The dynamical quantum group

$\mathcal{SU}_q(2)$

on the level of operator algebras

(work in progress)

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Joint project with Erik Koelink

Study the dynamical $\mathcal{SU}_q(2)$ on the level of operator algebras

Motivation/Aims

- *Erik's:* study relations between special functions and dynamical quantum groups beyond $\mathcal{SU}_q(2)$
- *mine:* study quantum groupoids in the C^*-/W^*-setting
  - establish a link to the setting of pure algebra
  - obtain fundamentally new examples
  - test case for a theory of proper C^*-quantum groupoids

Background from compact quantum groups

Each compact quantum group $\mathbb{G}$ has associated

1. an algebra $\mathcal{O}(\mathbb{G})$ of polynomial functions on $\mathbb{G}$
2. a universal C^*-algebra $C_u(\mathbb{G}) = C^*(\mathcal{O}(\mathbb{G}))$
3. a reduced C^*-algebra $C_r(\mathbb{G}) = \overline{\pi(\mathcal{O}(\mathbb{G}))} \subseteq \mathcal{L}(H)$, where $\pi\colon \mathcal{O}(\mathbb{G}) \to \mathcal{L}(H)$ is the GNS-rep. for the Haar state
4. a von Neumann algebra $L(\mathbb{G}) = \pi(\mathcal{O}(\mathbb{G}))'' \subseteq \mathcal{L}(H)$

Fundamental example: $\mathbb{G} = \mathrm{SU}_q(2)$

- $\mathcal{O}(\mathrm{SU}_q(2)) = \left\langle a, c \,\middle|\, u := \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \text{ is unitary} \right\rangle$
- $\Delta\colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2)) \odot \mathcal{O}(\mathrm{SU}_q(2))$ s.t. $u \xmapsto{M_2(\Delta)} u \boxtimes u$

Dynamical Hopf algebroids

Definition A *dynamical Hopf algebroid* consists of

- a commutative algebra B
- an algebra A with commuting inclusions $r, s\colon B \to A$
- an action of a group G on B and a $G \times G$-grading on A s.t. $\forall a \in A_{\gamma,\gamma'}, b \in B : ar(b) = r(\gamma(b))a$ and $as(b) = s(\gamma'(b))a$
- a homomorphism $\Delta\colon A \to A * A$ that is *coassociative*, where $A * A = \bigoplus_{\gamma,\gamma',\gamma'' \in G} (A_{\gamma,\gamma'})_s \underset{B}{\odot}{}_r (A_{\gamma',\gamma''})$
- an antipode and counit subject to further conditions

Example If a finite group G acts on a compact space X, we get

$$\underbrace{C(X)}_{=B} \overset{r,s}{\hookrightarrow} \underbrace{C(X) \rtimes G}_{=A} \text{ with } \Delta\colon f \rtimes U_\gamma \mapsto (f \rtimes U_\gamma) \underset{C(X)}{\odot} (1 \rtimes U_\gamma)$$

Definition of the dynamical $\mathcal{SU}_q(2)$ on the algebraic level

Let $q \in (0,1)$. Then $\mathcal{SU}_q(2)$ is a dynamical Hopf algebroid with

- $B = C_c(\mathbb{R})$, and $G = \mathbb{Z}$ acts via translations $f \mapsto f_{\pm k} := f(\cdot \pm k)$
- $A = \mathcal{O}(\mathcal{SU}_q(2)) = *$-algebra generated by $r(B), s(B), a, c$, subject to the following relations
  - $r(f)s(g)a = ar(f_{+1})s(g_{+1})$ and $r(f)s(g)c = cr(f_{-1})s(g_{+1})$
  - the matrix $u := \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \in M_2(A)$ satisfies

$$u\begin{pmatrix} s(F_{-1}) & 0 \\ 0 & 1 \end{pmatrix}^{-1} u^* \begin{pmatrix} r(F) & 0 \\ 0 & 1 \end{pmatrix} = 1 = u^* \begin{pmatrix} r(F_{-1}) & 0 \\ 0 & 1 \end{pmatrix} u \begin{pmatrix} s(F) & 0 \\ 0 & 1 \end{pmatrix}^{-1},$$

    where $F \in C_b(\mathbb{R})$ is given by $F(t) = \frac{q^{2t}+q^{-2}}{q^{2t}+1}$
- Δ on a, c given by the same formulas as for $\mathrm{SU}_q(2)$

Idea u is a corepresentation of $\mathcal{SU}_q(2)$ on a $\mathbb{C}^2$-bundle on $\mathbb{R}$

Relation to the usual definition of $\mathcal{SU}_q(2)$

Detailed study of $\mathcal{SU}_q(2)$ and its relations to special functions:

Koelink, Rosengren. *Harmonic analysis on the dynamical quantum group* $\mathcal{SU}_q(2)$*, 2001.*

Comments

1. The usual definition involves the choice $B = \mathfrak{M}(\mathbb{C})$; our modification allows us to pass to operator algebras
2. $\mathcal{SU}_q(2)$ can actually be defined over a smaller $B = Q$
3. There exists a diagram of algebras $C_b(\mathbb{R}) \leftarrow Q \rightarrow \mathfrak{M}(\mathbb{C})$ and a corresponding diagram of variants of $\mathcal{SU}_q(2)$:

$$\text{our } \mathcal{SU}_q(2)/_{C_c(\mathbb{R})} \xleftarrow[\text{change}]{\text{base}} \mathcal{SU}_q(2)/_Q \xrightarrow[\text{change}]{\text{base}} \text{usual } \mathcal{SU}_q(2)/_{\mathfrak{M}(\mathbb{C})}$$

Passage to the level of operator algebras

We shall construct a diagram of the following form:

algebraic level	universal C^*-level	reduced C^*-level	von Neumann level

$$\mathcal{O}(\mathcal{SU}_q(2)) \hookrightarrow C_u(\mathcal{SU}_q(2)) \twoheadrightarrow C_r(\mathcal{SU}_q(2)) \hookrightarrow L(\mathcal{SU}_q(2))$$

Hopf algebroid (Böhm et al.)		proper reduced C^*-qtm. groupoid (T.)	measured qtm. groupoid (Lesieur, Enock)

$$\begin{array}{ccccccc}
C_c(\mathbb{R}) & \hookrightarrow & C_0(\mathbb{R}) & \twoheadrightarrow & \overline{\pi_\mu(C_0(\mathbb{R}))} & \hookrightarrow & L^\infty(\mathbb{R}) \\
s\downarrow\downarrow r & & \downarrow\downarrow & & \downarrow\downarrow & & \bar{s}\downarrow\downarrow\bar{r} \\
\mathcal{O} & \hookrightarrow & C_u = C^*(\mathcal{O}) & \twoheadrightarrow & C_r = \overline{\pi_\nu(\mathcal{O})} & \hookrightarrow & L = \pi_\nu(\mathcal{O})'' \\
\Delta\downarrow & & \downarrow & & \downarrow & & \downarrow\bar{\Delta} \\
\mathcal{O} \underset{\text{alg}}{*} \mathcal{O} & \hookrightarrow & C_u \underset{\max}{*} C_u & \rightarrow & C_r \underset{\min}{*} C_r & \hookrightarrow & L \bar{*} L
\end{array}$$

The functional for the GNS-construction

We introduce $\nu\colon \mathcal{O}(\mathcal{SU}_q(2)) \to \mathbb{C}$ to define $C_r := \overline{\pi_\nu(\mathcal{O}(\mathcal{SU}_q(2)))}$:

Ingredient 1. Koelink, Rosengren:

- $\mathcal{O}(\mathcal{SU}_q(2))$ is a free $r(B)\otimes s(B)$-module with basis $(t^N_{i,j})_{N,i,j}$
- *Haar map* $h\colon \mathcal{O}(\mathcal{SU}_q(2)) \to B \otimes B$, $r(f)s(g)t^N_{i,j} \mapsto \delta_{N,0}(f \otimes g)$

Ingredient 2. We fix a measure on $\mathbb{R}$ that is (quasi-)invariant w.r.t. $\mathbb{Z}$ and obtain a positive map $\mu\colon C_c(\mathbb{R}) \to \mathbb{C}$ with bounded, nondegenerate GNS-construction (H_μ, π_μ)

Lemma: $\nu\colon \mathcal{O}(\mathcal{SU}_q(2)) \xrightarrow{h} C_c(\mathbb{R}) \otimes C_c(\mathbb{R}) \xrightarrow{\mu\otimes\mu} \mathbb{C}$ is positive

Proof: Use explicit formula for $h((t^N_{i,j})^* t^M_{k,l})$ [Koelink, Rosengren] and our modification of the definition of $\mathcal{SU}_q(2)$

The construction of a fundamental unitary

Let (H_ν, π_ν) be the GNS-construction for $\nu : \mathcal{O}(\mathcal{SU}_q(2)) \to \mathbb{C}$

Problem: Show that $\pi_\nu(a)$ is bounded for each $a \in \mathcal{O}(\mathcal{SU}_q(2))$

Lemma 1. $\pi_\nu(r(f)), \pi_\nu(s(f))$ are bounded for all $f \in C_c(\mathbb{R})$

2. $\exists$ extensions

$$\begin{array}{ccccc} C_c(\mathbb{R}) & & \xrightarrow{\pi_\nu \circ r,\, \pi_\nu \circ s} & & \mathcal{L}(H_\nu) \\ & \searrow^{\pi_\mu} & & \nearrow_{\bar{r}, \bar{s}} & \\ & & L^\infty(\mathbb{R}) & & \end{array}$$

2. $\exists \Lambda^{(\dagger)} : \mathcal{O}(\mathcal{SU}_q(2)) \to \mathcal{L}(H_\mu, H_\nu) : \Lambda(a)f = as(f), \Lambda^\dagger(a)f = s(f)a$

3. $E := \overline{\mathrm{Img}(\Lambda)}$ and $E^\dagger := \overline{\mathrm{Img}(\Lambda^\dagger)}$ are Hilbert C^*-modules over $C_0(\mathbb{R}) \subseteq \mathcal{L}(H_\mu)$, where $\langle \xi | \eta \rangle = \xi^* \eta$ and ξf is the composition

Theorem There exists a unitary $V : E \otimes_{\bar{s}} H_\nu \to E^\dagger \otimes_{\bar{r}} H_\nu$ such that

$$\Lambda(x) \otimes_{\bar{s}} y \mapsto \sum \Lambda^\dagger(x_{(1)}) \otimes_{\bar{r}} x_{(2)} y, \text{ where } \sum x_{(1)} \otimes x_{(2)} = \Delta(x)$$

Existence of the fundamental unitary

Theorem $\exists$ a unitary $V\colon \Lambda(x) \otimes_{\bar{s}} y \mapsto \sum \Lambda^\dagger(x_{(1)}) \otimes_{\bar{r}} x_{(2)} y$

Proof 1. isometric: $\psi\colon \mathcal{O}(\mathcal{SU}_q(2)) \xrightarrow{h} C_c(\mathbb{R}) \otimes C_c(\mathbb{R}) \xrightarrow{\mu\otimes\mathrm{id}} C_c(\mathbb{R})$

$$\|\Lambda(x) \otimes_{\bar{s}} y\|^2 = \langle y | s(\psi(x^* x)) y\rangle,$$

$$\|\sum \Lambda^\dagger(x_{(1)}) \otimes_{\bar{r}} x_{(2)} y\|^2 = \sum \langle y | r(\psi(x^*_{(1)} x_{(1')})) x_{(2)} x_{(2')} y\rangle,$$

both expressions coincide because ψ is *right-invariant*:

- $t^N_{i,j} \in \ker h \subseteq \ker \psi$ for $N \geq 1$ and $\Delta(t^N_{i,j}) = \sum_k t^N_{i,k} \odot t^N_{k,j}$
- $z = r(f)s(g)t^N_{i,j}$ for $N \geq 1 \Rightarrow s(\psi(z)) = 0 = r(\psi(z_{(1)}))z_{(2)}$
- $z = r(f)s(g) \Rightarrow \Delta(z) = r(f) \odot s(g)$

$$\Rightarrow s(\psi(z)) = s(\mu(f)g) = \sum r(\psi(z_{(1)}))s(z_{(2)})$$

2. surjective: the following map is inverse to V,

$$\Lambda^\dagger(x) \otimes_{\bar{r}} y \mapsto \sum \Lambda(x_{(1)}) \otimes_{\bar{s}} S(x_{(2)}) y$$

Boundedness of the GNS-construction for ν

Corollary $\pi_\nu(\mathcal{O}(\mathcal{SU}_q(2))) \subseteq \mathcal{L}(\mathcal{H})$. Thus, we can define

$$C_r(\mathcal{SU}_q(2)) := \overline{\pi_\nu(\mathcal{O}(\mathcal{SU}_q(2)))}, \quad L(\mathcal{SU}_q(2)) := C_r(\mathcal{SU}_q(2))''.$$

Sketch of the Proof Let $x, x' \in \mathcal{O}(\mathcal{SU}_q(2))$. Then

- there exist bounded linear operators

$$I(x)\colon H_\nu \to E \otimes_{\bar{s}} H_\nu, \quad \xi \mapsto \Lambda(x) \otimes_{\bar{s}} \xi,$$
$$I^\dagger(x')\colon H_\nu \to E^\dagger \otimes_{\bar{r}} H_\nu, \quad \xi \mapsto \Lambda^\dagger(x') \otimes_{\bar{r}} \xi,$$

- $I^\dagger(x')^* V I(x)\colon H_\nu \xrightarrow{I(x)} E \otimes_{\bar{s}} H_\nu \xrightarrow{V} E^\dagger \otimes_{\bar{r}} H_\nu \xrightarrow{I^\dagger(x')^*} H_\nu$ is equal to $\pi_\nu(z)$, where $z = \sum r\big(\Lambda^\dagger(x')^* \Lambda^\dagger(x_{(1)})\big) x_{(2)}$
- elements like z span $\mathcal{O}(\mathcal{SU}_q(2))$

The fundamental unitary is a pseudo-multiplicative unitary

l.c. quantum group $\mathbb{G}$ $\longrightarrow$ multiplicative unitary $W \in \mathcal{L}(K \otimes K)$, $W_{12}W_{13}W_{23} = W_{23}W_{12}$ $\longrightarrow$ $C_u(\mathbb{G}), C_r(\mathbb{G}), LG$; $C_u(\widehat{\mathbb{G}}), C_r(\widehat{\mathbb{G}}), L\widehat{\mathbb{G}}$

(C^*-)quantum groupoid $\mathbb{G}$ $\longrightarrow$ (C^*-)pseudo-multiplicative unitary $\longrightarrow$ $(C_r(\mathbb{G}), C_r(\widehat{\mathbb{G}}))$, $L\mathbb{G}, L\widehat{\mathbb{G}}$

Theorem V is a *regular pseudo-multiplicative unitary*

in the sense of Vallin (W^*-level) and of T. (C^*-level)

Lemma $\exists$ canonical identifications of $E \otimes_{\bar{s}} H_\nu$ and $E^\dagger \otimes_{\bar{r}} H_\nu$ with

Connes' fusions $H_\nu \,{}_{\bar{t}}\!\underset{\bar{\mu}}{\otimes}_{\bar{s}}\, H_\nu$ and $H_\nu \,{}_{\bar{s}}\!\underset{\bar{\mu}}{\otimes}_{\bar{r}}\, H_\nu$, where

- $\bar{t} \colon L^\infty(\mathbb{R}) \to \mathcal{L}(H_\nu)$ is given by $\bar{t}(f)a = as(f)$,
- $\bar{\mu}$ is the n.s.f. weight on $L^\infty(\mathbb{R})$ given by integration.

The C^*- and W^*-quantum groupoid

Corollary $\exists$ Hopf C^*-bimodule & Hopf-von Neumann bimodule

$$C_0(\mathbb{R}) \underset{\bar{s}}{\overset{\bar{r}}{\rightrightarrows}} C_r(\mathcal{SU}_q(2)) \xrightarrow{\bar{\Delta}} C_r(\mathcal{SU}_q(2)) * C_r(\mathcal{SU}_q(2)), \qquad (1)$$

$$L^\infty(\mathbb{R}) \underset{\bar{s}}{\overset{\bar{r}}{\rightrightarrows}} L(\mathcal{SU}_q(2)) \xrightarrow{\bar{\Delta}} L(\mathcal{SU}_q(2)) * L(\mathcal{SU}_q(2)), \qquad (2)$$

where $\bar{\Delta}$ is given by $x \mapsto V(x \,{}_{\bar{t}}\underset{\bar{\mu}}{\otimes}{}_{\bar{s}} 1) V^*$

Theorem (4) is a measured quantum groupoid [Lesieur,Enock]

((3) is a proper reduced C^*-quantum groupoid [T])

Key point $\psi \colon \mathcal{O}(\mathcal{SU}_q(2)) \to C_c(\mathbb{R})$ extends to a right-invariant

- regular C^*-valued weight $\tilde{\psi} \colon C_r(\mathcal{SU}_q(2)) \to C_0(\mathbb{R})$
- n.s.f. operator-valued weight $\bar{\psi} \colon L(\mathcal{SU}_q(2)) \to L^\infty(\mathbb{R})$

Kustermans inverse GNS-construction $C_r(\mathcal{SU}_q(2)) \to \mathcal{L}_{C_0(\mathbb{R})}(E)$

The dual of $\mathcal{SU}_q(2)$

Corollary Hopf C^*-bimodule & measured quantum groupoid

$$C_0(\mathbb{R}) \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} C_r(\widehat{\mathcal{SU}_q(2)}) \xrightarrow{\widehat{\Delta}} C_r(\widehat{\mathcal{SU}_q(2)}) * C_r(\widehat{\mathcal{SU}_q(2)}), \quad (3)$$

$$L^\infty(\mathbb{R}) \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} L(\widehat{\mathcal{SU}_q(2)}) \xrightarrow{\widehat{\Delta}} L(\widehat{\mathcal{SU}_q(2)}) * L(\widehat{\mathcal{SU}_q(2)}), \quad (4)$$

where $\widehat{\Delta}$ is given by $x \mapsto V^*(1_{\bar{s}} \underset{\bar{\mu}}{\otimes}{}_{\bar{r}} x)V$

Proof (4) follows from the work of Lesieur, (3) from work of T.

Proposition 1. $\exists F\colon \mathcal{O}(\mathcal{SU}_q(2)) \to L\big(\mathcal{O}(\mathcal{SU}_q(2)), C_c(\mathbb{R})\big)$ such that $\mathcal{O}(\widehat{\mathcal{SU}_q(2)}) := \mathrm{Img}(F)$ is a $*$-algebra w.r.t. convolution

2. $\exists$ representation $\lambda\colon \mathcal{O}(\widehat{\mathcal{SU}_q(2)}) \to \mathcal{L}(H_\nu)$ such that

$$C_r(\widehat{\mathcal{SU}_q(2)}) = \overline{\mathrm{Img}(\lambda)} \text{ and } L(\widehat{\mathcal{SU}_q(2)}) = \mathrm{Img}(\lambda)''$$

The next problems on the agenda

- Investigation of the construction for $\mathcal{SU}_q(2)$
  - computation of the structure maps of the C^*-/W^*-quantum groupoids in terms of special functions
  - explicit formula for V with respect to $H \cong l^2(\mathbb{N} \times \mathbb{N} \times \mathbb{N})$
  - relation to the algebraic dual of $\mathcal{SU}_q(2)$ by Rosengren
- Axiomatics of proper reduced C^*-quantum groupoids
- Extension to the non-proper case
- Dynamical Hopf algebroids on the universal C^*-level