# Integration on and duality of algebraic quantum groupoids

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## Aim and background

Aim An algebraic approach to quantum groupoids that

1. features a generalized Pontrjagin duality (done)
2. connects to the setting of operator algebras (open)

similar to Van Daele's theory of multiplier Hopf algebras w. integrals

Related work in this direction includes

- the full theory in the finite-dimensional case
  [Böhm-Nill-Szlachányi; Nikshych-Vainerman; Vallin; . . . ]
- integrals on and duality of Hopf algebroids [Böhm-Szlachányi]
  (integration only partial; duality only in fiber-wise finite case)
- integrals on and duality of weak multiplier Hopf algebras
  [Van Daele-Wang]
  (not yet published; base needs to be separable Frobenius)

## Sources of quantum groupoids

Idea: A *quantum groupoid* consists of a *total algebra* $A$, a *base algebra* $B$, *target* and *source maps* $B, B^{\mathrm{op}} \to A$ and a *comultiplication* $\Delta \colon A \to A \underset{B}{*} A$ satisfying certain conditions which depend on the setting.

Examples of quantum groupoids include the following:

- *linking quantum groupoids* for monoidally equivalent quantum groups [De Commer]
- *quantum transformation groupoids* $G \ltimes B$, where $G$ is a quantum group, $B$ a braided-comm. $G$-YD-algebra [Lu, Brzezinski-Militaru]
- *Tannaka-Krein duals* of fiber functors into a category of $B$-bimodules [Hayashi, Day, Street, Hai, Pfeiffer, . . . ]
- *dynamical quantum groups* associated to solutions of the dynamical Yang-Baxter equation [Etingov-Varchenko, . . . ]
- two-sided crossed products $B^{\mathrm{op}} \rtimes G \ltimes B$, where $G$ is a quantum group acting on an algebra $B$



## Plan

1. Regular multiplier Hopf algebroids
   - definition of bialgebroids and regular multiplier Hopf algebroids
   - examples: function and convolution algebra of an étale groupoid
2. Integration
   - ingredients needed
   - main results
   - example: two-sided crossed products of quantum group actions
3. Duality
   - the duality of measured regular multiplier Hopf algebroids
   - example: crossed products of braided-commutative YD-algebras
4. (Passage to operator algebras)

## Background — the notion of a bialgebroid

Definition A *bialgebroid* consists of

- a unital algebra $A$ and commuting unital subalgebras $B, C \subseteq A$ with anti-isomorphisms $B \underset{S}{\overset{S}{\leftrightarrows}} C$
  (we will write $a, b, c, \ldots \in A, \quad x, x', \ldots \in B, \quad y, y', \ldots \in C$)
- a *left* and a *right comultiplication*
  $$\Delta_B \colon A \to {}_BA \otimes {}_{S(B)}A \quad \text{and} \quad \Delta_C \colon A \to A_{S(C)} \otimes A_C$$
  satisfying
  - $\Delta_B(a)(x \otimes 1) = \Delta_B(a)(1 \otimes S(x))$ and multiplicativity
  - $\Delta_B(x) = (1 \otimes x)$, $\Delta_B(y) = (y \otimes 1)$ and co-associativity
  - similar conditions for $\Delta_C$
  - joint co-associativity relating $\Delta_B$ and $\Delta_C$
- a *left counit* ${}_B\varepsilon \colon A \to B$ and a *right counit* $\varepsilon_C \colon A \to C$

Note The inclusions $B \underset{S}{\overset{\mathrm{id}}{\rightrightarrows}} A$ correspond to functors ${}_A\mathrm{Mod} \to {}_B\mathrm{Mod}_B$ and the maps $\Delta_B$, ${}_B\varepsilon$ correspond to compatible monoidal structures on ${}_A\mathrm{Mod}$



## Regular multiplier Hopf algebroids

Definition A *multiplier bialgebroid* consists of

- an algebra $A$ and commuting subalgebras $B, C \subseteq M(A)$ with anti-isomorphisms $B \underset{S}{\overset{S}{\leftrightarrows}} C$, where we assume no units but suitable regularity properties
- a *left* and a *right comultiplication* $\Delta_B$ and $\Delta_C$ which take values in a left and a right multiplier algebra such that
  1. $\Delta_B(a)(1 \otimes b)$ and $\Delta_B(b)(a \otimes 1)$ lie in ${}_BA \otimes {}_{S(B)}A$
  2. $(a \otimes 1)\Delta_C(b)$ and $(1 \otimes b)\Delta_C(a)$ lie in $A_{S(C)} \otimes A_C$
  3. $\Delta_B, \Delta_C$ are co-associative, multiplicative, jointly co-associative

Theorem [T.-Van Daele '13] There exist left and right counits and an antipode if and only if the maps that send $a \otimes b \in A \otimes A$ to the products in 1. and 2. descend to bijections $A \underset{B}{\otimes} A \to {}_BA \otimes {}_{S(B)}A, \ldots$

Definition We call $(A, \Delta_B, \Delta_C)$ a *regular multiplier Hopf algebroid* if both conditions hold.

## Examples coming from étale groupoids

Consider a groupoid $X \underset{t}{\overset{s}{\leftleftarrows}} G \xleftarrow{m} G\,{}_s\!\times_t G$ that is étale in the sense that $s$ and $t$ are local homeomorphisms (with discrete fibers).

Example The function algebra as a multiplier Hopf algebroid:

- $A = C_c(G)$, $B = s^*(C_c(X))$, $C = t^*(C_c(X))$
  ($B$ and $C$ consist of functions that are constant along fibers of $s$ or $t$)
- the maps $B \underset{S}{\overset{S}{\leftrightarrows}} C$ are the transpose of the inversion, $(Sf)(\gamma) = f(\gamma^{-1})$
- $\Delta_B$ and $\Delta_C$ send $C_c(G)$ to $C_b(G\,{}_s\!\times_t G)$ and are transposes of the multiplication, $(\Delta_{B,C} f)(\gamma, \gamma') = f(\gamma\gamma')$

Example The convolution algebra as a multiplier Hopf algebroid:

- $A = C_c(G)$ with convolution and $B = C = C_c(X) \hookrightarrow A$
- $\Delta_B$ and $\Delta_C$ send $C_c(G)$ to $C_c(G_{(s,t)}\!\times_{(s,t)} G)$ and push forward along the diagonal map



## What do we need for integration?

Recall that a *left integral* on a multiplier Hopf algebra is a $\phi$ s.t.

$$(\mathrm{id} \otimes \phi)((a \otimes 1)\Delta(b)) = a\phi(b) \text{ and } (\mathrm{id} \otimes \phi)(\Delta(b)(c \otimes 1)) = \phi(b)c$$

Ansatz For a regular multiplier Hopf algebroid $\mathcal{A}$, we require

- a map ${}_C\phi_C \colon A \to C$ that is *left-invariant*: $\forall a, b, c \in A, y \in C$,
  1. ${}_C\phi_C(ay) = {}_C\phi_C(a)y$ and $(\mathrm{id} \underset{C}{\otimes} {}_C\phi_C)((a \otimes 1)\Delta_C(b)) = a\,{}_C\phi_C(b)$
  2. ${}_C\phi_C(ya) = y\,{}_C\phi_C(a)$ and $(\mathrm{id} \underset{B}{\otimes} {}_C\phi_C)(\Delta_B(b)(c \otimes 1)) = {}_C\phi_C(b)c$
- a map ${}_B\psi_B \colon A \to B$ that is *right-invariant*
- functionals $\mu_B, \mu_C$ on $B, C$ that are *relatively invariant* in the sense that the functionals

$$\phi \colon A \xrightarrow{{}_C\phi_C} C \xrightarrow{\mu_C} \mathbb{C} \quad \text{and} \quad \psi \colon A \xrightarrow{{}_B\psi_B} B \xrightarrow{\mu_B} \mathbb{C}$$

  are related by invertible multipliers $\delta, \delta'$ via $\psi = \phi(\delta -) = \phi(-\delta')$

## Adapted functionals and balanced slice maps

Fix a pair of faithful functionals $\mu = (\mu_B, \mu_C)$ on $B$ and $C$.

Definition A functional $\omega$ on $A$ is *$\mu$-adapted* if we can write

$$\omega = \mu_B \circ {}_B\omega = \mu_B \circ \omega_B = \mu_C \circ {}_C\omega = \mu_C \circ \omega_C$$

with ${}_B\omega \in \mathrm{Hom}({}_BA, {}_BB)$, $\omega_B \in \mathrm{Hom}(A_B, B_B)$, ...

Example $\psi = \mu_B \circ {}_B\psi_B = \mu_C \circ {}_C\phi_C(\delta -) = \mu_C \circ {}_C\phi_C(-\delta')$

In the theory of multiplier Hopf algebras, one frequently uses

- *slice* maps of the form $\upsilon \otimes \mathrm{id}, \mathrm{id} \otimes \omega \colon A \otimes A \to A$ and
- *tensor products* $\upsilon \otimes \omega \colon A \otimes A \to \mathbb{C}$

Key If $\upsilon, \omega$ are $\mu$-balanced, we can form *balanced analogues* $\upsilon \odot \mathrm{id}$, $\mathrm{id} \odot \omega$, $\upsilon \odot \omega$ on all kinds of balanced tensor products $A \odot A$, e.g.,

$$\upsilon \underset{B}{\otimes} \omega \colon A \underset{B}{\otimes} A \to \mathbb{C},\ a \otimes b \mapsto \mu_B(\upsilon_B(a)_B\omega(b)) = \upsilon(a_B\omega(b)) = \omega(\upsilon_B(a)b)$$

$$\text{so } \upsilon \underset{B}{\otimes} \omega = \mu_B \circ (\upsilon_B \otimes {}_B\omega) = \upsilon \circ (\mathrm{id} \otimes {}_B\omega) = \omega \circ (\upsilon_B \otimes \mathrm{id})$$



## The definition of integrals

We defined a functional $\omega$ on $A$ to be $\mu$-adapted if there exist

$${}_B\omega \in \mathrm{Hom}({}_BA, {}_BB),\ \ \omega_B \in \mathrm{Hom}(A_B, B_B),\ \ {}_C\omega,\ \ \omega_C$$

such that $\omega = \mu_B \circ {}_B\omega = \mu_B \circ \omega_B = \mu_C \circ {}_C\omega = \mu_C \circ \omega_C$.

Definition A *left integral* for $(\mathcal{A}, \mu)$ is a $\mu$-adapted functional $\phi$ s.t. ${}_C\phi = \phi_C =: {}_C\phi_C$ is left-invariant. We call $\phi$ *full* if ${}_B\phi(A) = B = \phi_B(A)$. We define *right integrals* and *full right integrals* similarly.

On $\mu = (\mu_B, \mu_C)$, we henceforth impose the following conditions:

1. faithfulness, i.e., if $\mu_B(xB) = 0$ or $\mu_B(Bx) = 0$, then $x \neq 0$
2. $\mu_B \circ S = \mu_C = \mu_B \circ S^{-1}$ and 3. $\mu_B \circ {}_B\varepsilon = \mu_C \circ \varepsilon_C$

Using $B$- and $C$-linearity of ${}_B\varepsilon$ and $\varepsilon_C$ and relation 3., one finds:
Proposition $\mu_B(x'x) = \mu_B(S^2(x)x')$ and $\mu_C(yy') = \mu_C(y'S^2(y))$

## The main results on integrals

Theorem [T.] Let $\mathcal{A}$ be a regular multiplier Hopf algebroid with base weight $\mu$ and full left integral $\phi$, where ${}_BA, A_B, {}_CA, A_C$ are flat.

1. If ${}_BA, A_B, {}_CA, A_C$ are projective, then $\phi$ is faithful.

Assume now that $\phi$ is faithful.

2. There exists a *modular automorphism* $\sigma^\phi$ of $A$ satisfying
$$\phi(ab) = \phi(b\sigma^\phi(a)) \text{ for all } a, b \in A.$$
Moreover, $\sigma^\phi(y) = S^2(y)$ for $y \in C$, and $\sigma^\phi(M(B)) = M(B)$.
3. Every left integral has the form $\phi(x-)$ with $x \in M(B)$.
4. Every right integral has the form $\phi(\delta-)$ with $\delta \in M(A)$.
5. There exist invertible *modular elements* $\delta, \delta^\dagger \in M(A)$ such that $\phi \circ S^{-1} = \phi(\delta-)$ and $\phi \circ S = \phi(-\delta^\dagger)$. These elements satisfy
$$\Delta_C(\delta) = \delta \otimes \delta, \quad \Delta_B(\delta) = \delta^\dagger \otimes \delta, \quad \Delta_B(\delta^\dagger) = \delta^\dagger \otimes \delta^\dagger, \; \Delta_C(\delta^\dagger) = \delta \otimes \delta^\dagger$$
$$S(\delta^\dagger) = \delta^{-1}, \quad \varepsilon(\delta a) = \varepsilon(a) = \varepsilon(a\delta^\dagger), \quad \text{and (in the } {}^*\text{-case) } \delta^\dagger = \delta^*.$$



## Measured regular multiplier Hopf algebroids and their duality

Definition A *measured regular multiplier Hopf algebroid* consists of

- a regular multiplier Hopf algebroid $A$ as above, where the modules ${}_BA, A_B, {}_CA, A_C$ are flat
- base weights $\mu_B, \mu_C$ on $B, C$ that satisfy the conditions above (both are faithful, $\mu_B \circ S = \mu_C = \mu_B \circ S^{-1}$, $\mu_B \circ {}_B\varepsilon = \mu_C \circ \varepsilon_C$)
- a left and a right integral $\phi$ and $\psi$ that are full and faithful

Example Let $G$ be a second countable, étale groupoid with a Radon measure on the unit space which has full support and is *continuously quasi-invariant*. Then the function and the convolution algebra of $G$ become measured regular multiplier Hopf algebroids.

## An example coming from quantum group actions

Example Assume that

- $H$ is a regular (mult.) Hopf algebra with integrals $\phi_H, \psi_H$,
- $B$ is an algebra with a right action of $H$, written $x \triangleleft h$
- $\mu_B$ is a faithful $H$-invariant trace on $B$.

Then $C = B^{\mathrm{op}}$ carries a left $H$-action and an $H$-invariant trace $\mu_C$ s.t.

$$h \triangleright x^{\mathrm{op}} = (x \triangleleft S_H^{-1}(h))^{\mathrm{op}} \quad \text{and} \quad \mu_C(x^{\mathrm{op}}) = \mu_B(x)$$

We obtain a measured regular multiplier Hopf algebroid, where

- $A = C \rtimes H \ltimes B$ is the space $C \otimes H \otimes B$ with the multiplication
$$(y \otimes h \otimes x)(y' \otimes h' \otimes x') = y(h_{(1)} \triangleright y') \otimes h_{(2)}h'_{(1)} \otimes (x \triangleleft h'_{(2)})x'$$
- the left and right comultiplication $\Delta_B$ and $\Delta_C$ are given by
$$\Delta_B(y \otimes h \otimes x)(a \otimes b) = yh_{(1)}a \otimes h_{(2)}xb$$
$$(a \otimes b)\Delta_C(y \otimes h \otimes x) = ayh_{(1)} \otimes bh_{(2)}x$$
- $\phi(y \otimes h \otimes x) = \mu_C(y)\phi_H(h)\mu_B(x), \ \psi(y \otimes h \otimes x) = \mu_C(y)\psi_H(h)\mu_B(x)$



## The dual convolution algebra

Let $(A, \mu, \phi, \psi)$ be a measured regular multiplier Hopf algebroid.

Lemma Consider the space $\hat{A} := \{\phi(a-) : a \in A\} \subseteq \mathrm{Hom}(A, \mathbb{C})$.

1. $\hat{A} = \{\phi(-a) : a \in A\} = \{\psi(a-) : a \in A\} = \{\psi(-a) : a \in A\}$.
2. Let $\upsilon, \omega \in \hat{A}$. Then the compositions
$$\upsilon *_B \omega := (\upsilon \otimes \omega) \circ \Delta_B \quad \text{and} \quad \upsilon *_C \omega := (\upsilon \otimes \omega) \circ \Delta_C$$
(a) are well-defined, (b) belong to $\hat{A}$ and (c) coincide.
3. $\hat{A}$ is a non-degenerate, idempotent algebra w.r.t. $(\upsilon, \omega) \mapsto \upsilon * \omega$.

Proof of assertion 2.(c):

- coassociativity $\Rightarrow (\upsilon *_B \theta) *_C \omega = \upsilon *_B (\theta *_C \omega)$ for all $\mu$-adapted $\theta$
- counit property $\Rightarrow \upsilon *_B \varepsilon = \upsilon$ and $\varepsilon *_C \omega = \omega$
- relations 1.+2. $\Rightarrow \upsilon *_B \omega = \upsilon *_B (\varepsilon *_C \omega) = (\upsilon *_B \varepsilon) *_C \omega = \upsilon *_C \omega$

## The duality of measured regular multiplier Hopf algebroids

Theorem [T.] Let $(A, \mu, \phi, \psi)$ be a m.r.m.H.a. Then there exists a *dual m.r.m.H.a.* $(\hat{A}, \hat{\mu}, \hat{\phi}, \hat{\psi})$, where $\hat{A}$ was defined above and

- $\hat{B} = C$ and $\hat{C} = B$ are embedded in $M(\hat{A})$ such that
$$y\omega = \omega(-y), \ \ \omega y = \omega(-S^{-1}(y)), \ \ x\omega = \omega(S^{-1}(x)-), \ \ \omega x = \omega(x-)$$
for all $y \in C$, $x \in B$, $\omega \in \hat{A}$
- the left and the right comultiplication $\hat{\Delta}_{\hat{B}}$ and $\hat{\Delta}_{\hat{C}}$ of $\hat{A}$ satisfy
$$(\hat{\Delta}_{\hat{B}}(\upsilon)(1 \otimes \omega)|a \otimes b) = (u \otimes \omega|(a \otimes 1)\Delta_C(b))$$
$$((\upsilon \otimes 1)\hat{\Delta}_{\hat{C}}(\omega)|a \otimes b) = (u \otimes \omega|\Delta_B(a)(1 \otimes b))$$
for all $a, b \in A$, $\upsilon, \omega \in \hat{A}$
- the dual counit $\hat{\varepsilon}$, antipode $\hat{S}$ and integrals $\hat{\phi}$ and $\hat{\psi}$ are given by
$$\hat{\varepsilon}(\phi(-a)) = \phi(a), \ \ \hat{S}(\omega) = \omega \circ S, \ \ \hat{\phi}(\psi(a-)) = \varepsilon(a) = \hat{\psi}(\phi(-a))$$

In the $*$-case, $\omega^* = \omega \circ * \circ S$ and $\hat{\psi}(\phi(-a)^*\phi(-a)) = \phi(a^*a)$.

Theorem [T.] Every m.r.m.H.a. is naturally isomorphic to its bidual.



## Outline of the construction of the dual comultiplications

By [T.-Van Daele], a r.m.H.a. is determined by the algebras $A$, $B, C \subseteq M(A)$, the anti-automorphisms $B \leftrightarrows C$, and the bijections
$$T_1 \colon A \underset{B}{\otimes} A \to A \underset{l}{\otimes} A, \ \ a \otimes b \mapsto \Delta_B(a)(1 \otimes b)$$
$$T_2 \colon A \underset{C}{\otimes} A \to A \underset{r}{\otimes} A, \ \ a \otimes b \mapsto (a \otimes 1)\Delta_C(b).$$

Starting from these maps, we obtain

- dual bijections $(T_1)^\vee$ and $(T_2)^\vee$, taking transposes
- various embeddings $\hat{A} \otimes \hat{A} \to (A \otimes A)^\vee$, using the fact that elements of $\hat{A}$ are $\mu$-adapted functionals and forming balanced tensor products
- bijections $\hat{T}_1$, $\hat{T}_2$, which then define the structure of a r.m.H.a. on $\hat{A}$

$$\begin{array}{ccc} (A \underset{r}{\otimes} A)^\vee & \xrightarrow{(T_2)^\vee} & (A \underset{C}{\otimes} A)^\vee \\ \uparrow & & \uparrow \\ \hat{A} \underset{\hat{B}}{\otimes} \hat{A} & \xrightarrow[\hat{T}_1]{} & \hat{A} \underset{\hat{l}}{\otimes} \hat{A} \end{array} \qquad \begin{array}{ccc} (A \underset{l}{\otimes} A)^\vee & \xrightarrow{(T_1)^\vee} & (A \underset{B}{\otimes} A)^\vee \\ \uparrow & & \uparrow \\ \hat{A} \underset{\hat{C}}{\otimes} \hat{A} & \xrightarrow[\hat{T}_2]{} & \hat{A} \underset{\hat{r}}{\otimes} \hat{A} \end{array}$$

## To do: examples from braided-commutative YD-algebras

Theorem [Lu '96; Brzeziński, Militaru '01] Let $B$ be a *braided-commutative Yetter-Drinfeld algebra* over a Hopf algebra $H$. Then the crossed product $A = B \rtimes H$ for the action is a Hopf algebroid.

Theorem [Neshveyev-Yamashita '13] Let $H$ be a compact quantum group.Then there exists an equivalence between

- unital braided-commutative Y.D.-algebras over $H$ and
- unitary tensor functors from Rep($H$) to $C^*$-tensor categories.

If we assume that $H$ is a regular multiplier Hopf algebra with integrals and that $B$ carries a faithful quasi-invariant KMS-functional,we expect $B \rtimes H$ and $B^{\mathrm{op}} \rtimes \hat{H}^{\mathrm{co}}$ to form mutually dual measured multiplier Hopf algebroids.

Theorem [Enock-T. '14] Let $N$ be a braided-commutative Y.D.-von Neumann-algebra over a l.c.q.gp $G$ with an invariant n.s.f. weight. Then $G \ltimes N$, $\hat{G} \ltimes N$ are mutually dual measured quantum groupoids.



## To do: passage to the setting of operator algebras

Let $(A, \mu, \phi, \psi)$ be a measured multiplier Hopf $*$-algebroid.

Aim We want to construct completions on the level of von Neumann algebras, to get a *measured quantum groupoid* [Enock, Lesieur, Vallin], and of $C^*$-algebras, where a full theory does not exist yet.

We will need additional assumptions, e.g.,

- $\mu_B$ and $\mu_C$ have associated GNS-representations $B, C \to \mathcal{L}(H_\mu)$
- the modular automorphisms of $\phi$ and $\psi$ commute
- (the modular element $\delta$ relating $\phi$ and $\psi$ has a square root $\delta^{1/2}$)

The key steps will be to show that

1. $\phi$ and $\psi$ admit a *bounded* GNS-representation $A \to \mathcal{L}(H)$
2. $\Delta_B$ extends to a comultiplication on $A'' \subseteq \mathcal{L}(H)$ rel. to $B'' \subseteq \mathcal{L}(H_\mu)$
3. $\phi$ and $\psi$ induce left- and right-invariant n.s.f. weights $A'' \to B'', C''$

Special case *proper dynamical quantum groups* treated before [T.]

## Steps for the passage to the setting of operator algebras

Theorem [T.] Let $(A, \mu, \phi, \psi)$ be a measured multiplier Hopf $*$-algebroid, where $\mu, \phi, \psi$ are positive. Assume that $\mu_B$ and $\mu_C$ admit bounded GNS-representations. Then:

1. $\phi$ and $\psi$ admit *bounded* GNS-representations $\pi_\phi \colon A \to \mathcal{L}(H_\phi)$ and $\pi_\psi \colon A \to \mathcal{L}(H_\psi)$
2. $\Delta_B$ extends to comultiplications on $\pi_\phi(A)'' \subseteq \mathcal{L}(H_\phi)$ and $\pi_\psi(A)'' \subseteq \mathcal{L}(H_\psi)$ relative to $B'' \subseteq \mathcal{L}(H_\mu)$ so that
   - $\pi_\phi(A)''$ and $\pi_\psi(A)''$ become *Hopf-von Neumann bimodules*
   - $\overline{\pi_\phi(A)}$ and $\overline{\pi_\psi(A)}$ become *concrete Hopf $C^*$-bimodules*
3. $\Lambda_\phi(A) \subseteq H_\phi$ and $\Lambda_\psi(A) \subseteq H_\psi$ are Hilbert algebras so that $\phi$ and $\psi$ extend to n.s.f. weights on $\pi_\phi(A)''$ and $\pi_\psi(A)''$

Idea of proof: use $(C^*)$pseudo-multiplicative unitaries [Vallin, T]:

- the map $a \otimes b \mapsto \Delta_B(b)(a \otimes 1)$ induces a unitary on suitable completions of the domain and range
- identify these completions with certain Connes' fusions of $H_\phi$ over $B''$
- show that $U^*$ is a pseudo-multiplicative unitary