The Fell compactification and non-Hausdorff groupoids

Thomas Timmermann

`timmermt@math.uni-muenster.de`
SFB 478 "Geometrische Strukturen in der Mathematik"
Hittorfstr. 27, 48149 Münster

September 25, 2017

Abstract

A compactification of Fell is applied to locally compact non-Hausdorff groupoids and yields locally compact Hausdorff groupoids. In the étale case, this construction provides a geometric picture for the left-regular representations introduced by Khoshkam and Skandalis.

Keywords: groupoid, compactification, regular representation
MSC 2000: 22A22

1   Introduction

In [1], Fell introduced a compactification of locally compact non-Hausdorff spaces. We apply this construction to groupoids with open range and source map and Hausdorff unit space and show that if one leaves out the point at infinity, one obtains locally compact Hausdorff groupoids. If the non-Hausdorff groupoid was étale, then also the Hausdorff groupoid is étale and provides a geometric picture for the construction of the left-regular representation of the non-Hausdorff groupoid given by Khoshkam and Skandalis in [2]. We also comment on related constructions of Tu [6]. In the remainder of the introduction, we recall the constructions and results of [1].

The Fell compactification   Let X be a topological space. We call a subset $K \subseteq X$ *quasi-compact* if it has the finite covering property, and *compact* if it is quasi-compact and Hausdorff.[1] We assume that X is *locally compact* in the sense that every point $x \in X$ has a compact neighbourhood. In that case, every neighbourhood of any point $x \in X$ contains a compact neighbourhood of x, and X is locally compact in the sense of [1].

Let $(x_\nu)_\nu$ be a net in X. We denote by $\operatorname{clust}_\nu x_\nu$ and $\lim_\nu x_\nu$ the set of all cluster points and the set of all limit points of this net, respectively, and call it *primitive* if it satisfies the following equivalent conditions:

1. Every cluster point of the net is a limit point: $\operatorname{clust}_\nu x_\nu = \lim_\nu x_\nu$.

2. Each $x \in X \setminus \lim_\nu x_\nu$ has a neighbourhood that is eventually left by the net.

---

[1] In [1], Fell calls compact what we call quasi-compact.

3. Every quasi-compact set that contains no limit point is eventually left by the net.

The *Fell compactification* of X is the set $\mathfrak{K}X$ of all limit sets of primitive nets in X, equipped with the topology induced by the subbasis that consists of all sets of the form $\mathcal{U}_V = \{A \in \mathfrak{K}X \mid A \cap V \neq \emptyset\}$ and $\mathcal{U}^Q = \{A \in \mathfrak{K}X \mid A \cap Q = \emptyset\}$, where $V \subseteq X$ is open and $Q \subseteq X$ is quasi-compact. The space $\mathfrak{K}X$ is compact [1], and X embeds into $\mathfrak{K}X$ since each constant net is primitive. This embedding is not necessarily continuous but has dense image. For each net $(x_\nu)_\nu$ in X, the net $(\{x_\nu\})_\nu$ converges in $\mathfrak{K}X$ to some set $A \subseteq X$ if and only if $(x_\nu)_\nu$ is primitive and $A = \lim_\nu x_\nu$ in X [1]. If X is quasi-compact, then $\emptyset \notin \mathfrak{K}X$ and we let $\mathfrak{H}X := \mathfrak{K}X$. Otherwise, $\emptyset \in \mathfrak{K}X$ and $\mathfrak{K}X$ is the one-point compactification of $\mathfrak{H}X := \mathfrak{K}X \setminus \{\emptyset\}$. If X is Hausdorff, then $\mathfrak{H}X = X$.

The Fell compactification is functorial in the following sense. Let X, Y be locally compact spaces and let $f \colon X \to Y$ be continuous and *proper* in the sense that the preimage of every quasi-compact subset is quasi-compact again. Using condition 3 above, one finds that the image of each primitive net in X is primitive in Y, so that the map $\mathfrak{K}f \colon \mathfrak{K}X \to \mathfrak{K}Y$, $A \mapsto f(A)$, is well-defined. One easily checks that this map is continuous and that the restriction $\mathfrak{H}f := \mathfrak{K}f|_{\mathfrak{H}X} \colon \mathfrak{H}X \to \mathfrak{H}Y$ is proper.

Quasi-continuous functions Let X be as above and let Y be a locally compact Hausdorff space. A map $f \colon X \to Y$ is *(w-)quasi-continuous* if for each primitive net $(x_\nu)_\nu$ in X (with non-empty limit set), the net $(f(x_\nu))_\nu$ converges in Y. Evidently, the restriction $f|_X$ of each continuous map $f \colon \mathfrak{H}X \to Y$ is w-quasi-continuous. Conversely, the following lemma implies that for each w-quasi-continuous function $f \colon X \to Y$, the extension $\tilde{f} \colon \mathfrak{H}X \to Y$ defined by $\tilde{f}(\lim_\nu x_\nu) = \lim_\nu f(x_\nu)$ for each primitive net $(x_\nu)_\nu$ in X is continuous.

Lemma 1.1. *Let Z be a topological space with a dense subset Z_0. A map $f \colon Z \to Y$ is continuous if and only if $f(z) = \lim_\nu f(z_\nu)$ for each limit point $z \in Z$ of each net $(z_\nu)_\nu$ in Z_0.*

Proof. Let $f \colon Z \to Y$ be some map, let $(z_\nu)_\nu$ be a net in Z with limit point $z \in Z$, and assume that $(f(z_\nu))_\nu$ does not converge to $f(z)$. Replacing Y by its one-point compactification Y^+ and $(z_\nu)_\nu$ by a subnet, we may assume that $(f(z_\nu))_\nu$ converges to some $y \in Y^+$. Choose disjoint neighbourhoods $V_y, V_{f(z)}$ of y and $f(z)$. For each neighbourhood U of z, we find some ν such that $z_\nu \in U$ and, using the assumption on $(z_\nu)_\nu$, a point $z_U \in U \cap Z_0$ such that $f(z_U) \in V_y$. Ordering the neighbourhoods of z by inclusion, we obtain a net $(z_U)_U$ in Z_0 that converges to z and such that $f(z_U) \notin V_{f(z)}$ for all U. This is a contradiction. $\quad\square$

Summarizing, we obtain a bijection between all w-quasi-continuous maps $X \to Y$ and all continuous maps $\mathfrak{H}X \to Y$, as already stated in [1] but without proof.

2 Application to locally compact groupoids

Let G be a locally compact groupoid such that the unit space G^0 is Hausdorff and the range and the source maps $r, s \colon G \to G^0$ are open [4, 5]. Recall that an *action* of G on a topological space X consists of continuous maps $\rho \colon X \to G^0$ and $\mu \colon G_s \times_\rho X \to X$,

written $(x,y) \mapsto xy$, such that $\rho(x)x = x$, $\rho(\gamma x) = r(\gamma)$, and $(\gamma'\gamma)x = \gamma'(\gamma x)$ for all $x \in X$, $\gamma \in G_{\rho(x)}$, $\gamma' \in G_{r(\gamma)}$.

Proposition 2.1. *Let (ρ, μ) be an action of G on a locally compact space X, where ρ is w-quasi-continuous. Then (ρ, μ) extends to an action $(\tilde{\rho}, \tilde{\mu})$ of G on $\mathfrak{H}X$.*

Proof. By assumption, ρ extends to a continuous map $\tilde{\rho} \colon \mathfrak{H}X \to G^0$, and we only need to show that the formula $(\gamma, A) \mapsto \gamma A$ defines a continuous map $\tilde{\mu} \colon G_s \times_{\tilde{\rho}} \mathfrak{H}X \to \mathfrak{H}X$. Let $(\gamma, A) \in G_s \times_{\tilde{\rho}} \mathfrak{H}X$, where A is the limit set of a primitive net $(x_\nu)_\nu$ in X. Denote by I the set of pairs (U, ν) such that U is a neighbourhood of γ and $\tilde{\rho}(x_{\nu'}) \in s(U)$ whenever $\nu' \geq \nu$, and equip I with an order such that $(U, \nu) \geq (U', \nu')$ if and only if $U \subseteq U'$ and $\nu \geq \nu'$. For each $(U, \nu) \in I$, choose $\gamma_{(U,\nu)} \in U$ such that $s(\gamma_{(U,\nu)}) = \rho(x_\nu)$. Then $\gamma = \lim_{(U,\nu)} \gamma_{(U,\nu)}$ and hence $\gamma A \subseteq \lim_{(U,\nu)} \gamma_{(U,\nu)} x_\nu$. If $x' \in \mathrm{clust}_{(U,\nu)} \gamma_{(U,\nu)} x_\nu$, then $\rho(x') = \lim_\nu r(\gamma_{(U,\nu)}) = r(\gamma)$ and $\gamma^{-1} x' \in \mathrm{clust}_{(U,\nu)} \gamma_{(U,\nu)}^{-1} \gamma_{(U,\nu)} x_\nu = \mathrm{clust}_\nu x_\nu = A$, whence $x' \in \gamma A$. Thus, the net $(\gamma_{(U,\nu)} x_\nu)_{(U,\nu)}$ is primitive and $\gamma A \in \mathfrak{H}X$. Therefore, the map $\tilde{\mu}$ is well defined. The argument above shows that the subset $G_s \times_\rho X \subseteq G_s \times_{\tilde{\rho}} \mathfrak{H}X$ is dense. If $((\gamma_\nu, x_\nu))_\nu$ is a net in $G_s \times_\rho X$ that converges to some $(\gamma, A) \in G_s \times_{\tilde{\rho}} \mathfrak{H}X$, then the net $(x_\nu)_\nu$ is primitive and a similar argument as above shows that the net $(\gamma_\nu x_\nu)_\nu$ is primitive and converges to γA. By Lemma 1.1, $\tilde{\mu}$ is continuous. $\square$

Theorem 2.2. *The space $\mathfrak{H}G$ carries the structure of a locally compact Hausdorff groupoid such that the unit space $(\mathfrak{H}G)^0$ is the closure of G^0 in $\mathfrak{H}G$ and the range map $\mathfrak{r}$, the source map $\mathfrak{s}$, the multiplication $\mathfrak{m}$ and the inversion $\mathfrak{i}$ are given by*

$$\mathfrak{r}(A) = AA^{-1}, \quad \mathfrak{s}(A) = A^{-1}A, \quad \mathfrak{m}((A,B)) = AB, \quad \mathfrak{i}(A) = A^{-1} \quad \text{for all } A, B \in \mathfrak{H}G.$$

The proof involves several lemmas:

Lemma 2.3. *Let $((x_\nu, y_\nu))_\nu$ be a net in $G_s \times_r G$ such that $(x_\nu)_\nu$ and $(y_\nu)_\nu$ are primitive with limit sets $A, B \in \mathfrak{H}G$. Then the net $(x_\nu y_\nu)_\nu$ is primitive with limit set AB.*

Proof. Clearly, $\mathrm{clust}_\nu x_\nu y_\nu \subseteq AA^{-1} \mathrm{clust}_\nu x_\nu y_\nu \subseteq A \mathrm{clust}_\nu x_\nu^{-1} x_\nu y_\nu = A \mathrm{clust}_\nu y_\nu = AB \subseteq \lim_\nu x_\nu y_\nu$. $\square$

Lemma 2.4. *1. $A = AA^{-1}A$ and $A = xA^{-1}A$ for each $A \in \mathfrak{H}G$ and $x \in A$.*

2. $AB = xyB^{-1}B$ for all $A, B \in \mathfrak{H}G$, $x \in A, y \in B$ satisfying $A^{-1}A = BB^{-1}$.

Proof. 1. If $A = \lim_\nu x_\nu$ for a primitive net $(x_\nu)_\nu$ in G, then $AA^{-1}A = \lim_\nu x_\nu x_\nu^{-1} x_\nu = \lim_\nu x_\nu = A$ by the lemma above, and if $x, y \in A$, then $y = xx^{-1}y \in xA^{-1}A$.
 2. By 1., $AB = xA^{-1}AB = xBB^{-1}B = xyB^{-1}B$. $\square$

Lemma 2.5. *The range and the source map of G are w-quasi-continuous.*

Proof. Let $(x_\nu)_\nu$ be a primitive net in G with non-empty limit set. By Lemma 2.3, the nets $(r(x_\nu))_\nu = (x_\nu x_\nu^{-1})_\nu$ and $(s(x_\nu))_\nu = (x_\nu^{-1} x_\nu)_\nu$ are primitive in G and therefore also in G^0, and they have limit points in G^0 because r, s are continuous. $\square$

The range map $r \colon G \to G^0$ and the multiplication map $m \colon G_s \times_r G \to G$ form an action on G. By Lemma 2.5 and Proposition 2.1, it extends to an action $(\tilde{r}, \tilde{m})$ on $\mathfrak{H}G$.

3

Proof of Theorem 2.2. By Lemma 2.3 and 1.1, the maps $\mathfrak{r}, \mathfrak{s}, \mathfrak{i}$ are well defined and continuous. The map $\mathfrak{m}$ is well defined because $AB = xB = \tilde{m}(x, B) \in \mathfrak{H}G$ for all $(A, B) \in \mathfrak{H}G_{\mathfrak{s}} \times_{\mathfrak{r}} \mathfrak{H}G$ and $x \in A$. Equipped with these maps, $\mathfrak{H}G$ becomes a groupoid, as one can easily check using Lemma 2.4. We show that the map $\mathfrak{m}$ is continuous. Assume that $((A_\nu, B_\nu))_\nu$ is a net in $\mathfrak{H}G_{\mathfrak{s}} \times_{\mathfrak{r}} \mathfrak{H}G$ that converges to some point (A, B). Choose $x \in A$ and $y \in B$. If U, V are neighbourhoods of x, y, then $\mathcal{U}_U, \mathcal{U}_V$ are neighbourhoods of A, B and hence there exists some ν such that $A_\nu \in \mathcal{U}_U$, $B_\nu \in \mathcal{U}_V$, that is, $A_\nu \cap U \neq \emptyset$, $B_\nu \cap V \neq \emptyset$. Denote by I the set of all triples (U, V, ν), where U, V are neighbourhoods of x, y and $A_\nu \cap U \neq 0$, $B_\nu \cap V \neq 0$. Order I such that $(U, V, \nu) \geq (U', V', \nu')$ if and only if $U \subseteq U'$, $V \subseteq V'$, $\nu \geq \nu'$. Let $(U, V, \nu) \in I$ and choose $x_\nu^U \in A_\nu \cap U$, $y_\nu^V \in B_\nu \cap V$. The nets $(x_\nu^U y_\nu^V)_{(U,V,\nu)}$ and $(B_\nu^{-1} B_\nu)_{(U,V,\nu)}$ converge to xy and $B^{-1}B$, respectively, and hence $A_\nu B_\nu = \tilde{m}(x_\nu^U y_\nu^V, B_\nu^{-1} B_\nu)$ converges to $\tilde{m}(xy, B^{-1}B) = AB$ in $\mathfrak{H}G$. $\square$

We call an open subset $V \subseteq G$ a *G-set* if $r(V) \subseteq G^0$ is open and $r|_V : V \to r(V)$ is a homeomorphism. Recall that G is étale if it is covered by its open G-sets.

Proposition 2.6. *If G is étale, then $\mathfrak{H}G$ is étale.*

Proof. Let $V \subseteq G$ be an open Hausdorff G-set and let $s_V := s|_V : V \to s(V)$. We show that $\mathfrak{s}(\mathcal{U}_V) = \mathcal{U}_{s(V)} \cap (\mathfrak{H}G)^0$ and construct a continuous map $\mathfrak{t} : \mathcal{U}_{s(V)} \cap (\mathfrak{H}G)^0 \to \mathcal{U}_V$ that is inverse to $\mathfrak{s}$, and then the claim follows because sets of the form $\mathcal{U}_V$ cover $\mathfrak{H}G$. First, $\mathfrak{s}(\mathcal{U}_V) \subseteq \mathcal{U}_{s(V)} \cap (\mathfrak{H}G)^0$ because $A \cap V \neq \emptyset$ implies that $A^{-1}A \cap s(V) \neq \emptyset$ for each $A \in \mathfrak{H}G$. Clearly, $\tilde{s}(B) \in s(V)$ for each $B \in \mathcal{U}_{s(V)}$. The map $\mathfrak{t} : \mathcal{U}_{s(V)} \cap (\mathfrak{H}G)^0 \to \mathfrak{H}G$ given by $B \mapsto s_V^{-1}(\tilde{s}(B))B = \tilde{m}(s_V^{-1}(\tilde{s}(B)), B)$ is continuous because the maps $s_V^{-1}, \tilde{s}, \tilde{m}$ are continuous, its image is contained in $\mathcal{U}_V$ because $s_V^{-1}(\tilde{s}(B)) \in s_V^{-1}(\tilde{s}(B))B$ for each $B \in \mathcal{U}_{s(V)}$, and $\mathfrak{t}(\mathfrak{s}(A)) = (A \cap V)A^{-1}A = A$ and $\mathfrak{s}(\mathfrak{t}(B)) = B^{-1}B = B$ for each $A \in \mathcal{U}_V$, $B \in \mathcal{U}_{s(V)} \cap (\mathfrak{H}G)^0$. $\square$

The groupoid $\mathfrak{H}G$ can be identified with the quotient of the transformation groupoid [4] associated to an adjoint action of G on $(\mathfrak{H}G)^0$ as follows.

Proposition 2.7.
1. *There exists an action $(\tilde{t}, \tilde{\mathrm{Ad}})$ of G on $(\mathfrak{H}G)^0$ such that $\tilde{t}(A) = \tilde{r}(A) = \tilde{s}(A)$ and $\tilde{\mathrm{Ad}}(x, A) = xAx^{-1}$ for all $A \in (\mathfrak{H}G)^0$, $x \in G_{\tilde{t}(A)}$.*

2. *The map $G \ltimes (\mathfrak{H}G)^0 \to \mathfrak{H}G$ given by $(x, A) \mapsto xA$ is a continuous surjective groupoid homomorphism with kernel $N = \{(x, A) \in G \ltimes (\mathfrak{H}G)^0 \mid x \in A\}$.*

3. *The induced map $(G \ltimes (\mathfrak{H}G)^0)/N \to \mathfrak{H}G$ is an isomorphism of topological groupoids.*

Proof. 1. The only nontrivial assertion is continuity of the map $\tilde{\mathrm{Ad}}$. Since s is open, $G_s \times_r G^0$ is dense in $G_s \rtimes_{\tilde{t}} (\mathfrak{H}G)^0$ (see the proof of Proposition 2.1), and continuity follows from Lemma 1.1 and 2.3.

2. The map is a groupoid homomorphism because for each pair of composable elements $(x, A), (x', A') \in G \ltimes (\mathfrak{H}G)^0$, we have $\mathfrak{s}(xA) = A^{-1}x^{-1}xA = A$, $\mathfrak{r}(x'A') = x'A'A'^{-1}x'^{-1} = x'A'x'^{-1}$, and $xAx'A' = xx'A'$. It is continuous because it is the restriction of $\tilde{m}$, and surjective because $A = xA^{-1}A$ for each $A \in \mathfrak{H}G$ and $x \in A$.

3. Denote by q the inverse of the induced map. We have to show that q is continuous and do so using Lemma 1.1. If $(x_\nu)_\nu$ is a primitive net in G with limit set A and $x \in A$, then the net $(q(x_\nu))_\nu = ((x_\nu, s(x_\nu))N)_\nu$ clearly converges to $q(A) = (x, A^{-1}A)N$. $\square$

4

Proposition 2.8. *Let (ρ, μ) be an action of G on a locally compact space X, where ρ is quasi-continuous. Then (ρ, μ) extends to an action $(\hat{\rho}, \hat{\mu})$ of $\mathfrak{H}G$ on $\mathfrak{H}X$ such that $\hat{\rho}(B) = \rho(B)$ and $\hat{\mu}(A, B) = AB$ for all $A \in \mathfrak{H}G$, $B \in \mathfrak{H}X$.*

Proof. For each primitive net $(x_\nu)_\nu$ in X, the net $(\rho(x_\nu))_\nu$ converges in G^0 and therefore is primitive in G and converges in $\mathfrak{H}G$. Hence, ρ is w-quasi-continuous as a map to $\mathfrak{H}G$ and extends as claimed. Similar arguments as in the proof of Theorem 2.2 show that the map $\hat{\mu}$ is well defined and continuous. Clearly, $(\hat{\rho}, \hat{\mu})$ is an action. $\square$

By functoriality of $\mathfrak{H}$, the assignment $G \mapsto \mathfrak{H}G$ extends to a functor from the category of locally compact groupoids with open range and source maps and Hausdorff unit space together with proper and continuous groupoid homomorphisms into the category of locally compact Hausdorff groupoids with proper continuous homomorphisms.

3   Relations to other constructions

A construction of Tu   Let X be a locally compact space. In [6], Jean-Louis Tu associated to X a Hausdorff space $\mathcal{H}X$ as follows. As a set, $\mathcal{H}X$ consists of all subsets $A \subset X$ satisfying the following condition: for every family $(V_x)_{x \in A}$ of open sets such that $x \in V_x$ for all $x \in A$ and $V_x = X$ except perhaps for finitely many $x \in A$, one has $\bigcap_{x \in A} V_x \neq \emptyset$. This set is endowed with the topology generated by all subsets of the form $\Omega_V = \{A \in \mathcal{H}X \mid A \cap V \neq \emptyset\}$ and $\Omega^Q = \{A \in \mathcal{H}X \mid A \cap Q = \emptyset\}$, where $V \subseteq X$ is open and $Q \subseteq X$ is quasi-compact.

Proposition 3.1. *$\mathfrak{H}X$ is a subspace of $\mathcal{H}X$.*

Proof. Let $A \in \mathfrak{H}X$ and let $(V_x)_{x \in A}$ be a family of open sets such that $x \in V_x$ for all $x \in A$ and $V_x = X$ for all but finitely many $V_1, \ldots, V_n$. Then $\mathcal{U}_{V_1} \cap \cdots \cap \mathcal{U}_{V_n}$ is a neighbourhood of A and therefore contains a point $x \in X$, that is, $V_1 \cap \cdots \cap V_n \neq \emptyset$. The topology on $\mathfrak{H}X$ coincides with the subspace topology inherited from $\mathcal{H}X$ by definition. $\square$

Note that Proposition 2.1 is an analogue of [6, Proposition 3.10].

A spectral picture of the Fell compactification   Let X be a locally compact space and denote by $\mathcal{B}(X)$ the C^*-algebra of all bounded Borel functions on X. Given an open Hausdorff subset $U \subseteq X$ and a function $f \in C_c(U)$, we identify f with a function on X extending it outside of U by 0, and denote by $\operatorname{supp} f$ the support of f inside U. Denote by $\mathcal{A}_c(X) \subseteq \mathcal{B}(X)$ the smallest subalgebra containing $C_c(U)$ for each open Hausdorff subset $U \subseteq X$, and by $\mathcal{A}_0(X)$ its norm closure.

Proposition 3.2. *Restriction of functions defines a $*$-isomorphism $C_0(\mathfrak{H}X) \to \mathcal{A}_0(X)$ that maps $C_c(\mathfrak{H}X)$ to $\mathcal{A}_c(X)$.*

Proof. Let $U \subseteq X$ be open and Hausdorff and let $f \in C_c(U)$. If $(x_\nu)_\nu$ is a primitive net in X with non-empty limit set, then either $\operatorname{supp} f$ contains a limit point of the net, x, say, or $\operatorname{supp} f$ is eventually left by the net. In the first case, $f(x_\nu)$ converges to $f(x)$ because f is continuous on U, and in the second case, $f(x_\nu)$ is constant 0. Thus, f is

5

w-quasi-continuous on X. Consequently, each function in $\mathcal{A}_0(X)$ is w-quasi-continuous and extension of functions defines an embedding $\iota\colon \mathcal{A}_0(X) \hookrightarrow C_0(\mathfrak{H}X)$. If $A, B \in \mathfrak{H}X$ and $x \in A \backslash B$, then there exists an open Hausdorff neighbourhood U of x that is disjoint to B, and for each $f \in C_c(U)$ we have $(\iota(f))(A) = f(x)$ and $(\iota(f))(B) = 0$. Therefore, $\iota(\mathcal{A}_c(X))$ separates the points of $\mathfrak{H}X$. By the Stone-Weierstrass theorem, ι is surjective, and it follows that $C_c(\mathfrak{H}X) \subseteq \iota(\mathcal{A}_c)$. The last inclusion is an equality because for each $f \in \mathcal{A}_c(X)$, there exists a quasi-compact set $Q \subseteq X$ such that $f|_{X \backslash Q} = 0$, and $\mathcal{U}^Q$ is a neighbourhood of $\emptyset$ in $\mathfrak{K}X$ such that $\iota(f)|_{\mathcal{U}^Q \cap \mathfrak{H}X} \equiv 0$. $\qquad\square$

The left-regular representation of Khoshkam and Skandalis Let G be an étale, locally compact groupoid with open range map and Hausdorff unit space. In [2], Mahmood Khoshkam and Georges Skandalis define a left-regular representation of G using Hilbert C^*-modules [3] and functional-analytic tools. The action $\tilde{m}\colon G_s \times_{\tilde{r}} \mathfrak{H}G \to \mathfrak{H}G$ provides the following geometric picture for their construction.

Let $C_c(G)$ be the linear span of all subspaces $C_c(U)$, where $U \subseteq G$ is open and Hausdorff. Then $C_c(G)$ is a $*$-algebra with respect to the operations

$$(f * g)(x) = \sum_{z \in G^{r(x)}} f(z) g(z^{-1}x), \quad f^*(x) = \overline{f(x^{-1})}, \quad \text{where } x \in G, f, g \in C_c(G). \quad (1)$$

Denote $\mathcal{B}(G^0)$ the C^*-algebra of bounded Borel functions. Let $D \subseteq \mathcal{B}(G^0)$ be the C^*-subalgebra generated by all restrictions $f|_{G^0}$, where $f \in C_c(G)$, let Y be the spectrum of D, and identify D with $C_0(Y)$ in the canonical way. Then the algebraic tensor product $C_c(G) \odot C_0(Y)$ is a pre-Hilbert C^*-module over $C_0(Y)$ with respect to the inner product and right module structure given by

$$\langle f \odot a | g \odot b \rangle = a^* \cdot (f^* * g)|_{G^0} \cdot b, \quad (f \odot a)b = f \odot ab, \quad \text{where } f, g \in C_c(G), a, b \in C_0(Y).$$

Denote by $L^2(G)$ the separated completion. There exists a representation $L_G\colon C_c(G) \to \mathcal{L}(L^2(G))$ given by $L_G(f)(g \odot a) = f * g \odot a$ for all $f, g \in C_c(G)$, $a \in C_0(Y)$. Denote by $C_r^*(G)$ the completion of $L_G(C_c(G))$. Proposition 3.2 applied to G immediately implies:

Lemma 3.3. *The map $f \mapsto f|_{G^0}$ is a $*$-isomorphism $C_0((\mathfrak{H}G)^0) \to C_0(Y) \subseteq \mathcal{B}(G^0)$. In particular, Y is homeomorphic to $(\mathfrak{H}G)^0$.* $\qquad\square$

The space $C_c(\mathfrak{H}G)$ carries the structure of a $*$-algebra, where the operations are defined similarly as in (1), and the structure of a pre-Hilbert C^*-module over $C_0((\mathfrak{H}G)^0)$, where $\langle f | g \rangle = (f^* * g)|_{(\mathfrak{H}G)^0}$ and $(fh)(A) = f(A)h(\mathfrak{s}(A))$ for all $f, g \in C_c(\mathfrak{H}G), h \in C_0((\mathfrak{H}G)^0), A \in \mathfrak{H}G$. Denote the completion of this pre-Hilbert C^*-module by $L^2(\mathfrak{H}G)$. Then there exists a representation $L_{\mathfrak{H}G}\colon C_c(\mathfrak{H}G) \to \mathcal{L}(L^2(\mathfrak{H}G))$ such that $L_{\mathfrak{H}G}(f)g = f * g$ for all $f, g \in C_c(G)$, and the closure $C_r^*(\mathfrak{H}G) = \overline{L_{\mathfrak{H}G}(C_c(G))}$ is the reduced C^*-algebra of $\mathfrak{H}G$ [4, 5]. Identify $C_0(Y)$ with $C_0((\mathfrak{H}G)^0)$ as in Lemma 3.3, and $C_c(G)$ with a subspace of $\mathcal{A}_c(G) = C_c(\mathfrak{H}G)$, see Proposition 3.2. Denote by $\mathfrak{r}^*\colon C_0((\mathfrak{H}G)^0) \to \mathcal{L}(L^2(\mathfrak{H}G))$ the representation given by $((\mathfrak{r}^*h)f)(A) = h(\mathfrak{r}(A))f(A)$ for all $h \in C_0((\mathfrak{H}G)^0)$, $f \in C_c(\mathfrak{H}G)$, $A \in \mathfrak{H}G$, and by s^* the pull-back of functions from G^0 to G via s.

Lemma 3.4. $\mathcal{A}_c(G) = C_c(G)s^*(C_0(Y))$.

Proof. Let $U, V \subseteq G$ be open Hausdorff G-sets and let $f \in C_c(U)$, $g \in C_c(V)$. Choose $h \in C_c(U)$ such that $h|_{\operatorname{supp} f} = 1$. Then $h^* * g, h * g \in C_c(G)$,

$$(h^* * g)(s(x)) = g(x) \text{ for all } x \in \operatorname{supp} f, \quad f(x)g(x) = f(x)(h^* * g)(s(x)) \text{ for all } x \in G,$$
$$g(s(x)) = (h * g)(x) \text{ for all } x \in \operatorname{supp} f, \quad f(x)g(s(x)) = f(x)(h * g)(x) \text{ for all } x \in G,$$

and hence $fg = fs^*((h^* * g)|_{G^0})$ and $fs^*(g|_{G^0}) = f(h * g)$. Using a partition of unity argument and the fact that G is étale, we can conclude that $C_c(G)C_c(G) = C_c(G)s^*(C_0(Y))$. By induction, we find that $C_c(G)^n = C_c(G)^{n-1}s^*(C_0(Y)) = \cdots = C_c(G)s^*(C_0(Y))$ for each $n \in \mathbb{N}$ and therefore $\mathcal{A}_c(G) = C_c(G)s^*(C_0(Y))$. $\qquad\square$

Proposition 3.5. *1. There exists an isomorphism $\Phi \colon L^2(G) \to L^2(\mathfrak{H}G)$ of Hilbert C^*-modules such that $\Phi(f \odot a) = fs^*(a)$ for all $f \in C_c(G) \subseteq C_c(\mathfrak{H}G)$, $a \in C_0(Y)$.*

2. The C^*-algebra generated by $\operatorname{Ad}_\Phi(C_r^*(G))$ and $\mathfrak{r}^*(C_0((\mathfrak{H}G)^0))$ is $C_r^*(\mathfrak{H}G)$.

Proof. Both assertions follow from Lemma 3.4 and the straightforward relations

$$\langle fs^*(a)|gs^*(b)\rangle_{L^2(\mathfrak{H}G)} = \langle f \odot a|g \odot b\rangle_{L^2(G)} \qquad \text{for all } f, g \in C_c(G), a, b \in C_0(Y),$$
$$\operatorname{Ad}_\Phi(L_G(f))\mathfrak{r}^*(h) = L_{\mathfrak{H}G}(fs^*(h)) \qquad \text{for all } f \in C_c(G), h \in C_0(Y). \qquad\square$$

References

[1] J. M. G. Fell. A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space. *Proc. Amer. Math. Soc.*, 13:472–476, 1962.

[2] M. Khoshkam and G. Skandalis. Regular representation of groupoid C^*-algebras and applications to inverse semigroups. *J. Reine Angew. Math.*, 546:47–72, 2002.

[3] E. C. Lance. *Hilbert C^*-modules*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists.

[4] A. L. T. Paterson. *Groupoids, inverse semigroups, and their operator algebras*, volume 170 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999.

[5] J. Renault. *A groupoid approach to C^*-algebras*, volume 793 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[6] J.-L. Tu. Non-Hausdorff groupoids, proper actions and K-theory. *Doc. Math.*, 9:565–597 (electronic), 2004.