *If δ is fine, then $\mathcal{F}$ is admissible.*

Proof. The proof is similar to the proof of Lemma 6.2 i). By 9.2 iii), $\Gamma_0(\mathcal{F}^0) \cong C_{G^0}$ is a continuous $C_0(G^0)$-algebra. Let $u \in G^0$, denote by $I_u \subset C_0(G^0)$ the ideal of all functions vanishing at u, and assume that $\mathcal{F}_u = 0$. Then $\Gamma_0(\mathcal{F}^0) = [\Gamma_0(\mathcal{F}^0)I_u]$ and $[C_r^*(\mathcal{F})] = [C_r^*(\mathcal{F})\Gamma_0(\mathcal{F}^0)] = [C_r^*(\mathcal{F})I_u]$, whence $C = [C\rho_\gamma(I_u)]$. Define $j(\chi_{G^0})$ as in the proof of Proposition 9.3. Then $[\delta(C)|\gamma\rangle_1 C_r^*(G)] = [|\gamma\rangle_1 C_r^*(G)]$ and

$$\begin{aligned}[r^*(C_0(G^0))C_r^*(G)] = [\langle\gamma|_1|\gamma\rangle_1 C_r^*(G)] &= [\langle\gamma|_1\delta(CI_u)|\gamma\rangle_1 C_r^*(G)] \\ &= [\langle\gamma|_1|\gamma\rangle_1 r^*(I_u)C_r^*(G)] = [r^*(I_u)C_r^*(G)],\end{aligned}$$

whence $[j(\chi_{G^0})^* C_r^*(G) j(\chi_{G^0})] = I_u \neq C_0(G^0)$, a contradiction. □

The construction of the Fell bundle is functorial with respect to the following class of morphisms.

Definition 9.8. *A morphism ρ of coactions (C_K^γ, δ_C) and (D_L^ϵ, δ_D) of $C_r^*(G)$ is* strongly nondegenerate *if $[\rho(C)D_{G^0}] = D$.*

Proposition 9.9. *Let π be a strongly nondegenerate morphism of fine coactions (C_K^γ, δ_C), (D_L^ϵ, δ_D) with associated Fell bundles $\mathcal{F}$, $\mathcal{G}$ and $*$-homomorphisms $\iota_\mathcal{F}$, $\iota_\mathcal{G}$. Then there exists a unique morphism T from $\mathcal{F}$ to $\mathcal{G}$ such that $\iota_\mathcal{G} \circ T_* = \pi \circ \iota_\mathcal{F}$.*

Proof. Let $U, V \in \mathfrak{G}$. Then $\pi(C_U)D_V \subseteq D_{UV}$ because

$$\begin{aligned}\delta_D(\pi(C_U)D_V)|\epsilon\rangle_1 &= ((\pi * \mathrm{id})(\delta_C(C_U)))\delta_D(D_V)|\epsilon\rangle_1 \\ &\subseteq ((\pi * \mathrm{id})(\delta_C(C_U)))|\epsilon\rangle_1 L(C_0(V)) \\ &\subseteq |\epsilon\rangle_1 L(C_0(U))L(C_0(V)) = |\epsilon\rangle_1 L(C_0(UV)) \\ \text{and } \pi(C_U)D_V &\subseteq [\pi(C\rho_\gamma(C_0(s(U))))D_V] \subseteq [\pi(C)D\rho_\epsilon(C_0(s(UV)))],\end{aligned}$$

where the last inclusion follows similarly as in the proof of Lemma 9.2 ii). Define a map $S_{U,V} \colon \Gamma_0(\mathcal{F}|_U) \times \Gamma_0(\mathcal{G}|_V) \to \Gamma_0(\mathcal{G}|_{UV})$ by $(f, g) \mapsto \iota_{\mathcal{G}}^{-1}(\pi(\iota_{\mathcal{F}}(f))\iota_{\mathcal{G}}(g))$, let $(x, y) \in (U \times V) \cap G_s \times_r G$, and denote by $I_x \subseteq \Gamma_0(\mathcal{F}|_U)$, $I_y \subseteq \Gamma_0(\mathcal{G}|_V)$, $I_{xy} \subseteq \Gamma_0(\mathcal{G}|_{UV})$ the subspaces of all sections vanishing at x, y, and xy, respectively. Using Lemma 9.2 i), one easily verifies that $S_{U,V}$ maps $I_x \times \Gamma_0(\mathcal{G}|_V)$ and $\Gamma_0(\mathcal{F}|_U) \times I_y$ into I_{xy}. Hence, there exists a unique map $S_{x,y} \colon \mathcal{F}_x \times \mathcal{G}_y \to \mathcal{G}_{xy}$ such that $S_{x,y}(f(x), g(y)) = (S_{U,V}(f, g))(xy)$ for all $f \in \Gamma_0(\mathcal{F}|_U)$, $g \in \Gamma_0(\mathcal{G}|_V)$, and this map depends on (x, y) but not on (U, V). For each $x \in G$ and $c \in \mathcal{F}_x$, define $T(c) \colon \mathcal{G}|_{G^{s(x)}} \to \mathcal{G}|_{G^{r(x)}}$ by $T(c)d = S_{x,y}(c, d)$ for each $y \in G^{s(x)}$, $d \in \mathcal{G}_y$. One easily checks that then T is a continuous map from $\mathcal{F}$ to $\mathcal{M}(\mathcal{G})$ which satisfies conditions i) and ii) of Definition 7.8, and that the representation $\tilde{\pi} := \iota_{\mathcal{G}}^{-1} \circ \pi \circ \iota_{\mathcal{F}} \colon C^*_r(\mathcal{F}) \to M(C^*_r(\mathcal{G}))$ satisfies $\tilde{\pi}(f)g = (T \circ f)g$ for all $f \in \Gamma_c(\mathcal{F})$, $g \in \Gamma_c(\mathcal{G})$. We show that T also satisfies condition iii) of Definition 7.8. Since π is strongly nondegenerate, $D = [\pi(C)D_{G^0}]$, that is, $C^*_r(\mathcal{G}) = [\tilde{\pi}(C^*_r(\mathcal{F}))\Gamma_0(\mathcal{G}^0)]$ and hence $\Gamma^2(\mathcal{G}, \lambda^{-1}) = [\tilde{\pi}(C^*_r(\mathcal{F}))\Gamma_0(\mathcal{G}^0)]$. In particular, $\mathcal{G}_x = [T(\mathcal{F}_x)\mathcal{G}_{s(x)}]$ for each $x \in G$ because $G_{s(x)}$ is discrete. □

The unit and counit of the adjunction Denote by $\mathbf{Coact}^{as}_{C^*_r(G)}$ the category of very fine left-full coactions of $C^*_r(G)$ with all strongly nondegenerate morphisms. Then the functor $\check{\mathbf{F}} \colon \mathbf{Fell}^a_G \to \mathbf{Coact}^a_{C^*_r(G)}$ constructed in the preceding section actually takes values in $\mathbf{Coact}^{as}_{C^*_r(G)}$:

Lemma 9.10. *Let T be a morphism of admissible Fell bundles $\mathcal{F}, \mathcal{G}$ on G. Then the morphism $\check{\mathbf{F}}T$ from $\check{\mathbf{F}}\mathcal{F}$ to $\check{\mathbf{F}}\mathcal{G}$ is strongly nondegenerate.*

Proof. Immediate from Proposition 7.9 ii). □

The constructions in Proposition 9.1 and 9.9 yield a functor $\check{\mathbf{G}} \colon \mathbf{Coact}^{as}_{C^*_r(G)} \to \mathbf{Fell}^a_G$. We now obtain an embedding $(\check{\mathbf{F}}, \check{\mathbf{G}}, \check{\eta}, \check{\epsilon})$ of $\mathbf{Fell}^a_G$ into $\mathbf{Coact}^{as}_{C^*_r(G)}$ as a full and coreflective subcategory.

Proposition 9.11. *Let $\mathcal{F}$ be an admissible Fell bundle, $(\pi_{\mathcal{F}}(C^*_r(\mathcal{F}))^{\gamma_{\mathcal{F}}}_{K_{\mathcal{F}}}, \delta_{\mathcal{F}}) = \check{\mathbf{F}}\mathcal{F}$ the associated fine coaction, and $\mathcal{G} = \check{\mathbf{G}}\check{\mathbf{F}}\mathcal{F}$ and $\iota_{\mathcal{G}} \colon C^*_r(\mathcal{G}) \to \pi_{\mathcal{F}}(C^*_r(\mathcal{F}))$ the Fell bundle and the $*$-homomorphism associated to this coaction as above. Then there exists a unique isomorphism $\check{\eta}_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{G}$ such that $\iota_{\mathcal{G}} \circ (\check{\eta}_{\mathcal{F}})_* = \pi_{\mathcal{F}}$.*

Proof. Let $(C^\gamma_K, \delta) = (\pi_{\mathcal{F}}(C^*_r(\mathcal{F}))^{\gamma_{\mathcal{F}}}_{K_{\mathcal{F}}}, \delta_{\mathcal{F}})$ and $U \in \mathfrak{G}$. We show that $C_U = \pi_{\mathcal{F}}(\Gamma_0(\mathcal{F}|_U))$. Note that $[\{h_{\xi,\xi'} \mid \xi \in C_c(r(U)), \xi' \in C_c(U)\}] = C_0(U)$, where the functions $h_{\xi,\xi'}$ were defined in (17). Using Lemma 9.4, we can conclude

$$C_U = [\langle j(C_c(r(U)))|_2 \delta(\pi_{\mathcal{F}}(\Gamma_c(\mathcal{F})))|j(C_c(U))\rangle_2].$$

By Lemma 8.12, we have for all $\xi \in C_c(r(U))$, $f \in \Gamma_c(\mathcal{F})$, $\xi' \in C_c(U)$,

$$\langle j(\xi)|_2\delta(\pi_{\mathcal{F}}(f))|j(\xi')\rangle_2 = \pi_{\mathcal{F}}(fh_{\xi,\xi'}) \in \pi_{\mathcal{F}}(\Gamma_0(\mathcal{F}|_U)),$$

where $fh_{\xi,\xi'}$ denotes the pointwise product. Consequently, $C_U = \pi_{\mathcal{F}}(\Gamma_0(\mathcal{F}|_U))$. Since $U \in \mathfrak{G}$ was arbitrary, we can conclude that there exists an isomorphism $\check{\eta}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{G}$ of Banach bundles such that $\iota_{\mathcal{G}} \circ (\check{\eta}_{\mathcal{F}})_* = \pi_{\mathcal{F}}\colon \Gamma_c(\mathcal{F}) \to C$. Using the fact that $(\check{\eta}_{\mathcal{F}})_*$ is a $*$-homomorphism and that G is étale, one easily concludes that $\check{\eta}_{\mathcal{F}}$ is an isomorphism of Fell bundles. ☐

Proposition 9.12. *Let $(\mathcal{C},\delta)$ be a very fine coaction of $C^*_r(G)$, where $\mathcal{C} = C^\gamma_K$, and let $\mathcal{F}, \iota\colon C^*_r(\mathcal{F}) \to C$ be the associated Fell bundle and $*$-isomorphism. Then there exists a unique strongly nondegenerate morphism $\check{\epsilon}_{(\mathcal{C},\delta)}$ from $(\pi_{\mathcal{F}}(C^*_r(\mathcal{F}))^{\gamma_{\mathcal{F}}}_{K_{\mathcal{F}}}, \delta_{\mathcal{F}})$ to $(\mathcal{C},\delta)$ such that $\check{\epsilon}_{(\mathcal{C},\delta)} \circ \pi_{\mathcal{F}} = \iota$.*

Lemma 9.13. *Let $U \in \mathfrak{G}$, $\xi \in C_c(U)$, $\eta \in \gamma$, and $\omega = |\eta\rangle_1 j(\xi) \in \gamma \triangleright \alpha \subseteq \mathcal{L}(\mathfrak{K}, K {}_\gamma\underset{\mathfrak{b}}{\otimes}{}_\beta H)$.*

i) There exists a $C_0(G^0)$-weight $\phi\colon \Gamma_0(\mathcal{F}^0) \to C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K})$, $f \mapsto \omega^*\delta(\iota(f))\omega$.

ii) There exists a unique isometry $S_\omega\colon K_\phi = \Gamma^2(\mathcal{F},\nu;\phi) \to K {}_\gamma\underset{\mathfrak{b}}{\otimes}{}_\beta H$ such that $S_\omega \hat{j}_\phi(f) = \delta(\iota(f))\omega$ for all $f \in \Gamma_c(\mathcal{F})$. Furthermore, $S_\omega\pi_\phi(f) = \delta(\iota(f))S_\omega$ for all $f \in \Gamma_c(\mathcal{F})$.

iii) $S_\omega j_\phi(\Gamma_c(\mathcal{F})) \subseteq \gamma \triangleright \alpha$.

Proof. i) First, note that $\omega^*\delta(C_{G^0})\omega \subseteq [\alpha^*\langle\gamma|_1|\gamma\rangle_1 L(C_0(G^0))\alpha] = [\alpha^*\langle\gamma|_1|\gamma\rangle_1\alpha] = C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K})$. Second, observe that for all $c \in C_{G^0}$, $f \in C_0(G^0)$,

$$\begin{aligned}\phi(cf) &= j(\xi)^*\langle\eta|_1\delta(cf)|\eta\rangle_1 j(\xi)\\ &= j(\xi)^*\langle\eta|_1\delta(c)|\eta\rangle_1 r^*(f)j(\xi) = j(\xi)^*\langle\eta|_1\delta(c)|\eta\rangle_1 j(\xi)f = \phi(c)f.\end{aligned}$$

ii) As before, denote by $p_0\colon \Gamma_f(\mathcal{F}) \to \Gamma_0(\mathcal{F}^0)$ the restriction. Let $U \in \mathfrak{G}$, $f, f' \in \Gamma_c(\mathcal{F})$, and $g = f^*f'$. Using the relation $\operatorname{supp} h_{\xi,\xi} \subseteq G^0$ and Lemma 9.4, we find

$$\begin{aligned}\omega^*\delta(\iota(f))^*\delta(\iota(f'))\omega &= \eta^*\langle j(\xi)|_2\delta(\iota(g))|j(\xi)\rangle_2\eta^*\\ &= \eta^*\iota(g \cdot h_{\xi,\xi})\eta\\ &= \omega^*\delta(\iota(p_0(g)))\omega^* = \phi(p_0(g)) = \langle f|f'\rangle_{\Gamma^2(\mathcal{F},\lambda^{-1},\phi)}.\end{aligned}$$

The existence of S_ω follows. Finally, $S_\omega\pi_\phi(f) = \delta(\iota(f))S_\omega$ because $S_\omega\pi_\phi(f)\hat{j}_\phi(g) = S_\omega\hat{j}_\phi(fg) = \delta(\iota(fg))\omega = \delta(\iota(f))S_\omega\hat{j}_\phi(g)$ for all $f, g \in \Gamma_c(\mathcal{F})$.

iii) Let $V \in \mathfrak{G}, f \in \Gamma_c(\mathcal{F}|_V), \zeta \in C_c(G^0)$, and define $\zeta' \in L^2(G^0,\mu)$ by $\zeta'(s(x)) = \zeta(r(x))D^{1/2}(x)$ for all $x \in V$ and $\zeta'(y) = 0$ for all $y \in G^0\backslash s(V)$. Then $(j_\phi(f)\zeta)(x) = f(x)\zeta(r(x)) = (\hat{j}_\phi(f)\zeta')(x)$ for all $x \in G$ and therefore

$$S_\omega j_\phi(f)\zeta = S_\omega\hat{j}_\phi(f)\zeta' = \delta(\iota(f))\omega\zeta' = \delta(\iota(f))|\eta\rangle_1 j(\xi)\zeta'.$$

Since $f \in \Gamma_c(\mathcal{F}|_V)$, there exist $f' \in L(C_0(V))$, $\eta' \in \gamma$ such that $\delta(\iota(f))|\eta\rangle_1 = |\eta'\rangle_1 L(f')$. Now,

$$S_\omega j_\phi(f)\zeta = \delta(\iota(f))|\eta\rangle_1 j(\xi)\zeta' = |\eta'\rangle_1 L(f')j(\xi)\zeta' = |\eta'\rangle j(L(f')\xi)\zeta$$

because $(L(f')j(\xi)\zeta')(z) = 0$ for $z \notin VU$ and

$$(L(f')j(\xi)\zeta')(xy) = D^{-1/2}(x)f'(x)\xi(y)\zeta'(r(y)) = f'(x)\zeta(r(x))\xi(y) = j(L(f')\xi)\zeta(xy)$$

for all $(x,y) \in (V \times U) \cap G_s \times_r G$. Thus, $S_\omega j_\phi(f)\zeta \in \gamma \triangleright \alpha$. The claim follows. □

Proof of Proposition 9.12. Since $\pi_{\mathcal{F}}$ is injective, we can define $\check{\epsilon} = \check{\epsilon}_{(\mathcal{C},\delta)} := \iota \circ \pi_{\mathcal{F}}^{-1}$. We show that $\delta \circ \check{\epsilon}$ is a morphism from $\pi_{\mathcal{F}}(C_r^*(\mathcal{F}))_{K_{\mathcal{F}}}^{\gamma_{\mathcal{F}}}$ to $\delta(C)_{K_\gamma \underset{\mathfrak{b}}{\otimes}_\beta H}^{\gamma \triangleright \alpha}$. For each $C_0(G^0)$-weight ϕ on $\Gamma_0(\mathcal{F}^0)$, denote by $p_\phi \colon K_{\mathcal{F}} \to K_\phi$ the canonical projection. Let $\mathcal{S} \subseteq \mathcal{L}(K_{\mathcal{F}}, K_\gamma \underset{\mathfrak{b}}{\otimes}_\beta H)$ be the closed linear span of all operators of the form $S_\omega p_\phi$, where $U, \xi, \eta, \omega, \phi$ are as in the lemma above. Then $Sa = \delta(\check{\epsilon}(a))$ for each $S \in \mathcal{S}$, $a \in \pi_{\mathcal{F}}(C_r^*(\mathcal{F}))$, and $[\mathcal{S}\gamma_{\mathcal{F}}] = [\delta(\iota(\Gamma_c(\mathcal{F})))(\gamma \triangleright \alpha)] = \gamma \triangleright \alpha$. The claim follows. Since δ is an isomorphism from $\mathcal{C}$ to $\delta(C)_{K_\gamma \underset{\mathfrak{b}}{\otimes}_\beta H}^{\gamma \triangleright \alpha}$, we can conclude that $\check{\epsilon}$ is a morphism from $\pi_{\mathcal{F}}(C_r^*(\mathcal{F}))_{K_{\mathcal{F}}}^{\gamma_{\mathcal{F}}}$ to $\mathcal{C}$. The relation $(\check{\epsilon} * \mathrm{id}) \circ \delta = \delta \circ \check{\epsilon}$ follows from the fact that

$$\begin{aligned}\langle j(\xi)|_2\delta(\check{\epsilon}(g) \cdot f)|j(\xi')\rangle_2 &= \check{\epsilon}(g) \cdot (f h_{\xi,\xi'}) \\ &= \check{\epsilon}(g \cdot (f h_{\xi,\xi'})) = \check{\epsilon}\big(\langle j(\xi)|_2\delta(\pi_{\mathcal{F}}(g))|j(\xi')\rangle_2\big)\end{aligned}$$

for all $U \in \mathfrak{G}$, $g \in \Gamma_c(\mathcal{F}|_U)$, $f \in C_c(U)$, $\xi, \xi' \in C_c(G)$ by Lemma 9.4. □

Corollary 9.14. *Every very fine coaction of $C_r^*(G)$ is left-full.*

Proof. Let $(\mathcal{C}, \delta)$ be a very fine coaction of $C_r^*(G)$, let $\check{\epsilon}_{(\mathcal{C},\delta)}$ and $(\pi_{\mathcal{F}}(C_r^*(\mathcal{F}))_{K_{\mathcal{F}}}^{\gamma_{\mathcal{F}}}, \delta_{\mathcal{F}})$ as above, and let $I := \{T \in \mathcal{L}_s((K_{\mathcal{F}}, \gamma_{\mathcal{F}}), (K, \gamma)) \mid Tx = \check{\epsilon}_{(\mathcal{C},\delta)}(x)T$ for all $x \in \pi_{\mathcal{F}}(C_r^*(\mathcal{F}))\}$. Then $\gamma = [I\gamma_{\mathcal{F}}]$ because $\check{\epsilon}_{(\mathcal{C},\delta)}$ is a morphism, and since $\delta_{\mathcal{F}}$ is left-full,

$$\begin{aligned}[\delta(C)|\gamma\rangle_1 C_r^*(G)] &= [(\check{\epsilon}_{(\mathcal{C},\delta)} * \mathrm{id})(\delta_{\mathcal{F}}(\pi_{\mathcal{F}}(C_r^*(\mathcal{F}))))(I \underset{\mathfrak{b}}{\otimes} \mathrm{id})|\gamma_{\mathcal{F}}\rangle_1 C_r^*(G)] \\ &= [(I \underset{\mathfrak{b}}{\otimes} \mathrm{id})\delta_{\mathcal{F}}(\pi_{\mathcal{F}}(C_r^*(\mathcal{F})))|\gamma_{\mathcal{F}}\rangle_1 C_r^*(G)] \\ &= [(I \underset{\mathfrak{b}}{\otimes} \mathrm{id})|\gamma_{\mathcal{F}}\rangle_1 C_r^*(G)] = [|\gamma\rangle_1 C_r^*(G)].\end{aligned}$$ □

Theorem 9.15. *$(\check{\mathbf{F}}, \check{\mathbf{G}}, \check{\eta}, \check{\epsilon})$ is an embedding of $\mathbf{Fell}_G^a$ into $\mathbf{Coact}_{C_r^*(G)}^{as}$ as a full and coreflective subcategory.*

Proof. One easily verifies that $\check{\mathbf{G}}$ is faithful and that the families $(\check{\eta}_{\mathcal{F}})_{\mathcal{F}}$ and $(\check{\epsilon}_{(\mathcal{C},\delta)})_{(\mathcal{C},\delta)}$ are natural transformations as desired. Since $\check{\eta}$ is a natural isomorphism, $\check{\mathbf{F}}$ is full and faithful [18, IV.3 Theorem 1]. □

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