MULTIPLIER HOPF ALGEBROIDS. BASIC THEORY AND EXAMPLES.

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Abstract. Multiplier Hopf algebroids are algebraic versions of quantum groupoids that generalize Hopf algebroids to the non-unital case and weak (multiplier) Hopf algebras to non-separable base algebras. The main structure maps of a multiplier Hopf algebroid are a left and a right comultiplication. We show that bijectivity of two associated canonical maps is equivalent to the existence of an antipode, discuss invertibility of the antipode, and present some examples and special cases.

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1. Introduction

Quantum groupoids have appeared in a variety of guises and mathematical contexts, for example, as generalized Galois symmetries for depth 2 inclusions of factors or algebras [4], [9], [10], [14], [15], [22], as dynamical quantum groups in connection with solutions to the quantum dynamical Yang-Baxter equation [7], [11], [16], or as Tannaka-Krein duals of certain tensor categories of bimodules [13], [18], [24]. Common to all approaches are the basic constituents of a quantum groupoid — a pair of anti-isomorphic algebras B and C with homomorphisms into an algebra A together with a comultiplication on A that takes values in a certain fiber product $A * A$ involving B and C. These ingredients are, in a sense, dual to the constituents of a groupoid, and satisfy corresponding conditions like co-associativity of the comultiplication.

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In this article, we extend the existing algebraic approaches to quantum groupoids via Hopf algebroids [1], [5], [17, 39], weak Hopf algebras [3], [23], [26] and weak multiplier Hopf algebras [2], [36], [38] by considering so-called multiplier Hopf algebroids, where the underlying algebras are no longer assumed to be unital.

The motivation to study multiplier versions of Hopf algebroids and weak Hopf algebras is two-fold. First, there are natural examples which exhibit all features of a quantum groupoid except that the underlying algebras are not unital and can not be made unital in a natural way, like algebras of functions on non-compact groupoids. Second, such examples appear as generalized Pontryagin duals of unital Hopf algebroids or weak Hopf algebras, and as in the case of Hopf algebras, one has to pass to a multiplier version to obtain a good duality theory beyond finite-dimensional cases [32]. In [29], we show that the multiplier Hopf algebroids introduced in this article provide a good algebraic setting for a generalised Pontryagin duality theory for quantum groupoids.

The theory of (multiplier) Hopf algebroids and the theory of weak (multiplier) Hopf algebras differ mainly in the target of the comultiplication and both have their advantages and draw-backs. Weak (multiplier) Hopf algebras may be easier to work with, but their base algebras are automatically separable and, in particular, semi-simple; see Proposition 2.11 [3]. Multiplier Hopf algebroids overcome this restriction and are not only more general, but also, in a sense, more natural than weak multiplier Hopf algebras. They may, however, appear more difficult because they involve two versions of the comultiplication simultaneously, as will be explained below. In the finite-dimensional case, both approaches are equivalent [21, 26]. In [30] and [35], we show that every regular weak multiplier Hopf algebra gives rise to a regular multiplier Hopf algebroid, but that even in the case where the base algebras are separable, the converse is not true.

Let us explain the main result of this article in some more detail.

Similarly like a bialgebroid, a *multiplier bialgebroid* is given by a total algebra A, two base algebras B, C with anti-isomorphisms $B \rightleftarrows C$, and a left and a right comultiplication Δ_C and Δ_B, respectively, related by a mixed co-associativity condition. In the unital case, these comultiplications take values in the left and the right Takeuchi product, respectively. In the non-unital case, the latter have to be replaced by certain left or right multiplier algebras such that all products of the form

$$\Delta_C(b)(a \otimes 1), \qquad \Delta_C(a)(1 \otimes b), \qquad (a \otimes 1)\Delta_B(b), \qquad (1 \otimes b)\Delta_B(a),$$

where $a, b \in A$, make sense as elements of certain tensor products of A with itself relative to B or C, respectively. The definition of a left and a right counit then carries over from the unital case.

The main result of this article is that a multiplier bialgebroid with a left and a right counit has an antipode if and only if the canonical maps

$$_{\lambda}T \colon a \otimes b \mapsto (a \otimes 1)\Delta_B(b), \qquad T_{\rho} \colon a \otimes b \mapsto \Delta_C(a)(1 \otimes b), \tag{1.1}$$

are bijective, where the ranges and domains are various tensor products of A with itself relative to B or C, respectively. In that case, we call the multiplier bialgebroid a *multiplier Hopf algebroid*. Its antipode is invertible if and only if the canonical maps

$$_{\rho}T \colon a \otimes b \mapsto (1 \otimes b)\Delta_B(a), \qquad T_{\lambda} \colon a \otimes b \mapsto \Delta_C(b)(a \otimes 1), \tag{1.2}$$

are bijective as well, and in that case, we call the multiplier Hopf algebroid *regular*. This result generalizes corresponding characterizations of multiplier Hopf algebras and Hopf algebroids among multiplier bialgebras or bialgebroids; see [31] and Proposition 4.2 in [5]. In the case of multiplier Hopf algebras, bijectivity of the maps ${}_{\lambda}T$ and T_{ρ} implies existence of a counit. In the case of multiplier bialgebroids, we can only prove existence of counits if the maps ${}_{\rho}T$ and T_{λ} are bijective as well.

The proof of the main result uses only the canonical maps in (1.1) and (1.2) and a few key relations between them that are equivalent to multiplicativity, co-associativity and compatibility of the comultiplications Δ_C and Δ_B. To a large extent, we adopt and refine the arguments in [31], but replace calculations involving the comultiplications by transparent commutative diagrams. This change of technique proves to be very helpful for keeping track of the module structures used for these tensor products and for ensuring that all maps involved are well-defined. More importantly, this method makes explicit the key relations of the maps T_{λ}, T_{ρ} and ${}_{\lambda}T, {}_{\rho}T$ used in the arguments and suggests to shift the perspective and to regard these canonical maps as the fundamental structure maps of a multiplier bialgebroid.

This article is organized as follows.

In §2, we introduce multiplier analogues of left bialgebroids, which are given by algebras A and C with a homomorphism $s\colon C \to M(A)$, an anti-homomorphism $t\colon C \to M(A)$, and a left-sided comultiplication Δ_C from A into a multiplier version of the Takeuchi product. The map Δ_C and its defining properties are described in terms of the canonical maps T_{λ} and T_{ρ}, see (1.1) and (1.2), and various commutative diagrams, which will be used extensively later on. The notation used for these diagrams is explained in 2.4.

In §3, we introduce counits of left multiplier bialgebroids, and prove uniqueness and existence in the case where the canonical maps are surjective or bijective, respectively. In contrast to the unital case, we do not include existence of a counit in the definition of a left multiplier bialgebroid, but consider them as additional structure.

In §4, we turn to right multiplier bialgebroids and briefly summarize the right-handed analogues of the left-handed concepts and results of §2 and §3.

In §5, we come to the main result of this article, which is the definition and characterization of multiplier Hopf algebroids. We first formulate the necessary compatibility relation for a left and a right multiplier bialgebroid to form a two-sided multiplier bialgebroid and then show that existence of an antipode is equivalent to bijectivity of the canonical maps (1.1). Along the way, we obtain many useful relations for the canonical maps and describe their inverses in terms of the antipode.

In §6, we show that the antipode is invertible if and only if the maps (1.2) are invertible as well, and derive further relations between the antipode and the canonical maps which hold in this case.

In §7, we present several special cases and examples, including multiplier Hopf algebroids arising from weak multiplier Hopf algebras, multiplier Hopf $*$-algebroids, the function algebras and convolution algebras of étale groupoids, two-sided crossed products which generalize constructions in [1], [23] and [37], and proper, co-commutative and étale multiplier Hopf algebroids.

We use the following conventions and terminology.

The identity map on a set X will be denoted by ι_X or simply ι. All algebras and modules will be complex vector spaces and all morphisms will be linear maps, but much of the theory developed in this article should apply in wider generality.

We denote the linear span of a subset X of a vector space V by $\operatorname{span} X$.

Let B be an algebra, not necessarily unital. We denote by B^{op} the *opposite algebra*, which has the same underlying vector space as B but the reversed multiplication. When necessary, we write b^{op} when we regard an element $b \in B$ as an element of B^{op} to avoid confusion.

Given a right module M over B, we write M_B if we want to emphasize that M is regarded as a right B-module. We call M_B *faithful* if for each non-zero $b \in B$ there exists an $m \in M$ such that mb is non-zero, *non-degenerate* if for each non-zero $m \in M$ there exists a $b \in B$ such that mb is non-zero, *idempotent* if $MB = M$, and we say that M_B *has local units in* B if for every finite subset $F \subset M$ there exists a $b \in B$ with $mb = m$ for all $m \in F$. Note that the last property implies the preceding two.

For left modules, we obtain the corresponding notation and terminology by identifying left B-modules with right B^{op}-modules.

We write B_B or ${}_BB$ when we regard B as a right or left module over itself with respect to right or left multiplication. We say that the algebra B is *non-degenerate*, *idempotent*, or *has local units* if the modules ${}_BB$ and B_B both are non-degenerate, idempotent or both have local units in B, respectively. Note that the last property again implies the preceding two.

Working with non-unital algebras, we frequently need to use multipliers.

A *left multiplier* of the algebra B is a linear map $T\colon B \to B$ satisfying $T(bb') = T(b)b'$ for all $b, b' \in B$, that is, an endomorphism of the right B-module B_B. We denote by $L(B) := \operatorname{End}(B_B)$ the algebra of all left multipliers of B.

A *right multiplier* of the algebra B is an endomorphism of the left B-module ${}_BB$. When we think of such an endomorphism T as a right multiplier, we write bT instead of $T(b)$ for the image of $b \in B$ under T. We denote by $R(B) := \operatorname{End}({}_BB)^{\mathrm{op}}$ the algebra of right multipliers of B, so that $b(TS) = (bT)S$ for all $b \in B$ and $T, S \in R(B)$.

Note that B_B or ${}_BB$ is non-degenerate if and only if the natural map from B to $L(B)$ or $R(B)$, respectively, is injective.

Suppose that B is non-degenerate. Then we define a *multiplier* of B to be a pair $T = (T_l, T_r)$ of maps $T_l, T_r\colon B \to B$ satisfying $bT_l(b') = T_r(b)b'$ for all $b, b' \in B$. We write $Tb := T_l(b)$ and $bT := T_r(b)$ for all $b \in B$, so that the preceding equation takes the form $b(Tb') = (bT)b'$ for all $b, b' \in B$. All multipliers form an algebra $M(B)$ with respect to the obvious addition and the multiplication given by $(T_l, T_r) \circ (S_l, S_r) = (T_l \circ S_l, S_r \circ T_r)$, that is, $(TS)b = T(Sb)$ and $b(TS) = (bT)S$ for all $b \in B$. A multiplier $T = (T_l, T_r)$ is uniquely determined by the components T_l and T_r, which are a left and a right multiplier of B, respectively, so that $M(B)$ can be identified with subalgebras of $L(B)$ and $R(B)$.

More generally, if B_B is non-degenerate, we define the multiplier algebra of B to be the subalgebra $M(B) := \{T \in L(B) : BT \subseteq B\} \subseteq L(B)$, where we identify B with its image in $L(B)$. Likewise we define $M(B) = \{T \in R(B) : BT \subseteq B\}$ if ${}_BB$ is non-degenerate, and both definitions coincide with the preceding one if B_B and ${}_BB$ are non-degenerate.

2. Left multiplier bialgebroids

Let A be an algebra, not necessarily unital. Regard A as a right module over itself via right multiplication, and denote this module by A_A. We impose the following assumption:

(A1) *The right module A_A is idempotent and non-degenerate.*

Then A embeds naturally into the algebra $L(A) = \mathrm{End}(A_A)$ of left multipliers and we can form the multiplier algebra $M(A) \subseteq L(A)$. If A has a unit 1_A, then the map $T \mapsto T1_A$ identifies $M(A) = L(A)$ with A as an algebra. We denote elements of A by $a, a', b, b', \ldots$

Let C be an algebra, not necessarily unital, with a homomorphism $s\colon C \to M(A)$ and an anti-homomorphism $t\colon C \to M(A)$ such that $s(C)$ and $t(C)$ commute. We denote elements of C by $x, x', y, y', \ldots$ We write ${}_CA$ and A^C when we regard A as a left or right C-module via left multiplication along s or t, respectively, that is, $x \cdot a = s(x)a$ and $a \cdot x = t(x)a$. Similarly, we write A_C and ${}^CA$ when we regard A as a right or left C-module via right multiplication along s or t, respectively. We make the following assumption:

(A2) *The modules ${}_CA$ and A^C are faithful and idempotent.*

This condition means that the maps s and t are injective and $s(C)A = A = t(C)A$. Note that then C is non-degenerate as an algebra. Indeed, if $xC = 0$, then $s(x)A = s(x)s(C)A = 0$ and hence $x = 0$, and if $Cx = 0$, then $t(x)A = t(x)t(C)A = t(Cx)A = 0$ and hence $x = 0$ again.

We next form the tensor product $A^C \otimes {}_CA$ of C-modules and regard it as a right module over $A \otimes 1$ or $1 \otimes A$ in the obvious way. We would like the following condition to hold:

(A3) *The space $A^C \otimes {}_CA$ is non-degenerate as a right module over $A \otimes 1$ and over $1 \otimes A$.*

We next list several cases in which this assumption is satisfied, and use the following terminology. We call a multiplier $E \in M(C^{\mathrm{op}} \otimes C)$ a *left separability multiplier* if for every element $x \in C$, we have

$$E(x^{\mathrm{op}} \otimes 1) = E(1 \otimes x) \in C^{\mathrm{op}} \otimes C,$$

and the linear map $y^{\mathrm{op}} \otimes z \mapsto yz$ sends this element above to x; see also [34, §1]. An algebra D is *firm* if the multiplication map $D \underset{D}{\otimes} D \to D$ is an isomorphism, and a module M over an algebra D is *locally projective* if for every finite subset $F \subseteq M$, there exist finitely many morphisms $\upsilon_i \in \mathrm{Hom}(M, D)$ and $m_i \in \mathrm{Hom}(D, M)$, where D is regarded as a D-module in the obvious way, such that

$$\sum_i m_i(\upsilon_i(m)) = m \quad \text{for all } m \in F;$$

see [40, Theorem 2.1]. In this case, M is also *universally torsionless* and a *trace module*, see [12, Theorem 3.2]. The module M is *projective* if there exist υ_i and m_i as above, but possibly infinitely many, such that the sum above is finite and equal to m for every $m \in M$.

2.1. **Lemma.** *Assume that (A1), (A2) and one of the following conditions holds:*

(1) *The right module A_A has local units in A.*

(2) *There exists a left separability multiplier $E \in M(C^{\mathrm{op}} \otimes C)$.*
(3) *The algebra C is firm and the C-modules ${}_CA$ and A^C are locally projective.*

Then condition (A3) is satisfied.

Proof. (1) Straightforward.

(2) The assumptions on E imply that the map $j\colon A^C \otimes_C A \to A \otimes A$ given by $a \otimes b \mapsto (t \otimes s)(E)(a \otimes b)$ is well-defined and that the canonical map $A \otimes A \to A^C \otimes_C A$ is a left inverse to j. Since $A \otimes A$ is non-degenerate as a right module over $A \otimes 1$ and over $1 \otimes A$, so is the image $j(A^C \otimes_C A)$ and hence also $A^C \otimes_C A$.

(3) Let $w = \sum_k a_k \otimes b_k \in A^C \otimes_C A$ and assume $w(c \otimes 1) = 0$ for all $c \in A$. Choose $\upsilon_i \in \mathrm{Hom}(A^C, C_C)$, $e_i \in \mathrm{Hom}(C_C, A^C)$ and $f_j \in \mathrm{Hom}({}_CA, {}_CC)$, $\omega_j \in \mathrm{Hom}({}_CC, {}_CA)$ such that $\sum_i e_i(\upsilon_i(a_k)) = a_k$ and $\sum_j f_j(\omega_j(b_k)) = b_k$ for all k. Fix i and j. Then $\sum_k t(\omega_j(b_k))a_k c = 0$ for all $c \in A$, whence $\sum_k t(\omega_j(b_k))a_k = 0$ by (A1) and hence $\sum_k \upsilon_i(a_k)\omega_j(b_k) = 0$. Since C is firm, we can conclude $\sum_k \upsilon_i(a_k) \otimes \omega_j(b_k) = 0$ in $C \underset{C}{\otimes} C$. We apply $e_i \otimes f_j$, sum over i and j, and get $w = 0$. Therefore, $A^C \otimes_C A$ is non-degenerate as a right module over $A \otimes 1$. A similar argument shows that it is non-degenerate over $1 \otimes A$ as well. □

Let A, C be algebras and $s, t\colon C \to M(A) \subseteq L(A)$ be maps with commuting images such that (A1)–(A3) hold.

2.2. **Remark.** Before we proceed, let us note that we choose a slightly different notation than in [30], where roles of s and t are switched and C is implicitly replaced by $B = C^{\mathrm{op}}$. With the present choice, the space $A^C \otimes_C A$, which carries the target of the comultiplication, is a balanced tensor product of a right module with a left module, whereas in [30] it was the balanced tensor product of a left with a right module which may lead to some confusion.

The left comultiplication on A takes values in the subspace

$$A^C \overline{\times}_C A \subseteq \mathrm{End}(A^C \otimes_C A)$$

formed by all endomorphisms T of $A^C \otimes_C A$ satisfying the following condition:

For every $a, b \in A$, there exist elements

$$T(a \otimes 1) \in A^C \otimes_C A \quad \textit{and} \quad T(1 \otimes b) \in A^C \otimes_C A$$

such that $T(a \otimes b) = (T(a \otimes 1))(1 \otimes b) = (T(1 \otimes b))(a \otimes 1)$.

This subspace is a subalgebra and commutes with the right $A \otimes A$-module action. Note that the elements $T(a \otimes 1)$ and $T(1 \otimes b)$ are uniquely determined thanks to the non-degeneracy assumption on $A^C \otimes_C A$.

If A has a unit 1_A, then the map $A^C \overline{\times}_C A \to A^C \otimes_C A$ given by $T \mapsto T(1_A \otimes 1_A)$ identifies $A^C \overline{\times}_C A$ with the left Takeuchi product, which is the algebra

$$A^C \times_C A = \left\{ w \in A^C \otimes_C A : w(t(x) \otimes 1) = w(1 \otimes s(x)) \text{ for all } x \in C \right\} \subseteq A^C \otimes_C A. \tag{2.1}$$

2.3. **Lemma.** *Let $\Delta\colon A \to A^C \overline{\times}_C A$ be a linear map. Then the linear maps*

$$\widetilde{T_\lambda}, \widetilde{T_\rho}\colon A \otimes A \to A^C \otimes_C A$$

given by

$$\widetilde{T_\lambda}(a\otimes b)=\Delta(b)(a\otimes 1), \qquad \widetilde{T_\rho}(a\otimes b)=\Delta(a)(1\otimes b), \tag{2.2}$$

for all $a,b\in A$ satisfy

$$\widetilde{T_\lambda}(t(x)a\otimes b)=\widetilde{T_\lambda}(a\otimes b)(1\otimes s(x)), \qquad \widetilde{T_\rho}(a\otimes s(y)b)=\widetilde{T_\rho}(a\otimes b)(t(y)\otimes 1) \tag{2.3}$$

for all $a,b\in A$ and $x,y\in B$ and make the following diagrams commute,

$$\begin{tikzcd} & A^C\otimes {}_CA\otimes A \arrow[dr, "\iota\otimes m"] & \\ A\otimes A\otimes A \arrow[ur, "\widetilde{T_\lambda}\otimes\iota"] \arrow[dr, "\iota\otimes\widetilde{T_\rho}"'] & & A^C\otimes {}_CA, \\ & A\otimes A^C\otimes {}_CA \arrow[ur, "m^{\mathrm{op}}\otimes\iota"'] & \end{tikzcd} \tag{2.4}$$

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\iota\otimes\widetilde{T_\lambda}"] \arrow[d, "m^{\mathrm{op}}\otimes\iota"'] & A\otimes A^C\otimes {}_CA \arrow[d, "m^{\mathrm{op}}\otimes\iota"] \\ A\otimes A \arrow[r, "\widetilde{T_\lambda}"] & A^C\otimes {}_CA, \end{tikzcd} \qquad \begin{tikzcd} A\otimes A\otimes A \arrow[r, "\widetilde{T_\rho}\otimes\iota"] \arrow[d, "\iota\otimes m"'] & A^C\otimes {}_CA\otimes A \arrow[d, "\iota\otimes m"] \\ A\otimes A \arrow[r, "\widetilde{T_\rho}"] & A^C\otimes {}_CA, \end{tikzcd} \tag{2.5}$$

where $m\colon A\otimes A\to A$ denotes the multiplication, $m^{\mathrm{op}}\colon A\otimes A\to A$ the opposite multiplication, and tensor products over $\mathbb{C}$ and over C appear side by side.

Conversely, every pair of linear maps $(\widetilde{T_\lambda},\widetilde{T_\rho})$ which make diagram (2.4) *commute and satisfy* (2.3) *determines a linear map $\Delta\colon A\to A^C\overline{\times}_CA$ by* (2.2)*, and each of the maps $\Delta,\widetilde{T_\lambda},\widetilde{T_\rho}$ determines the other two.*

Before we can list the defining properties of a left comultiplication Δ and the corresponding properties of the associated maps $\widetilde{T_\lambda}$ and $\widetilde{T_\rho}$, we need to fix some notation.

2.4. **Notation.** (1) We need to consider iterated tensor products of vector spaces, of C-modules and of C-bimodules. For example, we write

$$A^C\otimes A\otimes {}_CA \quad\text{and}\quad A^C\otimes {}^CA^C\otimes {}_CA$$

for the quotients of $A\otimes A\otimes A$ by the subspaces spanned by all elements of the form $t(x)a\otimes b\otimes c-a\otimes b\otimes s(x)a$ in the first case, or of the form $t(x)a\otimes b\otimes c-a\otimes bt(x)\otimes c$ or $a\otimes t(x)b\otimes c-a\otimes b\otimes s(x)c$ in the second case.

(2) We fix an algebra B with an anti-isomorphism $\kappa\colon B\to C$, use this anti-isomorphism to regard A as a B-module in various ways, and write

$$A^B\otimes {}^BA \quad\text{and}\quad A_B\otimes {}_BA$$

for the quotients of $A\otimes A$ by the subspaces spanned by all elements of the form $s(\kappa(x))a\otimes b-a\otimes bs(\kappa(x))$ in case of $A^B\otimes {}^BA$, or $t(\kappa(x))a\otimes b-a\otimes bt(\kappa(x))$ in case of $A_B\otimes {}_BA$. Note that this definition does not depend on the choice of B or κ. From Section 5 on, we shall fix a specific B and κ and explicitly define the underlying B-module structures on A, which do depend on this choice.

(3) Given vector spaces V and W, we denote by $\Sigma_{(V,W)}\colon V\otimes W\to W\otimes V$ the flip map. In case $V=W=A$, the flip map descends to isomorphisms

$$\Sigma_{(A_C,{}_CA)}\colon A_C\otimes {}_CA\to A^B\otimes {}^BA, \qquad \Sigma_{(A^C,{}^CA)}\colon A^C\otimes {}^CA\to A_B\otimes {}_BA.$$

(4) We adopt the usual leg notation for maps on tensor product. For example, we write $(\widetilde{T_\lambda})_{13}$ for the composition

$$A \otimes A \otimes A \xrightarrow{\iota\otimes\Sigma_{(A,A)}} A \otimes A \otimes A \xrightarrow{\widetilde{T_\lambda}\otimes\iota} A^C \otimes {}_C A \otimes A \xrightarrow{\iota\otimes\Sigma_{(A,A)}} A^C \otimes A \otimes {}_C A.$$

(5) The multiplication maps $m\colon A \otimes A \to A$ and $m^{\mathrm{op}} = m \circ \Sigma_{(A,A)}\colon A \otimes A \to A$ descend to maps

$$A_C \otimes {}_C A \xrightarrow{m_C} A, \qquad A_B \otimes {}_B A \xrightarrow{m_B} A, \qquad A^C \otimes {}^C A \xrightarrow{m^{\mathrm{op}}_C} A, \qquad A^B \otimes {}^B A \xrightarrow{m^{\mathrm{op}}_B} A.$$

With this notation at hand, we can write down the key conditions on Δ, $\widetilde{T_\lambda}$ and $\widetilde{T_\rho}$. Note that (2.3) is equivalent to saying that $\widetilde{T_\lambda}$ and $\widetilde{T_\rho}$ are maps of C-modules

$$\widetilde{T_\lambda}\colon A^C \otimes A \to A^C \otimes {}_C A_C, \qquad \widetilde{T_\rho}\colon A \otimes {}_C A \to {}^C A^C \otimes {}_C A. \tag{2.6}$$

2.5. **Lemma.** *Let $\Delta\colon A \to A^C \overline{\times}_C A$ and $\widetilde{T_\lambda}, \widetilde{T_\rho}\colon A \otimes A \to A^C \otimes_C A$ be linear maps related by* (2.2).

(1) *The map Δ is a homomorphism if and only if one (and then both) of the following diagrams commute:*

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\iota\otimes m"] \arrow[d, "(\widetilde{T_\lambda})_{13}"'] & A \otimes A \arrow[dd, "\widetilde{T_\lambda}"] \\
A^C \otimes A \otimes {}_C A \arrow[d, "\widetilde{T_\lambda}\otimes\iota"'] & \\
A^C \otimes {}_C A_C \otimes {}_C A \arrow[r, "\iota\otimes m_C"'] & A^C \otimes {}_C A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m^{\mathrm{op}}\otimes\iota"] \arrow[d, "(\widetilde{T_\rho})_{13}"'] & A \otimes A \arrow[dd, "\widetilde{T_\rho}"] \\
A^C \otimes A \otimes {}_C A \arrow[d, "\iota\otimes\widetilde{T_\rho}"'] & \\
A^C \otimes {}^C A^C \otimes {}_C A \arrow[r, "m^{\mathrm{op}}_C\otimes\iota"'] & A^C \otimes {}_C A
\end{tikzcd} \tag{2.7}$$

(2) *The map Δ satisfies*

$$\Delta(s(y)t(x)as(y')t(x')) = (s(y) \otimes t(x))\Delta(a)(s(y') \otimes t(x')) \tag{2.8}$$

if and only if one (and then both) of the following conditions hold:

$$\begin{aligned}
\widetilde{T_\lambda}(a \otimes t(x)s(y)bt(x')s(y')) &= (s(y) \otimes t(x))\widetilde{T_\lambda}(s(y')a \otimes b)(1 \otimes t(x')), \\
\widetilde{T_\rho}(t(x)s(y)at(x')s(y') \otimes b) &= (s(y) \otimes t(x))\widetilde{T_\rho}(a \otimes t(x')b)(s(y') \otimes 1).
\end{aligned} \tag{2.9}$$

If these conditions hold, then $\widetilde{T_\lambda}$ and $\widetilde{T_\rho}$ descend to maps

$$T_\lambda\colon A^B \otimes {}^B A \to A^C \otimes {}_C A, \qquad T_\rho\colon A_B \otimes {}_B A \to A^C \otimes {}_C A.$$

(3) *Assume that the conditions in (2) hold. Then Δ is coassociative in the sense that*

$$(\Delta \otimes \iota)(\Delta(b)(1 \otimes c))(a \otimes 1 \otimes 1) = (\iota \otimes \Delta)(\Delta(b)(a \otimes 1))(1 \otimes 1 \otimes c) \tag{2.10}$$

if and only if the following diagram commutes:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\iota\otimes\widetilde{T_\rho}"] \arrow[d, "\widetilde{T_\lambda}\otimes\iota"'] & A \otimes A^C \otimes {}_C A \arrow[d, "\widetilde{T_\lambda}\otimes\iota"] \\
A^C \otimes {}_C A \otimes A \arrow[r, "\iota\otimes\widetilde{T_\rho}"] & A^C \otimes {}_C A^C \otimes {}_C A.
\end{tikzcd} \tag{2.11}$$

Note that (2.9) implies that $\widetilde{T_\lambda}$ and $\widetilde{T_\rho}$ are maps of C-bimodules

$$\widetilde{T_\lambda}\colon A \otimes {}_C A^C \to {}_C A^C \otimes {}_C A^C, \qquad \widetilde{T_\rho}\colon {}_C A^C \otimes A \to {}_C A^C \otimes {}_C A^C. \tag{2.12}$$

If A is unital and $A^C \overline{\times}_C A$ is identified with the left Takeuchi product (2.1), then equations (2.8) and (2.10) reduce to the conditions

$$\Delta(t(x)s(y)) = s(y) \otimes t(x), \qquad (\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta. \tag{2.13}$$

If (A1)–(A3) and (2.8), (2.10) hold, we call (A, C, s, t, Δ) a left multiplier bialgebroid:

2.6. **Definition.** *A* left multiplier bialgebroid *is a tuple $\mathcal{A} = (A, C, s, t, \Delta)$ consisting of*

(1) *algebras A and C such that the right A-module A_A is non-degenerate and idempotent;*
(2) *a homomorphism $s\colon C \to M(A)$ and an anti-homomorphism $t\colon C \to M(A)$ such that the images of s and t commute, the C-modules ${}_C A$ and A^C are faithful and idempotent, and $A^C \otimes {}_C A$ is non-degenerate as a right module over $A \otimes 1$ and over $1 \otimes A$;*
(3) *a homomorphism $\Delta\colon A \to A^C \overline{\times}_C A$, called the* left comultiplication*, which satisfies the C-bilinearity condition* (2.8) *and the coassociativity condition* (2.10).

We call the maps T_λ and T_ρ defined above the canonical maps *associated to $\mathcal{A}$. We call a left multiplier bialgebroid as above* unital *if the algebras A, C and the maps s, t, Δ are unital.*

Given a left multiplier bialgebroid, one can reverse the comultiplication as follows.

2.7. **Proposition.** *Let $\mathcal{A} = (A, C, s, t, \Delta)$ be a left multiplier bialgebroid with associated maps $(\widetilde{T_\lambda}, \widetilde{T_\rho})$. Regard s as an anti-homomorphism and t as a homomorphism from C^{op} to $M(A)$. Write $A^{C^{\mathrm{op}}}$ and ${}_{C^{\mathrm{op}}}A$ for A, regarded as a C^{op}-module via $a \cdot y^{\mathrm{op}} := s(y)a$ and $y^{\mathrm{op}} \cdot a := t(y)a$, where $y \in C$ and $a \in A$. Then the flip map $\Sigma_{(A,A)}$ on $A \otimes A$ descends to an isomorphism $\Sigma_{(A^C, {}_C A)}$ from $A^C \otimes {}_C A$ to $A^{C^{\mathrm{op}}} \otimes {}_{C^{\mathrm{op}}} A$, there exists a well-defined homomorphism*

$$\Delta^{\mathrm{co}}\colon A \to A^{C^{\mathrm{op}}} \overline{\times}_{C^{\mathrm{op}}} A, \quad \Delta^{\mathrm{co}}(a)(b \otimes c) = \Sigma_{(A^C, {}_C A)}(\Delta(a)(c \otimes b)),$$

and $\mathcal{A}^{\mathrm{co}} := (C^{\mathrm{op}}, A, t, s, \Delta^{\mathrm{co}})$ is a left multiplier bialgebroid with associated maps

$$\widetilde{T_\lambda}^{\mathrm{co}} = \Sigma_{(A^C, {}_C A)} \circ \widetilde{T_\rho} \circ \Sigma_{(A,A)}, \qquad \widetilde{T_\rho}^{\mathrm{co}} = \Sigma_{(A^C, {}_C A)} \circ \widetilde{T_\lambda} \circ \Sigma_{(A,A)}.$$

The proof is straightforward and therefore omitted.

The canonical maps satisfy pentagonal relations:

2.8. **Proposition.** *Let (A, C, s, t, Δ) be a left multiplier bialgebroid. If $A^C \otimes {}_C A^C \otimes {}_C A$ is non-degenerate as a right module over $A \otimes 1 \otimes 1$ and over $1 \otimes 1 \otimes A$, then the following diagrams commute:*

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "(\widetilde{T_\lambda})_{12}(\widetilde{T_\lambda})_{23}"] \arrow[d, "(\widetilde{T_\lambda})_{12}"'] & A^C \otimes {}_C A^C \otimes {}_C A, \\
A^C \otimes {}_C A \otimes A \arrow[r, "(\widetilde{T_\lambda})_{13}"] & A^C \otimes A^B \otimes {}^B_C A \arrow[u, "(T_\lambda)_{23}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "(\widetilde{T_\rho})_{23}(\widetilde{T_\rho})_{12}"] \arrow[d, "(\widetilde{T_\rho})_{23}"'] & A^C \otimes {}_C A^C \otimes {}_C A. \\
A \otimes A^C \otimes {}_C A \arrow[r, "(\widetilde{T_\rho})_{13}"] & A^C_B \otimes {}_B A \otimes {}_C A \arrow[u, "(T_\rho)_{12}"']
\end{tikzcd}$$

Proof. The pentagonal relation for $\widetilde{T_\rho}$ follows from commutativity of the diagram

$$\begin{tikzcd}
A \otimes A \otimes A \otimes A \arrow[r, "(\widetilde{T_\rho})_{23}"] \arrow[d, "(\widetilde{T_\lambda})_{12}"'] \arrow[ddd, bend right, "(\widetilde{T_\rho})_{24}(\widetilde{T_\rho})_{34}"'] & A \otimes A^C \otimes_C A \otimes A \arrow[r, "(\widetilde{T_\rho})_{34}"] \arrow[d, "m^{\mathrm{op}} \otimes \iota \otimes \iota"] & A \otimes A^C \otimes_C A^C \otimes_C A \arrow[d, "m^{\mathrm{op}} \otimes \iota \otimes \iota"] \\
A^C \otimes_C A \otimes A \otimes A \arrow[r, "\iota \otimes m \otimes \iota"] \arrow[d, "(\widetilde{T_\rho})_{34}"'] & A^C \otimes_C A \otimes A \arrow[r, "(\widetilde{T_\rho})_{23}"] & A^C \otimes_C A^C \otimes_C A, \\
A^C \otimes_C A \otimes A^C \otimes_C A \arrow[r, "(\widetilde{T_\rho})_{24}"] & A^C \otimes_C A^C_B \otimes_B A \otimes_C A \arrow[ur, "\iota \otimes m_B \otimes \iota"] & A \otimes A^C \otimes_C A^C \otimes_C A \arrow[u, "m^{\mathrm{op}} \otimes \iota \otimes \iota"'] \\
 & A \otimes A^C_B \otimes_B A \otimes_C A \arrow[u, "(\widetilde{T_\lambda})_{12}"] \arrow[ur, "(T_\rho)_{23}"'] &
\end{tikzcd}$$

and the pentagonal relation for T_λ can be concluded similarly. □

2.9. **Remark.** Similar arguments as those used in the proof of Lemma 2.1 show that the assumption on the module $A^C \otimes_C A^C \otimes_C A$ holds if condition (1) or (2) of Lemma 2.1 is satisfied, or if the algebra C is firm and the modules ${}_CA$ and A^C are locally projective. In the latter case, one uses the fact that then ${}_CA \otimes A^C$ is projective with respect to the C-modules structure given by $x \cdot (a \otimes b) = a \otimes t(x)b$ and $(a \otimes b) \cdot x = s(x)a \otimes b$, respectively.

Throughout this article, we shall mainly use the canonical maps instead of the comultiplication itself. To make some formulas and calculations more accessible, we also write them out in a generalized Sweedler notation, which is more intuitive but a bit difficult to make precise. We shall not attempt to formalize it and note that for every expression involving this notation, one needs to check whether it is well-defined. In the context of multiplier Hopf algebras, the correct usage of this notation is explained in [31, 33]. With these words of warning, given a left multiplier bialgebroid $\mathcal{A} = (A, C, s, t, \Delta)$, we write

$$\Delta(a) = a_{(1)} \otimes a_{(2)} \in \mathrm{End}(A^C \otimes_C A)$$

for all $a \in A$, where the right hand sides are purely formal expressions. For example, we then have

$$(ab)_{(1)} \otimes (ab)_{(2)} = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \tag{2.14}$$

$$a_{(1)} \otimes a_{(2)}s(z) = a_{(1)}t(z) \otimes a_{(2)}, \tag{2.15}$$

$$T_\lambda(a \otimes b) = b_{(1)}a \otimes b_{(2)}, \quad T_\rho(a \otimes b) = a_{(1)} \otimes a_{(2)}b, \tag{2.16}$$

$$\Delta^{\mathrm{co}}(a) = a_{(2)} \otimes a_{(1)} \tag{2.17}$$

for all $a, b \in A$ and $z \in C$, and the pentagonal relation for $\widetilde{T_\rho}$ takes the form

$$a_{(1)} \otimes (a_{(2)}b)_{(1)} \otimes (a_{(2)}b)_{(2)}c = (a_{(1)})_{(1)} \otimes (a_{(1)})_{(2)}b_{(1)} \otimes a_{(2)}b_{(2)}c.$$

3. Counits for left multiplier bialgebroids

We next discuss counits of left multiplier bialgebroids, and prove uniqueness, some multiplicativity, and existence in the case where the canonical maps are surjective or bijective, respectively.

Let us fix some notation. Given a left multiplier bialgebroid (A, C, s, t, Δ) and morphisms $\phi \in \mathrm{Hom}(A^C, C_C)$ and $\psi \in \mathrm{Hom}({}_CA, {}_CC)$, we can form slice maps

$$\phi \odot \iota \colon A^C \otimes {}_CA \to A,\ a \otimes b \mapsto s(\phi(a))b, \quad \iota \odot \psi \colon A^C \otimes {}_CA \to A,\ a \otimes b \mapsto t(\psi(b))a.$$

3.1. **Definition.** *A* left counit *for a left multiplier bialgebroid* (A, C, s, t, Δ) *is a map* $\varepsilon \colon A \to C$ *that satisfies*

$$\varepsilon(s(y)a) = y\varepsilon(a) \qquad \textit{and} \qquad \varepsilon(t(x)a) = \varepsilon(a)x \qquad \textit{for all } a \in A, x, y \in C, \tag{3.1}$$

that is, $\varepsilon \in \mathrm{Hom}({}_CA, {}_CC) \cap \mathrm{Hom}(A^C, C_C)$*, and*

$$(\varepsilon \odot \iota)(T_\rho(a \otimes b)) = ab \qquad \textit{and} \qquad (\iota \odot \varepsilon)(T_\lambda(a \otimes b)) = ba \qquad \textit{for all } a, b \in A. \tag{3.2}$$

3.2. **Remark.** (1) In Sweedler notation, (3.2) takes the form

$$s(\varepsilon(a_{(1)}))a_{(2)}b = ab \quad \text{and} \quad t(\varepsilon(b_{(2)}))b_{(1)}a = ba \quad \text{for all } a, b \in A. \tag{3.3}$$

(2) Note that left counits for a left multiplier bialgebroid $\mathcal{A}$ and left counits for its co-opposite $\mathcal{A}^{\mathrm{co}}$ introduced in Proposition 2.7 coincide up to the canonical linear identification $C \to C^{\mathrm{op}}$.

If A has a unit 1_A, we can identify $A^C\overline{\times}_CA$ with the left Takeuchi product (2.1), and then commutativity of the diagrams above is equivalent to the equations

$$(\varepsilon \odot \iota) \circ \Delta = \iota_A = (\iota \odot \varepsilon) \circ \Delta. \tag{3.4}$$

From these equations, one can easily deduce that a left counit, if it exists, is unique. If it also is multiplicative in the sense that

$$\varepsilon(ab) = \varepsilon(as(\varepsilon(b))) = \varepsilon(at(\varepsilon(b))) \tag{3.5}$$

for all $a, b \in A$, then we obtain a left bialgebroid in the well-known sense as described in, for example, in [1], [17]:

3.3. **Proposition.** *Let* (A, C, s, t, Δ) *be a unital left multiplier bialgebroid with a left counit* ε *that is unital and satisfies* (3.5)*. Then we can regard* A *as an* $C \otimes C^{\mathrm{op}}$*-ring and as a* C*-bimodule via* $y \cdot a \cdot x = t(x)s(y)a$ *for all* $x, y \in C, a \in A$*, and* Δ *as a homomorphism from* A *to* $A^C\overline{\times}_CA \cong A_C{\times}A$*. The tuple* (A, Δ, ε) *is a* C*-coring and, together with the* $C \otimes C^{\mathrm{op}}$*-ring structure on* A*, forms a left bialgebroid. Conversely, every left bialgebroid arises this way from a unital left multiplier bialgebroid with a unital left counit satisfying* (3.5)*.*

Proof. Straightforward. □

In the non-unital case, we can prove uniqueness and multiplicativity of left counits only under additional assumptions which are analogues of the conditions in [38, Definition 1.4].

3.4. **Definition.** *We call a left multiplier bialgebroid* (A, C, s, t, Δ) left-full *if* A *is equal to the linear span of elements of the form* $(\iota \odot \psi)(\widetilde{T_\rho}(a \otimes b))$*, where* $\psi \in \mathrm{Hom}({}_CA, {}_CC)$ *and* $a, b \in A$*,* right-full *if* A *is equal to the linear span of elements of the form* $(\phi \odot \iota)(\widetilde{T_\lambda}(a \otimes b))$*, where* $\phi \in \mathrm{Hom}(A^C, C_C)$ *and* $a, b \in A$*, and* full *if it is both left-full and right-full.*

In the unital case, (3.4) shows that existence of a left counit implies fullness. In general, we only know the following:

3.5. **Remark.** If a left multiplier bialgebroid (A, C, s, t, Δ) has a left counit ε and its canonical map T_λ (or T_ρ) is surjective, then it is left-full (resp. right-full). To see this, take ϕ (or ψ) above to be equal to ε and use the relation $AA = A$.

If the left multiplier bialgebroid is full, then the left counit, if it exists, is unique:

3.6. **Proposition.** *Let $\mathcal{A} = (A, C, s, t, \Delta)$ be a left multiplier bialgebroid with a left counit ε.*

(1) *If $\mathcal{A}$ is left-full or right-full, then the left counit is unique.*
(2) *If the canonical map T_λ (or T_ρ) is surjective, then for all $a, b \in A$,*

$$\varepsilon(ab) = \varepsilon(as(\varepsilon(b))) \quad (\textit{or } \varepsilon(ab) = \varepsilon(at(\varepsilon(b))), \ \textit{respectively}). \tag{3.6}$$

Proof. (1) Assume that $\mathcal{A}$ is left-full and that ε is a left counit. Let $a, b \in A, \psi \in \mathrm{Hom}({}_CA, {}_CC)$ and write $\widetilde{T_\rho}(a \otimes b) = \sum_i c_i \otimes d_i$ with $c_i, d_i \in A$. Then (3.2) shows that $\sum_i s(\varepsilon(c_i))d_i = ab$ and hence

$$\varepsilon\Big(\sum_i t(\psi(d_i))c_i\Big) = \sum_i \varepsilon(c_i)\psi(d_i) = \sum_i \psi(s(\varepsilon(c_i))d_i) = \psi(ab).$$

But since (A, C, s, t, Δ) is assumed to be left-full, elements of the form $\sum_i t(\psi(d_i))c_i$ span A. If (A, C, s, t, Δ) is right-full, a similar argument applies.

(2) Assume that T_ρ is surjective and consider the following diagram:

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r,"\iota\otimes\widetilde{T_\rho}"] \arrow[dd,"m\otimes\iota"'] \arrow[dr,"\iota\otimes m"'] & A\otimes A^C \otimes_C A \arrow[r,"(\widetilde{T_\rho})_{13}"] \arrow[d,"\iota\otimes(\varepsilon\odot\iota)"] & A^C_B\otimes_B A\otimes_C A \arrow[d,"\iota\otimes(\varepsilon\odot\iota)"] \arrow[rdd,"m_B\otimes\iota"] & \\
& A\otimes A \arrow[r,"\widetilde{T_\rho}"] \arrow[d,"m"'] & A^C\otimes_C A \arrow[d,"\varepsilon\odot\iota"] & \\
A\otimes A \arrow[r,"m"] \arrow[rrr, bend right, "\widetilde{T_\rho}"'] & A \arrow[r, equal] & A & A^C\otimes_C A \arrow[l,"\varepsilon\odot\iota"']
\end{tikzcd}$$

The outer cell commutes by (2.5), and all other cells except for the right one commute as well. Since $\widetilde{T_\rho}$ is surjective, we can conclude that the right cell must commute. Therefore, $\varepsilon(ab) = \varepsilon(at(\varepsilon(b)))$ for all $a, b \in A$. If T_λ is surjective, a similar argument applies. $\square$

3.7. **Remark.** In Sweedler notation, the commutative diagram above amounts to the calculation

$$\begin{aligned}
s(\epsilon(a_{(1)}b_{(1)}))a_{(2)}b_{(2)}c &= s(\varepsilon((ab)_{(1)}))(ab)_{(2)}c && \text{(by (2.14))} \\
&= (ab)c && \text{(by (3.3))} \\
&= s(\varepsilon(a_{(1)}))a_{(2)}s(\epsilon(b_{(1)}))b_{(2)}c && \text{(by (3.3))} \\
&= s(\varepsilon(a_{(1)})t(\epsilon(b_{(1)})))a_{(2)}b_{(2)}c && \text{(by (2.15))},
\end{aligned}$$

which, thanks to surjectivity of T_ρ, implies $\varepsilon(a'b') = \varepsilon(a't(b'))$ for all $a', b' \in A$.

Let (A, C, s, t, Δ) be a left multiplier bialgebroid. We shall prove existence of a left counit provided that the canonical maps T_ρ and T_λ are bijective and a further technical condition holds. This condition involves the left ideal ${}_CI \subseteq C$ and the right ideal $I^C \subseteq C$ given by

$$\begin{aligned} {}_CI &:= \operatorname{span}\{\psi(a) : \psi \in \operatorname{Hom}({}_CA, {}_CC), a \in A\}, \\ I^C &:= \operatorname{span}\{\phi(a) : \phi \in \operatorname{Hom}(A^C, C_C), a \in A\}. \end{aligned}$$

Recall that a two-sided ideal I in C is *essential* if $yI \neq 0$ and $Iy \neq 0$ whenever $y \in C$ and $y \neq 0$.

3.8. **Lemma.** *Let (A, C, s, t, Δ) be a left multiplier bialgebroid.*

1. *Let (A, C, s, t, Δ) be a left multiplier bialgebroid with a left counit ε. Then the image $C_0 := \varepsilon(A)$ is an idempotent, essential two-sided ideal in C, contained in $I^C \cap {}_CI$, and $s(C_0)A = A = t(C_0)A$.*
2. *If $s(I^C)A = A = t({}_CI)A$, then $I^C = I^C \cdot {}_CI = {}_CI$ is an idempotent, essential two-sided ideal in C.*

Proof. (1) Equations (3.1) and (3.2) imply that C_0 is a two-sided ideal, contained in ${}_CI \cap I^C$, and that $s(C_0)A = AA = t(C_0)A$. But $AA = A$ by (A1). Applying (3.1), we conclude that $C_0C_0 = C_0$. The relations $s(C_0)A = A = t(C_0)A$ and injectivity of s and t imply that the ideal C_0 is essential.

(2) Applying elements of $\operatorname{Hom}({}_CA, {}_CC)$ or $\operatorname{Hom}(A^C, C_C)$ to the assumed equality, we find ${}_CI = I^C \cdot {}_CI = I^C$. If $z \in C$ is non-zero, then, using injectivity of s and t, we can conclude that $s(zy)$ and $t(xz)$ are non-zero for some $y \in I^C$ and $x \in {}_CI$. □

If $s(I^C)A = A = t({}_CI)A$, then we can assume $I^C = C = {}_CI$ without much loss of generality:

3.9. **Lemma.** *Let $\mathcal{A} = (A, C, s, t, \Delta)$ be a left multiplier bialgebroid with a two-sided ideal $C_0 \subseteq C$ such that $s(C_0)A = A = t(C_0)A$. Denote by s_0 and t_0 the restrictions of s and t, respectively, to C_0. Then the natural map $A^{C_0} \otimes {}_{C_0}A \to A^C \otimes {}_CA$ is an isomorphism. Denote by Δ_0 the composition of Δ with the induced isomorphism $A^C \overline{\times}_C A \to A^{C_0} \overline{\times}_{C_0} A$. Then $\mathcal{A}_0 := (A, C_0, s_0, t_0, \Delta_0)$ is a left multiplier bialgebroid, and every left counit for $\mathcal{A}$ takes values in C_0 and is a left counit for $\mathcal{A}_0$.*

Proof. We first show that the natural map $A^{C_0} \otimes {}_{C_0}A \to A^C \otimes {}_CA$ is an isomorphism. Given $a, b \in A$ and $x \in C$, we can write $a = \sum_i t(x_i)a_i$ with $x_i \in C_0$ and $a_i \in A$, and then $t(x)a \otimes b - a \otimes s(x)b$ is equal to

$$\sum_i (t(xx_i)a_i \otimes b - a_i \otimes s(xx_i)b + a_i \otimes s(x_i)s(x)b - t(x_i)a_i \otimes s(x)b)$$

and therefore lies in the space spanned by all elements of the form $t(x')a' \otimes b' - a' \otimes s(x')b'$, where $a', b' \in A$ and $x' \in C_0$. The first assertion follows.

It follows immediately that $\mathcal{A}_0$ is a left multiplier bialgebroid.

If ε is a left counit for $\mathcal{A}$, then the assumption $A = s(C_0)A$ and (3.1) imply that $\varepsilon(A)$ is contained in C_0, and clearly, ε also is a left counit for $\mathcal{A}_0$. □

Now, we can prove the existence result:

3.10. **Proposition.** *Let (A, C, s, t, Δ) be a left multiplier bialgebroid. If its canonical maps (T_λ, T_ρ) are bijective and $s(I^C)A = A = t({}_CI)A$,then it has a unique left counit.*

Proof. Suppose that the assumptions hold. Then uniqueness of a left counit follows from 3.6 (1). To prove existence, consider the linear maps

$$E_s := m_B \circ T_\rho^{-1} \colon A^C \otimes {}_C A \to A, \qquad E_t := m_B^{\mathrm{op}} \circ T_\lambda^{-1} \colon A^C \otimes {}_C A \to A,$$

Since $E_s(a \otimes bc) = E_s(a \otimes b)c$ for all $a, b, c \in A$ by (2.5), the formula $\varepsilon_s(a)b := E_s(a \otimes b)$ defines a map $\varepsilon_s \colon A \to L(A)$. By definition and by (2.9),

$$\varepsilon_s(t(z)s(y)a)b = (m \circ T_\rho^{-1})(s(y)a \otimes s(z)b) = s(y)\varepsilon_s(a)s(z)b \tag{3.7}$$

for all $y, z \in C$ and $a, b \in A$. Likewise, the formula $\varepsilon_t(a)b := E_t(b \otimes a)$ defines a map $\varepsilon_t \colon A \to L(A)$ satisfying

$$\varepsilon_t(t(x)s(z)a)b = t(x)\varepsilon_t(a)t(z)b \tag{3.8}$$

for all $x, z \in B$ and $a, b \in A$.

Consider the following diagram:

$$\begin{tikzcd}
 & A^C \otimes {}_C A^C \otimes {}_C A \arrow[dl, "T_\lambda^{-1} \otimes \iota"'] \arrow[dr, "\iota \otimes T_\rho^{-1}"] & \\
A^B \otimes {}^B A^C \otimes {}_C A \arrow[r, "m_B^{\mathrm{op}} \otimes \iota"] \arrow[dr, "\iota \otimes T_\rho^{-1}"'] & A^C \otimes {}_C A & A^C \otimes {}_C A_B \otimes {}_B A \arrow[l, "\iota \otimes m_B"'] \arrow[dl, "T_\lambda^{-1} \otimes \iota"] \\
 & A^B \otimes {}^B A_B \otimes {}_B A. &
\end{tikzcd}$$

The outer square and the lower cell commute by (2.4) and (2.11). Hence, the upper cell commutes, showing that for all $a, b, c \in A$,

$$\varepsilon_t(b)a \otimes c = E_t(a \otimes b) \otimes c = a \otimes E_s(b \otimes c) = a \otimes \varepsilon_s(b)c \text{ in } A^C \otimes {}_C A. \tag{3.9}$$

Applying $\iota \otimes \psi$ or $\phi \otimes \iota$ with $\phi \in \mathrm{Hom}(A^C, C_C)$ and $\psi \in \mathrm{Hom}({}_C A, {}_C C)$, we obtain

$$t(\psi(c))\varepsilon_t(b)a = t(\psi(\varepsilon_s(b)c))a \qquad \text{and} \qquad s(\phi(\varepsilon_t(b)a))c = s(\phi(a))\varepsilon_s(b)c.$$

Let us focus on the first equation. Since $a \in A$ is arbitrary, we can conclude $t(\psi(c))\varepsilon_t(b) = t(\psi(\varepsilon_s(b)c))$ for all $b, c \in A$ and hence $t({}_C I)\varepsilon_t(A) \subseteq t({}_C I)$. Using the assumption and equation (3.8), we conclude $\varepsilon_t(A) = \varepsilon_t(t({}_C I)A) = t({}_C I)\varepsilon_t(A) \subseteq t({}_C I)$. A similar argument applied to the second equation shows that $\varepsilon_s(A) \subseteq s(I^C)$. In particular, we get

$$(t^{-1} \circ \varepsilon_t(b))\psi(c) = \psi(\varepsilon_s(b)c) = (s^{-1} \circ \varepsilon_s(b))\psi(c) \quad \text{for all } b, c \in A, \psi \in \mathrm{Hom}({}_C A, {}_C C).$$

Using Lemma 3.8, we can conclude that $s^{-1} \circ \varepsilon_s = t^{-1} \circ \varepsilon_t$, and this map is a left counit by construction. □

4. Right multiplier bialgebroids

The definitions and results of sections §2 and §3 have natural right-handed analogues which are briefly summarized below and will be needed for the definition of multiplier Hopf algebroids in section §5. For proofs, explanations and comments, we refer to the corresponding left-handed versions.

Let A be an algebra, not necessarily unital. We write ${}_A A$ when we regard A as a left A-module, and assume that this module is non-degenerate and idempotent. Denote by $R(A) = \mathrm{End}({}_A A)^{\mathrm{op}}$ the algebra of right multipliers of A, and write the application of a $T \in R(A)$ to an $a \in A$ as aT, so that $a(TS) = (aT)S$ for all $a \in A$ and $S, T \in R(A)$.

Then A embeds into $R(A)$ and we can form the multiplier algebra $M(A) \subseteq R(A)$. If A is unital, then the map $M(A) \to A$ given by $T \mapsto 1_A T$ is an isomorphism.

Let B be an algebra with a homomorphism $s\colon B \to M(A) \subseteq R(A)$ and an anti-homomorphism $t\colon B \to M(A) \subseteq R(A)$ such that the images of s and t commute. We write A_B and ${}^BA$ if we regard A as a right or left B-module such that $a \cdot y = as(y)$ or $y \cdot a = at(y)$ for all $a \in A$ and $y \in B$. We similarly write ${}_BA$ and A^B when we use multiplication on the left hand side instead of the right hand side.

Assume that the tensor product $A_B \otimes {}^BA$ is non-degenerate as a left module over $A \otimes 1$ and over $1 \otimes A$, respectively. We consider the opposite algebra $\mathrm{End}(A_B \otimes {}^BA)^{\mathrm{op}}$ and write $(a \otimes b)T$ for the image of an element $a \otimes b$ under an element $T \in \mathrm{End}(A_B \otimes {}^BA)^{\mathrm{op}}$, so that $(a \otimes b)(ST) = ((a \otimes b)S)T$ for all $a, b \in A$ and $S, T \in \mathrm{End}(A_B \otimes {}^BA)^{\mathrm{op}}$. Denote by

$$A_B \overline{\times} {}^BA \subseteq \mathrm{End}(A_B \otimes {}^BA)^{\mathrm{op}}$$

the subspace formed by all endomorphisms T such that for all $a, b \in A$, there exist elements $(a \otimes 1)T \in A_B \otimes {}^BA$ and $(1 \otimes b)T \in A_B \otimes {}^BA$ such that

$$(a \otimes b)T = (1 \otimes b)((a \otimes 1)T) = (a \otimes 1)((1 \otimes b)T).$$

This subspace is a subalgebra. If A has a unit 1_A, then the map $T \mapsto (1_A \otimes 1_A)T$ identifies this algebra with the right Takeuchi product

$$A_B \times {}^BA = \big\{w \in A_B \otimes {}^BA : (s(y) \otimes 1)w = (1 \otimes t(y))w \text{ for all } y \in B\big\} \subseteq A_B \otimes {}^BA. \tag{4.1}$$

4.1. **Definition.** *A* right multiplier bialgebroid *is a tuple (A, B, s, t, Δ) consisting of*

1. *algebras A and B such that the left A-module ${}_AA$ is non-degenerate and idempotent;*
2. *a homomorphism $s\colon B \to M(A) \subseteq R(A)$ and an anti-homomorphism $t\colon B \to M(A) \subseteq R(A)$ such that the images of s and t commute, the B-modules A_B and ${}^BA$ are faithful and idempotent, and $A_B \otimes {}^BA$ is non-degenerate as a left module over $A \otimes 1$ and over $1 \otimes A$;*
3. *a homomorphism $\Delta\colon A \to A_B \overline{\times} {}^BA$, called the* right comultiplication*, satisfying*
$$\Delta(t(y)s(x)at(y')s(x')) = (t(y) \otimes s(x))\Delta(a)(t(y') \otimes s(x')), \tag{4.2}$$
$$(a \otimes 1 \otimes 1)((\Delta \otimes \iota)((1 \otimes c)\Delta(b))) = (1 \otimes 1 \otimes c)((\iota \otimes \Delta)((a \otimes 1)\Delta(b))) \tag{4.3}$$
for all $a, b, c \in A$ and $x, y \in B$.

We call a right multiplier bialgebroid as above unital if the algebras A, B and the maps s, t, Δ are unital.

Let (A, B, s, t, Δ) be a right multiplier bialgebroid. Then the linear maps

$$\widetilde{{}_\lambda T}\colon A \otimes A \to A_B \otimes {}^BA, \quad a \otimes b \mapsto (a \otimes 1)\Delta(b),$$
$$\widetilde{{}_\rho T}\colon A \otimes A \to A_B \otimes {}^BA, \quad a \otimes b \mapsto (1 \otimes b)\Delta(a),$$

satisfy the following analogues of relations (2.3) and (2.9),

$$\begin{aligned} \widetilde{{}_\lambda T}(as(z) \otimes s(x)bt(y')s(x')) &= (1 \otimes t(z)s(x))\widetilde{{}_\lambda T}(a \otimes b)(t(y') \otimes s(x')), \\ \widetilde{{}_\rho T}(t(y)at(y')s(x') \otimes bt(z)) &= (t(y)s(z) \otimes 1)\widetilde{{}_\rho T}(a \otimes b)(t(y') \otimes s(x')), \end{aligned} \tag{4.4}$$

and make the following analogues of (2.4), (2.5), (2.7) and (2.11) commute:

$$\begin{tikzcd}[column sep=small] & A_B \otimes {}^B A \otimes A \arrow[dr, "\iota\otimes m^{\mathrm{op}}"] & \\ A\otimes A\otimes A \arrow[ur, "\widetilde{{}_\lambda T}\otimes\iota"] \arrow[dr, "\iota\otimes\widetilde{{}_\rho T}"'] & & A_B \otimes {}^B A \\ & A\otimes A_B\otimes {}^B A, \arrow[ur, "m\otimes\iota"'] & \end{tikzcd} \tag{4.5}$$

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\iota\otimes\widetilde{{}_\lambda T}"] \arrow[d, "m\otimes\iota"'] & A\otimes A_B\otimes {}^B A \arrow[d, "m\otimes\iota"] \\ A\otimes A \arrow[r, "\widetilde{{}_\lambda T}"] & A_B\otimes {}^B A, \end{tikzcd} \qquad \begin{tikzcd} A\otimes A\otimes A \arrow[r, "\widetilde{{}_\rho T}\otimes\iota"] \arrow[d, "\iota\otimes m^{\mathrm{op}}"'] & A_B\otimes {}^B A\otimes A \arrow[d, "\iota\otimes m^{\mathrm{op}}"] \\ A\otimes A \arrow[r, "\widetilde{{}_\rho T}"] & A_B\otimes {}^B A, \end{tikzcd} \tag{4.6}$$

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\iota\otimes m^{\mathrm{op}}"] \arrow[d, "(\widetilde{{}_\lambda T})_{13}"'] & A\otimes A \arrow[dd, "\widetilde{{}_\lambda T}"] \\ A_B\otimes A\otimes {}^B A \arrow[d, "(\widetilde{{}_\lambda T})_{12}"'] & \\ A_B\otimes {}^B A^B\otimes {}^B A \arrow[r, "\iota\otimes m_B^{\mathrm{op}}"'] & A_B\otimes {}^B A \end{tikzcd} \qquad \begin{tikzcd} A\otimes A\otimes A \arrow[r, "m\otimes\iota"] \arrow[d, "(\widetilde{{}_\rho T})_{13}"'] & A\otimes A \arrow[dd, "\widetilde{{}_\rho T}"] \\ A_B\otimes A\otimes {}^B A \arrow[d, "(\widetilde{{}_\rho T})_{23}"'] & \\ A_B\otimes {}_B A_B\otimes {}^B A \arrow[r, "m_B\otimes\iota"'] & A_B\otimes {}^B A \end{tikzcd}$$

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\iota\otimes\widetilde{{}_\rho T}"] \arrow[d, "\widetilde{{}_\lambda T}\otimes\iota"'] & A\otimes A_B\otimes {}^B A \arrow[d, "\widetilde{{}_\lambda T}\otimes\iota"] \\ A_B\otimes {}^B A\otimes A \arrow[r, "\iota\otimes\widetilde{{}_\rho T}"] & A_B\otimes {}^B A_B\otimes {}^B A. \end{tikzcd} \tag{4.7}$$

4.2. **Notation.** Similarly as in Notation 2.4, we choose an algebra C with an anti-isomorphism $\kappa\colon C\to B$, use this to regard A as a C-module in various ways, and write

$$A_C\otimes {}_C A \quad \text{and} \quad A^C\otimes {}^C A$$

for the quotients of $A\otimes A$ by the subspaces spanned by all elements of the form $t(\kappa(x))a\otimes b - a\otimes bt(\lambda(x))$ in the first case and $as(\kappa(x))\otimes b - a\otimes s(\kappa(x))b$ in the second case. Again, the choice C and κ is irrelevant here and will only be fixed from Section 5 on. Note that the flip map $\Sigma_{(A,A)}$ descends to isomorphisms

$$\Sigma_{(A_B,{}_B A)}\colon A_B\otimes {}_B A \to A^C\otimes {}^C A \quad \text{and} \quad \Sigma_{(A^B,{}^B A)}\colon A^B\otimes {}^B A\to A_C\otimes {}_C A. \tag{4.8}$$

In the notation above, the maps $\widetilde{{}_\lambda T}$ and $\widetilde{{}_\rho T}$ descend to maps

$${}_\lambda T\colon A_C\otimes {}_C A\to A_B\otimes {}^B A \qquad \text{and} \qquad {}_\rho T\colon A^C\otimes {}^C A\to A_B\otimes {}^B A,$$

respectively, and these maps satisfy pentagonal relations similar to those given in Proposition 2.8 if the corresponding assumptions hold.

Given a right multiplier bialgebroid $\mathcal{A} = (A, B, s, t, \Delta)$, one can, similarly as in the case of left multiplier bialgebroids (see Proposition 2.7), reverse the comultiplication and obtain a *co-opposite* right multiplier bialgebroid

$$\mathcal{A}^{\mathrm{co}} = (B^{\mathrm{op}}, t, s, \Delta^{\mathrm{co}}).$$

One can also reverse the multiplication of the underlying algebra A to pass between left and right multiplier bialgebroids as follows.

4.3. **Proposition.** *Let $\mathcal{A} = (A, C, s, t, \Delta)$ be a left multiplier bialgebroid. Regard s as an anti-homomorphism and t as a homomorphism, respectively, from C to $M(A^{\mathrm{op}}) \subseteq R(A^{\mathrm{op}})$, and write $(A^{\mathrm{op}})_C$ and ${}^C(A^{\mathrm{op}})$ for A^{op}, regarded as a C-module via $x \cdot a^{\mathrm{op}} = (s(x)a)^{\mathrm{op}}$ and $a^{\mathrm{op}} \cdot x = (t(x)a)^{\mathrm{op}}$, respectively. Then the map $a \otimes b \mapsto a^{\mathrm{op}} \otimes b^{\mathrm{op}}$ descends to a linear isomorphism*

$$A^C \otimes {}_C A \to (A^{\mathrm{op}})_C \otimes {}^C(A^{\mathrm{op}}), \quad w \mapsto w^{\mathrm{op}\otimes\mathrm{op}},$$

there exists a well-defined homomorphism

$$\Delta^{\mathrm{op}} \colon A^{\mathrm{op}} \to (A^{\mathrm{op}})_C \overline{\times}^C (A^{\mathrm{op}}), \quad ((b^{\mathrm{op}} \otimes c^{\mathrm{op}})\Delta^{\mathrm{op}}(a^{\mathrm{op}})) := (\Delta(a)(b \otimes c))^{\mathrm{op}\otimes\mathrm{op}},$$

and $\mathcal{A}^{\mathrm{op}} := (A^{\mathrm{op}}, C, t, s, \Delta^{\mathrm{op}})$ is a right multiplier bialgebroid. Its associated maps $\widetilde{{}_\lambda T}^{\mathrm{op}}$ and $\widetilde{{}_\rho T}^{\mathrm{op}}$ are given by

$$\widetilde{{}_\lambda T}^{\mathrm{op}}(a^{\mathrm{op}} \otimes b^{\mathrm{op}}) = (\widetilde{T_\lambda}(a \otimes b))^{\mathrm{op}\otimes\mathrm{op}}, \qquad \widetilde{{}_\rho T}^{\mathrm{op}}(a^{\mathrm{op}} \otimes b^{\mathrm{op}}) = (\widetilde{T_\lambda}(a \otimes b))^{\mathrm{op}\otimes\mathrm{op}}.$$

Conversely, for every right multiplier bialgebroid $\mathcal{A}$, there exists a unique left multiplier bialgebroid $\mathcal{A}^{\mathrm{op}}$ such that $\mathcal{A} = (\mathcal{A}^{\mathrm{op}})^{\mathrm{op}}$.

The notion of a counit carries over as follows.

Given a right multiplier bialgebroid (A, B, s, t, Δ) and morphisms $\phi \in \mathrm{Hom}({}^B A, {}_B B)$ and $\psi \in \mathrm{Hom}(A_B, B_B)$, we can form slice maps

$$\phi \odot \iota \colon A_B \otimes {}^B A \to A, \ a \otimes b \mapsto bs(\phi(a)), \quad \iota \odot \psi \colon A_B \otimes {}^B A \to A, \ a \otimes b \mapsto at(\psi(b)).$$

4.4. **Definition.** *A* right counit *for a right multiplier bialgebroid $\mathcal{A} = (A, B, s, t, \Delta)$ is a map $\varepsilon \colon A \to B$ that satisfies*

$$\varepsilon(at(y)) = ya \qquad \text{and} \qquad \varepsilon(as(x)) = ax \qquad \text{for all } a \in A, x, y \in B, \tag{4.9}$$

that is, $\varepsilon \in \mathrm{Hom}(A_B, B_B) \cap \mathrm{Hom}({}^B A, {}_B B)$, and

$$(\varepsilon \odot \iota)({}_\rho T(a \otimes b)) = ba \quad \text{and} \quad (\iota \odot \varepsilon)({}_\lambda T(a \otimes b)) = ab \quad \text{for all } a, b \in A. \tag{4.10}$$

One easily verifies that right counits for a right multiplier bialgebroid $\mathcal{A} = (A, B, s, t, \Delta)$ coincide with right counits for the co-opposite $\mathcal{A}^{\mathrm{co}}$ up the canonical linear identification of B with B^{op}, and with left counits for the opposite $\mathcal{A}^{\mathrm{op}}$ up the canonical linear identification of A with A^{op}.

Proposition 3.6, Lemma **??** and Proposition 3.10 have the following right-handed counterparts. If $\mathcal{A} = (A, B, s, t, \Delta)$ is a right multiplier bialgebroid, then

(1) a right counit for $\mathcal{A}$ is unique if $\mathcal{A}$ is left- or right-full in a sense similar as it was defined for left multiplier bialgebroids in Definition 3.4;
(2) if the map ${}_\rho T$ (or T_λ) is surjective, then
$$\varepsilon(ab) = \varepsilon(s(\varepsilon(a))b) \quad (\text{or } \varepsilon(ab) = \varepsilon(t(\varepsilon(a))b), \text{ respectively}); \tag{4.11}$$
(3) without much loss of generality, one can assume ε to be surjective;
(4) if the maps $({}_\lambda T, {}_\rho T)$ are bijective and $A = As({}^t I) = At({}^s I)$, where ${}^s I, {}^t I \subseteq B$ denote the linear span of the images of all $\phi \in \mathrm{Hom}(A_B, B_B)$ and $\psi \in \mathrm{Hom}({}^B A, {}_B B)$, respectively, then $\mathcal{A}$ has a unique right counit.

In the unital case, right multiplier bialgebroids with right counits satisfying the equations in (4.11) correspond to right bialgebroids [1], [5], that is, an analogue of Proposition 3.3 holds.

Given a right multiplier bialgebroid (A, B, s, t, Δ), we use the generalized Sweedler notation as well, but put the subscripts in brackets, so that

$$\Delta(a) = a_{[1]} \otimes a_{[2]}, \qquad {}_{\lambda}T(a \otimes b) = ab_{[1]} \otimes b_{[2]}, \qquad {}_{\rho}T(a \otimes b) = a_{[1]} \otimes ba_{[2]}$$

and so on.

5. Multiplier Hopf algebroids

We now come to the main part of this article, where the left- and the right-handed concepts introduced above get assembled into a two-sided structure.

In detail, a multiplier bialgebroid will be given by a left multiplier bialgebroid and a right multiplier bialgebroid

$$\mathcal{A}_C = (A, C, s_C, t_C, \Delta_C) \quad \text{and} \quad \mathcal{A}_B = (A, B, s_B, t_B, \Delta_B),$$

respectively, subject to the following assumptions.

First, $\mathcal{A}_C$ and $\mathcal{A}_B$ have the same underlying total algebra A. By assumption, this algebra is non-degenerate on the left and on the right, so that we can form the two-sided multiplier algebra $M(A)$ which is the target of the maps s_B, t_B, s_C, t_C.

The second assumption will be used to make sense of the third one, and reads

$$s_B(B) = t_C(C), \qquad\qquad t_B(B) = s_C(C). \tag{5.1}$$

Then the maps $S_B := t_C^{-1} \circ s_B \colon B \to C$ and $S_C := t_B^{-1} \circ s_C \colon C \to B$ are anti-isomorphisms, but not necessarily inverse to each other. To simplify notation, we shall identify B with the image $s_B(B)$ and C with the image $s_C(C)$, that is, we assume $s_B = \iota_B$, $s_C = \iota_C$ and write S_B and S_C for t_C^{-1} and t_B^{-1}, respectively. Furthermore, we denote elements of B by $x, x', x'', \ldots$ and elements of C by $y, y', y'', \ldots$, and write ${}_BA$, A_B if we regard A as a left or right module over B via left or right multiplication, and A^B, ${}^BA$ if we regard A as a right or left module over B via $a \cdot x = t_B(x)a$ or $x \cdot a = at_B(x)$, respectively. Likewise, we use the notation ${}_CA$, A_C, A^C, ${}^CA$, respectively.

To formulate the third assumption, observe that C-bilinearity of Δ_C, see (2.8), and B-bilinearity of Δ_B, see (4.2), now take the form

$$\Delta_C(xyax'y') = (y \otimes x)\Delta_C(a)(y' \otimes x'), \quad \Delta_B(xyax'y') = (y \otimes x)\Delta_B(a)(y' \otimes x') \tag{5.2}$$

for all $a \in A$, $x, x' \in B$, $y, y' \in C$. Similarly, one can rewrite the relations (2.9) and (4.4) for the canonical maps $\widetilde{T_\lambda}, \widetilde{T_\rho}$ and $\widetilde{{}_\lambda T}, \widetilde{{}_\rho T}$. Now, the third assumption is the following mixed co-associativity,

$$\begin{aligned} ((\Delta_C \otimes \iota)((1 \otimes c)\Delta_B(b)))(a \otimes 1 \otimes 1) &= (1 \otimes 1 \otimes c)((\iota \otimes \Delta_B)(\Delta_C(b)(a \otimes 1))), \\ (a \otimes 1 \otimes 1)((\Delta_B \otimes \iota)(\Delta_C(b)(1 \otimes c))) &= ((\iota \otimes \Delta_C)((a \otimes 1)\Delta_B(b)))(1 \otimes 1 \otimes c) \end{aligned} \tag{5.3}$$

for all $a, b, c \in A$, which amounts to commutativity of the following diagrams,

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\iota\otimes\widetilde{T_\rho}"] \arrow[d, "{}_\lambda\widetilde{T}\otimes\iota"'] & A \otimes A^C \otimes {}_C A \arrow[d, "{}_\lambda\widetilde{T}\otimes\iota"] \\ A_B \otimes {}^B A \otimes A \arrow[r, "\iota\otimes\widetilde{T_\rho}"] & A_B \otimes {}^B A^C \otimes {}_C A, \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \otimes A \arrow[r, "\iota\otimes{}_\rho\widetilde{T}"] \arrow[d, "\widetilde{T_\lambda}\otimes\iota"'] & A \otimes {}^C A \otimes A_C \arrow[d, "\widetilde{T_\lambda}\otimes\iota"] \\ A^C \otimes {}_C A \arrow[r, "\iota\otimes{}_\rho\widetilde{T}"] & A^C \otimes {}_C A_B \otimes {}^B A, \end{tikzcd} \tag{5.4}$$

and in Sweedler notation to the relations

$$(a_{(1)})_{[1]} \otimes (a_{(1)})_{[2]} \otimes a_{(2)} = a_{[1]} \otimes (a_{[2]})_{(1)} \otimes (a_{[2]})_{(2)}, \tag{5.5}$$

$$a_{(1)} \otimes (a_{(2)})_{[1]} \otimes (a_{(2)})_{[2]} = (a_{[1]})_{(1)} \otimes (a_{[1]})_{(2)} \otimes a_{[2]} \tag{5.6}$$

for all $a \in A$.

5.1. **Definition.** *A* multiplier bialgebroid $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ *consists of*

(1) *a non-degenerate, idempotent algebra A,*
(2) *subalgebras $B, C \subseteq M(A)$ with anti-isomorphisms $t_B \colon B \to C$ and $t_C \colon C \to B$,*
(3) *maps $\Delta_C \colon A \to A^C \overline{\times}_C A$ and $\Delta_B \colon A \to A_B \overline{\times}^B A$*

such that

(4) *$\mathcal{A}_B = (A, B, \iota_B, t_B, \Delta_B)$ is a right multiplier bialgebroid,*
(5) *$\mathcal{A}_C = (A, C, \iota_C, t_C, \Delta_C)$ is a left multiplier bialgebroid, and*
(6) *the mixed co-associativity conditions* (5.3) *hold.*

We call left counits of $\mathcal{A}_C$ and right counits of $\mathcal{A}_B$ just left and right counits, respectively, of $\mathcal{A}$. Likewise, we call the canonical maps T_λ, T_ρ of $\mathcal{A}_C$ and ${}_\lambda T, {}_\rho T$ of $\mathcal{A}_B$ just the canonical maps of $\mathcal{A}$.

We call such a multiplier bialgebroid $\mathcal{A}$ unital *if the algebras A, B, C, the inclusions $B, C \hookrightarrow A$ and the maps Δ_C, Δ_B are unital, that is, if $\mathcal{A}_B$ and $\mathcal{A}_C$ are unital.*

Note that we do not assume existence of counits.

To establish our main result and the key properties multiplier bialgebroids, we need to perform a fair amount of calculations involving the associated canonical maps. We present these calculations and the key relations satisfied by the canonical maps in the form of commutative diagrams, where one can verify that all of the maps involved are well-defined on the underling tensor products. Additionally, we write out the key relations in the generalised Sweedler notation introduced at the end of §2 and of §4, respectively.

5.2. **Definition.** *An* antipode *for a multiplier bialgebroid $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ is a linear map $S \colon A \to M(A)$ satisfying the following conditions:*

(1) *S is an anti-homomorphism such that $S(A)A = A = AS(A)$;*
(2) *the extension of S to $M(A)$ satisfies $S \circ t_B = \iota_B$ and $S \circ t_C = \iota_C$;*
(3) *there exist a left counit ${}_C\varepsilon$ and a right counit ε_B for $\mathcal{A}$ such that the following diagrams commute, where the unlabelled maps are given by multiplication:*

$$\begin{tikzcd} A \otimes A \arrow[r, "\varepsilon_B\otimes\iota"] \arrow[d, "\widetilde{T_\rho}"'] & B \otimes A \arrow[r] & A, \\ A^C \otimes {}_C A \arrow[rr, "S\otimes\iota"'] & & M(A)_C \otimes {}_C A \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "\iota\otimes{}_C\varepsilon"] \arrow[d, "{}_\lambda\widetilde{T}"'] & A \otimes C \arrow[r] & A. \\ A_B \otimes {}^B A \arrow[rr, "\iota\otimes S"'] & & A_B \otimes {}_B M(A) \arrow[u] \end{tikzcd} \tag{5.7}$$

We call such an antipode invertible *if it maps A bijectively to $A \subseteq M(A)$.*

Condition (1) is equivalent to saying that S is a morphism of bimodules

$$S\colon {}^{B}A^{B} \to {}_{B}M(A)_{B} \quad \text{and} \quad S\colon {}^{C}A^{C} \to {}_{C}M(A)_{C},$$

and in Sweedler notation, commutativity of the diagrams (5.7) amounts to the equations

$$\varepsilon_B(a)b = S(a_{(1)})a_{(2)}b, \qquad\qquad a_C\varepsilon(b) = ab_{[1]}S(b_{[2]}) \tag{5.8}$$

for all $a, b \in A$.

We now come to the first main result of this article. Our proof follows a similar strategy as the proof of the implication (iv)⇒(i) of Proposition 4.2 in [5], but will be purely diagrammatic.

5.3. **Proposition.** *Let $\mathcal{A}$ be a multiplier bialgebroid. Suppose that it has counits and that its canonical maps ${}_{\lambda}T$ and T_{ρ} are bijective. Then it has a unique antipode S.*

Proof. Consider the compositions

$$S_{\rho}\colon A^{C} \otimes {}_{C}A \xrightarrow{T_{\rho}^{-1}} A_{B} \otimes {}_{B}A \xrightarrow{\varepsilon_B \otimes \iota} B_{B} \otimes {}_{B}A \to A,$$
$${}_{\lambda}S\colon A^{C} \otimes {}_{C}A \xrightarrow{{}_{\lambda}T^{-1}} A_{C} \otimes {}_{C}A \xrightarrow{\iota \otimes_C \varepsilon} A_{C} \otimes {}_{C}C \to A.$$

If S is an antipode for $\mathcal{A}$, then 5.2 (3) implies that $S(a)b = S_{\rho}(a \otimes b)$ and $aS(b) = {}_{\lambda}S(a \otimes b)$ for all $a, b \in A$. Therefore, the antipode is unique. Let us prove existence. The maps S_{ρ} and ${}_{\lambda}S$ satisfy $aS_{\rho}(b \otimes c) = {}_{\lambda}S(a \otimes b)c$ for all $a, b, c \in A$ because the following diagram commutes,

$$\begin{tikzcd}
A_B \otimes {}^{B}A^{C} \otimes {}_{C}A \arrow[r, "\iota \otimes T_{\rho}^{-1}"] \arrow[d, "{}_{\lambda}T^{-1} \otimes \iota"'] & A_B \otimes {}^{B}A_{B} \otimes {}_{B}A \arrow[r, "\iota \otimes \varepsilon_B \otimes \iota"] \arrow[d, "{}_{\lambda}T^{-1} \otimes \iota"'] & A_B \otimes {}_{B}B_{B} \otimes {}_{B}A \arrow[d] \\
A_C \otimes {}_{C}A^{C} \otimes {}_{C}A \arrow[r, "\iota \otimes T_{\rho}^{-1}"] \arrow[d, "\iota \otimes_C \varepsilon \otimes \iota"'] & A_C \otimes {}_{C}A_{B} \otimes {}_{B}A \arrow[r, "m_C \otimes \iota"] \arrow[d, "\iota \otimes m_B"] & A_B \otimes {}_{B}A \arrow[d, "m_B"] \\
A_C \otimes {}_{C}C_{C} \otimes {}_{C}A \arrow[r] & A_C \otimes {}_{C}A \arrow[r, "m_C"] & A.
\end{tikzcd} \tag{5.9}$$

Consequently, there exists a linear map $S\colon A \to M(A)$ such that

$$S(b)c = S_{\rho}(b \otimes c) \quad \text{and} \quad aS(b) = {}_{\lambda}S(a \otimes b)$$

for all $a, b, c \in A$. By construction,

$$\begin{aligned} S(t_C(y)a)b &= S_{\rho}(t_C(y)a \otimes b) = S_{\rho}(a \otimes yb) = S(a)yb, \\ aS(bt_B(x)) &= S_{\lambda}(a \otimes bt_B(x)) = S_{\lambda}(ax \otimes b) = axS(b) \end{aligned} \tag{5.10}$$

for all $x \in B$, $y \in C$ and $a, b \in A$.

To see that S is an anti-homomorphism, observe that the following diagram commutes,

$$\begin{tikzcd}
A^C \otimes {}^C A^C \otimes {}_C A \arrow[rrr, "m_C^{\mathrm{op}}\otimes\iota"] \arrow[d, "(T_\rho^{-1})_{23}"'] & & & A^C \otimes {}_C A \arrow[d, "T_\rho^{-1}"] \\
A^C \otimes A_B \otimes {}_{B,C} A \arrow[r, "(T_\rho^{-1})_{13}"] \arrow[d, "\iota\otimes\varepsilon_B\otimes\iota"'] & A_B^C \otimes {}^C A \otimes {}_B A \arrow[rr, "m_C^{\mathrm{op}}\otimes\iota"] \arrow[d, "\iota\otimes S_B\circ\varepsilon_B\otimes\iota"'] & & A_B \otimes {}_B A \arrow[d, "\varepsilon_B\otimes\iota"] \\
A^C \otimes B_B \otimes {}_{B,C} A \arrow[d] & A_B^C \otimes {}_C C \otimes {}_B A \arrow[d] & & B_B \otimes {}_B A \arrow[d] \\
{}_B A \otimes A^B \arrow[r, "T_\rho^{-1}"] & A_B \otimes {}_B A \arrow[r, "\varepsilon_B\otimes\iota"] & B_B \otimes {}_B A \arrow[r] & A.
\end{tikzcd}$$

Indeed, the upper rectangle commutes by (2.7), and the lower right square commutes because $\varepsilon_B(ab) = \varepsilon_B(\varepsilon_B(a)b)$ for all $a, b \in A$; see (4.11). Thus, $S_\rho(ba)c = S_\rho(a)S_\rho(b)c$ for all $a, b, c \in A$.

We claim that $AS(A) = A = S(A)A$. Indeed, diagram (5.9) shows that

$$(\iota \otimes S_\rho)(A_B \otimes {}^B A^C \otimes {}_C A) = A_B \otimes {}_B A, \quad ({}_\lambda S \otimes \iota)(A_B \otimes {}^B A^C \otimes {}_C A) = A_C \otimes {}_C A$$

because the maps $T_\rho^{-1}, {}_\lambda T^{-1}$ and m are surjective. We apply $\varepsilon_B \otimes \iota$ or $\iota \otimes_C \varepsilon$, respectively, use Lemma 3.8 and (5.10), and conclude

$$A = \varepsilon_B(A)A \subseteq S_\rho(A^C \otimes {}_C A) = S(A)A, \quad A = A_C\varepsilon(A) \subseteq {}_\lambda S(A_B \otimes {}^B A) = AS(A).$$

Finally, the diagrams in (5.7) commute by construction. □

Conversely, given a multiplier bialgebroid $\mathcal{A}$ with an antipode S, we shall construct an inverse to the canonical map T_ρ. A similar construction will give the inverse of ${}_\lambda T$.

5.4. **Lemma.** *There exists a unique linear map*

$$T_\rho^\dagger \colon A^C \otimes {}_C A \to A_B \otimes {}_B A$$

such that the following diagrams commute:

$$\begin{tikzcd}
A^C \otimes {}^C A \otimes A \arrow[r, "\iota\otimes m(S\otimes\iota)"] \arrow[d, "{}_\rho T\otimes\iota"'] & A^C \otimes {}_C A \arrow[d, "T_\rho^\dagger"] \\
A_B \otimes {}^B A \otimes A \arrow[r, "\iota\otimes m(S\otimes\iota)"] & A_B \otimes {}_B A
\end{tikzcd}
\qquad
\begin{tikzcd}
A_C \otimes {}_C A^C \otimes {}_C A \arrow[r, "\iota\otimes T_\rho^\dagger"] \arrow[d, "{}_\lambda T\otimes\iota"'] & A_C \otimes {}_C A_B \otimes {}_B A \arrow[d, "m\otimes\iota"] \\
A_B \otimes {}^B A^C \otimes {}_C A \arrow[r, "\iota\otimes m(S\otimes\iota)"] & A_B \otimes {}_B A.
\end{tikzcd} \tag{5.11}$$

In the unital case, we could just let $T_\rho^\dagger = (\iota \otimes S) \circ {}_\rho T$ so that $T_\rho^\dagger(a) = a_{[1]} \otimes S(a_{[2]})$. In the non-unital case, this relation is not well-defined but captured by commutativity of the diagrams above. Indeed, in Sweedler notation, these diagrams amount to the relations

$$T_\rho^\dagger(a \otimes S(b)c) = a_{[1]} \otimes S(ba_{[2]})c \quad \text{and} \quad ab_{[1]} \otimes S(b_{[2]})c = (a \otimes 1)T_\rho^\dagger(b \otimes c).$$

Proof of Lemma 5.4. Let $S_\rho = m(S \otimes \iota)$ as before. To prove existence of a map $T_\rho^\dagger$ that makes the first diagram commute, we need to show that whenever we have an element

$\omega = \sum_i b_i \otimes c_i \otimes d_i \in A^C \otimes {}^C A \otimes A$ such that

$$(\iota \otimes S_\rho)(\omega) = \sum_i b_i \otimes S(c_i)d_i$$

is zero in $A^C \otimes {}_C A$, then also

$$(\iota \otimes S_\rho)({}_\rho T \otimes \iota)(\omega) = \sum_i b_{i[1]} \otimes S(c_i b_{i[2]})d_i \tag{5.12}$$

is zero in $A_B \otimes {}_B A$. To this end, consider the following diagram:

$$\begin{tikzcd}
A_C \otimes {}_C A^C \otimes {}^C A \otimes A \arrow[rrr, "\iota\otimes\iota\otimes S_\rho"] \arrow[dr, "\iota\otimes{}_\rho T\otimes\iota"] \arrow[ddd, "{}_\lambda T\otimes\iota\otimes\iota"'] & & & A_C \otimes {}_C A^C \otimes {}_C A \arrow[dl, dashed, "\iota\otimes T_\rho^\dagger"'] \arrow[ddd, "{}_\lambda T\otimes\iota"] \\
{} \arrow[dr, phantom, "(1)"] & A_C \otimes {}_C A_B \otimes {}^B A \arrow[r, "\iota\otimes\iota\otimes S_\rho"] \arrow[d, "m\otimes\iota\otimes\iota"] \arrow[dr, phantom, "(2)"] & A_C \otimes {}_C A_B \otimes {}_B A \arrow[d, "m\otimes\iota"] \arrow[dr, phantom, "(5)"] & {} \\
{} & A_B \otimes {}^B A \otimes A \arrow[r, "\iota\otimes S_\rho"'] \arrow[dr, phantom, "(3)"] & A_B \otimes {}_B A & {} \\
A_B \otimes {}^B A^C \otimes {}^C A \otimes A \arrow[ur, "\iota\otimes m^{\mathrm{op}}\otimes\iota"] \arrow[rrr, "\iota\otimes\iota\otimes S_\rho"'] & {} & {} & A_B \otimes {}^B A^C \otimes {}_C A \arrow[ul, "\iota\otimes S_\rho"']
\end{tikzcd}$$

Cell (1) commutes by (4.5), (2) trivially, and (3) because S is an anti-homomorphism. Now, we make the argument above precise. Suppose that $w = \sum_i b_i \otimes c_i \otimes d_i \in A^C \otimes^C A \otimes A$ and $(\iota \otimes S_\rho)(w)$ is zero. Since the outer square commutes, we can conclude that

$$(\iota \otimes \iota \otimes S_\rho)({}_\lambda T \otimes \iota \otimes \iota)(a \otimes w) = \sum_i ab_{i[1]} \otimes b_{i[2]} \otimes S(c_i)d_i$$

is zero for all $a \in A$, and since cells (1)–(3) commute, also

$$(m \otimes \iota)(\iota \otimes \iota \otimes S_\rho)(\iota \otimes {}_\rho T \otimes \iota)(a \otimes w) = \sum_i ab_{i[1]} \otimes S(b_{i[2]})S(c_i)d_i$$

is zero for all $a \in A$. But then also the element (5.12) is zero because S is anti-multiplicative and $A_B \otimes {}_B A$ is non-degenerate as a left module over $A \otimes 1$. Hence, we can deduce that there exists a unique map $T_\rho^\dagger$ that makes the first diagram in (5.11) and cell (4) in the diagram above commute. As S_ρ is surjective, we can deduce that cell (5) and hence also the second diagram in (5.11) commute. $\square$

5.5. **Lemma.** *The following diagram commutes:*

$$\begin{tikzcd}
A^B \otimes {}^B A^C \otimes {}_C A \arrow[r, "\iota\otimes T_\rho^\dagger"] \arrow[d, "T_\lambda\otimes\iota"'] & A^B \otimes {}^B A_B \otimes {}_B A \arrow[d, "T_\lambda\otimes\iota"] \\
A^C \otimes {}_C A^C \otimes {}_C A \arrow[r, "\iota\otimes T_\rho^\dagger"] & A^C \otimes {}_C A_B \otimes {}_B A.
\end{tikzcd}$$

Proof. This follows easily from commutativity of the first diagram in Lemma 5.4, commutativity of (2.11), and surjectity of the map S_ρ. $\square$

5.6. **Lemma.** *We have* $(m \circ T_\rho^\dagger)(a \otimes b) = {}_C\varepsilon(a)b$ *for all* $a, b \in A$.

Proof. In Sweedler notation, the idea is that $a_C\varepsilon(b)c = ab_{[1]}S(b_{[2]})c = a((m \circ T_\rho^\dagger)(b \otimes c))$ for all $a, b, c \in A$ by (5.11) and (5.7). More formally, consider the following diagram:

$$\begin{tikzcd}
A_C \otimes {}_C A_B \otimes {}_B A \arrow[r, "m\otimes\iota"] & A_C \otimes {}_C A^C \otimes {}_C A \arrow[dr, "m"] & \\
A_C \otimes {}_C A^C \otimes {}_C A \arrow[u, "\iota\otimes T_\rho^\dagger"] \arrow[r, "{}_\lambda T\otimes\iota"] \arrow[d, "\iota\otimes {}_C\varepsilon\otimes\iota"'] & A_B \otimes {}^B A^C \otimes {}_C A \arrow[u, "\iota\otimes m(S\otimes\iota)"'] \arrow[d, "m(\iota\otimes S)\otimes\iota"] & A \\
A_C \otimes {}_C C_C \otimes {}_C A \arrow[r] & A_C \otimes {}_C A \arrow[ur, "m"'] &
\end{tikzcd}$$

The upper and the lower rectangle commute because of (5.11) and (5.7), respectively, and the rectangle on the right hand side commutes as well. Hence so does the outer cell. □

5.7. **Proposition.** *Let $\mathcal{A}$ be a multiplier bialgebroid with an antipode S. Then its canonical maps ${}_\lambda T$ and T_ρ are bijective.*

Proof. We show that the map $T_\rho^\dagger$ defined above is inverse to T_ρ.

In the unital case, we can use (5.5), (5.8), and find

$$\begin{aligned}(T_\rho^\dagger \circ T_\rho)(a \otimes b) = (a_{(1)})_{[1]} \otimes S((a_{(1)})_{[2]})a_{(2)}b &= a_{[1]} \otimes S((a_{[2]})_{(1)})(a_{[2]})_{(2)}b \\ &= a_{[1]} \otimes \varepsilon_B(a_{[2]})b = a_{[1]}\varepsilon_B(a_{[2]}) \otimes b = a \otimes b\end{aligned}$$

for all $a, b \in A$. Thus, $T_\rho^\dagger \circ T_\rho = \iota$, and a similar calculation shows that $T_\rho \circ T_\rho^\dagger = \iota$.

In the general case, we proceed as follows. We first claim that $T_\rho^\dagger \circ T_\rho = \iota$. Diagrams (5.4) and (5.11) imply that the following diagram commutes:

$$\begin{tikzcd}
A_C \otimes {}_C A_B \otimes {}_B A \arrow[r, "\iota\otimes T_\rho"] \arrow[d, "{}_\lambda T\otimes\iota"'] & A_C \otimes {}_C A^C \otimes {}_C A \arrow[r, "\iota\otimes T_\rho^\dagger"] \arrow[d, "{}_\lambda T\otimes\iota"'] & A_C \otimes {}_C A_B \otimes {}_B A \arrow[d, "m\otimes\iota"] \\
A_B \otimes {}^B A_B \otimes {}_B A \arrow[r, "\iota\otimes T_\rho"] & A_B \otimes {}^B A^C \otimes {}_C A \arrow[r, "\iota\otimes m(S\otimes\iota)"] & A_B \otimes {}_B A
\end{tikzcd}$$

By (5.7), the lower composition maps $a\otimes b\otimes c$ to $a\otimes\varepsilon_B(b)c = a\varepsilon_B(b)\otimes c$, and by definition of the counit, precomposition with ${}_\lambda T\otimes\iota$ gives $m\otimes\iota$. Therefore, $(m\otimes\iota)(\iota\otimes T_\rho^\dagger)(\iota\otimes T_\rho) = m \otimes \iota$. Since $A_B \otimes {}_B A$ is non-degenerate as a left $A \otimes 1$-module, the claim follows.

To see that $T_\rho \circ T_\rho^\dagger = \iota$, consider the following diagram:

$$\begin{tikzcd}
A^B \otimes {}^B A^C \otimes {}_C A \arrow[r, "\iota\otimes T_\rho^\dagger"] \arrow[d, "T_\lambda\otimes\iota"'] & A^B \otimes {}^B A_B \otimes {}_B A \arrow[r, "\iota\otimes T_\rho"] \arrow[d, "T_\lambda\otimes\iota"'] & A^B \otimes {}^B A_B \otimes {}_B A \arrow[d, "m^{\mathrm{op}}\otimes\iota"] \\
A^C \otimes {}_C A^C \otimes {}_C A \arrow[r, "\iota\otimes T_\rho^\dagger"] & A^C \otimes {}_C A_B \otimes {}_B A \arrow[r, "\iota\otimes m"] & A^C \otimes {}_C A
\end{tikzcd}$$

The right cell commutes by (2.4). To see that the left cell commutes, use commutativity of the first diagram in (5.11), commutativity of (2.11), and surjectity of the map S_ρ. By Lemma 5.6, the lower composition maps $a \otimes b \otimes c$ to $a \otimes {}_C\varepsilon(b)c = t_C({}_C\varepsilon(b))a \otimes c$, and now a similar argument as above shows that $T_\rho \circ T_\rho^\dagger = \iota$.

A similar argument shows that the map ${}_\lambda T$ is invertible. □

Summarising, we find:

5.8. **Theorem.** *Let $\mathcal{A}$ be a multiplier bialgebroid. Then the following two conditions are equivalent:*

(1) *$\mathcal{A}$ has an antipode.*
(2) *$\mathcal{A}$ has counits and its canonical maps T_ρ and ${}_\lambda T$ are bijective.*

If these conditions hold, then the antipode is unique and the following diagrams commute:

$$\begin{tikzcd} A^C \otimes {}^C A \arrow[r, "\iota\otimes S"] \arrow[d, "{}_\rho T"'] & A^C \otimes {}_C A \arrow[d, "T_\rho^{-1}"] \\ A_B \otimes {}^B A \arrow[r, "\iota\otimes S"] & A_B \otimes {}_B A, \end{tikzcd} \qquad \begin{tikzcd} A^B \otimes {}^B A \arrow[r, "S\otimes\iota"] \arrow[d, "T_\lambda"'] & A_B \otimes {}^B A \arrow[d, "{}_\lambda T^{-1}"] \\ A^C \otimes {}_C A \arrow[r, "S\otimes\iota"] & A_C \otimes {}_C A. \end{tikzcd} \tag{5.13}$$

In Sweedler notation, commutativity of (5.13) amounts to the equations

$$T_\rho^{-1}(a \otimes S(b)) = a_{[1]} \otimes S(ba_{[2]}), \qquad {}_\lambda T^{-1}(S(a) \otimes b) = S(b_{(1)}a) \otimes b_{(2)}$$

for all $a, b \in A$.

Proof of Theorem 5.8. Propositions 5.3 and 5.7 imply equivalence of (1) and (2). Suppose that both conditions hold. To see that the first diagram above commutes, use (5.11) and the fact that S is anti-multiplicative. Similar arguments imply that the second diagram commutes as well. □

We adopt the following terminology:

5.9. **Definition.** *A* multiplier Hopf algebroid *is a multiplier bialgebroid with an antipode.*

In the next result, we use the flip maps defined in (4.8) and their inverses, but omit subscripts to simplify notation.

5.10. **Proposition.** *Let $\mathcal{A}$ be a multiplier Hopf algebroid with antipode S. Then the following diagrams commute:*

$$\begin{tikzcd} A_C \otimes {}_C A \arrow[r, "\Sigma"] \arrow[d, "{}_\lambda T"'] & A^B \otimes {}^B A \arrow[r, "T_\lambda"] & A^C \otimes {}_C A \\ A_B \otimes {}^B A \arrow[rr, "\iota\otimes S"'] & & A_B \otimes {}_B A \arrow[u, "T_\rho"'] \end{tikzcd} \qquad \begin{tikzcd} A_B \otimes {}_B A \arrow[r, "\Sigma"] \arrow[d, "T_\rho"'] & A^C \otimes {}^C A \arrow[r, "{}_\rho T"] & A_B \otimes {}^B A, \\ A^C \otimes {}_C A \arrow[rr, "S\otimes\iota"'] & & A_C \otimes {}_C A. \arrow[u, "{}_\lambda T"'] \end{tikzcd}$$

Proof. We only prove commutativity of the first diagram. Using Sweedler notation, we can use (5.6), (5.8) and (3.3) to conclude that

$$\begin{aligned} a_{(1)}(b_{[1]})_{(1)} \otimes a_{(2)}(b_{[1]})_{(2)}S(b_{[2]}) &= a_{(1)}b_{(1)} \otimes a_{(2)}(b_{(2)})_{[1]}S((b_{(2)})_{[2]}) \\ &= a_{(1)}b_{(1)} \otimes a_{(2)\,C}\varepsilon(b_{(2)}) \\ &= a_{(1)}t_C({}_C\varepsilon(b_{(2)}))b_{(1)} \otimes a_{(2)} = a_{(1)}b \otimes a_{(2)} \end{aligned}$$

for all $a,b \in A$. More formally, consider following diagram,

$$\begin{tikzcd}
 & A_B \otimes {}^B A^C \otimes {}_C A \arrow[r, "\iota\otimes S\otimes\iota"] \arrow[dr, "\iota\otimes S_\rho"'] & A_B \otimes {}_B A_C \otimes {}_C A \arrow[r, "T_\rho\otimes\iota"] \arrow[d, "\iota\otimes m_C"] & A^C \otimes {}_C A_C \otimes {}_C A \arrow[d, "\iota\otimes m_C"] \\
A_C \otimes {}_C A^C \otimes {}_C A \arrow[ur, "{}_\lambda T\otimes\iota"] \arrow[r, "\iota\otimes T_\rho^{-1}"] \arrow[dr, "\Sigma\otimes\iota"'] & A_C \otimes {}_C A_B \otimes {}_B A \arrow[r, "m_C\otimes\iota"'] & A_B \otimes {}_B A \arrow[r, "T_\rho"'] & A^C \otimes {}_C A, \\
 & A^{C,B} \otimes {}^B A \otimes {}_C A \arrow[r, "T_\lambda\otimes\iota"'] & A^C \otimes {}_C A_C \otimes {}_C A \arrow[ur, "\iota\otimes m_C"'] &
\end{tikzcd}$$

where m_C denotes the multiplication map from $A_C \otimes {}_C A$ to A. The lower cell commutes by (2.7) and (2.4), the upper left cell by (5.11), and the upper right cell by (2.5). Hence, the entire diagram commutes, showing that $(T_\rho \otimes \iota)(\iota \otimes S)({}_\lambda T \otimes \iota) = (T_\lambda \otimes \iota)(\Sigma \otimes \iota)$. $\square$

6. The regular case

The antipode of a multiplier Hopf algebroid turns out to be invertible if and only if a certain co-opposite multiplier bialgebroid is a multiplier Hopf algebroid as well or, equivalently, if all four canonical maps are bijective and some minor technical condition holds. We prove the equivalence of these conditions and derive further relations between the canonical maps and the antipode, most importantly, that the antipode reverses the comultiplications.

Given a multiplier bialgebroid $(A, B, C, t_B, t_C, \Delta_B, \Delta_C)$, define ${}_C I, I^C \subseteq C$ by

$$\begin{aligned} {}_C I &:= \operatorname{span}\{\psi(a) : \psi \in \operatorname{Hom}({}_C A, {}_C C), a \in A\}, \\ I^C &:= \operatorname{span}\{\phi(a) : \phi \in \operatorname{Hom}(A^C, C_C), a \in A\} \end{aligned}$$

as before, and similarly define $I_B, {}^B I \subseteq B$.

6.1. **Definition.** *We call a multiplier bialgebroid* $(A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ *a* regular multiplier Hopf algebroid *if the following conditions hold:*

1. *the subspaces $t_C({}_C I)A$, $I^C A$, $At_B([A_B])$ and $A[{}^B A]$ are equal to A;*
2. *the canonical maps $T_\lambda, T_\rho, {}_\lambda T, {}_\rho T$ are bijective.*

This terminology is justified:

6.2. **Remark.** Every regular multiplier Hopf algebroid has counits by Proposition 3.10 and hence is a multiplier Hopf algebroid by Theorem 5.8. Conversely, if $\mathcal{A}$ is a multiplier Hopf algebroid, then condition (1) holds by Lemma 3.8 and the right-handed counterpart, and the maps ${}_\lambda T$ and T_ρ are bijective, but not necessarily T_λ nor ${}_\rho T$.

To establish the main result stated above, we will make use of the following four-fold symmetry of multiplier bialgebroids.

Let $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ be a multiplier bialgebroid. Write $(A^{\mathrm{co}})^B$, ${}_B(A^{\mathrm{co}})$ and $(A^{\mathrm{co}})_C$, ${}^C(A^{\mathrm{co}})$ for A, regarded as a B-module or C-module such that

$$a \cdot x := t_C^{-1}(x)a, \qquad x \cdot a := xa, \qquad a \cdot y := ay, \qquad y \cdot a := at_B^{-1}(y)$$

for all $a \in A$, $x \in B$, $y \in C$. Then we can define flip maps

$$\Sigma_{(A^C, {}_C A)} \colon A^C \otimes {}_C A \to (A^{\mathrm{co}})^B \otimes {}_B(A^{\mathrm{co}}), \quad \Sigma_{(A_B, {}^B A)} \colon A_B \otimes {}^B A \to (A^{\mathrm{co}})_C \otimes {}^C(A^{\mathrm{co}})$$

and homomorphisms

$$(\Delta_C)^{\mathrm{co}}\colon A \to (A^{\mathrm{co}})^B \overline{\times}_B (A^{\mathrm{co}}), \qquad ((\Delta_C)^{\mathrm{co}}(a))(b\otimes c) = \Sigma_{(A^C, {}_CA)}(\Delta_C(a)(c\otimes b)),$$

$$(\Delta_B)^{\mathrm{co}}\colon A \to (A^{\mathrm{co}})_C \overline{\times}^C (A^{\mathrm{co}}), \qquad (b\otimes c)((\Delta_B)^{\mathrm{co}}(a)) = \Sigma_{(A_B, {}^BA)}((c\otimes b)\Delta_B(a)).$$

By Proposition 2.7 and the right-handed analogue, the tuple

$$\mathcal{A}^{\mathrm{co}} := (A, C, B, t_B^{-1}, t_C^{-1}, (\Delta_B)^{\mathrm{co}}, (\Delta_C)^{\mathrm{co}})$$

is a multiplier bialgebroid again. We call it the *co-opposite* of $\mathcal{A}$.

Next, denote by $C^{\mathrm{op}}, B^{\mathrm{op}}$ the images of C and B under the canonical identification $M(A)^{\mathrm{op}} \cong M(A^{\mathrm{op}})$, regard t_C and t_B as anti-isomorphisms between C^{op} and B^{op}, denote by $a \mapsto a^{\mathrm{op}}$ the canonical anti-isomorphism from A to A^{op}, and write $(A^{\mathrm{op}})_{B^{\mathrm{op}}}$, ${}^{B^{\mathrm{op}}}(A^{\mathrm{op}})$ and $(A^{\mathrm{op}})^{C^{\mathrm{op}}}$, ${}_{C^{\mathrm{op}}}(A^{\mathrm{op}})$ for A^{op}, regarded as a B^{op}-module or C^{op}-module such that

$$a^{\mathrm{op}}\cdot x^{\mathrm{op}} = (xa)^{\mathrm{op}}, \quad x^{\mathrm{op}}\cdot a^{\mathrm{op}} = (t_C^{-1}(x)a)^{\mathrm{op}}, \quad a^{\mathrm{op}}\cdot y^{\mathrm{op}} = (at_B^{-1}(y))^{\mathrm{op}}, \quad y^{\mathrm{op}}\cdot a^{\mathrm{op}} = (ay)^{\mathrm{op}}$$

for all $x \in B$, $y \in C$, $a \in A$. Then the map $a\otimes b \mapsto a^{\mathrm{op}}\otimes b^{\mathrm{op}}$ descends to isomorphisms

$$A^C \otimes {}_CA \to (A^{\mathrm{op}})_{B^{\mathrm{op}}} \otimes {}^{B^{\mathrm{op}}}(A^{\mathrm{op}}), \qquad A_B \otimes {}^BA \to (A^{\mathrm{op}})^{C^{\mathrm{op}}} \otimes {}_{C^{\mathrm{op}}}(A^{\mathrm{op}}),$$

which we write as $w \mapsto w^{(\mathrm{op}\otimes\mathrm{op})}$. Using these isomorphisms, we define homomorphisms

$$(\Delta_C)^{\mathrm{op}}\colon A^{\mathrm{op}} \to (A^{\mathrm{op}})_{B^{\mathrm{op}}} \overline{\times}^{B^{\mathrm{op}}} (A^{\mathrm{op}}), \quad (b^{\mathrm{op}}\otimes c^{\mathrm{op}})((\Delta_C)^{\mathrm{op}}(a^{\mathrm{op}})) = (\Delta_C(a)(b\otimes c))^{\mathrm{op}\otimes\mathrm{op}},$$

$$(\Delta_B)^{\mathrm{op}}\colon A^{\mathrm{op}} \to (A^{\mathrm{op}})^{C^{\mathrm{op}}} \overline{\times}_{C^{\mathrm{op}}} (A^{\mathrm{op}}), \quad ((\Delta_B)^{\mathrm{op}}(a^{\mathrm{op}})(b^{\mathrm{op}}\otimes c^{\mathrm{op}}) = ((b\otimes c)\Delta_B(a))^{\mathrm{op}\otimes\mathrm{op}}.$$

Using Proposition 4.3, one verifies that

$$\mathcal{A}^{\mathrm{op}} = (A^{\mathrm{op}}, B^{\mathrm{op}}, C^{\mathrm{op}}, t_C^{-1}, t_B^{-1}, (\Delta_C)^{\mathrm{op}}, (\Delta_B)^{\mathrm{op}})$$

is a multiplier bialgebroid again. We call it the *opposite* of $\mathcal{A}$.

Composing the two constructions, we obtain the *bi-opposite* multiplier bialgebroid

$$\mathcal{A}^{\mathrm{op,co}} = (A^{\mathrm{op}}, C^{\mathrm{op}}, B^{\mathrm{op}}, t_C, t_B, (\Delta_C)^{\mathrm{op,co}}, (\Delta_B)^{\mathrm{op,co}}).$$

In Sweedler notation,

$$(\Delta_C)^{\mathrm{co}}(a) = a_{(2)}\otimes a_{(1)}, \quad (\Delta_B)^{\mathrm{op}}(a^{\mathrm{op}}) = a^{\mathrm{op}}_{[1]}\otimes a^{\mathrm{op}}_{[2]}, \quad (\Delta_B)^{\mathrm{op,co}}(a^{\mathrm{op}}) = a^{\mathrm{op}}_{[2]}\otimes a^{\mathrm{op}}_{[1]},$$

$$(\Delta_B)^{\mathrm{co}}(a) = a_{[2]}\otimes a_{[1]}, \quad (\Delta_C)^{\mathrm{op}}(a^{\mathrm{op}}) = a^{\mathrm{op}}_{(1)}\otimes a^{\mathrm{op}}_{(2)}, \quad (\Delta_C)^{\mathrm{op,co}}(a^{\mathrm{op}}) = a^{\mathrm{op}}_{(2)}\otimes a^{\mathrm{op}}_{(1)}.$$

If ${}_C\varepsilon$ and ε_B are a left and a right counit of $\mathcal{A}$, then left and right counits of $\mathcal{A}^{\mathrm{co}}$, $\mathcal{A}^{\mathrm{op}}$ and $\mathcal{A}^{\mathrm{op,co}}$ are given by

$$({}_C\varepsilon)^{\mathrm{co}} = S_C\circ {}_C\varepsilon, \qquad (\varepsilon_B)^{\mathrm{op}} = S_B\circ\varepsilon_B, \qquad (\varepsilon_B)^{\mathrm{op,co}} = \varepsilon_B, \tag{6.1}$$

$$(\varepsilon_B)^{\mathrm{co}} = S_B\circ\varepsilon_B, \qquad ({}_C\varepsilon)^{\mathrm{op}} = S_C\circ {}_C\varepsilon, \qquad ({}_C\varepsilon)^{\mathrm{op,co}} = {}_C\varepsilon, \tag{6.2}$$

respectively, as one can easily check using Propositions 2.7 and 4.3.

The proof of the following result is straightforward and left to the reader.

6.3. **Lemma.** *Let $\mathcal{A}$ be a multiplier bialgebroid.*

(1) *If S is an antipode for $\mathcal{A}$, then S is an antipode for $\mathcal{A}^{\mathrm{op,co}}$.*
(2) *If $\mathcal{A}$ is a regular multiplier Hopf algebroid, then so are $\mathcal{A}^{\mathrm{co}}$, $\mathcal{A}^{\mathrm{op}}$ and $\mathcal{A}^{\mathrm{op,co}}$.*
(3) *If S is an invertible antipode for $\mathcal{A}$, then the inverse S^{-1} is an invertible antipode for $\mathcal{A}^{\mathrm{co}}$ and for $\mathcal{A}^{\mathrm{op}}$.*

We can now state and the second main result of this section.

6.4. **Theorem.** *Let $\mathcal{A}$ be a multiplier bialgebroid. Then the following conditions are equivalent:*

(1) *$\mathcal{A}$ has an invertible antipode;*
(2) *$\mathcal{A}$ is a regular multiplier Hopf algebroid;*
(3) *$\mathcal{A}$ is a multiplier Hopf algebroid and $\mathcal{A}^{\mathrm{op}}$ is a multiplier Hopf algebroid;*
(4) *$\mathcal{A}$ is a multiplier Hopf algebroid and $\mathcal{A}^{\mathrm{co}}$ is a multiplier Hopf algebroid.*

Proof. (1)⇒(2): By Theorem 5.8, T_ρ and ${}_\lambda T$ are invertible. Lemma 6.3, and the same theorem, applied to $\mathcal{A}^{\mathrm{op}}$ or $\mathcal{A}^{\mathrm{co}}$, imply that ${}_\rho T$ and T_λ are invertible as well. Condition (1) in Definition 6.1 holds by Lemma 3.8.

(2)⇒(3): Use Lemma 6.3 and apply Remark 6.2 to $\mathcal{A}$ and to $\mathcal{A}^{\mathrm{op}}$.

(3)⇒(4): By Lemma 6.3, $\mathcal{A}^{\mathrm{co}} = (\mathcal{A}^{\mathrm{op}})^{\mathrm{op,co}}$ is a multiplier Hopf algebroid.

(4)⇒(1): Denote by S and by S^{co} the antipodes of $\mathcal{A}$ and of $\mathcal{A}^{\mathrm{co}}$, respectively, and write $S_\rho(a \otimes b) = S(a)b$ for all $a, b \in A$ as before. Then the following diagram commutes,

$$\begin{tikzcd}
A^C \otimes {}_C A_B \otimes {}_B A \arrow[r, "\iota\otimes m_B"] \arrow[d, "T_\lambda^{-1}\otimes\iota"'] & A^C \otimes {}_C A \arrow[r, "S_\rho"] & A \\
A^B \otimes {}^B A_B \otimes {}_B A \arrow[r, "\iota\otimes T_\rho"] \arrow[dr, "\iota\otimes\varepsilon_B\otimes\iota"] & A^B \otimes {}^B A^C \otimes {}_C A \arrow[r, "\iota\otimes S_\rho"] \arrow[u, "m_B^{\mathrm{op}}\otimes\iota"'] & A^B \otimes {}_B A \arrow[u, "S_\rho"'] \\
 & A^B \otimes {}_B B_B \otimes {}_B A \arrow[ur] &
\end{tikzcd}$$

showing that $S(a)bc = S(S^{\mathrm{co}}(b)a)c = S(a)S(S^{\mathrm{co}}(b))c$ for all $a, b, c \in A$. We first conclude $S(S^{\mathrm{co}}(A)A) = S(A)A = A$, whence $S(A) = A$, and $S(S^{\mathrm{co}}(b)) = b$, and then by symmetry $S^{\mathrm{co}}(A) = A$ and $S^{\mathrm{co}}(S(b)) = b$. Thus, $S \circ S^{\mathrm{co}} = \iota_A$, and likewise $S^{\mathrm{co}} \circ S = \iota_A$. □

The canonical maps and the antipode satisfy the following useful relations.

6.5. **Corollary.** *Let $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ be a regular multiplier Hopf algebroid with antipode S and canonical maps $T_\lambda, T_\rho, {}_\lambda T, {}_\rho T$. Then the following diagrams commute, where we omitted the subscripts on Σ for better legibility:*

$$\begin{tikzcd}
A_C \otimes {}_C A \arrow[rr, "(S\otimes\iota)\Sigma"] \arrow[dd, "{}_\lambda T"'] \arrow[dr, "T_\lambda\Sigma"] & & A_B \otimes {}^B A \arrow[dd, "{}_\rho T^{-1}"] \\
 & A^C \otimes {}_C A \arrow[dr, "\Sigma T_\rho^{-1}"'] & \\
A_B \otimes {}^B A \arrow[rr, "(S\otimes\iota)\Sigma"'] & & A^C \otimes {}^C A
\end{tikzcd}
\qquad
\begin{tikzcd}
A_B \otimes {}_B A \arrow[rr, "(\iota\otimes S)\Sigma"] \arrow[dd, "T_\rho"'] \arrow[dr, "{}_\rho T\Sigma"] & & A^C \otimes {}_C A \arrow[dd, "T_\lambda^{-1}"] \\
 & A_B \otimes {}^B A \arrow[dr, "\Sigma {}_\lambda T^{-1}"'] & \\
A^C \otimes {}_C A \arrow[rr, "(\iota\otimes S)\Sigma"'] & & A^B \otimes {}^B A
\end{tikzcd}$$

Proof. The lower left triangles commute by Proposition 5.10 in the first square commutes. The same result applied to $\mathcal{A}^{\mathrm{co}}$ implies that the remaining triangles commute. □

6.6. **Remark.** In Sweedler notation, commutativity of the first diagram in Proposition 6.5 amounts to equivalence of the following conditions for arbitrary elements $a_i, b_i, c_j, d_j \in A$,

(1) $\sum_i S(b_i) \otimes a_i = \sum_j c_{j[1]} \otimes d_j c_{j[2]}$,
(2) $\sum_i a_{i(1)} b \otimes a_{i(2)} = \sum_j d_{j(1)} \otimes d_{j(2)} c_j$,
(3) $\sum_i a_i b_{i[1]} \otimes b_{i[2]} = \sum_j d_j \otimes S^{-1}(c_j)$,

and commutativity of the second diagram amounts to equivalence of the following conditions,

(1') $\sum_i b_i \otimes S(a_i) = \sum_j d_{j(1)} c_j \otimes d_{j(2)}$,
(2') $\sum_i b_{[1]} \otimes ab_{[2]} = \sum_j d_j c_{j[1]} \otimes c_{j[2]}$,
(3') $\sum_i a_{i(1)} \otimes a_{i(2)} b_i = \sum_j S^{-1}(d_j) \otimes c_j$.

The antipode does not only reverse the multiplication, but also the comultiplication:

6.7. **Proposition.** *Let $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ be a regular multiplier Hopf algebroid with antipode S and canonical maps $T_\lambda, T_\rho, {}_\lambda T, {}_\rho T$. Then the following diagrams commute, where we omitted the subscripts on Σ for better legibility:*

$$\begin{tikzcd}
A^B \otimes {}^B A \arrow[r, "\Sigma(S\otimes S)"] \arrow[d, "T_\lambda"'] & A^C \otimes {}^C A \arrow[d, "{}_\rho T"] \\
A^C \otimes {}_C A \arrow[r, "\Sigma(S\otimes S)"'] & A_B \otimes {}^B A
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
A_C \otimes {}_C A \arrow[r, "\Sigma(S\otimes S)"] \arrow[d, "{}_\lambda T"'] & A_B \otimes {}_B A \arrow[d, "T_\rho"] \\
A_B \otimes {}^B A \arrow[r, "\Sigma(S\otimes S)"'] & A^C \otimes {}_C A,
\end{tikzcd}$$

In Sweedler notation, commutativity of the diagrams above amounts to the relations

$$S(b)_{[1]} \otimes S(a)S(b)_{[2]} = S(b_{(2)}) \otimes S(b_{(1)}a) = S(b_{(2)}) \otimes S(a)S(b_{(1)}),$$
$$S(b)_{(1)} \otimes S(b)_{(2)}S(a) = S(b_{[2]}) \otimes S(ab_{[1]}) = S(b_{[2]}) \otimes S(b_{[1]})S(a)$$

for all $a, b \in A$. Note that in the non-unital case, the expressions on the right hand side require a suitable interpretation, which is given by the expressions in the middle.

Proof of Proposition 6.7. Combining the preceding result with the diagrams (5.13), we find that the following diagram and hence the first square commute:

$$\begin{tikzcd}
A^B \otimes {}^B A \arrow[r, "S\otimes\iota"] \arrow[d, "T_\lambda"'] & A_B \otimes {}^B A \arrow[r, "(S\otimes\iota)\Sigma"] & A^C \otimes {}^C A \arrow[d, "{}_\rho T"] \\
A^C \otimes {}_C A \arrow[r, "S\otimes\iota"'] & A_C \otimes {}_C A \arrow[r, "(S\otimes\iota)\Sigma"'] \arrow[u, "{}_\lambda T"] & A_B \otimes {}^B A
\end{tikzcd}$$

To obtain the second square, apply the same argument to $\mathcal{A}^{\mathrm{co}}$. $\square$

The definition of an isomorphism between multiplier bialgebroids is straightforward and left to the reader.

6.8. **Corollary.** *The antipode of a multiplier Hopf algebroid $\mathcal{A}$ is an isomorphism between $\mathcal{A}$ and the bi-opposite $\mathcal{A}^{\mathrm{op,co}}$. In particular, the counits and antipode of $\mathcal{A}$ are related by $S_C \circ {}_C\varepsilon = \varepsilon_B \circ S$ and $S_B \circ \varepsilon_B = {}_C\varepsilon \circ S$.*

Proof. The first assertion follows easily from Proposition 6.7, and implies that the composition $S_B \circ \varepsilon_B \circ S^{-1}$ is a left counit and $S_C \circ {}_C\varepsilon \circ S^{-1}$ is a right counit of $\mathcal{A}^{\mathrm{op,co}}$. But by (6.1), the counits of $\mathcal{A}^{\mathrm{op,co}}$ are just ${}_C\varepsilon$ and ε_B, respectively. $\square$

Let us finally comment on the relation to Hopf algebroids.

6.9. **Proposition.** *Let $\mathcal{A}$ be a unital regular multiplier Hopf algebroid with antipode S. Then the left and the right bialgebroid associated to $\mathcal{A}_B$ and $\mathcal{A}_C$, respectively, form a Hopf algebroid. Conversely, every Hopf algebroid with invertible antipode arises this way from a unital regular multiplier Hopf algebroid.*

Proof. Use Proposition 3.3 and its right-handed analogue, and note that the conditions (1) and (2) in Definition 5.2 are equivalent to conditions (iii) and (iv) of Definition 4.1 in [1]. □

7. Special cases and examples

To keep this article moderately sized, we only discuss a few special cases and examples. Further examples related to dynamical quantum groups [27], crossed products for braided-commutative Yetter-Drinfeld algebras [6], and Pontrjagin duality can be found in [28] and [29].

7.1. **Multiplier Hopf algebroids associated with weak multiplier Hopf algebras.** Weak multiplier Hopf algebras were introduced by the second author and Wang in [36], [37], [38] as non-unital versions of weak Hopf algebras. The precise relation between regular weak multiplier Hopf algebras and regular multiplier Hopf algebroids is studied in [30]. Briefly, one can associate to every regular weak multiplier Hopf algebra a regular multiplier Hopf algebroid as follows.

A *weak multiplier Hopf algebra* as defined in [38, Definition 1.14] consists of a non-degenerate, idempotent algebra A and a homomorphism $\Delta\colon A \to M(A \otimes A)$ satisfying the following conditions:

(1) for all $a, b \in A$, the elements $(a \otimes 1)\Delta(A)$ and $\Delta(a)(1 \otimes b)$ belong to $A \otimes A$;
(2) Δ is coassociative in the sense that for all $a, b, c \in A$,
$$(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c);$$
(3) Δ is full in the sense that there are no strict subspaces $V, W \subset A$ satisfying
$$\Delta(A)(1 \otimes A) \subseteq V \otimes A \qquad \text{or} \qquad (A \otimes 1)\Delta(A) \subseteq A \otimes W;$$
(4) there exists a linear map $\varepsilon\colon A \to \mathbb{C}$ called the *counit* such that for all $a, b \in A$,
$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab = (\iota \otimes \varepsilon)((a \otimes 1)\Delta(b));$$
(5) there exists an idempotent $E \in M(A \otimes A)$ such that
$$\Delta(A)(1 \otimes A) = E(A \otimes A) \qquad \text{and} \qquad (A \otimes 1)\Delta(A) = (A \otimes A)E,$$
(6) the idempotent E in condition (4) satisfies
$$(\Delta \otimes \iota)(E) = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1) = (\iota \otimes \Delta)(E),$$
where $\Delta \otimes \iota$ and $\iota \otimes \Delta$ are extended to homomorphisms $M(A \otimes A) \to M(A \otimes A \otimes A)$ such that $1 \mapsto E \otimes 1$ or $1 \mapsto 1 \otimes E$, respectively;
(7) the kernels of the linear maps
$$T_1\colon A \otimes A \to A \otimes A, \quad a \otimes b \mapsto \Delta(a)(1 \otimes b),$$
$$T_2\colon A \otimes A \to A \otimes A, \quad a \otimes b \mapsto (a \otimes 1)\Delta(b),$$
are given by
$$\ker T_1 = (1 - G_1)(A \otimes A) \quad \text{and} \quad \ker T_2 = (1 - G_2)(A \otimes A),$$

where $G_1, G_2 \colon A \otimes A \to A \otimes A$ are characterized by

$$(G_1 \otimes \iota)(\Delta_{13}(a)(1 \otimes b \otimes c)) = \Delta_{13}(a)(1 \otimes E)(1 \otimes b \otimes c),$$
$$(\iota \otimes G_2)((a \otimes b \otimes 1)\Delta_{13}(c)) = (a \otimes b \otimes 1)(E \otimes 1)\Delta_{13}(c)$$

for all $a, b, c \in A$, see [38, Proposition 1.11].

Given a weak multiplier Hopf algebra (A, Δ) as above, there exists an antipode, which is a linear map $S \colon A \to M(A)$ such that the maps

$$R_1 \colon A \otimes A \to A \otimes A, \quad a \otimes b \mapsto a_{(1)} \otimes S(a_{(2)})b,$$
$$R_2 \colon A \otimes A \to A \otimes A, \quad a \otimes b \mapsto aS(b_{(1)}) \otimes b_{(2)}$$

are well-defined and satisfy $T_i R_i T_i = T_i$ and $R_i T_i R_i = R_i$ for $i = 1, 2$; see [38, Propositions 2.4, 2.7]. Using this antipode, one defines source and target maps $\varepsilon_s, \varepsilon_t \colon A \to M(A)$ by

$$\varepsilon_s(a) = S(a_{(1)})a_{(2)}, \qquad \varepsilon_t(a) = a_{(1)}S(a_{(2)}).$$

Let (A, Δ) be a weak multiplier Hopf algebra (A, Δ). Then Δ is *regular* if $\Delta(A)(A \otimes 1)$ and $(1 \otimes A)\Delta(A)$ lie in $A \otimes A$ [38, Definition 1.1], and (A, Δ) is *regular* if the antipode S is bijective [38, Theorem 4.10].

7.1. **Theorem.** *Let (A, Δ) be a weak multiplier Hopf algebra, where Δ is regular. Then there exists a multiplier Hopf algebroid $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ such that*

$$B = \varepsilon_s(A), \qquad C = \varepsilon_t(A), \qquad t_B = S^{-1} \circ \iota_B, \qquad t_C = S^{-1} \circ \iota_C$$

and, denoting by $\pi_C \colon A \otimes A \to A^C \otimes_C A$ and $\pi_B \colon A \otimes A \to A_B \otimes {}^B A$ the natural quotient maps,

$$\Delta_C(a)(1 \otimes b) = \pi_C(\Delta(a)(1 \otimes b)), \qquad (a \otimes 1)\Delta_B(b) = \pi_B((a \otimes 1)\Delta(b))$$

for all $a, b \in A$. If (A, Δ) is regular, then so is $\mathcal{A}$.

Proof. For the regular case, the assertion is proven in [30, §4]. This proof carries over to the general case. □

In [30, §5], we also give necessary conditions for a regular multiplier Hopf algebroid to arise from a regular weak multiplier Hopf algebra this way.

7.2. **Involutions.** Let us briefly discuss involutions on multiplier bialgebroids and show that they behave with respect to counits and antipodes as one should expect from the theory of (weak) multiplier Hopf algebras [31], [38].

Suppose that $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ is a multiplier bialgebroid and A is a *-algebra, so that $M(A)$ is a *-algebra with respect to the involution given by $T^*a = (aT^*)^*$ and $aT^* = (Ta^*)^*$. Assume that B and C are *-subalgebras of $M(A)$ and that

$$t_B \circ * \circ t_C \circ * = \iota_C, \qquad t_C \circ * \circ t_B \circ * = \iota_B. \tag{7.1}$$

Then the formula $a \otimes b \mapsto a^* \otimes b^*$ defines mutually inverse conjugate-linear maps

$$A_B \otimes {}_B A \rightleftarrows A^C \otimes {}^C A, \qquad A_C \otimes {}_C A \rightleftarrows A^B \otimes {}^B A, \qquad A^C \otimes {}_C A \rightleftarrows A_B \otimes {}^B A,$$

and conjugation by $*\otimes*$ yields mutually inverse conjugate-linear, multiplicative bijections $\mathrm{End}(A^C \otimes {}_C A) \rightleftarrows \mathrm{End}(A_B \otimes {}^B A)$, which restrict to mutually inverse conjugate-linear, anti-multiplicative bijections

$$A^C \overline{\times}_C A \rightleftarrows A_B \overline{\times}^B A.$$

We write these bijections as $T \mapsto T^*$. Then

$$\Delta_C(a^*) = \Delta_B(a)^* \quad \text{for all } a \in A \tag{7.2}$$

if and only if the associated canonical maps satisfy

$$(*\otimes*) \circ T_\lambda = {}_\lambda T \circ (*\otimes*), \qquad\qquad (*\otimes*) \circ T_\rho = {}_\rho T \circ (*\otimes*). \tag{7.3}$$

7.2. **Definition.** *We call a multiplier bialgebroid $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ with an involution on the underlying algebra A a* multiplier *-bialgebroid *if B and C are *-subalgebras of $M(A)$ and the relations* (7.1) *and* (7.2) *hold. If $\mathcal{A}$ is also a multiplier Hopf algebroid, we call $\mathcal{A}$ a multiplier Hopf *-algebroid.*

A multiplier Hopf $*$-algebroid is automatically regular. This follows from Theorem 5.8 and (7.3), but also from the following relation for the antipode:

7.3. **Proposition.** *Let $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ be a multiplier Hopf *-algebroid. Then its left and right counits ${}_C\varepsilon$ and ε_B and its antipode S satisfy*

$$\varepsilon_B \circ * = * \circ S_C \circ {}_C\varepsilon, \qquad {}_C\varepsilon \circ * = * \circ S_B \circ \varepsilon_B, \qquad S \circ * \circ S \circ * = \iota_A.$$

Proof. Denote by $\overline{V}$ the complex-conjugate of a vector space V and by $\overline{f} \colon \overline{V} \to \overline{W}$ the complex-conjugate of a linear map $f \colon V \to W$ of complex vector spaces. Then we obtain a regular multiplier Hopf algebroid $\overline{\mathcal{A}} = (\overline{A}, \overline{B}, \overline{C}, \overline{t_B}, \overline{t_C}, \overline{\Delta_B}, \overline{\Delta_C})$ with counits $\overline{{}_C\varepsilon}, \overline{\varepsilon_B}$ and antipode $\overline{S}$. The relations (7.1) and (7.2) imply that the involution $*$ on A defines an isomorphism from the complex-conjugate $\overline{\mathcal{A}}$ to the opposite $\mathcal{A}^{\mathrm{op}}$ of $\mathcal{A}$. Now, (6.1) and Lemma 6.3 imply $\overline{{}_C\varepsilon} = * \circ ({}_C\varepsilon)^{\mathrm{co}} \circ * = * \circ S_B \circ \varepsilon_B \circ *$, $\overline{\varepsilon_B} = * \circ (\varepsilon_B)^{\mathrm{co}} \circ * = * \circ S_C \circ {}_C\varepsilon \circ *$ and $\overline{S} = * \circ S^{-1} \circ *$, whence the claim follows. □

7.3. **The function algebra and the convolution algebra of an étale groupoid.** Let G be a locally compact, Hausdorff groupoid that is étale in the sense that the source and the target map s and t from G to the space of units G^0 are local homeomorphisms; see, for example, [25]. Then the function algebra and the convolution algebra of G can be endowed with the structure of regular multiplier Hopf algebroids as follows.

Denote by $C_c(G)$ and $C_c(G^0)$ the algebras of compactly supported continuous functions on G and on G^0, respectively, and denote by $s^*, t^* \colon C_c(G^0) \to M(C_c(G))$ the pull-back of functions along s and t, respectively, that is,

$$(t^*(f)w)(\gamma) = f(t(\gamma))w(\gamma), \qquad\qquad (s^*(f)w)(\gamma) = f(s(\gamma))w(\gamma)$$

for all $f \in C_c(G^0)$, $w \in C_c(G)$ and $\gamma \in G$. Let

$$A = C_c(G), \qquad B = s^*(C_c(G^0)), \qquad C = t^*(C_c(G^0)),$$

and denote by t_B, t_C the isomorphisms $B \rightleftarrows C$ mapping $s^*(f)$ to $t^*(f)$ and vice versa. Since G is étale, the natural map $A \otimes A \to C_c(G \times G)$ factorizes to an isomorphism

$$A^C \otimes {}_C A = A_B \otimes {}^B A \to C_c(G\,{}_s\!\times_t G),$$

where $G_s{\times}_tG$ denotes the composable pairs of elements of G. Denote by $\Delta_C, \Delta_B \colon C_c(G) \to M(C_c(G_s{\times}_tG))$ the pull-back of functions along the groupoid multiplication, that is,

$$(\Delta_C(u)(v \otimes w))(\gamma, \gamma') = u(\gamma\gamma')v(\gamma)w(\gamma') = ((v \otimes w)\Delta_B(u))(\gamma, \gamma')$$

for all $u, v, w \in A, \gamma, \gamma' \in G$. The associated canonical maps $T_\lambda = {}_\lambda T$ and $T_\rho = {}_\rho T$ are the transposes of the maps

$$G_s{\times}_tG \to G_t{\times}_tG,\ (\gamma, \gamma') \mapsto (\gamma, \gamma\gamma'), \qquad G_s{\times}_tG \to G_s{\times}_sG,\ (\gamma, \gamma') \mapsto (\gamma\gamma', \gamma),$$

respectively, and therefore bijective, where $G_p{\times}_qG = \{(\gamma, \gamma') \in G \times G : p(\gamma) = q(\gamma')\}$. The tuple $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ is a multiplier Hopf *-algebroid with counits and antipode given by

$$_C\varepsilon(w) = t^*(w|_{G^0}), \qquad \varepsilon_B(w) = s^*(w|_{G^0}), \qquad (S(w))(\gamma) = w(\gamma^{-1})$$

for all $w \in C_c(G)$, as one can easily check. Note that this multiplier Hopf *-algebroid is unital if and only if the groupoid G is compact.

The space $C_c(G)$ can also be regarded as a *-algebra with respect to the convolution product and involution given by

$$(u * v)(\gamma) = \sum_{\gamma=\gamma'\gamma''} u(\gamma')v(\gamma''), \qquad u^*(\gamma) = \overline{u(\gamma^{-1})}.$$

Since G is étale, G^0 is closed and open in G, and the function algebra $C_c(G^0)$ embeds into the convolution algebra $C_c(G)$. Denote by $\hat{A}$ this convolution algebra, let $\hat{B} = \hat{C} = C_c(G^0) \subseteq \hat{A}$ and let $\hat{t}_{\hat{B}} = \hat{t}_{\hat{C}} = \iota_{C_c(G^0)}$. Then the natural map $A \otimes A \to C_c(G \times G)$ factorizes to isomorphisms

$$\hat{A}^{\hat{C}} \otimes {}_{\hat{C}}\hat{A} \to C_c(G_t{\times}_tG), \qquad \hat{A}_{\hat{B}} \otimes {}^{\hat{B}}\hat{A} \to C_c(G_s{\times}_sG).$$

Define $\hat{\Delta}_{\hat{C}} \colon C_c(G) \to \mathrm{End}(C_c(G_t{\times}_tG))$ and $\hat{\Delta}_{\hat{B}} \colon C_c(G) \to \mathrm{End}(C_c(G_s{\times}_sG))^{\mathrm{op}}$ by

$$(\hat{\Delta}_{\hat{C}}(u)(v \otimes w))(\gamma', \gamma'') = \sum_{t(\gamma)=t(\gamma')} u(\gamma)v(\gamma^{-1}\gamma')w(\gamma^{-1}\gamma''),$$
$$((v \otimes w)\hat{\Delta}_{\hat{B}}(u))(\gamma', \gamma'') = \sum_{s(\gamma)=s(\gamma')} v(\gamma'\gamma^{-1})w(\gamma''\gamma^{-1})u(\gamma).$$

Then $\widehat{\mathcal{A}} = (\hat{A}, \hat{B}, \hat{C}, \hat{t}_{\hat{B}}, \hat{t}_{\hat{C}}, \hat{\Delta}_{\hat{B}}, \hat{\Delta}_{\hat{C}})$ is a multiplier Hopf *-algebroid, and its counits and antipode are given by

$$({}_{\hat{C}}\hat{\varepsilon}(w))(\gamma) = \sum_{r(\gamma')=\gamma} w(\gamma'), \quad (\hat{\varepsilon}_{\hat{B}}(w))(\gamma) = \sum_{s(\gamma')=\gamma} w(\gamma'), \quad (\hat{S}(w))(\gamma'') = w(\gamma''^{-1})$$

for all $w \in C_c(G)$, $\gamma \in G^0$ and $\gamma'' \in G$, as one can easily check.

If G is discrete, then $C_c(G)$ can also be regarded as a weak multiplier Hopf algebra with respect to the pointwise multiplication or convolution product, see Examples 1.15 and 1.16 in [38], and then the multiplier Hopf algebroids $\mathcal{A}$ and $\widehat{\mathcal{A}}$ coincide with the ones obtained in Theorem 7.1.

7.4. **The tensor product** $C \otimes B$**.** Let B and C be non-degenerate and idempotent algebras with anti-isomorphisms $S_B \colon B \to C$ and $S_C \colon C \to B$. Then the tensor product $A := C \otimes B$ is non-degenerate and idempotent again. Identify B and C with their images in $M(A)$ under the canonical inclusions and define $\Delta_C \colon A \to \mathrm{End}(A^C \otimes {}_C A)$ and $\Delta_B \colon A \to \mathrm{End}(A_B \otimes {}^B A)^{\mathrm{op}}$ by

$$\Delta_C(y \otimes x)(a \otimes a') = ya \otimes xa', \qquad (a \otimes a')\Delta_B(y \otimes x) = ay \otimes a'x$$

for all $a, a' \in A$, $x \in B$, $y \in C$. Then $\mathcal{A} = (A, B, C, S_C^{-1}, S_B^{-1}, \Delta_B, \Delta_C)$ is a regular multiplier Hopf algebroid with counits and antipode given by

$${}_C\varepsilon(y \otimes x) = yS_B(x), \qquad \varepsilon_B(y \otimes x) = S_C(y)x, \qquad S(y \otimes x) = S_B(x) \otimes S_C(y)$$

for all $x \in B$, $y \in C$. The verification is straightforward, for example, the diagrams (5.7) commute because for all $a \in A$, $x \in B$, $y \in C$,

$$(m_C \circ (S \otimes \iota) \circ T_\rho)((y \otimes x) \otimes a) = S_C(y)xa = \varepsilon_B(y \otimes x)a,$$
$$(m_B \circ (\iota \otimes S) \circ {}_\lambda T)(a \otimes (y \otimes x)) = ayS_B(x) = a{}_C\varepsilon(y \otimes x).$$

If there exists a regular separability idempotent in $M(B \otimes C)$, then the algebra A can be equipped with the structure of a weak multiplier Hopf algebra, see [37], and again the multiplier Hopf algebroid $\mathcal{A}$ is isomorphic to the one obtained in Theorem 7.1.

7.5. **A two-sided crossed product.** The following construction generalizes Example 2.6 in [23], Example 3.4.6 in [1] and the preceding example, and involves actions of regular multiplier Hopf algebras, for which we refer to [8].

Let B and C be non-degenerate, idempotent algebras with anti-isomorphisms $S_B \colon B \to C$ and $S_C \colon C \to B$ again, and let H be a regular multiplier Hopf algebra with a unital left action on C and a unital right action on B such that the following conditions hold:

(1) B and C are H-module algebras, that is, for all $h \in H$, $x, x' \in B$, $y, y' \in C$,

$$(xx') \triangleleft h = (x \triangleleft h_{(1)})(x' \triangleleft h_{(2)}) \quad \text{and} \quad h \triangleright (yy') = (h_{(1)} \triangleright y)(h_{(2)} \triangleright y');$$

(2) if S_H denotes the antipode of H, then for all $x \in B, y \in C, h \in H$,

$$S_B(x \triangleleft h) = S_H(h) \triangleright S_B(x) \qquad \text{and} \qquad S_C(h \triangleright y) = S_C(y) \triangleleft S_H(h).$$

Then the space $A = C \otimes H \otimes B$ becomes a non-degenerate, idempotent algebra with respect to the product

$$(y \otimes h \otimes x)(y' \otimes h' \otimes x') = y(h_{(1)} \triangleright y') \otimes h_{(2)}h'_{(1)} \otimes (x \triangleleft h'_{(2)})x', \tag{7.4}$$

as can be seen using similar arguments as in [8, §5]. The algebras C, H, B embed naturally into $M(A)$. We identify them with their images in $M(A)$, and then the products

$$yhx = y \otimes h \otimes x, \qquad yxh = y \otimes h_{(1)} \otimes (x \triangleleft h_{(2)}), \qquad hyx = (h_{(1)} \triangleright y) \otimes h_{(2)} \otimes x$$

lie in $A \subseteq M(A)$ for all $x \in B$, $y \in C$ and $h \in H$. Define $\Delta_C \colon A \to \mathrm{End}(A^C \otimes {}_C A)$ and $\Delta_B \colon A \to \mathrm{End}(A_B \otimes {}^B A)^{\mathrm{op}}$ by

$$\Delta_C(yhx)(a \otimes a') = yh_{(1)}a \otimes h_{(2)}xa', \qquad (a \otimes a')\Delta_B(yhx) = ayh_{(1)} \otimes a'h_{(2)}x$$

for all $x \in B$, $y \in C$, $h \in H$, $a, a' \in A$. Note that here, the legs of h are covered by a or a', respectively. Then $\mathcal{A} = (A, B, C, S_C^{-1}, S_B^{-1}, \Delta_B, \Delta_C)$ is a regular multiplier Hopf algebroid with counits and antipode given by

$$_C\varepsilon(yxh) = yS_B(x)\varepsilon_H(h), \quad \varepsilon_B(hyx) = S_C(y)x\varepsilon_H(h), \quad S(yhx) = S_B(x)S_H(h)S_C(y)$$

for all $x \in B$, $y \in C$, $h \in H$.

The verification is a bit more tedious than in the previous examples, but straightforward again. For example, for all $x \in B$, $y \in C$, $a \in A$,

$$\begin{aligned}
({}_C\varepsilon \odot \iota)(\widetilde{T_\rho}(yxh \otimes a)) &= y\varepsilon_H(h_{(1)}) \otimes h_{(2)}xa = y \otimes xha \mapsto yxha, \\
(\iota \odot \varepsilon_B)(\widetilde{{}_\lambda T}(a \otimes hyx)) &= ah_{(1)}y \otimes \varepsilon_H(h_{(2)})x = ahy \otimes x \mapsto ahyx, \\
(m \circ (S \otimes \iota) \circ T_\rho)(hxy \otimes a) &= S(h_{(1)}y)h_{(2)}xa = S_C(y)S_H(h_{(1)})h_{(2)}xa = \varepsilon_B(hxy)a, \\
(m \circ (\iota \otimes S) \circ {}_\lambda T)(a \otimes yxh) &= ayh_{(1)}S(xh_{(2)}) = ayh_{(1)}S_H(h_{(2)})S_B(x) = a_C\varepsilon(yxh).
\end{aligned}$$

If there exists a regular separability idempotent in $M(B \otimes C)$ that is compatible with the actions of H on B and C, then the algebra A can also be equipped with the structure of a weak multiplier Hopf algebra, see [37], and again the multiplier Hopf algebroid $\mathcal{A}$ is isomorphic to the one obtained in Theorem 7.1.

Co-commutative, proper and étale multiplier Hopf algebroids. A special class of multiplier Hopf algebroids which includes the convolution algebras of étale Hausdorff groupoids was introduced in [19] under the name *étale Hopf algebroids.* We show that these are precisely the co-commutative and proper multiplier Hopf algebroids. Let us use the notation introduced in the beginning of section 6.

7.4. **Definition.** *A multiplier bialgebroid* $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ *is* co-commutative *if it is equal to its co-opposite* $\mathcal{A}^{\mathrm{co}} = (A, C, B, t_B^{-1}, t_C^{-1}, (\Delta_B)^{\mathrm{co}}, (\Delta_C)^{\mathrm{co}})$.

7.5. **Remarks.** Let $\mathcal{A}$ be a co-commutative multiplier bialgebroid as above.

(1) Evidently $B = C$, and this algebra is commutative.
(2) The maps t_B and t_C are involutive in the sense that $t_B = t_B^{-1}$ and $t_C = t_C^{-1}$. If $\mathcal{A}$ has counits $_C\varepsilon$ and ε_B, then $t_B = \iota_B$ and $t_C = \iota_C$. For example,

$$z_C\varepsilon(a)b = {}_C\varepsilon(za)b = {}_C\varepsilon(a)t_C^{-1}(z)b = t_C^{-1}(z)_C\varepsilon(a)b$$

for all $a, b \in A$ and $z \in C = B$, whence $z = t_C^{-1}(z)$ for all $z \in C$ by Lemma 3.8.
(3) If $\mathcal{A}$ is a multiplier Hopf algebroid, then it is regular by Theorem 6.4, and its antipode S is involutive in the sense that $S^2 = \iota$ by Lemma 6.3.

Recall that a groupoid G is *proper* if the map $G \to G^0 \times G^0$ given by $\gamma \mapsto (t(\gamma), s(\gamma))$ is proper. For a multiplier bialgebroid, we define the corresponding property as follows:

7.6. **Definition.** *A multiplier bialgebroid* $(A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ *is* proper *if* $BC \subseteq A$.

Given a multiplier bialgebroid, we define the Takeuchi products $A^C \times {}_CA \subseteq A^C \otimes {}_CA$ and $A_B \times {}^BA \subseteq A_B \otimes {}^BA$ as in the unital case, see (2.1) and (4.1), respectively, and identify these with subalgebras of $A^C \overline{\times}_C A$ and $A_B \overline{\times}^B A$ in the natural way.

7.7. **Lemma.** *Let $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ be a proper, co-commutative bialgebroid. Then*

$$B = C \subseteq A, \qquad \Delta_C(A) \subseteq A^C \times {}_C A, \qquad \Delta_B(A) \subseteq A_B \times {}^B A.$$

If $\mathcal{A}$ has counits, then they restrict to the identity on $B = C \subseteq A$.

Proof. Clearly, $B = C = BC \subseteq A$. We show that $\Delta_C(A) \subseteq A^C \times {}_C A$. Let $a, b \in A$ and $y \in C$. Then $\Delta_C(ay)(1 \otimes b) = \Delta_C(a)(y \otimes 1)(1 \otimes b)$, and since $\Delta_C(a)(y \otimes 1) \in \Delta_C(A)(A \otimes 1) = A^C \otimes {}_C A$, we can conclude that $\Delta_C(ay) \in A^C \times {}_C A$. But $AC = A$ and hence $\Delta_C(A) \subseteq A^C \times {}_C A$. A similar argument shows that $\Delta_B(A) \subseteq A_B \times {}^B A$. Finally, suppose that ${}_C\varepsilon$ and ε_B are counits for $\mathcal{A}$. Taking $a = y \in C$ in (3.2), we find

$$yb = ({}_C\varepsilon \otimes \iota)(T_\rho(y \otimes b)) = {}_C\varepsilon(y \otimes b) = {}_C\varepsilon(y)b$$

for all $b \in A$ and hence $({}_C\varepsilon)|_C = \iota$. A similar argument shows that $(\varepsilon_B)|_B = \iota$. □

Recall that an *étale Hopf algebroid* [19, 20] consists of

(E1) a total algebra A with a commutative subalgebra $A_0 \subseteq A$ in which A has local units,
(E2) a co-commutative coalgebra structure (Δ, ε) on A, regarded as an A_0-module with respect to right multiplication,
(E3) a linear involution $S \colon A \to A$

such that

(E4) $\varepsilon|_{A_0} = \iota$, and $\varepsilon(a'a) = \varepsilon(\varepsilon(a')a)$ for all $a, a' \in A$;
(E5) $\Delta(y) = y \otimes 1 = 1 \otimes y$ for all $y \in A_0$, and $\Delta(a'a) = \Delta(a')\Delta(a)$ for all $a, a' \in A$;
(E6) $S|_{A_0} = \iota$, and $S(a'a) = S(a)S(a')$ for all $a, a' \in A$;
(E7) if $\Delta(a) = \sum_i a'_i \otimes a''_i$, then $\Delta(S(a)) = \sum_i S(a'_i) \otimes S(a''_i)$;
(E8) $(\iota \otimes S) \circ {}_\lambda T \circ (\iota \otimes S) \circ {}_\lambda T = \iota$, where ${}_\lambda T \colon A_{A_0} \otimes {}_{A_0}A \to A_{A_0} \otimes A_{A_0}$ is given by $a \otimes b \mapsto (a \otimes 1)\Delta(b)$.

7.8. **Proposition.** *Let $\mathcal{A} = (A, B, C, t_B, t_C, \Delta_B, \Delta_C)$ be a proper, co-commutative multiplier Hopf algebroid, where A has local units in B. Denote by ε_B and S its right counit and its antipode, respectively. Then $(A, B, \Delta_B, \varepsilon_B, S)$ is an étale Hopf algebroid. Conversely, every étale Hopf algebroid arises this way.*

Proof. We first show that $(A, B, \Delta_B, \varepsilon_B, S)$ is an étale Hopf algebroid. Lemma 7.7 implies $\Delta_B(A) \subseteq A_B \times {}^B A \subseteq A_B \otimes {}^B A = A_B \otimes A_B$. Clearly, $(\Delta_B, \varepsilon_B)$ forms a co-commutative coalgebra structure on A_B satisfying (E5). Assumption (E4) holds by Lemma 7.7 and (4.11), (E3) and (E6) by Remarks 7.5, (E7) by Proposition 6.7 and (E8) by Theorem 5.8.

Conversely, let $(A, A_0, \Delta, \varepsilon, S)$ be an étale Hopf algebroid. Then (E1), (E2) and (E5) imply that $(A, A_0, \iota_{A_0}, \Delta)$ is a right multiplier bialgebroid, and (E3) and (E6) imply that with $\Delta' := (S \otimes S) \circ \Delta \circ S^{-1}$, the tuple $(A, A_0, \iota_{A_0}, \Delta')$ is a left multiplier bialgebroid. Now, $\mathcal{A} = (A, A_0, A_0, \iota_{A_0}, \iota_{A_0}, \Delta, \Delta')$ satisfies the mixed co-associativity conditions by (E7) and therefore is a multiplier bialgebroid. It is proper by (E1), co-commutative by (E2) and (E7), and its canonical map ${}_\lambda T$ is invertible by (E8). By co-commutativity and definition of Δ', the other three canonical maps of $\mathcal{A}$ are invertible as well. Finally, (E1), (E4) and Theorem 6.4 imply that $\mathcal{A}$ is a multiplier Hopf algebroid. □

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