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Pseudo-multiplicative unitaries and pseudo-Kac systems on C^*-modules

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Abstract

We study pseudo-multiplicative unitaries and pseudo-Kac systems on C^*-modules in general and examples arising from locally compact groupoids in particular. Multiplicative unitaries on Hilbert spaces were introduced by Saad Baaj and Georges Skandalis as a framework for generalisations of Pontrjagin duality of locally compact abelian groups: each locally compact quantum group gives rise to a multiplicative unitary from which one can construct two Hopf C^*-algebras representing the initial quantum group and its generalised Pontrjagin dual [3, 28]. Building on the notion of a Kac system, they define reduced crossed products for coactions of Hopf C^*-algebras and prove a generalisation of the Takesaki-Takai duality theorem [3].

For measurable quantum groupoids, a von Neumann-algebraic duality theory building on pseudo-multiplicative unitaries on Hilbert spaces exists [59, 12, 32], but up to now, the treatment of topological groupoids by C^*-algebraic means remained restricted to the finite case [5, 36, 40]. Aiming at this deficiency, we introduce pseudo-multiplicative unitaries on C^*-modules. The results cited above [3] do not carry over. Problems arise from the internal tensor product of C^*-bimodules and adjointability questions for linear operators on C^*-modules. Motivated by r-discrete groupoids, we introduce a decomposability condition and obtain the first main result: a decomposable regular pseudo-multiplicative unitary gives rise to two Hopf C^*-families. The construction rests on a number of new concepts: homogeneous operators on C^*-bimodules, the category of C^*-families and the internal tensor product of C^*-families.

The second main result is the axiomatisation of pseudo-Kac systems on C^*-modules and the generalisation of the reduced crossed product construction and the associated Baaj-Skandalis duality theorem to decomposable pseudo-Kac systems. This implies a corresponding duality theorem for r-discrete groupoids.

Finally, we study decomposable groupoids and their associated pseudo-Kac systems. We show that coactions of the function algebra of the groupoid coincide with actions of the groupoid and that coactions of the dual Hopf C^*-family coincide with upper semi-continuous Fell bundles on the groupoid, provided the latter is r-discrete. This is the third main result. A discussion of non-Hausdorff groupoids and the approaches of Mahmood Khoshkam and Georges Skandalis [23], Jean-Louis-Tu [55] and a Hausdorff compactification introduced by James Fell [16] completes the thesis.

Zusammenfassung

Wir untersuchen pseudo-multiplikative Unitäre und Pseudo-Kac-Systeme auf C^*-Moduln sowie Beispiele aus dem Bereich lokal-kompakter Gruppoide. Multiplikative Unitäre auf Hilbert-Räumen wurden von Saad Baaj und Georges Skandalis als Rahmen für Verallgemeinerungen der Pontrjagin-Dualität lokal-kompakter abelscher Gruppen eingeführt: zu jeder Quantengruppe kann ein multiplikatives Unitäres und aus diesem wiederum können zwei Hopf-C^*-Algebren konstruiert werden, welche die Ausgangsgruppe und ihr verallgemeinertes Duales repräsentieren [3, 63, 28]. Aufbauend auf dem Begriff eines Kac-Systems konstruieren die Autoren ferner verschränkte Produkte für Kowirkungen von Hopf-C^*-Algebren und verallgemeinern den Takesaki-Takai-Dualitätssatz [3].

Für messbare Quantengruppoide existiert eine Dualitätstheorie, welche auf pseudo-multiplikativen Unitären auf Hilberträumen und der räumlichen Theorie der von-Neumann-Algebren beruht [59, 12, 32]; die Behandlung topologischer Gruppoide mit C^*-algebraischen Methoden aber blieb bisher beschränkt auf den diskreten Fall [5, 36, 40]. Um diese Einschränkung zu überwinden, führen wir in dieser Arbeit den Begriff eines pseudo-multiplikativen Unitären auf C^*-Moduln ein. Die direkte Übertragung der Ergebnisse aus [3] scheitert an den Eigenschaften des internen Tensorprodukts von C^*-Bimoduln und der Adjungierbarkeitsfrage für lineare Operatoren auf C^*-Moduln. In Anlehnung an den Begriff eines r-diskreten Gruppoids formulieren wir eine Zerlegbarkeits-Bedingung und erhalten das erste Hauptergebnis – die Konstruktion zweier Hopf-C^*-Familien aus einem zerlegbaren regulären pseudo-multiplikativen Unitären. Dieses Ergebnis beruht auf einer Reihe neuer Konzepte: homogene Operatoren auf C^*-Bimoduln, die Kategorie der C^*-Familien und das interne Tensorprodukt von C^*-Familien.

Das zweite Hauptergebnis ist die Axiomatisierung von Pseudo-Kac-Systemen auf C^*-Moduln und die Übertragung der Konstruktion reduzierter verschränkter Produkte sowie des entsprechenden Dualitätssatzes auf zerlegbare pseudo-Kac-Systeme. Letzterer impliziert ein entsprechendes Ergebnis für Wirkungen von r-diskreten Gruppoiden.

Beispiele zerlegbarer Pseudo-Kac-Systeme liefert die Klasse der zerlegbaren Gruppoide, welche die r-diskreten Gruppoide verallgemeinern. Als drittes Hauptergebnis zeigen wir, dass Kowirkungen der Funktionenalgebra eines solchen Gruppoids und Kowirkungen der dazu dualen Hopf-C^*-Familie mit Wirkungen des Gruppoids beziehungsweise oberhalb-stetigen Fell-Bündeln auf dem Gruppoid korrespondieren – letzteres unter der Einschränkung, dass das Gruppoid r-diskret ist. Dies verallgemeinert entsprechende Resultate über diskrete Gruppen [2]. Zum Schluss betrachten wir Gruppoide, die nicht Hausdorffsch sind, und verknüpfen die Zugänge von Mahmood Khoshkam, Georges Skandalis [23] und Jean-Louis-Tu [55] mit einer von James Fell eingeführten Hausdorff-Kompaktifizierung [16].

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Contents

Introduction

Overview

In this thesis, we introduce and study pseudo-multiplicative unitaries and pseudo-Kac systems on C^*-modules in general and examples arising from locally compact groupoids in particular. Although these notions are modelled after (pseudo)-multiplicative unitaries and Kac systems on Hilbert spaces [3, 59, 60], the new setting presents several difficulties.

Multiplicative unitaries on Hilbert spaces are fundamental to the study of locally compact quantum groups and their duality which generalises the Pontrjagin duality of locally compact abelian groups [28]. In the von Neumann-algebraic theory of measured quantum groupoids, an analogue rôle is played by pseudo-multiplicative unitaries on Hilbert spaces [32, 12]. In both theories, the construction of the generalised Pontrjagin dual proceeds in two steps: the quantum group(oid), which is some kind of Hopf algebra equipped with a Haar measure, gives rise to a (pseudo-)multiplicative unitary. From this unitary, one constructs a pair of operator-algebraic Hopf algebras [3, 63, 28]. One of these coincides with the initial quantum group(oid) and the other is its Pontrjagin dual.

Although examples of pseudo-multiplicative unitaries on C^*-bimodules have already been discussed in the literature [38, 39], the construction of the two associated Hopf objects remained elusive [40]. The main obstacles arise from the internal tensor product of C^*-modules and its functorial properties which – on the level of operators – are poorly understood. An additional difficulty comes from the fact that adjointability of maps is much more subtle for C^*-modules than it is for Hilbert spaces, where every bounded linear map is automatically adjointable. Whereas the legs of a (pseudo)-multiplicative unitary consist of adjointable operators on Hilbert spaces, in the new setting, adjointability is lost.

In this thesis, we carry out the second step of generalised Pontrjagin duality – associating two generalised Hopf algebras to a pseudo-multiplicative unitary – under a certain decomposability condition on the underlying C^*-bimodules. In the situation of locally compact groupoids, this assumption is close to r-discreteness. Tailored to this situation, we introduce semigroup grading techniques on C^*-bimodules and their operators. The general idea is that relations which are trivial for Hilbert spaces but need not be satisfied for C^*-bimodules are now demanded to hold up to twists by partial automorphisms which are kept track of by additional book-keeping. We introduce the notion of a C^*-pre-family of "twisted operators" on a C^*-bimodule, internal tensor products of such families and morphisms between them. A main feature of this theory is a satisfactory functorial behaviour of the internal tensor product of C^*-pre-families. The notion of a Hopf C^*-pre-family follows immediately, and our generalisation of the Baaj-Skandalis construction associates to each decomposable pseudo-multiplicative unitary on C^*-bimodules a pair of such Hopf C^*-pre-families.

Kac systems are fundamental to duality theorems for C^*-dynamical systems which are (co)actions of locally compact (quantum) groups on C^*-algebras. Such systems are often studied via their covariant representations or their associated crossed products. It is interesting to ask whether a C^*-dynamical system can be reconstructed from its associated crossed product [30, 42]. In the case of a locally compact abelian group, the crossed product carries a natural action of the Pontrjagin dual, and the Takesaki-Takai duality theorem says that the iterated crossed product is Morita equivalent to the initial system [50, 51]. Based on the notion of a Kac system, Saad Baaj and Georges Skandalis have generalised the crossed product construction and the duality theorem to coactions of Hopf C^*-algebras [3]. Related crossed product constructions and duality theorems for coactions of groupoids on von Neumann algebras have been obtained by Takehiko Yamanouchi [66].

We introduce the notion of a pseudo-Kac system on C^*-modules and extend the crossed product construction as well as the Baaj-Skandalis duality theorem to the new setting. Here, the main difficulty is that of axiomatisation. A Kac system consists of a multiplicative unitary and a symmetry on the underlying Hilbert space. In the new setting, the example of groupoids shows that the symmetry arising from the inversion of the groupoid introduces an additional C^*-module. Besides examples, conceptual arguments show that one has to introduce a family of C^*-modules and operators in the axiom system. Once the axioms are fixed, the constructions necessary for Baaj-Skandalis duality carry over with minor modifications and a lot of book-keeping. As an application of the generalised Takesaki-Takai-Baaj-Skandalis duality theorem, one obtains a C^*-algebraic duality result for coactions and crossed products of r-discrete groupoids on C^*-algebras.

The initial motivation for this work came from locally compact Hausdorff groupoids. Unfortunately, the decomposability condition mentioned above restricts us to decomposable groupoids which, roughly speaking, are extensions of r-discrete groupoids by group bundles. To each such groupoid, we associate a pseudo-Kac system and identify the notions of coactions arising from the two associated Hopf C^*-pre-families. For one of them, injective non-degenerate coactions coincide with actions of the groupoid [31] on a C^*-algebra. For the other Hopf C^*-pre-family, such coactions coincide with upper semi-continuous Fell bundles on the groupoid, provided it is r-discrete. This generalises corresponding results for groups [35, 44, 2].

With non-Hausdorff groupoids, problems arise from the very beginning in the definition of the C^*-module underlying the pseudo-multiplicative unitary. We study a Hausdorff compactification functor introduced by James Fell [16] and show that its application to locally compact non-Hausdorff groupoids yields locally compact groupoids, which are, of course, Hausdorff. In the r-discrete case, this groupoid again is r-discrete and gives rise to the same C^*-module as the one associated to the initial groupoid by M. Khoshkam and G. Skandalis [23]. Thus, from the point of view of pseudo-multiplicative unitaries, each r-discrete groupoid can be replaced by its Hausdorff compactification.

The treatment of general locally compact groupoids still seems out of reach.

Organisation of the thesis

The thesis consists of three chapters, an appendix, and an introductory section which explains the mathematical context of this work.

The first chapter is concerned with semigroup grading techniques which are needed for the treatment of decomposable pseudo-multiplicative unitaries. We introduce *homogeneous operators* on C^*-bimodules. These are maps which – up to partial automorphisms of the underlying C^*-algebras – are adjointable and commute with left multiplication. They are closely related to ordinary operators on C^*-bimodules with suitably twisted structure maps. The point is that we consider simultaneously whole *C^*-pre-families* of such operators, indexed by the respective partial automorphisms. Roughly, a C^*-pre-family is a $*$-algebra of homogeneous operators whose subspace of operators of a fixed degree is closed for each degree. The control provided by the grading allows the formation of an internal tensor product of such families. The internal tensor product is used for the definition of morphisms of C^*-pre-families. This approach implies bi-functoriality of the internal tensor product by the very definition. The notion of multipliers, tensor products and morphisms lead to the definition of a *Hopf C^*-pre-family* and *coactions* of such a Hopf C^*-pre-family on C^*-pre-families.

In the remaining part of the chapter, we consider homogeneous elements of C^*-bimodules and introduce the notion of decomposability. We collect some useful facts which are needed later on and obtain particularly satisfying results in the case that the underlying C^*-algebra is commutative. The notion of decomposable C^*-algebras gives rise to the notion of an C^*-family which has better properties than a C^*-pre-family with respect to the grading.

In the second chapter, we introduce the notion of a pseudo-multiplicative unitary on C^*-modules and discuss its relation to other notions of multiplicative transformations [3, 4, 38, 59, 12]. Using ideas introduced in the first chapter, we formulate a decomposability and regularity condition on pseudo-multiplicative unitaries. The first main result of the thesis is the construction of the legs of a decomposable regular pseudo-multiplicative unitary which are Hopf C^*-families. The approach follows [3], but the setting of C^*-modules introduces many new technicalities. The section ends with a discussion of coactions of Hopf C^*-families on C^*-algebras. The definition raises many questions which suggest that it is more natural to consider coactions on C^*-families. For the latter ones, the property of regularity is introduced, which will be important for the proof of the generalised Baaj-Skandalis duality theorem. Finally, we study examples of coactions which arise from legs of coaction unitaries. They will turn up in the context of pseudo-Kac systems again.

Next, we introduce the notion of a pseudo-Kac system on C^*-modules. The complex axiom system is carefully motivated by conceptual reasoning and a discussion of the example of locally compact groupoids. The left and the right leg of a pseudo-Kac system are pairs consisting of a Hopf C^*-family and a canonical coaction on a C^*-algebra. Building on [3] and keeping track of the multitude of C^*-modules and operators involved, we define reduced crossed products for coactions of the legs and dual coactions on them. The chapter ends with a generalisation of the Baaj-Skandalis duality theorem which is the second main result of the thesis.

In the third chapter, we apply the material developed so far to the class of decomposable groupoids. We give a precise definition and study Haar systems on such groupoids. Then we illustrate the notions of C^*-families and internal tensor products by examples associated to continuous representations of decomposable groupoids.

The topic of the second section is the pseudo-Kac system associated to a decomposable groupoid. We determine the Hopf C^*-families comprising the legs of this system and study coactions of both legs. For the left leg, non-degenerate injective coactions coincide with groupoid actions on C^*-algebras. In the r-discrete case, injective coactions of the right leg coincide with upper semi-continuous Fell bundles on the groupoid. To prove the latter result, we carry over the construction of Haar means [2].

In the last section, we consider non-Hausdorff groupoids. First, we discuss a Hausdorff compactification introduced by James Fell [16] and show that the Hausdorff compactification of a non-Hausdorff locally compact groupoid is a locally compact groupoid. We explain how this Hausdorff groupoid facilitates a geometric description of a substitute for the fundamental C^*-module $L^2(G, \lambda)$ introduced in [23].

The appendix contains standard notions and results which are used freely throughout the text. An index of frequently used notation and terminology completes the thesis.

Background

In the following, we outline the mathematical and historical route from Pontrjagin duality to the objects of study of this thesis.

Groups, Hopf algebras and multiplicative unitaries

Pontrjagin duality [43, 53, 21, 14, 28] Let G be a group. A character on G is a group homomorphism $G \to S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$. If G is abelian, the set of characters on G, equipped with pointwise multiplication and inversion, forms *the dual group of* G. If G is locally compact, the compact-open topology on the subset $\hat{G}$ of all continuous characters endows $\hat{G}$ with the structure of a locally compact group. The Pontrjagin duality theorem says that the natural map $G \to \hat{\hat{G}}$ is an isomorphism of locally compact abelian groups.

Much work has been spent on generalisations of this duality. The characters of a non-abelian group see only its abelian quotient, so higher-dimensional representations have to be taken into account. These are encoded by the corresponding group (C^*-/von Neumann) algebra. In the abelian case, this group algebra is isomorphic to the function algebra on the dual group via the Fourier transform.

Hopf (C^*-)algebras [49, 1, 19, 3, 24, 57] Replacing the underlying space and maps of a group by its function algebra and the induced homomorphisms, one is lead to the the notion of a *Hopf algebra*. A Hopf algebra consists of an associative algebra A over some field, a coproduct $\Delta\colon A \to A \otimes A$ and an antipode $\kappa\colon A \to A$ subject to a number of axioms which are precisely the transposes of the group axioms.[1]

If A has finite dimension, the Hopf algebra (A, Δ, κ) has a *dual Hopf algebra* $(\hat{A}, \hat{\Delta}, \hat{\kappa})$ where $\hat{A} := A'$ is the dual space of A and the structure maps are defined by

$$(\phi \cdot \psi)(f) := (\phi \otimes \psi)\Delta(f), \qquad (\hat{\Delta}\phi)(f \otimes g) := \phi(fg), \qquad (\hat{\kappa}\phi)(f) := \phi(\kappa f).$$

In the formula for $\hat{\Delta}$, we identify $A' \otimes A'$ with $(A \otimes A)'$. The natural map $A \to \hat{\hat{A}}$ is an isomorphism of Hopf algebras.

This duality is an analogue of Pontrjagin duality for finite non-abelian groups: If G is a finite group, the function algebra $C(G)$ and the group algebra $\mathbb{C}G \cong C(G)'$ are dual Hopf algebras. Their structure maps are given by

$$\Delta(\delta_z) := \sum_{z=xy} \delta_x \otimes \delta_y, \quad \kappa(\delta_z) := \delta_{z^{-1}}, \quad \hat{\Delta}(\lambda_z) := \lambda_z \otimes \lambda_z, \quad \hat{\kappa}(\lambda_z) := \lambda_{z^{-1}},$$

where $\delta_z \in C(G)$ and $\lambda_z \in \mathbb{C}G$, $z \in G$, denote the operators of pointwise multiplication and left translation, respectively. If, in addition, the group G is abelian, the Hopf algebra $(\mathbb{C}G, \hat{\Delta}, \hat{\kappa})$ can be identified with the Hopf algebra of functions on $\hat{G}$.

A *Hopf C^*-algebra* consists of a C^*-algebra S and a coassociative $*$-homomorphism $\Delta\colon S \to M(S \otimes S)$. The antipode is replaced by the condition that the spaces $\Delta(S)(1 \otimes S)$ and $\Delta(S)(S \otimes 1)$ should be dense in $S \otimes S$. If G is a locally compact group, then $C_0(G)$ and $C^*_{(r)}(G)$ are Hopf C^*-algebras in a canonical way.

[1] We omit mentioning of the counit which is redundant.

Multiplicative unitaries and the Baaj-Skandalis construction [3] Given a Hilbert space H, a unitary V on $H \otimes H$ is called *multiplicative* if it satisfies the *pentagon equation* $V_{12}V_{13}V_{23} = V_{23}V_{12}$. Under additional regularity assumptions, one can show that the *right leg* $\big(S(V), \Delta\big)$ and the *left leg* $\big(\hat{S}(V), \hat{\Delta}\big)$ of V, given by

$$S(V) := \overline{\operatorname{span}}\{(\omega \otimes 1)(V) \mid \omega \in \mathcal{B}(H)_*\}, \qquad \Delta\colon s \mapsto V(s \otimes 1)V^*,$$
$$\hat{S}(V) := \overline{\operatorname{span}}\{(1 \otimes \omega)(V) \mid \omega \in \mathcal{B}(H)_*\}, \qquad \hat{\Delta}\colon \hat{s} \mapsto V^*(1 \otimes \hat{s})V,$$

are Hopf C^*-algebras (here, $\mathcal{B}(H)_*$ denotes the predual of $\mathcal{B}(H)$). We call this result the *Baaj-Skandalis construction*. Its relevance to generalisations of Pontrjagin duality lies in the fact that a (regular) multiplicative unitary encodes a pair of Hopf C^*-algebras simultaneously and therefore has duality "built in".

As an example, consider a locally compact group G with a left-invariant Haar measure λ. Then the formula $(Wf)(x, y) := f(x, x^{-1}y)$ defines a multiplicative unitary on $L^2(G, \lambda) \otimes L^2(G, \lambda)$. If G is finite, V is equal to the sum $\sum_{x \in G} \delta_x \otimes \lambda_x$, and the left and right leg of V coincide – up to a switched $\hat{}$-sign – with the Hopf algebras associated to G in the previous section. In the general case, $\hat{S}(V) = C_0(G)$ and $S(V) = C^*_r(G)$.

The notion of *multiplicative unitaries* is central to C^*-algebraic generalisations of Pontrjagin duality beyond finite Hopf algebras.

(Locally compact) quantum groups [61, 64, 26, 27, 28] A widely accepted slogan is that a *quantum group* is some kind of Hopf algebra equipped with a *Haar functional*. A left-invariant Haar functional on a complex Hopf algebra (A, Δ, κ) is a linear map $\phi\colon A \to \mathbb{C}$ satisfying $(1 \otimes \phi)\Delta(a) = \phi(a) \cdot 1$. Of course, this definition has to be adapted if one considers e.g. Hopf C^*-algebras or Hopf-von Neumann algebras. In the case of locally compact groups and compact quantum groups, the existence of a Haar functional can be deduced from the axioms.

Theories of quantum groups with a nice duality include multiplier Hopf algebras, compact and discrete quantum groups and locally compact quantum groups. We briefly outline the construction of the Pontrjagin dual in the latter approach. Given a locally compact quantum group consisting of a Hopf C^*-algebra (S, Δ) and a left-invariant Haar weight with corresponding GNS-space H, one constructs a unitary V^* on $H \otimes H$ via the map $S \times S \to S \otimes S$ given by $(s, t) \mapsto \Delta(t)(s \otimes 1)$. Then the unitary V is multiplicative, and the Hopf C^*-algebra of the dual locally compact quantum group is just the right leg of V.

Coactions, crossed products and duality

C^*-dynamical systems [50, 51, 42] A central topic in the theory of C^*-algebras is the study of automorphism groups and *C^*-dynamical systems*. Such a system consists of a C^*-algebra C, a locally compact group G and a strongly continuous action α of G on C. Usually, such a system is studied via its *covariant representations* which consist of a representation π of the C^*-algebra and a unitary representation U of the group on the same Hilbert space, related by the formula $U_x\pi(c)U_x^* = \pi(\alpha_x(c))$ for all $c \in C$ and $x \in G$. The covariant representations correspond bijectively to representations of the crossed product $C \rtimes_\alpha G$ which is the C^*-algebra generated by the universal covariant representation. This C^*-algebra has nice functorial properties, but its definition is very abstract. It is complemented by a *reduced crossed product* $C \rtimes_{\alpha,r} G$ which has a concrete description and coincides with the universal crossed product if G is amenable. In the following, we concentrate on the reduced crossed product.

It is natural to ask whether the system (C, G, α) can be reconstructed from the C^*-algebra $C \rtimes_{\alpha,r} G$. If G is locally compact and abelian, this question is answered by the Takesaki-Takai duality theorem. In that case, $C \rtimes_{\alpha,r} G$ carries a natural action $\hat{\alpha}$ of the dual group $\hat{G}$, and the iterated crossed product $(C \rtimes_{\alpha,r} G) \rtimes_{\hat{\alpha},r} \hat{G}$ is equivariantly isomorphic to the C^*-algebra $C \otimes \mathcal{K}\big(L^2(G,\lambda)\big)$, equipped with a natural action induced by α.

Coactions of Hopf (C^*-)algebras [33, 2, 24] The first step towards a generalisation of the Takesaki-Takai duality to arbitrary locally compact (quantum) groups is a reformulation of group actions in terms of coactions of Hopf C^*-algebras.

A *coaction* of a Hopf C^*-algebra (S, Δ) on a C^*-algebra C is a coassociative $*$-homomorphism $\delta\colon C \to M(C \otimes S)$ such that $\delta(C)(1 \otimes S)$ is contained in $C \otimes S$. If δ is injective and $\delta(C)(1 \otimes S)$ is dense in $C \otimes S$, the pair (C, δ) is called an (S, Δ)-algebra.

As an example, consider the Hopf C^*-algebras $(C_0(G), \Delta)$ and $(C_r^*(G), \hat{\Delta})$ associated to a locally compact group. The C^*-algebra $M(C \otimes C_0(G))$ can be identified with the algebra of bounded strictly continuous functions $G \to M(C)$. Then $C_0(G)$-algebras (C, δ) correspond to actions (C, α) of G via the formulas $\alpha_x(c) := (1 \otimes ev_x)\delta(c)$ and $(\delta(c))(x) := \alpha_x(c)$, where ev_x denotes evaluation at a point x of G. If G is discrete, $C_r^*(G)$-algebras $(C, \hat{\delta})$ correspond to G-graded C^*-algebras $C = \bigoplus_{x \in G} C_x$ (strictly speaking, Fell bundles on G) via the formulas $C_x = \{c \in C \mid \hat{\delta}(c) = c \otimes \lambda_x\}$ and $\hat{\delta}(c) = c \otimes \lambda_x$ for $x \in G$ and $c \in C_x$.

Kac systems [3] The generalisation of the Takesaki-Takai duality theorem to coactions of Hopf C^*-algebras rests on the notion of a *Kac system* which consists of a Hilbert space H, a multiplicative unitary V on $H \otimes H$ and a symmetry U on H, subject to a number of axioms. A prototypical example is the multiplicative unitary V of a locally compact group G together with the symmetry U on $L^2(G, \lambda)$ given by $(Uf)(x) := \Delta^{1/2}(x)f(x^{-1})$, where $\Delta = d\lambda^{-1}/d\lambda$ is the modular function of the group.

Given a coaction (C, δ) of the right leg $(S(V), \Delta)$ of V, the associated *reduced crossed product* $C \rtimes_{\delta,r} \hat{S}$ is the C^*-algebra of operators on the C^*-module $C \otimes H$

generated by $\delta(C)(1 \otimes \hat{S}(V))$. This reduced crossed product carries a dual coaction $\hat{\delta}$ of the left leg $(\hat{S}(V), \hat{\Delta})$ via $\delta(c)(1 \otimes \hat{s}) \mapsto (\delta(c) \otimes 1)(1 \otimes \hat{\Delta}(\hat{s}))$. One similarly defines reduced crossed products and dual coactions for coactions of the left leg $(\hat{S}, \hat{\Delta})$ of V, and the Baaj-Skandalis duality theorem says that the iterated crossed product $\left(C \rtimes_{\delta,r} \hat{S}\right) \rtimes_{\hat{\delta},r} S$ is equivariantly isomorphic to the C^*-algebra $C \otimes \mathcal{K}(H)$, equipped with a coaction naturally induced by δ.

From quantum groups to quantum groupoids

In many geometric situations, symmetries are not governed by a global group but rather by a groupoid. A groupoid G is a small category in which every morphism is invertible: it consists of a morphism set G, a set of objects (or units) G^0, range and source maps $r, s\colon G \to G^0$, an inversion map $i\colon G \to G$, and a composition law $\circ\colon G\,{}_s{\times}_r\, G \to G$, where $G\,{}_s{\times}_r\, G = \{(x, y) \in G \times G \mid s(x) = r(y)\}$, subject to a number of axioms. A left Haar system λ on G is a family $(\lambda^u)_{u \in G^0}$, where λ^u is a measure on the fibre $r^{-1}(u)$, subject to a number of axioms including left invariance.

Replacing spaces by algebras again, the definition of a groupoid suggests that a "quantum groupoid" should consist of a "base algebra" R over some field k, an algebra A with two commuting maps $\rho, \sigma\colon R \to A$, and a coproduct subject to several axioms. The difficulty lies in the definition of the target of the coproduct. If one translates the groupoid situation, the source of the multiplication map $G\,{}_s{\times}_r G$ should be replaced by the tensor product $A_\sigma \otimes_R {}_\rho A$. If R is non-commutative, one of the homomorphisms ρ, σ has to be replaced by an anti-homomorphism. However, this choice of the target does not lead to a self-dual concept of quantum groupoids on an algebraic level because there exists no pairing of $A \otimes_R A$ with $A' \otimes_R A'$. Hence, the tensor product over R has to be replaced by a "self-dual" product.

Hopf $\times_R$-algebras [52, 48] In the purely algebraic setting, the problem indicated above is solved by the $\times_R$-product. Denote by $\bar{R}$ the opposite of R and by $r \mapsto \bar{r}$ the natural anti-homomorphism $R \to \bar{R}$. Put $R^e := R \otimes \bar{R}$. The $\times_R$-product of two R^e-bimodules M and N is defined by $M \times_R N := \int^s \int_r {}_{\bar{r}}M_{\bar{s}} \otimes {}_r N_s$, where

$$\int_r {}_{\bar{r}}M \otimes {}_r N := M \otimes N / \big\langle \bar{r}m \otimes n - m \otimes rn \mid r \in R, m \in M, n \in N \big\rangle,$$
$$\int^s M_{\bar{s}} \otimes N_s := \big\{ \Sigma_i\, m_i \otimes n_i \in M \otimes N \mid \forall s \in R : \Sigma_i\, m_i \bar{s} \otimes n_i = \Sigma_i\, m_i \otimes n_i s \big\}.$$

The $\times_R$-product has the following characteristic property: If P is a right and Q a left R-module, the spaces $\mathrm{End}_k(P)$ and $\mathrm{End}_k(Q)$ are algebras over the ring R^e – shortly called R^e-algebras – and one has a natural map $\mathrm{End}_k(P) \times_R \mathrm{End}_k(Q) \to \mathrm{End}_k(P \otimes_R Q)$.

A $\times_R$-*Hopf algebra* A is an R^e-algebra equipped with a coproduct $\Delta\colon A \to A \times_R A$ and a counit $\epsilon\colon A \to \mathrm{End}_k(R)$ which are homomorphisms of R^e-algebras, subject to a number of axioms.

As an example, consider a finite groupoid G. Put $R := C(G^0)$. Consider the function algebra $C(G)$ and the convolution algebra $\mathbb{C}G$ as R^e-algebras via

$$(f \otimes \bar{g})\delta_x := f(r(x))\delta_x g(s(x)), \qquad (f \otimes \bar{g})\lambda_x := f(r(x))g(r(x))\lambda_x, \quad x \in G.$$

The representations $C(G) \hookrightarrow \mathrm{End}_{\mathbb{C}}(C(G))$ and $\mathbb{C}G \hookrightarrow \mathrm{End}_{\mathbb{C}}(C(G))$ given by point-wise multiplication and convolution, respectively, extend to inclusions

$$\begin{aligned} C(G) \times_R C(G) &= \int^g \int_f {}_{\bar{f}}C(G)_{\bar{g}} \otimes {}_f C(G)_g \\ &= C(G\,{}_s\!\times_r G) \hookrightarrow \mathrm{End}_{\mathbb{C}}\left(C(G\,{}_s\!\times_r G)\right) \end{aligned}$$

and

$$\begin{aligned} \mathbb{C}G \times_R \mathbb{C}G &= \int^g \int_f {}_{\bar{f}}\mathbb{C}G_{\bar{g}} \otimes {}_f\mathbb{C}G_g \\ &= \left\langle \lambda_x \otimes \lambda_y \middle| s(x) = s(y) \right\rangle \Big/ \left(\lambda_x \otimes \lambda_y \middle| \begin{matrix} r(x) = r(y), \\ s(x) = s(y) \end{matrix} \right) \hookrightarrow \mathrm{End}_{\mathbb{C}}\left(C(G\,{}_r\!\times_r G)\right). \end{aligned}$$

Now, $(C(G), \Delta, \epsilon)$ and $(\mathbb{C}G, \hat{\Delta}, \hat{\epsilon})$ are $\times_R$-Hopf algebras with the structure maps given by

$$\Delta(\delta_z) = \sum_{z=xy} \delta_x \otimes \delta_y, \quad \hat{\Delta}(\lambda_z) = \lambda_z \otimes \lambda_z, \quad \epsilon(\delta_z) = \hat{\epsilon}(\lambda_z) = \begin{cases} 0, & z \notin G^0, \\ \delta_z, & z \in G^0. \end{cases}$$

Weak Hopf (C^*-)algebras [5, 6, 36, 37] An alternative approach to the problem with the tensor product $\otimes_R$ indicated above is to restrict to cases where a coproduct can be defined as a map $\Delta\colon A \to A \otimes A$. For groupoids, this corresponds to the case where $G\,{}_s\!\times_r G \subset G \times G$ is open which is a severe restriction. This leads to the notion of a weak Hopf (C^*-)algebra which has been studied by Gabriella Böhm, Kornél Szlachányi, Dmitri Nikshych and others. For such objects, a theory including duality, Haar functionals, coactions, crossed or smashed products, duality of iterated smashed products and a relation to multiplicative isometries has been developed. However, the notion of a weak Hopf C^*-algebra is inherently restricted to finite-dimensional C^*-algebras.

Measured quantum groupoids [58, 59, 60, 12, 32] In the von Neumann-algebraic setting, Jean-Michel Vallin, Michel Enock and Franck Lesieur introduced the notion of measured quantum groupoids and pseudo-multiplicative unitaries, including a generalisation of Pontrjagin duality. In this setting, the tensor product $\otimes_R$ is replaced by the von Neumann algebra fibre product which builds on Connes' fusion product.

The pseudo-multiplicative unitary of a locally compact groupoid

A proto-typical example of the kind of objects we want to study is the pseudo-multiplicative unitary associated to a locally compact groupoid. For notational convenience, we describe the construction of the pseudo-multiplicative unitary V^{op} which is the "opposite" of the pseudo-multiplicative unitary V we usually associate to a groupoid. Let G be a locally compact groupoid with left Haar system λ. Denote by $L^2(G, \lambda)$ the associated C^*-module over $C_0(G^0)$. It is the completion of the pre-C^*-module $C_c(G)$ with structure maps given by

$$\begin{aligned} \langle f|g\rangle(u) &:= \int_{G^u} \overline{f(x)} g(x) d\lambda^u(x), \quad u \in G^0,\ f, g \in C_c(G), \\ (fh)(x) &:= f(x)h(r(x)), \quad x \in G,\ f \in C_c(G),\ h \in C_0(G^0), \end{aligned}$$

and corresponds to the continuous Hilbert bundle $\left(L^2(G^u, \lambda^u)\right)_u$ on G^0 . The range map and the source map of G induce representations π_r, π_s of $C_0(G^0)$ on $L^2(G, \lambda)$ given by

$$\begin{aligned} (\pi_r(h)f)(x) := f(x)h(r(x)), \qquad (\pi_s(h)f)(x) &:= f(x)h(s(x)), \\ x \in G,\ f \in C_c(G),\ h &\in C_0(G^0). \end{aligned}$$

The corresponding internal tensor products $L^2(G, \lambda) \otimes_{\pi_r} L^2(G, \lambda)$ and $L^2(G, \lambda) \otimes_{\pi_s} L^2(G, \lambda)$ can be identified with completions of the spaces $C_c(G\,{}_r\!\times_r G)$ and $C_c(G\,{}_r\!\times_s G)$, respectively. Consider the map $W_0\colon C_c(G\,{}_r\!\times_r G) \to C_c(G\,{}_r\!\times_s G)$ given by $(W_0 f)(x, y) := f(yx, y)$. By left-invariance of the Haar system λ, it extends a unitary operator $W\colon L^2(G, \lambda) \otimes_{\pi_r} L^2(G, \lambda) \to L^2(G, \lambda) \otimes_{\pi_s} L^2(G, \lambda)$. This unitary satisfies the pentagon equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$,

$$\begin{aligned} (W_{12}W_{13}W_{23}f)(x, y, z) &= (W_{13}W_{23}f)(yx, x, z) \\ &= (W_{23}f)(zyx, y, x) = f(zyx, zy, y), \\ (W_{23}W_{12}f)(x, y, z) &= (W_{12}f)(x, zy, z) = f(zyx, zy, z), \end{aligned}$$

in a sense which is made precise in section 2.2.

Notation

Apart from standard conventions, we use the following notation:

Let C be a normed space. Given a subset $A \subset C$, we denote by $\overline{\operatorname{span}} A$ the closed linear span of A. Given a family $\mathscr{C} = (\mathscr{C}_x)_x$ of subsets of C, we denote by

$$\overline{\operatorname{span}} \mathscr{C} := \overline{\operatorname{span}}_x \mathscr{C}_x := \overline{\operatorname{span}}\Big(\bigcup_x \mathscr{C}_x\Big)$$

the *closed linear span of the union of the subsets in the family.* Let C' and C'' be normed spaces with a continuous map $\cdot\colon C' \times C'' \to C$. Given subsets $A \subset C'$ and

$B \subset C''$, we denote by

$$A \cdot B := \text{closure of } \{a \cdot b \mid a \in A, b \in B\} \subset C$$

the *closure of the setwise product of A and B.* Likewise, for $a \in A$ and $b \in B$, we denote by $aB := \{a\}B$ and $Ab := A\{b\}$ the *closures* of the setwise products.

To differentiate between the algebraic tensor product and completions thereof, we denote the algebraic tensor product by $\odot$.

The index at the end of this thesis contains an extensive list of further notation used and introduced throughout the text.

Chapter 1

Semigroup grading techniques for C^*-bimodules

This chapter lays the foundation for the study of decomposable pseudo-multiplicative unitaries on C^*-bimodules.

1.1 The category of C^*-pre-families

The main objective of this section is the definition of Hopf C^*-pre-families which generalise Hopf C^*-algebras and provide the right concept for the generalisation of the Baaj-Skandalis construction. This notion builds on a number of new concepts and constructions – homogeneous operators on C^*-bimodules, C^*-pre-families of such operators, internal tensor products and morphisms of C^*-pre-families – each of which is carefully motivated by a discussion of the ultimate application in the Baaj-Skandalis construction presented in the second chapter.

Throughout this section, let A and B be C^*-algebras and let E and E' be C^*-B-A-bimodules, i.e. C^*-modules over A equipped with representations $B \to L_A(E)$ and $B \to L_A(E')$ which are written as left multiplication by elements of B.

1.1.1 Homogeneous operators on C^*-modules

This subsection introduces *homogeneous operators* on C^*-bimodules: maps which – up to partial automorphisms of the underlying C^*-algebras – are adjointable and commute with left multiplication. They are fundamental to the generalisation of the Baaj-Skandalis construction carried out in the second chapter. Their appearance and their rôle in this context is explained in the introductory section. The precise definition, a collection of some immediate properties and several examples follow. Finally, the relation to ordinary operators on C^*-bimodules is indicated.

Throughout this subsection, let α and β be partial automorphisms of A and B, respectively.

Motivation

Ordinary operators on C^*-bimodules do not suffice for the adaptation of the Baaj-Skandalis construction to pseudo-multiplicative unitaries on C^*-bimodules. We explain the problems which arise by a proto-typical example. Let G be a locally compact groupoid with a left Haar system λ and consider the pseudo-multiplicative unitary

$$W\colon L^2(G,\lambda) \otimes_{\pi_r} L^2(G,\lambda) \to L^2(G,\lambda) \otimes_{\pi_s} L^2(G,\lambda), \quad (Wf)(x,y) = f(yx,y),$$

defined in the introductory section.

To adapt the Baaj-Skandalis construction to the operator W, we rewrite the classical construction as follows. Let H be a Hilbert space and let V be a multiplicative unitary on $H \otimes H$. Then the right and the left leg of V are equal to

$$S(V) = \overline{\operatorname{span}}\,\{\theta_\eta^* V \theta_\xi \mid \xi,\eta \in H\} \quad \text{and} \quad \hat{S}(V) = \overline{\operatorname{span}}\,\{\theta^{\eta *} V \theta^\xi \mid \xi,\eta \in H\},$$

respectively, where the operators $\theta_\xi\colon H \to H \otimes H$ and $\theta^\xi\colon H \to H \otimes H$ associated to an element $\xi \in H$ are given by $\theta_\xi\zeta := \xi \otimes \zeta$ and $\theta^\xi\zeta := \zeta \otimes \xi$ for all $\zeta \in H$. It is easy to see that they are bounded.

Let us adapt these definitions to the operator W. For each element $\xi \in L^2(G,\lambda)$, the map $\theta_\xi\colon L^2(G,\lambda) \to L^2(G,\lambda) \otimes_\pi L^2(G,\lambda)$, $\pi = \pi_r$ or $\pi = \pi_s$, given by the same formula as before, is adjointable:

$$\langle \zeta' \otimes_\pi \zeta'' | \theta_\xi \zeta\rangle = \langle \zeta' \otimes_\pi \zeta'' | \xi \otimes_\pi \zeta\rangle = \big\langle \zeta'' \big| \pi(\langle \zeta' | \xi\rangle)\zeta\big\rangle = \langle \theta_\xi^*(\zeta' \otimes_\pi \zeta'') | \zeta\rangle$$

for all $\zeta \in L^2(G,\lambda)$, where $\theta_\xi^*(\zeta' \otimes_\pi \zeta'') = \pi(\langle \xi | \zeta'\rangle)\zeta''$ for all $\zeta',\zeta'' \in L^2(G,\lambda)$. Hence, $S(W)$ can be defined by the same formula as $S(V)$, and a short calculation shows that $S(W) = C_0(G)$. Difficulties occur with the left leg of W. In general, the map $\theta^\xi\colon L^2(G,\lambda) \to L^2(G,\lambda) \otimes_{\pi_s} L^2(G,\lambda)$ given by the formula above is *not adjointable*. If it were, it would have to commute with right multiplication by elements of $C_0(G^0)$. Then

$$\theta^\xi(\zeta h) = \zeta h \otimes_{\pi_s} \xi = \zeta \otimes_{\pi_s} \pi_s(h)\xi = \zeta \otimes_{\pi_s} \xi h = \theta^\xi(\zeta)h$$

for all $\zeta \in L^2(G,\lambda)$, whence $\pi_s(h)\xi = \xi h = \pi_r(h)\xi$ for all $h \in C_0(G^0)$. This condition is only satisfied if the support of ξ is contained in the subset $\{x \in G \mid r(x) = s(x)\}$ of G, which is a severe restriction.

Summarising, it is not clear how to define the left leg of W. This problem is related to the fact that the left leg of W encodes the left regular representation of G which – on $L^2(G,\lambda)$ – does not act via adjointable operators. In general, left multiplication on G does not commute with the range map and therefore the left translation operators are not even $C_0(G^0)$-linear.

The problems originate from the discrepancy between the range and the source map of the groupoid G. Let us assume that the groupoid G is r-discrete, i.e. that it has a cover consisting of open subsets $U \subset G$ such that the restrictions of the range and source map to U both are homeomorphisms onto open subsets of G^0. Locally on each such subset U, the discrepancy between the range and the source map of G can be measured by the partial homeomorphism $q_U \colon s(U) \to r(U)$ of G^0 given by $s(x) \mapsto r(x)$ for all $x \in U$: one has $r|_U = q_U \circ s|_U$. This relation provides a control on the non-adjointable maps which occurred above. Given an element $\xi \in C_c(U) \subset L^2(G, \lambda)$, we have

$$\begin{aligned}\big(\pi_s(h)\xi\big)(x) &= h(s(x))\xi(x) \\ &= h\big(q_U^{-1}(r(x))\big)\xi(x) = \big(\pi_r\big(q_{U*}(h)\big)\xi\big)(x), \quad x \in U,\, h \in C_0(s(U)),\end{aligned}$$

where $q_{U*} \colon C_0(s(U)) \to C_0(r(U))$ denotes the partial automorphism of $C_0(G^0)$ given by $(q_{U*}h)(v) = h(q_U^{-1}(v))$ for all $v \in r(U)$ and $h \in C_0(s(U))$. Now, consider the operator θ^ξ. For simplicity, let us assume that q_U extends to a homeomorphism of G^0. Then for all $\zeta, \zeta', \zeta'' \in L^2(G, \lambda)$, we have

$$\langle \zeta' \otimes_{\pi_s} \zeta'' | \theta^\xi \zeta \rangle = \big\langle \zeta'' \big| \pi_s\big(\langle \zeta' | \zeta \rangle\big)\xi \big\rangle = \langle \zeta'' | \xi \rangle q_{U*}\big(\langle \zeta' | \zeta \rangle\big) = q_{U*}\big(\langle \zeta' q_U^*(\langle \xi | \zeta'' \rangle) \big| \zeta \rangle\big),$$

and therefore the map $\zeta' \otimes_{\pi_s} \zeta'' \mapsto \zeta' q_U^*\big(\langle \xi | \zeta'' \rangle\big)$ is adjoint to the map θ^ξ up to a twist which is controlled by the (partial) automorphism q_{U*}.

The left regular representation of G can be treated similarly. For simplicity, let us assume that the left Haar system λ is given by the family of counting measures on the discrete fibres $G^v := r^{-1}(v), v \in G^0$. Let $U \subset G$ be an open subset as above. Then for each function $f \in C_c(U)$, the associated left convolution operator $\lambda_f \colon L^2(G, \lambda) \to L^2(G, \lambda)$ given by

$$(\lambda_f \zeta)(x) := \sum_{y \in G^{r(x)}} f(y)\zeta(y^{-1}x) = \begin{cases} 0, & r(x) \notin r(U), \\ f(y)\zeta(y^{-1}x), & y = r^{-1}(r(x)) \cap U, \end{cases}$$

where $x \in G$ and $\zeta \in L^2(G, \lambda)$, is adjointable up to the partial automorphism q_{U*}:

$$\begin{aligned}\langle \zeta' | \lambda_f \zeta \rangle(r(y)) &= \sum_{x \in G^{r(x)}} \overline{\zeta'(x)} f(y)\zeta(y^{-1}x) \\ &= \sum_{z \in G^{s(x)}} f(y)\overline{\zeta'(yz)}\zeta(z) = \big\langle \lambda_{f^*}\zeta' \big| \zeta \big\rangle(s(y)), \quad y \in U,\end{aligned}$$

$$\text{where} \quad f^*(y) := \overline{f(y^{-1})}, \quad y \in G,$$

i.e. $\langle \zeta' | \lambda_f \zeta \rangle = q_{U*}\big(\langle \lambda_{f^*} \zeta' | \zeta \rangle\big)$ for all $\zeta, \zeta' \in L^2(G, \lambda)$.

The preceding discussion motivated the consideration of operators which are adjointable up to a partial automorphism of the underlying C^*-algebra. For the formation of internal tensor products of operators on C^*-bimodules, it becomes necessary to also keep track of the covariance of the operators involved with respect to left multiplication. Operators of the form θ^ξ as above always commute with the representation, but an operator of the form

$$\theta_\xi\colon \big(L^2(G,\lambda),\pi_s\big) \to \big(L^2(G,\lambda),\pi_s\big) \otimes_{C_0(G^0)} \big(L^2(G,\lambda),\pi_s\big), \quad \zeta \mapsto \xi \otimes_{\pi_s} \zeta,$$

need not:

$$(\theta_\xi h)\zeta = \theta_\xi(\pi_s(h)\zeta) = \xi \otimes_{\pi_s} \pi_s(h)\zeta = \xi h \otimes_{\pi_s} \zeta, \qquad h(\theta_\xi\zeta) = \pi_s(h)\xi \otimes_{\pi_s} \zeta.$$

Again, if G is r-discrete, U is a subset of G as above and ξ belongs to $C_c(U)$, the preceding discussion shows that $h\theta_\xi = \theta_\xi q_{U^*}(h)$ for all $h \in C_0(s(U))$.

Definition and first properties

The following definition is fundamental to everything which follows:

Definition 1.1. *A map $T\colon E \to E'$ is a (β,α)-*homogeneous operator *if*

i) there exists a map $S\colon E' \to E$ such that $\langle SE'|E\rangle \subset \mathrm{Dom}(\alpha)$ and $\langle \eta|T\xi\rangle = \alpha\big(\langle S\eta|\xi\rangle\big)$ for all $\xi \in E$ and $\eta \in E'$,

ii) $\mathrm{Im}(T) \subset \mathrm{Im}(\beta)E'$ and $Tb\xi = \beta(b)T\xi$ for all $b \in \mathrm{Dom}(\beta)$ and all $\xi \in E$.

The (β,α)-homogeneous operators share many properties of ordinary operators on C^*-modules.

Proposition 1.2. *Let T and S as above. Then*

i) T is linear and bounded,

ii) the map S is uniquely determined by T and α,

iii) S is a (β^*,α^*)-homogeneous operator,

iv) if $\alpha' \in \mathrm{PAut}(A)$ extends α, then $T(\xi a') = (T\xi)\alpha(a')$ for all $a' \in \mathrm{Dom}(\alpha')$ and $\xi \in E$,

v) $T(\xi u_\nu) \to T\xi$ for each approximate unit $(u_\nu)_\nu$ of the ideal $\mathrm{Dom}(\alpha)$ and each $\xi \in E$,

vi) $T(v_\mu\xi) \to T\xi$ for each approximate unit $(v_\mu)_\mu$ of the ideal $\mathrm{Dom}(\beta)$ and each $\xi \in E$,

vii) if $\beta' \in \mathrm{PAut}(B)$ extends β, then $Tb\xi = \beta'(b)T\xi$ for all $b' \in \mathrm{Dom}(\beta')$ and $\xi \in E$,

viii) $\mathrm{Im}\,T \subset \mathrm{Im}(\beta)E'\,\mathrm{Im}(\alpha)$,

ix) the space $\langle E'|TE\rangle$ is an ideal in A, *x) if* $(\beta', \alpha') \in \mathrm{PAut}(B) \times \mathrm{PAut}(A)$ *extends* (β, α)*, then* T *is also* (β', α')*-homogeneous.*

Proof. Let $(u_\nu)_\nu$ and $(v_\mu)_\mu$ be bounded approximate units for the ideals $\mathrm{Dom}(\alpha)$ and $\mathrm{Dom}(\beta)$, respectively.

i,ii) If α is equal to the identity on A, the first two statements reduce to well-known facts about maps on C^*-modules. Their proofs carry over to the general case easily.

iii) Clearly, $\langle TE|E\rangle \subset \mathrm{Dom}(\alpha^*)$ and $\langle \xi|S\eta\rangle = \alpha^*\big(\langle T\xi|\eta\rangle\big)$ for all $\eta \in E'$ and $\xi \in E$. Furthermore, for all $b \in \mathrm{Dom}(\beta^*)$ and $\eta \in E'$, one has

$$\langle Sb\eta|\xi\rangle = \alpha^*\big(\langle \eta|b^*T\xi\rangle\big) = \alpha^*\big(\big\langle \eta\big|T\beta^*(b^*)\xi\big\rangle\big) = \big\langle \beta^*(b)S\big|\xi\big\rangle, \quad \xi \in E,$$

and therefore $Sb\eta = \beta^*(b)S\eta$. Finally, let us show that the image of S is contained in $\mathrm{Im}(\beta^*)E$. For each element $\eta \in E'$, the net $(v_\mu S\eta)_\mu$ converges to $S\eta$ because

$$\begin{aligned}\big\langle (1-v_\mu)S\eta\big|(1-v_\mu)S\eta\big\rangle &= \alpha^*\big(\big\langle \eta\big|T(1-v_\mu)^*(1-v_\mu)S\eta\big\rangle\big)\\ &= \alpha^*\big(\big\langle \eta\big|(1-\beta(v_\mu))^*(1-\beta(v_\mu))TS\eta\big\rangle\big)\end{aligned}$$

and $\lim_\mu (1-\beta(v_\mu))^*(1-\beta(v_\mu))TS\eta = 0$ because the image of T is contained in $\mathrm{Im}(\beta)E'$.

iv) For all $a' \in \mathrm{Dom}(\alpha')$ and all $\xi \in E, \eta \in E'$, one has

$$\big\langle \eta\big|T(\xi a')\big\rangle = \alpha\big(\langle S\eta|\xi a'\rangle\big) = \alpha\big(\langle S\eta|\xi\rangle\big)\alpha'(a') = \langle \eta|T\xi\rangle\alpha'(a') = \big\langle \eta\big|(T\xi)\alpha'(a')\big\rangle.$$

v) For each $\xi \in E$ one has

$$\big\langle T(\xi - \xi u_\nu)\big|T(\xi - \xi u_\nu)\big\rangle = (1-\alpha(u_\nu))^* \langle T\xi|T\xi\rangle (1-\alpha(u_\nu)) \xrightarrow{\nu\to\infty} 0$$

because $\langle T\xi|T\xi\rangle$ is contained in the ideal $\mathrm{Im}(\alpha)$.

vi) From part ii) of the previous definition, $\beta(v_\mu)(T\xi) = T(v_\mu\xi)$ converges to $T\xi$ for each $\xi \in E$.

vii) For each $b' \in \mathrm{Dom}(\beta')$ and $\xi \in E$, one has

$$Tb'\xi = \lim_\mu Tv_\mu b'\xi = \lim_\mu \beta(v_\mu b')T\xi = \lim_\mu \beta(v_\mu)\beta'(b')T\xi = \beta'(b')T\xi.$$

Here, we used part vi) and the fact that $\beta'(b')T\xi$ is contained in $\mathrm{Im}(\beta)E'$.

viii) One has $\mathrm{Im}\, T \subset E'\,\mathrm{Im}(\alpha)$ by iv) and v) and $\mathrm{Im}\, T \subset \mathrm{Im}(\beta)E'$ by definition.

ix) It is clear that $\langle E'|TE\rangle$ is a left-sided ideal in A. On the other hand, for all $\xi \in E, \eta \in E'$ and all $a \in A$, one has

$$\langle \eta|T\xi\rangle a = \lim_\nu \langle \eta|T\xi\rangle a u_\nu = \lim_\nu \langle \eta|(T\xi)au_\nu\rangle = \lim_\nu \langle \eta|T(\xi\alpha^*(au_\nu))\rangle \in \langle E'|TE\rangle.$$

x) This is obvious from the part vii) and the definition. □

Definition 1.3. *The* α-adjoint *of a* (β,α)*-homogeneous operator* $T\colon E \to E'$ *is the unique map* $S\colon E' \to E$ *satisfying the conditions of definition 1.1.*

First examples of homogeneous operators already occurred in the introductory motivation. The following example will reappear in subsection 1.2.3.

Example 1.4. Given unitaries $v \in B$ and $u \in A$, the map $O_{v,u}\colon E \to E$ given by $\xi \mapsto v\xi u^*$ is (β,α)-homogeneous, where $\beta := \mathrm{Ad}_v$ and $\alpha := \mathrm{Ad}_u$ are the inner automorphisms of B and A given by $b \mapsto vbv^*$ and $a \mapsto uau^*$, respectively.

This follows from the equations

$$\begin{aligned} &O_{v,u}(b\xi) = vb\xi u^* = (vbv^*)v\xi u^* = \mathrm{Ad}_v(b)O_{v,u}\xi, \\ &\langle \eta | O_{v,u}\xi\rangle = \langle \eta | v\xi u^*\rangle = uu^*\langle v^*\eta|\xi\rangle u^* = \mathrm{Ad}_u\big(\langle v^*\eta u|\xi\rangle\big), \quad \xi,\eta \in E. \end{aligned}$$

The following construction will be used in subsection 1.1.4.

Example 1.5. Consider the space $\mathrm{Im}(\alpha) \oplus \mathrm{Dom}(\alpha)$ with the maps $T := \left(\begin{smallmatrix} 0 & \alpha \\ 0 & 0 \end{smallmatrix}\right)$ and $S := \left(\begin{smallmatrix} 0 & 0 \\ \alpha^* & 0 \end{smallmatrix}\right)$. These are prototypes of homogeneous operators.

i) Consider the space $E_\alpha := \mathrm{Im}(\alpha) \oplus \mathrm{Dom}(\alpha)$ as a C^*-$\mathbb{C}$-A-bimodule in the canonical way:

$$\big\langle (a,b)\big|(a',b')\big\rangle := a^*a' + b^*b', \quad (a,b)c := (ac,bc), \quad \lambda(a,b) := (\lambda a, \lambda b).$$

Then $T_\alpha := T$ is an (id,α)-homogeneous operator with adjoint $S_\alpha := S$:

$$\big\langle (a,b)\big| \underbrace{T(a',b')}_{(\alpha(b'),0)} \big\rangle = a^*\alpha(b') = \alpha(\alpha^*(a)^*b') = \alpha\big(\big\langle \underbrace{S(a,b)}_{(0,\alpha^*(a))} \big|(a',b')\big\rangle\big)$$

for all $a,a' \in \mathrm{Im}(\alpha)$ and all $b,b' \in \mathrm{Dom}(\alpha)$.

ii) Consider $E^\alpha := \mathrm{Im}(\alpha) \oplus \mathrm{Dom}(\alpha)$ as a C^*-A-$\mathrm{Im}(\alpha)$-bimodule via

$$\big\langle (a,b)\big|(a',b')\big\rangle := a^*a' + \alpha(b^*b'), \quad (a,b)c := (ac, b\alpha^*(c)), \quad c(a,b) := (ca,cb).$$

Then $T^\alpha := T$ is an (α,id)-homogeneous operator with adjoint $S^\alpha := S$:

$$\big\langle (a,b)\big|T(a',b')\big\rangle = a^*\alpha(b') = \alpha(\alpha^*(a)^*b') = \big\langle S(a,b)\big|(a',b')\big\rangle$$

and

$$\mathrm{Im}\,T = \mathrm{Im}(\alpha) \oplus 0 \subset \mathrm{Im}(\alpha)E^\alpha, \qquad Tc(a,b) = \big(\alpha(cb),0\big) = \alpha(c)T(a,b)$$

for all $a,a' \in \mathrm{Im}(\alpha)$, $b,b',c \in \mathrm{Dom}(\alpha)$.

Uniqueness of the adjoint

We show that the adjoint of a homogeneous operator is uniquely determined by the operator itself and does not depend on the partial automorphism occurring in the adjunction equation.

Proposition 1.6. *Let $T\colon E \to E'$ be a (β,α)-homogeneous operator. Then the set $\Sigma := \{\alpha' \in \mathrm{PAut}(A) \mid T$ is (β',α')-homogeneous, $\beta' \in \mathrm{PAut}(B)\}$ has a smallest element.*

Proof. For each $\alpha' \in \Sigma$, denote the restriction of α' to the ideal $\alpha'^*\big(\langle E'|TE\rangle\big)$ by α'_0. Clearly, α'_0 belongs to Σ again. We show that α'_0 does not depend on the choice of α' and therefore is the smallest element in Σ.

Denote the α_0-adjoint and the α'_0-adjoint of T by S^α and $S^{\alpha'}$, respectively. Then

$$\alpha_0\big(\langle S^\alpha\eta|\xi\rangle b\big) = \alpha_0\big(\langle S^\alpha\eta|\xi b\rangle\big) = \big\langle\eta\big|T(\xi b)\big\rangle = \alpha'_0\big(\langle S^{\alpha'}\eta|\xi b\rangle\big) = \alpha'_0\big(\langle S^{\alpha'}\eta|\xi\rangle b\big)$$

for all $\xi \in E, \eta \in E'$ and all $b \in A$. Therefore,

$$\alpha_0\big(\alpha_0^*(a)b\big) = \alpha'_0\big(\alpha_0'^*(a)b\big) \quad \text{for all} \quad a \in \langle E'|TE\rangle,\, b \in A.$$

Let $(u_\nu)_\nu$ be an approximate unit for the ideal $\langle E'|TE\rangle$. Then $\alpha_0'^*(u_\nu)$ and $\alpha_0^*(u_\nu)$ form approximate units for the ideals $\mathrm{Dom}(\alpha'_0)$ and $\mathrm{Dom}(\alpha_0)$, respectively. Let $b \in \mathrm{Dom}(\alpha_0)$. Then the equation above implies that the limit $c := \lim_\nu \alpha_0'^*(u_\nu)b$ exists and that $\alpha_0(b) = \alpha'_0(c)$. Clearly $c \in \mathrm{Dom}(\alpha_0) \cap \mathrm{Dom}(\alpha'_0)$ and hence

$$\alpha_0(c) = \lim_\nu \alpha_0\big(\alpha_0^*(u_\nu)c\big) = \lim_\nu \alpha'_0\big(\alpha_0'^*(u_\nu)c\big) = \alpha'_0(c) = \alpha_0(b).$$

In particular, $c = b$. Therefore, $\mathrm{Dom}(\alpha_0)$ is contained in $\mathrm{Dom}(\alpha'_0)$, and α'_0 extends α_0. Hence, $\alpha'_0 = \alpha_0$. □

Corollary 1.7. *If $T\colon E \to E'$ is simultaneously (β,α)- and (β',α')-homogeneous for some (β,α) and $(\beta',\alpha') \in \mathrm{PAut}(B)\times\mathrm{PAut}(A)$, its α-adjoint and its α'-adjoint are equal.*

Notation 1.8. *We call the α-adjoint of a (β,α)-homogeneous operator $T\colon E \to E'$ simply* its adjoint *and denote it by T^*.*

Relation to ordinary operators

The (β,α)-homogeneous operators $E \to E'$ are closely related to ordinary operators on C^*-bimodules with suitably twisted structure maps, as will be explained in the following. Consider the subspaces

$$E_{(\beta,\alpha)} := \mathrm{Dom}(\beta)E\,\mathrm{Dom}(\alpha) \subset E, \qquad E'^{(\beta,\alpha)} := \mathrm{Im}(\beta)E'\,\mathrm{Im}(\alpha) \subset E'$$

as C^*-Dom(β)-Dom(α)-modules, where the structure maps of $E_{(\beta,\alpha)}$ are inherited from E and the structure maps of $E'^{(\beta,\alpha)}$ are twisted by α and β in the obvious way: Denote by $\iota\colon E'^{(\beta,\alpha)} \hookrightarrow E'$ the inclusion, then

$$\langle\eta|\xi\rangle_{E'^{(\beta,\alpha)}} := \alpha\big(\langle\iota(\eta)|\iota(\xi)\rangle_E\big), \quad \xi \cdot a := \iota^{-1}\big(\iota(\xi)\alpha(a)\big), \quad b \cdot \xi := \iota^{-1}\big(\beta(b)\iota(\xi)\big)$$

for all $\xi, \eta \in E'^{(\beta,\alpha)}$ and $b \in \mathrm{Dom}(\beta)$.

Proposition 1.9. *Let $T\colon E \to E'$ be a (β,α)-homogeneous operator. Then T and T^* restrict to a pair of adjoint operators $\tilde{T}\colon E_{(\beta,\alpha)} \to E'^{(\beta,\alpha)}$ and $\tilde{T}^*\colon E'^{(\beta,\alpha)} \to E_{(\beta,\alpha)}$ of C^*-modules which intertwine the representations of* Dom(β). *These restrictions uniquely determine T and T^*, and $\|\tilde{T}\| = \|T\|$.*

Proof. Parts iii) and viii) of proposition 1.2 imply that T and T^* restrict to linear maps $\tilde{T}\colon E_{(\beta,\alpha)} \to E'^{(\beta,\alpha)}$ and $\tilde{T}^*\colon E'^{(\beta,\alpha)} \to E_{(\beta,\alpha)}$. By definition, one has

$$\big\langle\eta\big|\tilde{T}\xi\big\rangle_{(E'^{(\beta,\alpha)})} = \alpha^*\big(\langle\eta|T\xi\rangle_{E'}\big) = \langle T^*\eta|\xi\rangle_E = \big\langle\tilde{T}^*\eta\big|\xi\big\rangle_{(E_{(\beta,\alpha)})}$$

for all $\xi \in E_{(\beta,\alpha)}$ and $\eta \in E'^{(\beta,\alpha)}$. It is easy to see that the restrictions $\tilde{T}$ and $\tilde{T}^*$ intertwine the representations of Dom(β). The fact that T is determined by $\tilde{T}$ and has the same norm follows from parts v) and vi) of proposition 1.2. □

In general, not every ordinary operator $E_{(\beta,\alpha)} \to E'^{(\beta,\alpha)}$ extends to a (β,α)-homogeneous operator from E to E'. This is illustrated by the following example.

Example 1.10. Let α be the identity on an ideal J of a unital C^*-algebra A and put $E = E' = A$. Let $B = \mathbb{C}$ act by scalar multiplication and put $\beta = \mathrm{id}_{\mathbb{C}}$. Then $L_{\mathrm{Dom}(\alpha)}\big(E_{(\alpha,\mathrm{id})}, E'^{(\alpha,\mathrm{id})}\big) = M(J)$, but every (id,α)-homogeneous operator $E \to E'$ is an ordinary operator on A and hence given by left multiplication by an element in A. In general, $A \subsetneqq M(J)$.

1.1.2 C^*-pre-families of homogeneous operators

This subsection introduces the notion of C^*-pre-families of homogeneous operators. In our generalisation of the Baaj-Skandalis construction, the legs of a decomposable pseudo-multiplicative unitary will turn out to be such C^*-pre-families.

As a preparation, we collect some easy properties of the family of all homogeneous operators on a C^*-bimodule. It resembles a graded C^*-algebra, but – as the ensuing discussion shows – the involution does not extend to the closed linear span of all homogeneous operators. We give the precise definition of a C^*-pre-family and carry over some notions from C^*-algebras to C^*-pre-families. The subsection ends with a discussion of the C^*-pre-family of all compact homogeneous operators.

The family of all homogeneous operators on a C^*-bimodule

Notation 1.11. *Let α and β be partial automorphisms of A and B, respectively. We denote the set of all (β,α)-homogeneous operators $E \to E'$ by $\mathscr{L}_{\alpha}^{\beta}(E,E')$, and the family $\big(\mathscr{L}_{\alpha'}^{\beta'}(E,E')\big)_{\alpha',\beta'}$ by $\mathscr{L}(E,E')$. Furthermore, we denote by $\mathscr{L}^{\beta}(E,E')$ the sub-family consisting of the members with fixed index β and varying index α.*

The next proposition summarises some easy properties of the family of all homogeneous operators $E \to E'$. Let E'' be an additional C^*-B-A-bimodule.

Proposition 1.12. *For all $\alpha,\alpha' \in \mathrm{PAut}(A)$ and all $\beta,\beta' \in \mathrm{PAut}(B)$, one has:*

i) $\mathscr{L}_{\alpha'}^{\beta'}(E',E'')\mathscr{L}_{\alpha}^{\beta}(E,E') \subset \mathscr{L}_{\alpha'\alpha}^{\beta'\beta}(E,E'')$.

ii) $\big(\mathscr{L}_{\alpha}^{\beta}(E,E')\big)^* = \mathscr{L}_{\alpha^*}^{\beta^*}(E',E)$. *The map $T \mapsto T^*$ is anti-linear, isometric and anti-multiplicative in the sense that $(T'T)^* = T^*T'^*$ for all $T' \in \mathscr{L}_{\alpha'}^{\beta'}(E',E'')$ and $T \in \mathscr{L}_{\alpha}^{\beta}(E,E')$. Furthermore, $\|T^*T\| = \|T\|^2$ for all $T \in \mathscr{L}_{\alpha}^{\beta}(E,E')$.*

iii) For each pair of partial identities $\epsilon \in \mathrm{PAut}(A)$ and $\epsilon' \in \mathrm{PAut}(B)$, the space $\mathscr{L}_{\epsilon}^{\epsilon'}(E)$ is a C^*-subalgebra of $L_A(E)$.

iv) The subset $\mathscr{L}_{\alpha}^{\beta}(E,E')$ of $L(E,E')$ is a C^*-bimodule over $\mathscr{L}_{\alpha\alpha^*}^{\beta\beta^*}(E')$ and $\mathscr{L}_{\alpha^*\alpha}^{\beta^*\beta}(E)$.

v) If (β',α') extends (β,α), then $\mathscr{L}_{\alpha}^{\beta}(E,E') \subset \mathscr{L}_{\alpha'}^{\beta'}(E,E')$.

Proof. Many of these statements are natural generalisations of the corresponding facts about ordinary operators on C^*-bimodules and can be proved in a similar way or follow directly from proposition 1.2. Therefore, we only prove part i).

i) Let $T \in \mathscr{L}_{\alpha}^{\beta}(E,E')$ and $T' \in \mathscr{L}_{\alpha'}^{\beta'}(E',E'')$. We check that $T'T$ satisfies condition i) of definition 1.1. One has inclusions

$$\langle E''|T'TE\rangle = \alpha'\big(\langle T'^*E''|TE\rangle\big) \subset \alpha'\big(\mathrm{Dom}(\alpha') \cap \mathrm{Im}(\alpha)\big) = \mathrm{Im}(\alpha'\alpha),$$
$$\langle T^*T'^*E''|E\rangle = \alpha^*\big(\langle T'^*E''|TE\rangle\big) \subset \alpha^*\big(\mathrm{Dom}(\alpha') \cap \mathrm{Im}(\alpha)\big) = \mathrm{Dom}(\alpha'\alpha).$$

Furthermore, for all $\xi \in E$ and $\eta \in E''$, one has

$$\langle \eta|T'T\xi\rangle = \alpha'\big(\langle T'^*\eta|T\xi\rangle\big) = (\alpha'\alpha)\big(\langle T^*T'^*\eta|\xi\rangle\big).$$

Next, we check that $T'T$ satisfies condition ii) of definition 1.1. The image of the composition $T'T$ is contained in

$$T'\,\mathrm{Im}(\beta)E = T'\big(\mathrm{Dom}(\beta') \cap \mathrm{Im}(\beta)\big)E \subset \beta'\big(\mathrm{Dom}(\beta') \cap \mathrm{Im}(\beta)\big)E = \mathrm{Im}(\beta'\beta)E.$$

Furthermore, for each $b \in \mathrm{Dom}(\beta'\beta)$ and $\xi \in E$, one has $T'Tb\xi = T'\beta(b)T\xi = \beta'(\beta(b))T'T\xi$. Therefore, $T'T$ is $(\beta'\beta,\alpha'\alpha)$-homogeneous with adjoint $T^*T'^*$. □

Why homogeneous operators do not form C^*-algebras

It is tempting to form the $*$-algebra generated by all homogeneous operators on a C^*-bimodule and to complete it to obtain a C^*-algebra. However, the involution does not extend to an isometry on this linear span and therefore the operator norm does not give rise to a C^*-norm on the closed linear span. We illustrate this phenomenon by the following example.

Regard $\mathbb{N}$ as a discrete space and let $A = C_0(\mathbb{N})$. For each $k \in \mathbb{N}$, denote by δ_k the element of A defined by $\delta_k(l) := \delta_{kl}$. For all $k, l \in \mathbb{N}$, denote by α_{kl} the partial automorphism with domain $\mathbb{C}\delta_l$ and range $\mathbb{C}\delta_k$ mapping δ_l to δ_k. Denote by e_{kl} the map $A \to A$ given by $(e_{kl}f)(m) = \delta_{km}f(l)$.

Consider A as a C^*-module over itself and let $B := \mathbb{C}$ act by scalar multiplication. Observe that the map e_{kl} is a well-defined adjointable operator on the Hilbert space $l^2(\mathbb{N})$, but is not an adjointable operator on the C^*-module $A = C_0(\mathbb{N})$ unless $k = l$. However, for each k and l in $\mathbb{N}$, the map e_{kl} is an $(\mathrm{id}, \alpha_{kl})$-homogeneous operator with adjoint e_{lk}. Let $(c_m)_m$ be a sequence of complex numbers converging to 0. Then the sum $\sum c_m e_{m1}$ converges in $L(A)$, but the sum of the adjoints $\sum \overline{c_m} e_{1m}$ converges if and only if the sequence $(c_m)_m$ is summable. Therefore, the involution $*$ does not extend from the set of all homogeneous operators to the closed linear span.

Likewise, the family of homogeneous operators comprising the left regular representation of a groupoid need not be contained in a C^*-algebra consisting of bounded linear maps on the Banach space $L^2(G, \lambda)$. For an example, consider the discrete groupoid of the full equivalence relation on $\mathbb{N}$, i.e. $G = \mathbb{N} \times \mathbb{N}$, $G^0 = \mathbb{N}$, and the structure maps are given by

$$r(k,l) := k, \qquad s(k,l) := l, \qquad (k,l)\cdot(l,m) := (k,m), \quad k,l,m \in \mathbb{N},$$

and reconsider the preceding discussion.

Summarising, we see that the notion of a C^*-algebra has to be replaced by a notion of a *C^*-pre-family* which takes the gradings of the homogeneous operators into account.

Definition and first properties

Definition 1.13. *Let $\mathscr{C}, \mathscr{C}' \subset \mathscr{L}(E, E')$ be families of closed subspaces. We call $\mathscr{C}$ a* sub-family *of $\mathscr{C}'$ and write $\mathscr{C} \subset \mathscr{C}'$ if $\mathscr{C}^\beta_\alpha \subset \mathscr{C}'^\beta_\alpha$ for all $\alpha \in \mathrm{PAut}(A)$ and all $\beta \in \mathrm{PAut}(B)$. We denote by $\mathscr{C}^{\mathrm{id}}$ and $\mathscr{C}_{\mathrm{id}}$ the families $\big(\mathscr{C}^{\mathrm{id}}_\alpha\big)_{\alpha \in \mathrm{PAut}(A)}$ and $\big(\mathscr{C}^\beta_{\mathrm{id}}\big)_{\beta \in \mathrm{PAut}(B)}$, respectively. In particular, we apply this notation to the family $\mathscr{L}(E, E')$ and $\mathscr{K}(E, E')$. By a useful slight abuse of notation, we identify $\mathscr{C}^{\mathrm{id}}$ and $\mathscr{C}_{\mathrm{id}}$ with the families given by*

$$(\mathscr{C}^{\mathrm{id}})^\beta_\alpha := \begin{cases} \mathscr{C}^{\mathrm{id}}_\alpha, & \beta = \mathrm{id}, \\ \{0\}, & \beta \neq \mathrm{id}, \end{cases} \qquad (\mathscr{C}_{\mathrm{id}})^\beta_\alpha := \begin{cases} \mathscr{C}^\beta_{\mathrm{id}}, & \alpha = \mathrm{id}, \\ \{0\}, & \alpha \neq \mathrm{id}, \end{cases}$$

respectively. We denote by $\mathscr{C}^* \subset \mathscr{L}(E', E)$ *the family of closed subspaces given by*

$$(\mathscr{C}^*)^\beta_\alpha := \left(\mathscr{C}^{\beta^*}_{\alpha^*}\right)^*, \qquad \alpha \in \mathrm{PAut}(A),\, \beta \in \mathrm{PAut}(B).$$

Let $\mathscr{D} \subset \mathscr{L}(E', E'')$ *be another family of closed subspaces. The* product $\mathscr{D}\mathscr{C} \subset \mathscr{L}(E, E'')$ *is the family of closed subspaces given by*

$$(\mathscr{D}\mathscr{C})^{\beta''}_{\alpha''} := \overline{\operatorname{span}}_{\beta'\beta \leq \beta'',\, \alpha'\alpha \leq \alpha''} \mathscr{D}^{\beta'}_{\alpha'}\mathscr{C}^\beta_\alpha, \quad \alpha'' \in \mathrm{PAut}(A),\, \beta'' \in \mathrm{PAut}(B).$$

Likewise, given a homogeneous operator $T \in \mathscr{L}^{\beta'}_{\alpha'}(E', E'')$ *where* $\alpha' \in \mathrm{PAut}(A)$ *and* $\beta' \in \mathrm{PAut}(B)$*, the* product $T\mathscr{C} \subset \mathscr{L}(E, E'')$ *is defined as*

$$(T\mathscr{C})^{\beta''}_{\alpha''} := \overline{\operatorname{span}}_{\beta'\beta \leq \beta'',\, \alpha'\alpha \leq \alpha''} T\mathscr{C}^\beta_\alpha, \quad \alpha'' \in \mathrm{PAut}(A),\, \beta'' \in \mathrm{PAut}(B).$$

Products of the form $\mathscr{D}S$ *are defined similarly.*

It is easy to see that the product of families thus defined is associative.

Definition 1.14. *A* C^*-pre-family on E *is a family* $\mathscr{C} \subset \mathscr{L}(E)$ *of closed subspaces satisfying* $\mathscr{C}^*\mathscr{C} \subset \mathscr{C}$ *and* $\mathscr{C}^\beta_\alpha \subset \mathscr{C}^{\beta'}_{\alpha'}$ *for all* $(\beta, \alpha), (\beta', \alpha') \in \mathrm{PAut}(B) \times \mathrm{PAut}(A)$ *such that* $(\beta, \alpha) \leq (\beta', \alpha')$*. A* C^*-pre-family module from E to E' *is a family* $\mathscr{D} \subset \mathscr{L}(E, E')$ *of closed subspaces satisfying* $\mathscr{D}\mathscr{D}^*\mathscr{D} \subset \mathscr{D}$ *and* $\mathscr{D}^\beta_\alpha \subset \mathscr{D}^{\beta'}_{\alpha'}$ *whenever* $(\beta, \alpha) \leq (\beta', \alpha')$.

Remarks 1.15. Let $\mathscr{D}$ be a C^*-pre-family module and let $\mathscr{C}$ be a C^*-pre-family.

i) The products $\mathscr{D}^*\mathscr{D}$ and $\mathscr{D}\mathscr{D}^*$ are C^*-pre-families.

ii) For each pair of partial identities $\epsilon \in \mathrm{PAut}(A)$ and $\epsilon' \in \mathrm{PAut}(B)$, the space $\mathscr{C}^{\epsilon'}_\epsilon$ is a C^*-subalgebra of $\mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(E) \subset L_A(E)$ – it is closed with respect to the norm and the algebraic operations:

$$(\mathscr{C}^{\epsilon'}_\epsilon)^* = \mathscr{C}^{\epsilon'^*}_{\epsilon^*} = \mathscr{C}^{\epsilon'}_\epsilon, \qquad\qquad \mathscr{C}^{\epsilon'}_\epsilon \cdot \mathscr{C}^{\epsilon'}_\epsilon \subset \mathscr{C}^{\epsilon'\epsilon'}_{\epsilon\epsilon} = \mathscr{C}^{\epsilon'}_\epsilon.$$

iii) For each $\beta \in \mathrm{PAut}(B)$ and $\alpha \in \mathrm{PAut}(A)$, the space $\mathscr{C}^\beta_\alpha$ is a C^*-module over the C^*-algebra $\mathscr{C}^{\beta^*\beta}_{\alpha^*\alpha}$:

$$\left(\mathscr{C}^\beta_\alpha\right)^* \cdot \mathscr{C}^\beta_\alpha = \mathscr{C}^{\beta^*}_{\alpha^*}\mathscr{C}^\beta_\alpha \subset \mathscr{C}^{\beta^*\beta}_{\alpha^*\alpha}, \qquad \mathscr{C}^\beta_\alpha \cdot \mathscr{C}^{\beta^*\beta}_{\alpha^*\alpha} \subset \mathscr{C}^{\beta\beta^*\beta}_{\alpha\alpha^*\alpha} = \mathscr{C}^\beta_\alpha.$$

It is simultaneously a left C^*-module over the C^*-algebra $\mathscr{C}^{\beta\beta^*}_{\alpha\alpha^*}$ and in fact a C^*-bimodule over both C^*-algebras in the sense of [17]. As a result of the first observation, we obtain $\mathscr{C}^\beta_\alpha = \mathscr{C}^\beta_\alpha \cdot \mathscr{C}^{\mathrm{id}}_{\mathrm{id}}$. Hence, $\mathscr{C}\,\mathscr{C}^{\mathrm{id}}_{\mathrm{id}} = \mathscr{C}$. In particular, $\mathscr{C}^*\mathscr{C} = \mathscr{C}$. A similar argument shows that for each C^*-pre-family module $\mathscr{D}$, one has $\mathscr{D}\mathscr{D}^*\mathscr{D} = \mathscr{D}$.

iv) For each family $\mathscr{C} \subset \mathscr{L}(E)$ of subsets of homogeneous operators, there exists a smallest C^*-pre-family containing $\mathscr{C}$. This smallest family is said to be *generated* by the family $\mathscr{C}$. Likewise, for each family $\mathscr{D} \subset \mathscr{L}(E, E')$ of subsets of homogeneous operators, there exists a smallest C^*-pre-family module containing $\mathscr{D}$.

Several notions and constructions known from C^*-algebras or C^*-categories have natural analogues for C^*-pre-families.

Definition 1.16. *Let $\mathscr{C}$ be a C^*-pre-family on E.*

i) $\mathscr{C}$ is non-degenerate *if* $\overline{\operatorname{span}}_{\alpha,\beta}(\mathscr{C}_\alpha^\beta E) = E$.

ii) The multiplier family *of $\mathscr{C}$ is the family of subspaces $\mathscr{M}(\mathscr{C}) \subset \mathscr{L}(E)$ given by*

$$\mathscr{M}(\mathscr{C})_\alpha^\beta := \{T \in \mathscr{L}_\alpha^\beta(E) \mid T\mathscr{C}, \mathscr{C}T \subset \mathscr{C}\}, \quad \alpha \in \mathrm{PAut}(A),\ \beta \in \mathrm{PAut}(B).$$

It is easy to see that this is a C^*-pre-family.

By remark 1.15 iii), these definitions can be reformulated as follows.

Remarks 1.17. Let $\mathscr{C}$ be a C^*-pre-family.

i) An operator T belongs to $\mathscr{M}(\mathscr{C})$ if and only if $T\mathscr{C}_{\mathrm{id}}^{\mathrm{id}}$ and $\mathscr{C}_{\mathrm{id}}^{\mathrm{id}}T$ are contained in $\mathscr{C}$.

ii) The C^*-pre-family $\mathscr{C}$ is non-degenerate if and only if the C^*-subalgebra $\mathscr{C}_{\mathrm{id}_A}^{\mathrm{id}_B} \subset L_A(E)$ is non-degenerate.

Intuitively, one should think of a C^*-pre-family $\mathscr{C}$ as a "C^*-algebra graded by an inverse semigroup" or rather as "sections of a Fell bundle over an inverse semigroup". The problem with the latter analogy is that the inclusion condition $\mathscr{C}_\alpha^\beta \subset \mathscr{C}_{\alpha'}^{\beta'}$ for $(\beta, \alpha) \leq (\beta', \alpha')$ is not complemented by a decomposition condition or a sheaf-like condition. This deficiency is remedied by the notion of a C^*-family which is introduced in subsection 1.2.3.

The C^*-pre-family of compact homogeneous operators

A standard example of a C^*-pre-family is the family of all compact homogeneous operators. To us, it is of minor relevance because it may be very small and even consist of the zero operator only. Replacements for this family are considered in the more appropriate setting of decomposable C^*-bimodules in section 1.2.

Lemma 1.18. *Consider E and E' as C^*-$\mathbb{C}$-A-bimodules. For each $\xi \in E\,\mathrm{Dom}(\alpha)$ and $\eta \in E'\,\mathrm{Im}(\alpha)$, the map $T_{\eta,\xi}^\alpha \colon E \to E'$ given by $\zeta \mapsto \eta\alpha(\langle\xi|\zeta\rangle)$ defines an (id, α)-homogeneous operator. One has $\left(T_{\eta,\xi}^\alpha\right)^* = T_{\xi,\eta}^{\alpha^*}$ and $\|T_{\eta,\xi}^\alpha\| \leq \|\xi\|\|\eta\|$.*

Proof. Let $\zeta \in E$ and $\zeta' \in E'$. Then the inner products $\langle\xi|\zeta\rangle$ and $\langle\zeta'|\eta\rangle$ are contained in $\mathrm{Dom}(\alpha)$ and $\mathrm{Im}(\alpha)$, respectively, and

$$\begin{aligned}\langle\zeta'|T^{\alpha}_{\eta,\xi}\zeta\rangle &= \langle\zeta'|\eta\rangle\alpha\big(\langle\xi|\zeta\rangle\big) \\ &= \alpha\big(\alpha^*(\langle\zeta'|\eta\rangle)\langle\xi|\zeta\rangle\big) = \alpha\big(\big\langle\xi\alpha^*(\langle\eta|\zeta'\rangle)\big|\zeta\big\rangle\big) = \alpha\big(\langle T^{\alpha^*}_{\xi,\eta}\zeta'|\zeta\rangle\big).\end{aligned}$$

Therefore, the operators $T := T^{\alpha}_{\eta,\xi}$ and $S := T^{\alpha*}_{\xi,\eta}$ satisfy the conditions of definition 1.1. □

Definition 1.19. *An operator $T\colon E \to E'$ is* elementary compact *if it is of the form $T = T^{\alpha}_{\eta,\xi}$ as in the proposition above, and* (β,α)-homogeneous compact *if it is (β,α)-homogeneous and contained in the closed linear span of all elementary compact operators with fixed α. We denote the space of all (β,α)-homogeneous compact operators by $\mathscr{K}^{\beta}_{\alpha}(E,E')$ and the family $\big(\mathscr{K}^{\beta}_{\alpha}(E,E')\big)_{\alpha,\beta}$ by $\mathscr{K}(E,E')$.*

Proposition 1.20. *i) The families $\mathscr{K}(E)$ and $\mathscr{K}(E,E')$ are a C^*-pre-family and a C^*-pre-family module, respectively.*

ii) For all partial automorphisms $\alpha,\alpha' \in \mathrm{PAut}(A)$ and $\beta,\beta' \in \mathrm{PAut}(B)$, the products $\mathscr{K}^{\beta'}_{\alpha'}(E',E'')\mathscr{L}^{\beta}_{\alpha}(E,E')$ and $\mathscr{L}^{\beta'}_{\alpha'}(E',E'')\mathscr{K}^{\beta}_{\alpha}(E,E')$ are contained in $\mathscr{K}^{\beta'\beta}_{\alpha'\alpha}(E,E'')$.

iii) If the C^*-pre-family $\mathscr{K}(E)$ is non-degenerate, the subspace $\mathscr{K}^{\beta}_{\alpha}(E,E')$ is strictly dense in $\mathscr{L}^{\beta}_{\alpha}(E,E')$ for all $\alpha \in \mathrm{PAut}(A), \beta \in \mathrm{PAut}(B)$.

Proof. i) By lemma 1.18, the adjoint of a compact operator is compact, too. The fact that the composition of two compact operators is compact again will follow from part ii). It is clear that $\mathscr{K}^{\beta}_{\alpha}(E,E')$ is contained in $\mathscr{K}^{\beta'}_{\alpha'}(E,E')$ if $(\beta,\alpha) \leq (\beta',\alpha')$.

ii) It is enough to show that the product of an elementary compact operator with another homogeneous operator can be approximated in norm by elementary compact operators which have the right homogeneity degree. Let $T^{\alpha'}_{\eta,\xi}\colon E' \to E''$ be an elementary compact operator and let $S \in \mathscr{L}^{\beta}_{\alpha}(E,E')$. Then ξ is contained in $E\,\mathrm{Dom}(\alpha')$, and by proposition 1.2, the element $S^*\xi$ is contained in $E\alpha^*\big(\mathrm{Dom}(\alpha') \cap \mathrm{Im}(\alpha)\big) = E\,\mathrm{Dom}(\alpha'\alpha)$. Let $(u_\nu)_\nu$ be an approximate unit for the ideal $\mathrm{Im}(\alpha'\alpha)$ and put $v_\nu := (\alpha'\alpha)^*(u_\nu)$. Consider the net of operators $T_\nu := T^{\alpha'\alpha}_{\eta u_\nu, S^*\xi}$. For each $\zeta \in E$, one has

$$T_\nu\zeta = \eta u_\nu \cdot (\alpha'\alpha)\big(\langle S^*\xi|\zeta\rangle\big) = \eta \cdot (\alpha'\alpha)\big(\big\langle(S^*\xi)v_\nu^*\big|\zeta\big\rangle\big).$$

Since $S^*\xi = \lim_\nu (S^*\xi)v_\nu^*$ and $(\alpha'\alpha)\big(\big\langle S^*\xi\big|\zeta\big\rangle\big) = \alpha\big(\langle\xi|S\zeta\rangle\big)$, the net $\big(T_\nu\big)_\nu$ converges to $T^{\alpha}_{\eta,\xi}S$ in norm. Hence, $\mathscr{K}^{\beta'}_{\alpha'}(E',E'')\mathscr{L}^{\beta}_{\alpha}(E,E')$ is contained in $\mathscr{K}^{\beta'\beta}_{\alpha'\alpha}(E,E'')$.

The second inclusion is obtained by taking adjoints.

iii) If the assumption is satisfied, the C^*-algebra $\mathscr{K}^{\mathrm{id}}_{\mathrm{id}}(E)$ acts non-degenerately on E, whence there exists a net $(u_\nu)_\nu \in \mathscr{K}^{\mathrm{id}}_{\mathrm{id}}(E)$ which converges strictly to id_E.

Then for each operator $T \in \mathscr{L}_\alpha^\beta(E, E')$, the net $(Tu_\nu)_\nu$ lies in $\mathscr{K}_\alpha^\beta(E, E')$ and converges strictly to T. □

It is easy to see that the relation between homogeneous operators on C^*-bimodules and ordinary operators on C^*-bimodules with twisted structure maps is bijective for compact operators.

Proposition 1.21. *The restriction $T \mapsto \tilde{T} \in L_{\mathrm{Dom}(\alpha)}(E_{(\beta,\alpha)}, E'^{(\beta,\alpha)})$ introduced in proposition 1.9 defines a bijection between the set of all (β,α)-homogeneous compact operators $E \to E'$ and ordinary compact operators $E_{(\beta,\alpha)} \to E'^{(\beta,\alpha)}$ of C^*-bimodules which commute with left multiplication.* □

1.1.3 The internal tensor product of C^*-pre-families

This section introduces the internal tensor product of C^*-pre-families, which is needed to define the notion of a Hopf C^*-pre-family and of a coaction: by analogy to other Hopf-like notions, a Hopf C^*-pre-family $\mathscr{S}$ should be equipped with a coproduct Δ which is some kind of morphism from $\mathscr{S}$ to $\mathscr{M}(\mathscr{S} \otimes_* \mathscr{S})$, and the internal tensor product $\mathscr{S} \otimes_* \mathscr{S}$ occuring in the range still has to be defined.

Given an internal tensor product $E \otimes_B F$ of suitable C^*-bimodules and two operators $S \in L(E)$, $T \in L(F)$, the equation $\xi \otimes_B b\eta = \xi b \otimes_B \eta$ implies that the map $\xi \otimes_B \eta \mapsto S\xi \otimes_B T\eta$ is only well-defined if S intertwines right multiplication by elements of B in a way which matches up with the way in which T intertwines left multiplication by elements of B. First, we formulate this compatibility condition.

Definition 1.22. *Two partial automorphisms β and β' of B are* compatible*, written $\beta \curlyvee \beta'$, if and only if $\beta^*\beta' \leq \mathrm{id}$ and $\beta\beta'^* \leq \mathrm{id}$.*

In general, compatibility is not an equivalence relation because it is not transitive: the trivial partial automorphism defined on the 0-ideal of B is compatible to every other partial automorphism. The following observation is immediate.

Lemma 1.23. *Let $\beta, \beta' \in \mathrm{PAut}(B)$ such that $\beta \curlyvee \beta'$. Then $\beta^* \curlyvee \beta'^*$. For each $b \in \mathrm{Dom}(\beta') \cap \mathrm{Dom}(\beta)$, one has $\beta(b) = \beta'(b)$.* □

Let C be a C^*-algebra and let F and F' be C^*-C-B-bimodules.

Proposition 1.24. *Let $T \in L_{\beta'}^\gamma(F, F')$ and $S \in L_\alpha^\beta(E, E')$ where $\gamma \in \mathrm{PAut}(C)$, $\alpha \in \mathrm{PAut}(A)$ and $\beta, \beta' \in \mathrm{PAut}(B)$ such that $\beta \curlyvee \beta'$. Then the map $T \odot S \colon \eta \odot \xi \mapsto T\eta \otimes_* S\xi$ defines a (γ,α)-homogeneous operator $T \otimes_* S \colon F \otimes_* E \to F' \otimes_* E'$ with norm $\|T \otimes_* S\| \leq \|S\|\|T\|$ and adjoint $(T \otimes_* S)^* = T^* \otimes_* S^*$.*

Proof. For each $n \in \mathbb{N}$, consider $\mathbb{C}^n$ with its standard inner product as a C^*-module over $\mathbb{C}$. Denote by $\mathbb{C}_n$ its conjugate which is a C^*-module over $M_n(\mathbb{C})$.

We write elements of $\mathbb{C}^n$ as column vectors and elements of $\mathbb{C}_n$ as row vectors. Put $M_n(B) := B \otimes M_n(\mathbb{C})$ and

$$\begin{aligned} \boldsymbol{E}^{(n)} &:= E \otimes \mathbb{C}^n, & \boldsymbol{E'}^{(n)} &:= E' \otimes \mathbb{C}^n, & \boldsymbol{S}^{(n)} &:= S \otimes \mathrm{id}_{\mathbb{C}^n}, \\ \boldsymbol{F}_{(n)} &:= F \otimes \mathbb{C}_n, & \boldsymbol{F'}_{(n)} &:= F' \otimes \mathbb{C}_n, & \boldsymbol{T}_{(n)} &:= T \otimes \mathrm{id}_{\mathbb{C}_n}. \end{aligned}$$

Then $\boldsymbol{F}_{(n)}, \boldsymbol{F'}_{(n)}$ are C^*-C-$M_n(B)$-bimodules, and $\boldsymbol{E}^{(n)}, \boldsymbol{E'}^{(n)}$ are C^*-$M_n(B)$-A-bimodules. The operators $\boldsymbol{T}$ and $\boldsymbol{S}$ are $(\gamma, \boldsymbol{\beta'})$- and $(\boldsymbol{\beta}, \alpha)$-homogeneous, respectively, where $\boldsymbol{\beta} := \beta \otimes \mathrm{id}_{M_n(\mathbb{C})}$ and $\boldsymbol{\beta'} := \beta \otimes \mathrm{id}_{M_n(\mathbb{C})}$.

First, we prove the adjunction equation for finite sums of elementary tensors. In the following, we fix n and omit the index n in the notation introduced above. Let $\zeta = \sum_{i=1}^n \eta_i \otimes_* \xi_i$ in $F \otimes_* E$ and $\zeta' = \sum_{i=1}^n \eta'_i \otimes_* \xi'_i$ in $F' \otimes_* E'$. Put

$$\begin{aligned} \boldsymbol{\xi} &= (\xi_1, \ldots, \xi_n)^T \in \boldsymbol{E}, & \boldsymbol{\eta} &= (\eta_1, \ldots, \eta_n) \in \boldsymbol{F}, \\ \boldsymbol{\xi'} &= (\xi'_1, \ldots, \xi'_n)^T \in \boldsymbol{E'}, & \boldsymbol{\eta'} &= (\eta'_1, \ldots, \eta'_n) \in \boldsymbol{F'}. \end{aligned}$$

The inner products on $F \otimes_* E$ and $\boldsymbol{F} \otimes_* \boldsymbol{E}$ are related as follows:

$$\begin{aligned} \langle \zeta | \zeta \rangle_{(F \otimes_* E)} &= \sum_{i,j} \big\langle \xi_i \big| \langle \eta_i | \eta_j \rangle_F \cdot \xi_j \big\rangle_E \\ &= \big\langle \boldsymbol{\xi} \big| \langle \boldsymbol{\eta} | \boldsymbol{\eta} \rangle_{\boldsymbol{F}} \cdot \boldsymbol{\xi} \big\rangle_{\boldsymbol{E}} = \langle \boldsymbol{\eta} \otimes_* \boldsymbol{\xi} | \boldsymbol{\eta} \otimes_* \boldsymbol{\xi} \rangle_{(\boldsymbol{F} \otimes_* \boldsymbol{E})} \end{aligned}$$

We use similar relations between $F' \otimes_* E'$ and $\boldsymbol{F'} \otimes_* \boldsymbol{E'}$ in the following calculations.

$$\big\langle \zeta' \big| (T \odot S) \zeta \big\rangle_{(F' \otimes_* E')} = \big\langle \boldsymbol{\xi'} \big| \langle \boldsymbol{\eta'} | \boldsymbol{T} \boldsymbol{\eta} \rangle_{\boldsymbol{F'}} \boldsymbol{S} \boldsymbol{\xi} \big\rangle_{\boldsymbol{E'}} = \big\langle \boldsymbol{\xi'} \big| \boldsymbol{\beta'} \big(\langle \boldsymbol{T}^* \boldsymbol{\eta'} | \boldsymbol{\eta} \rangle_{\boldsymbol{F}} \big) \boldsymbol{S} \boldsymbol{\xi} \big\rangle_{\boldsymbol{E'}}$$

Let $(u_\nu)_\nu$ and $(u'_{\nu'})_{\nu'}$ be approximate units for $\mathrm{Dom}(\beta)$ and $\mathrm{Dom}(\beta')$, respectively, and put $\boldsymbol{v}_{\nu',\nu} := \beta'(u'_{\nu'}) \beta(u_\nu) \otimes 1_{M_n(\mathbb{C})}$. Then the homogeneity of $\boldsymbol{T}$ and $\boldsymbol{S}$ implies

$$\big\langle \zeta' \big| (T \odot S) \zeta \big\rangle_{(F' \otimes_* E')} = \lim_{\nu,\nu'} \big\langle \boldsymbol{\xi'} \big| \boldsymbol{\beta'} \big(\langle \boldsymbol{T}^* \boldsymbol{\eta'} | \boldsymbol{\eta} \rangle_{\boldsymbol{F}} \big) \boldsymbol{v}_{\nu',\nu} \boldsymbol{S} \boldsymbol{\xi} \big\rangle_{\boldsymbol{E'}}.$$

Since $\boldsymbol{\beta'}(\langle \boldsymbol{T}^* \boldsymbol{\eta'} | \boldsymbol{\eta} \rangle) \boldsymbol{v}_{\nu',\nu}$ is contained in $\mathrm{Im}(\boldsymbol{\beta'}) \, \mathrm{Im}(\boldsymbol{\beta}) = \boldsymbol{\beta'}(\mathrm{Dom}(\boldsymbol{\beta}^* \boldsymbol{\beta'}))$ and $\boldsymbol{\beta}^* \boldsymbol{\beta'} \leq \mathrm{id}$, we have

$$\boldsymbol{\beta'}(\langle \boldsymbol{T}^* \boldsymbol{\eta'} | \boldsymbol{\eta} \rangle) \boldsymbol{v}_{\nu',\nu} \boldsymbol{S} = \boldsymbol{S} \boldsymbol{\beta}^* \big(\boldsymbol{\beta'}(\langle \boldsymbol{T}^* \boldsymbol{\eta'} | \boldsymbol{\eta} \rangle) \boldsymbol{v}_{\nu',\nu} \big) = \boldsymbol{S} \langle \boldsymbol{T}^* \boldsymbol{\eta'} | \boldsymbol{\eta} \rangle \boldsymbol{\beta'}^* (\boldsymbol{v}_{\nu',\nu}).$$

Thus, $\big\langle \zeta' \big| (T \odot S) \zeta \big\rangle$ is equal to

$$\lim_{\nu,\nu'} \big\langle \boldsymbol{\xi'} \big| \boldsymbol{S} \langle \boldsymbol{T}^* \boldsymbol{\eta'} | \boldsymbol{\eta} \rangle \boldsymbol{\beta'}^* (\boldsymbol{v}_{\nu',\nu}) \boldsymbol{\xi} \big\rangle_{\boldsymbol{E'}} = \lim_{\nu,\nu'} \alpha \big(\big\langle \boldsymbol{\beta'}^* (\boldsymbol{v}^*_{\nu',\nu}) \langle \boldsymbol{T}^* \boldsymbol{\eta'} | \boldsymbol{\eta} \rangle^* \boldsymbol{S}^* \boldsymbol{\xi'} \big| \boldsymbol{\xi} \big\rangle_{\boldsymbol{E}} \big).$$

Since $\langle \boldsymbol{T}^* \boldsymbol{\eta'} | \boldsymbol{\eta} \rangle^* \boldsymbol{S}^* \boldsymbol{\xi'}$ is contained in $\mathrm{Dom}(\boldsymbol{\beta'}) \, \mathrm{Dom}(\boldsymbol{\beta}) \boldsymbol{E}$, we have

$$\lim_{\nu,\nu'} \boldsymbol{\beta'}^* (\boldsymbol{v}^*_{\nu',\nu}) \langle \boldsymbol{T}^* \boldsymbol{\eta'} | \boldsymbol{\eta} \rangle^* \boldsymbol{S}^* \boldsymbol{\xi'} = \langle \boldsymbol{T}^* \boldsymbol{\eta'} | \boldsymbol{\eta} \rangle^* \boldsymbol{S}^* \boldsymbol{\xi'}.$$

Summarising, we obtain the adjunction equation:

$$\begin{aligned}\big\langle\zeta'\big|(T\odot S)\zeta\big\rangle &= \alpha\big(\big\langle\langle \boldsymbol{T}^*\boldsymbol{\eta'}|\boldsymbol{\eta}\rangle_{\boldsymbol{F}}^*\boldsymbol{S}^*\boldsymbol{\xi'}|\boldsymbol{\xi}\big\rangle_{\boldsymbol{E}}\big)\\ &= \alpha\big(\big\langle \boldsymbol{S}^*\boldsymbol{\xi'}\big|\langle \boldsymbol{T}^*\boldsymbol{\eta'}|\boldsymbol{\eta}\rangle_{\boldsymbol{F}}\boldsymbol{\xi}\big\rangle_{\boldsymbol{E}}\big) = \alpha\big(\big\langle (T^*\odot S^*)\zeta'\big|\zeta\big\rangle\big).\end{aligned}$$

Now, let us show that $T\odot S$ extends to a bounded linear map. By the preceding discussion, we have

$$\big\langle (T\odot S)\zeta\big|(T\odot S)\zeta\big\rangle = \alpha\big(\langle (T^*\odot S^*)\zeta|\zeta\rangle = \alpha\big(\langle (T^*T\odot S^*S)\zeta|\zeta\rangle\big).$$

By the discussion above and since $\boldsymbol{S}^*\boldsymbol{S}$ commutes with $M_n(B)$, we find

$$\langle\zeta|(T^*T\odot S^*S)\zeta\rangle = \big\langle \boldsymbol{\xi}\big|\langle\boldsymbol{\eta}|\boldsymbol{T}^*\boldsymbol{T}\boldsymbol{\eta}\rangle_{\boldsymbol{F}}\boldsymbol{S}^*\boldsymbol{S}\boldsymbol{\xi}\big\rangle_{\boldsymbol{E}}.$$

In the C^*-algebra $M_n(B)$, we have $\langle \boldsymbol{T}^*\boldsymbol{T}\boldsymbol{\eta}|\boldsymbol{\eta}\rangle \leq \|\boldsymbol{T}\|^2\langle\boldsymbol{\eta}|\boldsymbol{\eta}\rangle$. Furthermore, since $\boldsymbol{S}^*\boldsymbol{S}$ commutes with $\langle\boldsymbol{\eta}|\boldsymbol{\eta}\rangle$, we have $\boldsymbol{S}^*\boldsymbol{S}\langle\boldsymbol{\eta}|\boldsymbol{\eta}\rangle \leq \|\boldsymbol{S}^*\boldsymbol{S}\|\langle\boldsymbol{\eta}|\boldsymbol{\eta}\rangle$. By proposition 1.12, $\|\boldsymbol{S}^*\boldsymbol{S}\| = \|\boldsymbol{S}\|^2$ and $\|\boldsymbol{T}^*\boldsymbol{T}\| = \|\boldsymbol{T}\|^2$, and by [29, lemma 4.2], $\|\boldsymbol{T}\| = \|T\|$ and $\|\boldsymbol{S}\| = \|S\|$. Therefore,

$$\big\langle (T\odot S)\zeta\big|(T\odot S)\zeta\big\rangle \leq \|\boldsymbol{S}\|^2\|\boldsymbol{T}\|^2\big\langle\boldsymbol{\xi}|\langle\boldsymbol{\eta}|\boldsymbol{\eta}\rangle_{\boldsymbol{F}}\boldsymbol{\xi}\big\rangle_{\boldsymbol{E}} = \|S\|^2\|T\|^2\langle\zeta|\zeta\rangle,$$

whence $T\odot S$ extends to a linear map $T\otimes_* S\colon F\otimes_* E\to F'\otimes_* E'$ of norm less than or equal to $\|T\|\|S\|$. With the adjunction equation proved above, it is easy to see that the this map is a (γ,α)-homogeneous operator with adjoint $T^*\otimes_* S^*$. ☐

Note that the preceding proof substantially simplifies if $\beta=\beta'$. The preceding proposition generalises the "balanced tensor product" defined in [11, proposition 1.34] which corresponds to the case where α,β,β' and γ are the identity.

Lemma 1.25. *Adjoints and products of elementary operators are elementary again.*

Proof. The statement concerning adjoints follows from the previous proposition. Let

$$\begin{aligned}&T\in\mathscr{L}^{\gamma}_{\beta'}(F),\quad S\in\mathscr{L}^{\beta}_{\alpha}(E),\quad \gamma\in\mathrm{PAut}(C),\,\alpha\in\mathrm{PAut}(A),\,\beta,\beta'\in\mathrm{PAut}(B),\\ &T'\in\mathscr{L}^{\gamma'}_{\delta}(F),\quad S'\in\mathscr{L}^{\delta}_{\alpha'}(E),\quad \gamma'\in\mathrm{PAut}(C),\,\alpha'\in\mathrm{PAut}(A),\,\delta,\delta'\in\mathrm{PAut}(B),\end{aligned}$$

such that $\beta\curlyvee\beta'$ and $\delta\curlyvee\delta'$. Then $(\delta\beta)\curlyvee(\delta'\beta')$ since

$$(\delta\beta)^*(\delta'\beta') = \beta^*(\delta^*\delta')\beta' \leq \beta^*\beta' \leq \mathrm{id},\quad (\delta\beta)(\delta'\beta')^* = \delta(\beta\beta'^*)\delta'^* \leq \delta\delta'^* \leq \mathrm{id},$$

and $T'T\in\mathscr{L}^{\gamma'\gamma}_{\delta'\beta'}(F)$, $S'S\in\mathscr{L}^{\delta\beta}_{\alpha'\alpha}(E)$. Therefore, the internal tensor product $T'T\otimes_* S'S$ is well defined, and the formula $(T'\odot S')(T\odot S) = T'T\odot S'S$ implies $(T'\otimes_* S')(T\otimes_* S) = T'T\otimes_* S'S$. ☐

Now, the following definition and proposition are immediate.

Definition 1.26. *The* internal tensor product of families of closed subspaces $\mathscr{C} \subset \mathscr{L}(E, E')$ and $\mathscr{D} \subset \mathscr{L}(F, F')$ *is the family* $\mathscr{D} \otimes_* \mathscr{C} \subset \mathscr{L}(F \otimes_* E, F' \otimes_* E')$ *of closed subspaces given by*

$$(\mathscr{D} \otimes_* \mathscr{C})_\alpha^\gamma := \overline{\operatorname{span}}_{\beta \curlyvee \beta'} \big(\mathscr{D}_{\beta'}^\gamma \otimes_* \mathscr{C}_\alpha^\beta\big), \quad \alpha \in \mathrm{PAut}(A),\, \gamma \in \mathrm{PAut}(C),$$

where the closed span is taken over all compatible partial automorphisms β and β' of B. An operator of the form $T \otimes_* S \in \mathscr{D} \otimes_* \mathscr{C}$ is elementary.

Remark 1.27. Let $\mathscr{C}, \mathscr{C}' \subset \mathscr{L}(E)$ and $\mathscr{D}, \mathscr{D}' \subset \mathscr{L}(F)$ be families of closed subspaces. Then $(\mathscr{D}' \otimes_* \mathscr{C}')(\mathscr{D} \otimes_* \mathscr{C}) \subset (\mathscr{D}'\mathscr{D}) \otimes_* (\mathscr{C}'\mathscr{C})$. However, *this inclusion may be strict and fail to be an equality.* As a simple example, consider the case where all spaces comprising the families $\mathscr{C}$ and $\mathscr{D}$ are 0 except for $\mathscr{D}_{\beta'}^\gamma$ and $\mathscr{C}_\alpha^\beta$, where $\alpha \in \mathrm{PAut}(A)$, $\gamma \in \mathrm{PAut}(C)$ and $\beta, \beta' \in \mathrm{PAut}(B)$ are fixed and β and β' are not compatible. Then $\mathscr{D}' \otimes_* \mathscr{C}' = \mathscr{D} \otimes_* \mathscr{C} = 0$, but $(\mathscr{D}'\mathscr{D} \otimes_* \mathscr{C}'\mathscr{C})_{\mathrm{id}}^{\mathrm{id}} = \mathscr{D}_{\beta'^*\beta'}^{\gamma^*\gamma} \otimes_* \mathscr{C}_{\alpha^*\alpha}^{\beta^*\beta}$ need not be 0.

However, it is easy to see that $(\mathscr{D}' \otimes_* \mathscr{C}')(\mathscr{D} \otimes_* \mathscr{C}) = (\mathscr{D}'\mathscr{D}) \otimes_* (\mathscr{C}'\mathscr{C})$ if $\mathscr{D}' \subset \mathscr{L}_{\mathrm{id}}^{\mathrm{id}}(F)$ and $\mathscr{C}' \subset \mathscr{L}_{\mathrm{id}}^{\mathrm{id}}(E)$.

Proposition 1.28. *Let $\mathscr{C} \subset \mathscr{L}(E, E')$ and $\mathscr{D} \subset \mathscr{L}(F, F')$ be C^*-pre-family modules. Then the internal tensor product $\mathscr{D} \otimes_* \mathscr{C}$ is a C^*-pre-family module again. If $E = E', F = F'$ and $\mathscr{C}, \mathscr{D}$ are C^*-pre-families, then $\mathscr{D} \otimes_* \mathscr{C}$ is a C^*-pre-family. In that case, one has a natural inclusion $\mathscr{M}(\mathscr{D}) \otimes_* \mathscr{M}(\mathscr{C}) \subset \mathscr{M}(\mathscr{D} \otimes_* \mathscr{C})$.*

Proof. If $\mathscr{C}$ and $\mathscr{D}$ are C^*-pre-family modules, then

$$(\mathscr{D} \otimes_* \mathscr{C})(\mathscr{D} \otimes_* \mathscr{C})^*(\mathscr{D} \otimes_* \mathscr{C}) \subset (\mathscr{D}\mathscr{D}^*\mathscr{D}) \otimes_* (\mathscr{C}\mathscr{C}^*\mathscr{C}) = \mathscr{D} \otimes_* \mathscr{C},$$

and for all extensions $(\gamma, \alpha) \leq (\gamma', \alpha')$ in $\mathrm{PAut}(C) \times \mathrm{PAut}(A)$, one has

$$(\mathscr{D} \otimes_* \mathscr{C})_\alpha^\gamma = \overline{\operatorname{span}}_{\beta \curlyvee \beta'} \big(\mathscr{D}_{\beta'}^\gamma \otimes_* \mathscr{C}_\alpha^\beta\big) \subset \overline{\operatorname{span}}_{\beta \curlyvee \beta'} \big(\mathscr{D}_{\beta'}^{\gamma'} \otimes_* \mathscr{C}_{\alpha'}^\beta\big) = (\mathscr{D} \otimes_* \mathscr{C})_{\alpha'}^{\gamma'}.$$

Therefore, $\mathscr{D} \otimes_* \mathscr{C}$ is a C^*-pre-family module. The proof of the second statement is similar. The last statement follows from the inclusion

$$\big(\mathscr{M}(\mathscr{D}) \otimes_* \mathscr{M}(\mathscr{C})\big) \cdot (\mathscr{D} \otimes_* \mathscr{C}) \subset \mathscr{M}(\mathscr{D})\mathscr{D} \otimes_* \mathscr{M}(\mathscr{C})\mathscr{C} = \mathscr{D} \otimes_* \mathscr{C}. \qquad \square$$

The internal tensor product operation is not quite associative on the level of operators. It may happen that R, S and T are operators such that the internal tensor products $T \otimes_* S$ and $(T \otimes_* S) \otimes_* R$ are well-defined but the internal tensor product $S \otimes_* R$ is not well-defined at all. This phenomenon occurs e.g. if $T = 0$ and S and R are not compatible, but it is not restricted to trivial cases like this one and difficult to analyse. For the precise formulation of semigroup gradings on families of operators, the associativity of the internal tensor product on the level of C^*-pre-families was a crucial test.

Proposition 1.29. *The internal tensor product operation on families of closed subspaces is associative.*

Proof. Let D be a C^*-algebra, let G, G' be C^*-D-C-bimodules and let $\mathscr{C} \subset \mathscr{L}(E, E')$, $\mathscr{D} \subset \mathscr{L}(F, F')$ and $\mathscr{E} \subset \mathscr{L}(G, G')$ be families of closed subspaces. For each $\delta \in \mathrm{PAut}(D)$ and each $\alpha \in \mathrm{PAut}(A)$, the spaces $\big((\mathscr{E} \otimes_* \mathscr{D}) \otimes_* \mathscr{C}\big)_\alpha^\delta$ and $\big(\mathscr{E} \otimes_* (\mathscr{D} \otimes_* \mathscr{C})\big)_\alpha^\delta$ both are equal to

$$\overline{\mathrm{span}}\{T \otimes_* S \otimes_* R \mid T \in \mathscr{E}_{\gamma'}^\delta,\, S \in \mathscr{D}_{\beta'}^\gamma,\, R \in \mathscr{C}_{\alpha'}^\beta, \gamma \curlyvee \gamma', \beta \curlyvee \beta'\}. \qquad \square$$

1.1.4 Morphisms of C^*-pre-families

This subsection introduces the notion of a morphism between C^*-pre-families. Like the internal tensor product, this is needed in order to define the notion of a Hopf C^*-pre-family. The definition which we give may look strange but is well motivated. We show that it implies the expected properties and that non-degenerate morphisms extend to multiplier families. The subsection ends with a proof of the bi-functoriality of the internal tensor product of C^*-pre-families. The latter two results are needed in order to make sense of equations like $(\Delta \otimes_* 1)\Delta = (1 \otimes_* \Delta)\Delta$ and $(\delta' \otimes_* 1)\phi = (\phi \otimes_* 1)\delta$ which express coassociativity of the coproduct Δ of a Hopf C^*-pre-family and equivariance of a morphism ϕ between C^*-pre-families with coactions δ and δ', respectively.

Motivation

For the coproducts on the legs of a multiplicative unitary, the Baaj-Skandalis construction gives formulas which – on a set-theoretic level – still make sense in the setting pseudo-multiplicative unitaries. Apart from the set-theoretic level, it is clear that a morphism of C^*-pre-families $\phi\colon \mathscr{C} \to \mathscr{D}$ should consist of a family of linear maps $\phi_\alpha^\beta\colon \mathscr{C}_\alpha^\beta \to \mathscr{D}_\alpha^\beta$ given for all $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$, which are multiplicative and commute with the involution in the obvious sense. As in the case of $*$-homomorphisms, these properties imply that for every α and β, the map ϕ_α^β is norm-decreasing. It is easy to check that the formulas for the coproducts mentioned above give rise to such families.

The problem is that it is not clear whether the internal tensor product is functorial with respect to such families of maps. More precisely, given two morphisms $\phi\colon \mathscr{C} \to \mathscr{C}'$ and $\psi\colon \mathscr{D} \to \mathscr{D}'$ of C^*-pre-families on suitable C^*-bimodules, it is difficult to compare the norm of an operator of the form $\phi(S) \otimes_B \psi(T)$, where $S \in \mathscr{C}$ and $T \in \mathscr{D}$, to the norm of the operator $S \otimes_B T$.

We turn the problem upside down and incorporate the functoriality requirement in the definition of a morphism. Then the proof of bi-functoriality of the internal tensor product reduces to an easy check.

Definition and first properties

Definition 1.30. *Let $\mathscr{C}$ and $\mathscr{D}$ be C^*-pre-families on E and E', respectively. A* morphism $\phi\colon \mathscr{C} \to \mathscr{D}$ *is a family of maps $\phi^\beta_\alpha\colon \mathscr{C}^\beta_\alpha \to \mathscr{D}^\beta_\alpha$ given for all $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$, subject to the following condition: for each C^*-module F' over B and each C^*-A-C-bimodule F over some C^*-algebra C, the family of maps*

$$\begin{aligned} \mathrm{id} \times \phi^\beta_{\alpha'} \times \mathrm{id}\colon \mathscr{L}^{\mathrm{id}}_{\beta'}(F') \times \mathscr{C}^\beta_{\alpha'} \times \mathscr{L}^\alpha_{\mathrm{id}}(F) \to \mathscr{L}^{\mathrm{id}}_{\beta'}(F') \otimes_* \mathscr{D}^\beta_{\alpha'} \otimes_* \mathscr{L}^\alpha_{\mathrm{id}}(F), \\ (R, S, T) \mapsto R \otimes_* \phi^\beta_{\alpha'}(S) \otimes_* T, \end{aligned}$$

where $\alpha \curlyvee \alpha' \in \mathrm{PAut}(A)$ and $\beta \curlyvee \beta' \in \mathrm{PAut}(B)$, induces a $*$-homomorphism of C^*-algebras

$$\mathrm{id} \otimes_* \phi \otimes_* \mathrm{id}\colon \mathscr{L}^{\mathrm{id}}(F') \otimes_* \mathscr{C} \otimes_* \mathscr{L}_C(F) \to \mathscr{L}^{\mathrm{id}}(F') \otimes_* \mathscr{D} \otimes_* \mathscr{L}_C(F).$$

Remark 1.31. In the definition above, one has inclusions

$$\begin{aligned} \mathscr{L}^{\mathrm{id}}(F') \otimes_* \mathscr{C} \otimes_* \mathscr{L}_{\mathrm{id}}(F) \subset \mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(F' \otimes_* E \otimes_* F) \subset L_C(F' \otimes_* E \otimes_* F), \\ \mathscr{L}^{\mathrm{id}}(F') \otimes_* \mathscr{D} \otimes_* \mathscr{L}_{\mathrm{id}}(F) \subset \mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(F' \otimes_* E' \otimes_* F) \subset L_C(F' \otimes_* E' \otimes_* F). \end{aligned}$$

Remark 1.32. It is easy to see that the composition of two morphisms is a morphism again. Hence, the collection of all C^*-pre-families on C^*-B-A-bimodules with morphisms as above forms a category.

Next, we show that the family of maps comprising a morphism has all the expected properties. The proof depends on an embedding construction which involves example 1.5 and requires some preparation.

Lemma 1.33. *Let F be a C^*-C-B-bimodule, where C is some C^*-algebra.*

i) If $\langle F|F\rangle E = E$, then for each $T \in \mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(E)$, the norm of the operator $1 \otimes_* T$ on $F \otimes_* E$ is equal to the norm of T.

ii) If the representation of B on E is faithful, then for each $S \in \mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(F)$, the norm of the operator $S \otimes_* 1$ on $F \otimes_* E$ is equal to the norm of S.

Proof. i) It is enough to show that the $*$-homomorphism $\mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(E) \to \mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(F \otimes_* E)$ given by $T \mapsto 1 \otimes_* T$ is injective. But if $T \neq 0$, then $\langle F \otimes_* E | F \otimes_* TE\rangle = \big\langle E \big| T\langle F|F\rangle E\big\rangle \neq 0$.

ii) Likewise, the map $S \mapsto S \otimes_* 1$ is injective since $\langle F \otimes_* E | SF \otimes_* E\rangle = \big\langle E \big| \langle F|SF\rangle E\big\rangle \neq 0$ if $S \neq 0$. ☐

Lemma 1.34. *For all $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$, there exist a C^*-A-$\mathrm{Im}(\alpha)$-bimodule E^α and a C^*-$\mathbb{C}$-B-bimodule E_β with operators $T^\alpha \in \mathscr{L}^\alpha_{\mathrm{id}}(E^\alpha)$ and $T_\beta \in \mathscr{L}^{\mathrm{id}}_\beta(E_\beta)$ such that the map $\mathscr{L}^\beta_\alpha(E) \to \mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(E_\beta \otimes_* E \otimes_* E^\alpha)$ given by $S \mapsto T_\beta \otimes_* S \otimes_* T^\alpha$ is an isometric embedding.*

Proof. Let E^α, T^α and E_β, T_β be as in example 1.5. Then $\|T^\alpha\| = \|T_\beta\| = 1$, and hence the map $S \mapsto T_\beta \otimes_* S \otimes_* T^\alpha$ is norm-decreasing. On the other hand, one has

$$\|T_\beta \otimes_* S \otimes_* T^\alpha\|^2 = \|T_\beta^* T_\beta \otimes_* S^* S \otimes_* T^{\alpha *} T^\alpha\|.$$

Here, $T^{\alpha *} T^\alpha$ and $T_\beta^* T_\beta$ are equal to the identity on the subspaces $\mathrm{Dom}(\alpha) \subset E^\alpha$ and $\mathrm{Dom}(\beta) \subset E_\beta$, respectively. By the previous lemma and proposition 1.9, $\|T_\beta \otimes_* S \otimes_* T^\alpha\|^2 = \|S^* S\|$. □

Proposition 1.35. *Let $\phi\colon \mathscr{C} \to \mathscr{D}$ be a morphism of C^*-pre-families on E and E', respectively. For all $\alpha, \alpha' \in \mathrm{PAut}(A)$ and $\beta, \beta' \in \mathrm{PAut}(B)$, the map ϕ_α^β is linear and norm-decreasing, $\phi_{\alpha^*}^{\beta^*}(c^*) = \left(\phi_\alpha^\beta(c)\right)^*$ and $\phi_\alpha^\beta(c)\phi_{\alpha'}^{\beta'}(c') = \phi_{\alpha\alpha'}^{\beta\beta'}(cc')$ for all $c \in \mathscr{C}_\alpha^\beta$, $c' \in \mathscr{C}_{\alpha'}^{\beta'}$.*

Proof. Let $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$. Choose E_β, T_β and E^α, T^α as in the lemma above. Then the embeddings constructed above yield a commutative diagram

$$\begin{array}{ccl} \mathscr{C}_\alpha^\beta & \hookrightarrow & \mathscr{L}^{\mathrm{id}}(E_\beta) \otimes_* \mathscr{C} \otimes_* \mathscr{L}_{\mathrm{id}}(E^\alpha) \subset L_{\mathrm{Im}(\alpha)}(E_\beta \otimes_* E \otimes_* E^\alpha) \\ \downarrow{\scriptstyle \phi_\alpha^\beta} & & \quad \downarrow{\scriptstyle \mathrm{id} \otimes_* \phi \otimes_* \mathrm{id}} \\ \mathscr{D}_\alpha^\beta & \hookrightarrow & \mathscr{L}^{\mathrm{id}}(E_\beta) \otimes_* \mathscr{D} \otimes_* \mathscr{L}_{\mathrm{id}}(E^\alpha) \subset L_{\mathrm{Im}(\alpha)}(E_\beta \otimes_* E' \otimes_* E^\alpha). \end{array}$$

This implies that ϕ_α^β is norm-decreasing, linear, and that $* \circ \phi_\alpha^\beta = \phi_{\alpha^*}^{\beta^*} \circ *$.

Multiplicativity can be proved by a similar embedding technique, using the space $\mathrm{Im}(\alpha\alpha') \oplus (\mathrm{Dom}(\alpha) \cap \mathrm{Im}(\alpha')) \oplus \mathrm{Dom}(\alpha\alpha')$ and the maps

$$\begin{pmatrix} 0 & 0 & \alpha\alpha' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha' \\ 0 & 0 & 0 \end{pmatrix},$$

and a similar construction for β, β'. □

It is natural to ask whether morphisms of C^*-pre-families on C^*-B-A-bimodules are linear with respect to B and A in suitable sense. This question will be taken up in subsection 1.2.3.

Definition 1.36. *A morphism $\phi\colon \mathscr{C} \to \mathscr{D}$ is* injective *if each component ϕ_α^β, $\alpha \in \mathrm{PAut}(A)$, $\beta \in \mathrm{PAut}(B)$, is injective.*

Remark 1.37. From the previous proposition and remark 1.15 iii), it follows that ϕ is injective if and only if the component $\phi_{\mathrm{id}}^{\mathrm{id}}$ is an injective $*$-homomorphism.

Non-degenerate morphisms and extension to multiplier C^*-pre-families

In the setting of Hopf C^*-algebras, the extension of non-degenerate $*$-homomorphisms to multiplier algebras is a fundamental technique. For instance, the coassociativity condition on a coproduct $\Delta\colon S \to M(S \otimes S)$ given by $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ involves the extension of the maps $\Delta \otimes 1$ and $1 \otimes \Delta$ – which are defined on $S \otimes S$ – to the multiplier algebra $M(S \otimes S)$. In the following, we establish the corresponding technique for C^*-pre-families.

Definition 1.38. *A morphism $\phi\colon \mathscr{C} \to \mathscr{M}(\mathscr{D})$ of C^*-pre-families is* non-degenerate *if $\phi(\mathscr{C})\mathscr{D} = \mathscr{D}$.*

Lemma 1.39. *A morphism $\phi\colon \mathscr{C} \to \mathscr{M}(\mathscr{D})$ of C^*-pre-families is non-degenerate if and only if its component $\phi^{\mathrm{id}}_{\mathrm{id}}\colon \mathscr{C}^{\mathrm{id}}_{\mathrm{id}} \to M(\mathscr{D}^{\mathrm{id}}_{\mathrm{id}})$ is non-degenerate as a $*$-homomorphism.*

Proof. This is an easy consequence of remark 1.15 iii). If the $*$-homomorphism $\phi^{\mathrm{id}}_{\mathrm{id}}$ is non-degenerate, the family $\phi(\mathscr{C})\mathscr{D}$ contains

$$\phi^{\mathrm{id}}_{\mathrm{id}}(\mathscr{C}^{\mathrm{id}}_{\mathrm{id}})\,\mathscr{D} = \phi^{\mathrm{id}}_{\mathrm{id}}(\mathscr{C}^{\mathrm{id}}_{\mathrm{id}})\,\mathscr{D}^{\mathrm{id}}_{\mathrm{id}}\,\mathscr{D} = \mathscr{D}^{\mathrm{id}}_{\mathrm{id}}\,\mathscr{D} = \mathscr{D}.$$

Conversely, assume that ϕ is non-degenerate. Then $\mathscr{D}^{\mathrm{id}}_{\mathrm{id}}$ is the closed linear span of products of the form $\phi^{\beta}_{\alpha}(c')d'$ where $c' \in \mathscr{C}^{\beta}_{\alpha}$ and $d' \in \mathscr{D}^{\beta'}_{\alpha'}$ for some (β, α) and (β', α') in $\mathrm{PAut}(B) \times \mathrm{PAut}(A)$ satisfying $\alpha\alpha' \leq \mathrm{id}$ and $\beta\beta' \leq \mathrm{id}$. Write $c' = c_0 c''$ with $c_0 \in \mathscr{C}^{\mathrm{id}}_{\mathrm{id}}$ and $c'' \in \mathscr{C}^{\alpha}_{\beta}$. Then

$$\phi^{\beta}_{\alpha}(c')d' = \phi^{\mathrm{id}}_{\mathrm{id}}(c_0)\big(\phi^{\beta}_{\alpha}(c'')d'\big) \in \phi(\mathscr{C}^{\mathrm{id}}_{\mathrm{id}})\,\mathscr{D}^{\mathrm{id}}_{\mathrm{id}}. \qquad \square$$

Lemma 1.40. *Let $\alpha \in \mathrm{PAut}(A)$, $\beta \in \mathrm{PAut}(B)$ and let $(T_\lambda)_\lambda$ be a net in $\mathscr{L}^{\beta}_{\alpha}(E, E')$ such that for each $\xi \in E$ and each $\eta \in E'$, the nets $(T_\lambda \xi)_\lambda$ and $(T^*_\lambda \eta)_\lambda$ converge in E' and E, respectively. Then the map $\xi \mapsto \lim_\lambda T_\lambda \xi$ defines a (β, α)-homogeneous operator whose adjoint is given by the map $\eta \mapsto \lim_\lambda T^*_\lambda \eta$.*

Proof. By the Banach-Steinhaus theorem, the assignment of the limits defines bounded linear maps $T\colon E \to E'$ and $S\colon E' \to E$. Let $\eta \in E'$ and $\xi \in E$. Since $\mathrm{Dom}(\alpha)$ is closed, the inner product $\langle S\eta|\xi\rangle$ is contained in $\mathrm{Dom}(\alpha)$, and one has $\alpha\big(\langle S\eta|\xi\rangle\big) = \lim_\lambda \alpha\big(\langle T^*_\lambda \eta|\xi\rangle\big) = \lim_\lambda \langle \eta|T_\lambda \xi\rangle = \langle \eta|T\xi\rangle$. Likewise, $\mathrm{Im}\, T \subset \overline{\mathrm{span}}_\lambda\big(\mathrm{Im}\, T_\lambda\big) \subset \mathrm{Im}(\beta)E'$ and $Tb\xi = \lim_\lambda T_\lambda(b\xi) = \lim_\lambda \beta(b)T_\lambda \xi = \beta(b)T\xi$ for all $b \in \mathrm{Dom}(\beta)$ and $\xi \in E$. $\square$

Proposition 1.41. *Let $\phi\colon \mathscr{C} \to \mathscr{M}(\mathscr{D})$ be a non-degenerate morphism of C^*-pre-families. If the C^*-pre-family $\mathscr{D}$ is non-degenerate, then ϕ extends uniquely to a morphism $\mathscr{M}(\mathscr{C}) \to \mathscr{M}(\mathscr{D})$.*

Proof. Let $\mathscr{C}$ and $\mathscr{D}$ act on E and E', respectively. Let $\alpha \in \mathrm{PAut}(A)$, $\beta \in \mathrm{PAut}(B)$ and consider an element $c \in \mathscr{M}(\mathscr{C})^{\beta}_{\alpha}$. Choose a bounded positive approximate unit $(u_\nu)_\nu$ for the C^*-algebra $\mathscr{C}^{\mathrm{id}}_{\mathrm{id}}$. Since ϕ and $\mathscr{D}$ are non-degenerate, the image $\phi(\mathscr{C})$ is non-degenerate on E'. Therefore, the nets $\big(\phi^{\alpha}_{\beta}(cu_\nu)\big)_\nu$ and $\big(\phi^{\alpha^*}_{\beta^*}(u_\nu c^*)\big)_\nu$ converge pointwise. Denote the pointwise limits by $\tilde{\phi}^{\beta}_{\alpha}(c)$ and $\tilde{\phi}^{\beta^*}_{\alpha^*}(c)$, respectively. By lemma 1.40, they form a pair of adjoint operators in $\mathscr{L}^{\beta}_{\alpha}(E')$ and $\mathscr{L}^{\beta^*}_{\alpha^*}(E')$, respectively. A standard argument shows that they do not depend on the choice of the approximate unit.

Let $\alpha' \in \mathrm{PAut}(A)$, $\beta' \in \mathrm{PAut}(B)$ and $d \in \mathscr{D}^{\beta'}_{\alpha'}$. Since ϕ is non-degenerate, $\tilde{\phi}^{\beta}_{\alpha}(c)d$ is the norm limit of the products $\phi^{\beta}_{\alpha}(cu_\nu)d \in \mathscr{D}^{\alpha\alpha'}_{\beta\beta'}$. Hence, it belongs to $\mathscr{D}^{\alpha\alpha'}_{\beta\beta'}$. A symmetric argument shows that $\tilde{\phi}^{\beta^*}_{\alpha^*}(c)d$ belongs to $\mathscr{D}^{\beta^*\beta'}_{\alpha^*\alpha'}$. Thus, $\tilde{\phi}^{\beta}_{\alpha}(c)$ is contained in $\mathscr{M}(\mathscr{D})^{\beta}_{\alpha}$.

Now, let us show that the collection $\tilde{\phi}$ defines a morphism $\mathscr{M}(\mathscr{C}) \to \mathscr{M}(\mathscr{D})$. Let F' be a C^*-module over B and F be a C^*-A-C-module, where C is some C^*-algebra. Then ϕ induces a $*$-homomorphism

$$\begin{aligned}\mathscr{L}^{\mathrm{id}}(F') \otimes_* \mathscr{C} \otimes_* \mathscr{L}_{\mathrm{id}}(F) \xrightarrow{\mathrm{id} \otimes_* \phi \otimes_* \mathrm{id}} \mathscr{L}^{\mathrm{id}}(F') &\otimes_* \mathscr{M}(\mathscr{D}) \otimes_* \mathscr{L}_{\mathrm{id}}(F) \\ &\subset M\big(\mathscr{L}^{\mathrm{id}}(F') \otimes_* \mathscr{D} \otimes_* \mathscr{L}_{\mathrm{id}}(F)\big).\end{aligned}$$

Since $\phi(\mathscr{C})\mathscr{D} = \mathscr{D}$, this $*$-homomorphism is non-degenerate. Hence, it extends to the C^*-subalgebra $\mathscr{L}^{\mathrm{id}}(F') \otimes_* \mathscr{M}(\mathscr{C}) \otimes_* \mathscr{L}_{\mathrm{id}}(F)$ of $M\big(\mathscr{L}^{\mathrm{id}}(F') \otimes_* \mathscr{C} \otimes_* \mathscr{L}_{\mathrm{id}}(F)\big)$. By construction, this extension is induced by the collection of maps $\big(\mathrm{id} \times \tilde{\phi}^{\beta}_{\alpha} \times \mathrm{id}\big)^{\beta}_{\alpha}$, $\alpha \in \mathrm{PAut}(A)$, $\beta \in \mathrm{PAut}(B)$. ☐

In the following, we give some examples of morphisms.

Examples 1.42. Let $\mathscr{C}$ be a C^*-pre-family on E.

i) An inclusion of C^*-pre-families is a morphism.

ii) Let $V\colon E \to E'$ be a unitary which intertwines the representations of B on E and E'. Then the family $\mathrm{Ad}_V(\mathscr{C})$ given by $\mathrm{Ad}_V(\mathscr{C})^{\beta}_{\alpha} := \mathrm{Ad}_V(\mathscr{C}^{\beta}_{\alpha})$ for all $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$ is a C^*-pre-family, and the map Ad_V defines a non-degenerate morphism $\mathscr{C} \to \mathrm{Ad}_V(\mathscr{C})$. If the C^*-pre-family $\mathscr{C}$ is non-degenerate, so is $\mathrm{Ad}_V(\mathscr{C})$.

iii) Let E'' be a C^*-module over some C^*-algebra C with a representation $\pi\colon C \to \mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(E)$. Consider $E' = E'' \otimes_* E$ as a C^*-B-A-bimodule via the action of B on E. Then the family $1 \otimes_* \mathscr{C}$ given by $(1 \otimes_* \mathscr{C})^{\beta}_{\alpha} := 1 \otimes_* \mathscr{C}^{\beta}_{\alpha}$ for all $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$ is a C^*-pre-family, and the map $T \mapsto 1 \otimes_* T$ defines a non-degenerate morphism $\mathscr{C} \to 1 \otimes_* \mathscr{C}$. If the C^*-pre-family $\mathscr{C}$ is non-degenerate, so is $1 \otimes_* \mathscr{C}$.

The internal tensor product of morphisms

The characterising property of our definition of morphisms is that it makes the internal tensor product of C^*-pre-families bi-functorial.

Proposition 1.43. *Let C be a C^*-algebra and let*

- $\phi\colon \mathscr{C} \to \mathscr{C}'$ *be a morphism of C^*-pre-families on C^*-B-A-bimodules and*
- $\psi\colon \mathscr{D} \to \mathscr{D}'$ *be a morphism of C^*-pre-families on C^*-C-B-bimodules.*

Then for all $\alpha \in \mathrm{PAut}(A)$ and $\gamma \in \mathrm{PAut}(C)$, the family $\left(\psi^{\gamma}_{\beta'} \times \phi^{\beta}_{\alpha}\right)_{\beta\curlyvee\beta'}$ defines a map $(\psi \otimes_* \phi)^{\gamma}_{\alpha}\colon (\mathscr{D} \otimes_* \mathscr{C})^{\gamma}_{\alpha} \to (\mathscr{D}' \otimes_* \mathscr{C}')^{\gamma}_{\alpha}$, and the family $(\psi \otimes_* \phi)$ defines a morphism $\psi \otimes_* \phi\colon \mathscr{D} \otimes_* \mathscr{C} \to \mathscr{D}' \otimes_* \mathscr{C}'$. Furthermore, the following diagram commutes.

$$\begin{array}{ccc} \mathscr{D} \otimes_* \mathscr{C} & \xrightarrow{\psi\otimes_* 1} & \mathscr{D}' \otimes_* \mathscr{C} \\ {\scriptstyle 1\otimes_*\phi}\downarrow & \searrow^{\psi\otimes_*\phi} & \downarrow{\scriptstyle 1\otimes_*\phi} \\ \mathscr{D} \otimes_* \mathscr{C}' & \xrightarrow[\psi\otimes_* 1]{} & \mathscr{D}' \otimes_* \mathscr{C}'. \end{array}$$

Proof. Let $\alpha \in \mathrm{PAut}(A)$ and $\gamma \in \mathrm{PAut}(C)$. We first consider the case $\mathscr{D}' = \mathscr{D}$ and $\psi = \mathrm{id}_{\mathscr{D}}$. Choose E_γ, T_γ and E^α, T^α as in lemma 1.34, and consider the embeddings

$$(\mathscr{D} \otimes_* \mathscr{C})^{\gamma}_{\alpha} \hookrightarrow \mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(E_\gamma \otimes_* F \otimes_* E \otimes_* E^\alpha),$$
$$(\mathscr{D} \otimes_* \mathscr{C}')^{\gamma}_{\alpha} \hookrightarrow \mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(E_\gamma \otimes_* F \otimes_* E' \otimes_* E^\alpha)$$

given by $x \mapsto T_\gamma \otimes_* x \otimes_* T^\alpha$. Since ϕ is a morphism, it induces a $*$-homomorphism $\mathrm{id} \otimes_* \phi \otimes_* \mathrm{id}\colon \mathscr{L}^{\mathrm{id}}(E_\gamma \otimes_* F) \otimes_* \mathscr{C} \otimes_* \mathscr{L}_{\mathrm{id}}(E^\alpha) \to \mathscr{L}^{\mathrm{id}}(E_\gamma \otimes_* F) \otimes_* \mathscr{C} \otimes_* \mathscr{L}_{\mathrm{id}}(E^\alpha)$. Therefore, the collection $\left(\mathrm{id} \times \phi^{\beta}_{\alpha}\right)_{\beta\curlyvee\beta'}$ yields a well-defined map $(\mathrm{id} \otimes_* \phi)^{\gamma}_{\alpha}\colon (\mathscr{D} \otimes_* \mathscr{C})^{\gamma}_{\alpha} \to (\mathscr{D} \otimes_* \mathscr{C})^{\gamma}_{\alpha}$.

Let G' be an arbitrary C^*-module over C and let G be an arbitrary C^*-A-D-module, where D is some C^*-algebra. By definition, ϕ induces a $*$-homomorphism $\mathrm{id} \otimes_* \phi \otimes_* \mathrm{id}\colon \mathscr{L}^{\mathrm{id}}(G' \otimes_* F) \otimes_* \mathscr{C} \otimes_* \mathscr{L}_{\mathrm{id}}(G) \to \mathscr{L}^{\mathrm{id}}(G' \otimes_* F) \otimes_* \mathscr{C} \otimes_* \mathscr{L}_{\mathrm{id}}(G)$. Restricting to the C^*-subalgebra $\mathscr{L}^{\mathrm{id}}(G') \otimes_* \mathscr{D} \otimes_* \mathscr{C} \otimes_* \mathscr{L}_{\mathrm{id}}(G)$, we see that $\mathrm{id} \otimes_* \phi$ induces a $*$-homomorphism as desired.

A symmetric argument shows that the morphism $\psi \otimes_* 1_{\mathscr{C}'}$ is well-defined. Repeating the argument for the composition $(\psi \otimes_* 1_{\mathscr{C}'})(1_{\mathscr{D}} \otimes_* \phi)$, we obtain the morphism $\psi \otimes_* \phi$.

The last statement follows from lemma 1.39. □

Proposition 1.44. *Let C be a C^*-algebra and let*

- $\phi\colon \mathscr{C} \to \mathscr{M}(\mathscr{C}')$ *be a non-degenerate morphism of C^*-pre-families on C^*-B-A-bimodules and*
- $\psi\colon \mathscr{D} \to \mathscr{M}(\mathscr{D}')$ *be a non-degenerate morphism of C^*-pre-families on C^*-C-B-bimodules.*

Then the internal tensor product $\psi \otimes_* \phi\colon \mathscr{D} \otimes_* \mathscr{C} \to \mathscr{M}(\mathscr{D}' \otimes_* \mathscr{C}')$ is non-degenerate.

Proof. The assumption implies

$$\left(\psi^{\mathrm{id}}_{\mathrm{id}}(\mathscr{D}^{\mathrm{id}}_{\mathrm{id}}) \otimes_* \phi^{\mathrm{id}}_{\mathrm{id}}(\mathscr{C}^{\mathrm{id}}_{\mathrm{id}})\right) \cdot (\mathscr{D}' \otimes_* \mathscr{C}')^{\mathrm{id}}_{\mathrm{id}} = (\mathscr{D}' \otimes_* \mathscr{C}')^{\mathrm{id}}_{\mathrm{id}}.$$ □

1.1.5 Left Hopf C^*-pre-families

The notion of a Hopf C^*-pre-family and a coaction is the straight-forward generalisation of the corresponding C^*-algebraic notions introduced in [2] and [3]. Stefaan Vaes and Alfons Van Daele gave a refined definition of a Hopf C^*-algebra [57] which involves the Haagerup tensor product of C^*-algebras and therefore does not easily carry over to C^*-pre-families.

Let E be a C^*-bimodule over A.

Definition 1.45. *A* left Hopf C^*-pre-family *is a non-degenerate C^*-pre-family on E with an injective morphism $\Delta\colon \mathscr{S} \to \mathscr{M}(\mathscr{S} \otimes_* \mathscr{S})$ such that*

$$\Delta(\mathscr{S})(1 \otimes_* \mathscr{S}) = \mathscr{S} \otimes_* \mathscr{S} = \Delta(\mathscr{S})(\mathscr{S} \otimes_* 1)$$

and the following diagram commutes.

$$\begin{CD} \mathscr{S} @>\Delta>> \mathscr{M}(\mathscr{S} \otimes_* \mathscr{S}) \\ @V\Delta VV @VV1\otimes_*\Delta V \\ \mathscr{M}(\mathscr{S} \otimes_* \mathscr{S}) @>>\Delta\otimes_* 1> \mathscr{M}(\mathscr{S} \otimes_* \mathscr{S} \otimes_* \mathscr{S}). \end{CD}$$

Remark 1.46. If $(\mathscr{S}, \Delta)$ is a left Hopf C^*-pre-family, the morphism Δ is non-degenerate. In the diagram above, we used the extension of non-degenerate morphisms to multiplier families.

Definition 1.47. *Let $(\mathscr{S}, \Delta)$ be a left Hopf C^*-pre-family on E.*

i) Let $\mathscr{C}$ be a non-degenerate C^*-pre-family on a C^*-A-B-bimodule E'. A left coaction *of $(\mathscr{S}, \Delta)$ on $\mathscr{C}$ is a non-degenerate morphism $\delta\colon \mathscr{C} \to \mathscr{M}(\mathscr{S} \otimes_* \mathscr{C})$ such that $\delta(\mathscr{C})(\mathscr{S} \otimes_* 1) \subset \mathscr{S} \otimes_* \mathscr{C}$ and the following diagram commutes.*

$$\begin{CD} \mathscr{C} @>\delta>> \mathscr{M}(\mathscr{S} \otimes_* \mathscr{C}) \\ @V\delta VV @VV1\otimes_*\delta V \\ \mathscr{M}(\mathscr{S} \otimes_* \mathscr{C}) @>>\Delta\otimes_* 1> \mathscr{M}(\mathscr{S} \otimes_* \mathscr{S} \otimes_* \mathscr{C}). \end{CD}$$

If δ is injective and $\delta(\mathscr{C})(\mathscr{S} \otimes_* 1) = \mathscr{S} \otimes_* \mathscr{C}$, the pair $(\mathscr{C}, \delta)$ is a left $(\mathscr{S}, \Delta)$-pre-family.

ii) Let $\mathscr{C}$ and $\mathscr{C}'$ be non-degenerate C^*-pre-families on C^*-A-B-bimodules with left coactions δ and δ', respectively. A non-degenerate morphism $\phi\colon \mathscr{C} \to \mathscr{M}(\mathscr{C}')$ is equivariant *if the following diagram commutes.*

$$\begin{CD} \mathscr{C} @>\delta>> \mathscr{M}(\mathscr{S} \otimes_* \mathscr{C}) \\ @V\phi VV @VV1\otimes_*\phi V \\ \mathscr{M}(\mathscr{C}') @>>\delta'> \mathscr{M}(\mathscr{S} \otimes_* \mathscr{C}'). \end{CD}$$

iii) *Let $\mathscr{D}$ be a non-degenerate C^*-pre-family on a C^*-B-A-bimodule E'. A* right coaction *of $(\mathscr{S},\Delta)$ on $\mathscr{C}$ is a non-degenerate morphism $\delta\colon \mathscr{D}\to\mathscr{M}(\mathscr{D}\otimes_*\mathscr{S})$ such that $\delta(\mathscr{D})(1\otimes_*\mathscr{S})\subset\mathscr{D}\otimes_*\mathscr{S}$ and $(\delta\otimes_* 1)\delta=(1\otimes_*\Delta)\delta$. If δ is injective and $\delta(\mathscr{D})(1\otimes_*\mathscr{S})=\mathscr{D}\otimes_*\mathscr{S}$, the pair $(\mathscr{D},\delta)$ is a* right $(\mathscr{S},\Delta)$-pre-family.

iv) *Let $\mathscr{D}$ and $\mathscr{D}'$ be non-degenerate C^*-pre-families on C^*-B-A-bimodules with left coactions δ and δ', respectively. A non-degenerate morphism $\psi\colon\mathscr{D}\to\mathscr{M}(\mathscr{D}')$ is* equivariant *if $\delta'\circ\psi=(\psi\otimes_* 1)\circ\delta$.*

Clearly, the class of left coactions and the class of right coactions, together with non-degenerate equivariant morphisms, forms a category.

1.2 Semigroup gradings on C^*-bimodules

Homogenous elements of C^*-bimodules play a decisive rôle in our generalisation of the Baaj-Skandalis construction. The introduction of homogeneous operators on C^*-bimodules is partially motivated by the occurrence of maps of the form $\theta^\xi\colon E\to E\otimes_* E$, $\zeta\mapsto\zeta\otimes_*\xi$ and $\theta_\xi\colon E\to E\otimes_* E$, $\zeta\mapsto\xi\otimes_*\zeta$. In general, maps of the first kind are not adjointable and maps of the second kind do not commute with left multiplication. However, they are homogeneous if the element $\xi\in E$ is *homogeneous.* The Baaj-Skandalis construction can be adapted to pseudo-multiplicative unitaries whose underlying C^*-bimodules are decomposable in the sense that they are the closed linear span of their homogeneous elements.

The material in this section splits up into two parts. We collect some easy properties of homogeneous elements of C^*-bimodules and C^*-algebras which are used throughout the calculations in chapter 2. These include nice factorisation theorems for compact homogeneous operators on C^*-bimodules over commutative C^*-algebras. Additionally, the underlying concepts are studied for their own sake: we characterise decomposability of a C^*-algebra A as being equivalent to the condition $A=Z(A)A$, introduce the notion of a C^*-family which refines the notion of a C^*-pre-family and give precise bundle-theoretic descriptions of decomposable C^*-bimodules and of C^*-families in the case that the underlying C^*-algebras are commutative.

Throughout this section, let A be a C^*-algebra and let E and E' be C^*-bimodules over A, i.e. C^*-A-A-bimodules.

1.2.1 The family of homogeneous elements of a C^*-bimodule

We introduce the notion of homogeneity for elements of C^*-bimodules and collect some easy properties. Furthermore, we consider operators which are related to the maps θ^ξ and θ_ξ discussed above. The following definition is fundamental.

Definition 1.48. *An element $\xi\in E$ is* γ-homogeneous/γ-covariant *for some $\gamma\in$* $\mathrm{PAut}(A)$ *if it is contained in the subspace $E\,\mathrm{Dom}(\gamma)$ and satisfies $\xi a=\gamma(a)\xi$ for all*

$a \in \mathrm{Dom}(\gamma)$. *We denote the set of all γ-homogeneous elements of E by $\mathscr{C}ov_\gamma(E)$, and the family $\big(\mathscr{C}ov_\gamma(E)\big)_\gamma$ by $\mathscr{C}ov(E)$. The C^*-bimodule E is* decomposable *if the closed linear span of the family $\mathscr{C}ov(E)$ is equal to E. We use the same notation for a C^*-algebra, considering it as a C^*-bimodule over itself in the canonical way.*

Examples. Examples of homogeneous elements of C^*-bimodules occurred already in the introductory section preceding the definition of homogeneous operators. For each r-discrete groupoid G, the associated C^*-bimodule $\big(L^2(G,\lambda), \pi_s\big)$ over $C_0(G^0)$ is decomposable, see proposition 3.5.

If X is a locally compact space and $u \in A$ is a unitary, then each element of the form $f \otimes u$ in the C^*-algebra $C_0(X) \otimes A$ is $(\mathrm{id} \otimes \mathrm{Ad}_u)$-homogeneous:

$$(f \otimes u) \cdot (g \otimes a) = (g \otimes uau^*) \cdot (f \otimes u) = \big((\mathrm{id} \otimes \mathrm{Ad}_u)(g \otimes a)\big) \cdot (f \otimes u)$$

for all $g \otimes a \in C_0(X) \otimes A$.

In proposition 1.55, we will show that a C^*-algebra B is decomposable if and only if $B = Z(B)B$.

For each element $\xi \in E$, the map $\theta_\xi \colon A \to E$ given by $a \mapsto \xi a$ defines a compact operator, and the map $\theta_- \colon E \to K_A(A,E)$ given by $\xi \mapsto \theta_\xi$ is an isometric isomorphism $E \xrightarrow{\cong} K_A(A,E)$ of Banach spaces. In presence of a representation of A on E, this result can be refined as follows.

Proposition 1.49. *i) For each $\xi \in \mathscr{C}ov_\gamma(E)$, $\gamma \in \mathrm{PAut}(A)$, the map $\theta_\xi \colon A \to E$ given by $a \mapsto \xi a$ is a (γ, id)-homogeneous compact operator. Its adjoint is given by $\theta_\xi^*(\eta) = \langle \xi | \eta \rangle$ for all $\eta \in E$.*

ii) For each $\gamma \in \mathrm{PAut}(A)$, the map $\theta_- \colon \mathscr{C}ov_\gamma(E) \to \mathscr{K}_{\mathrm{id}}^\gamma(A,E)$ given by $\xi \mapsto \theta_\xi$ defines an isomorphism $\mathscr{C}ov_\gamma(E) \xrightarrow{\cong} \mathscr{K}_{\mathrm{id}}^\gamma(A,E)$.

Proof. i) For all $a \in \mathrm{Dom}(\gamma)$ and $a' \in A$, one has $\theta_\xi(aa') = \xi aa' = \gamma(a)\theta_\xi a'$. The image of θ_ξ is contained in $\mathrm{Im}(\gamma)E$ because ξ belongs to $\mathrm{Im}(\gamma)E$. This proves that θ_ξ is a (γ, id)-homogeneous operator. It is homogeneous compact by the remark preceding the proposition.

ii) We only need to show that the map θ_- is surjective. Let $T \in \mathscr{K}_{\mathrm{id}}^\gamma(A,E)$. Then there exists an element $\xi \in E$ such that $Ta = \xi a$ for all $a \in A$. Since ξ is contained in $\xi A = \mathrm{Im}\, T$, it belongs to $\mathrm{Im}(\gamma)E$. For all $a \in \mathrm{Dom}(\gamma)$ and $b \in A$, one has

$$(\xi a - \gamma(a)\xi)b = \xi(ab) - \gamma(a)\xi b = T(ab) - \gamma(a)(Tb) = 0$$

and therefore $\xi a = \gamma(a)\xi$. Hence ξ belongs to $\mathscr{C}ov_\gamma(E)$. ∎

The formulation of the next lemma involves the following notation.

Notation 1.50. *For each pair of partial automorphisms $\gamma, \gamma' \in \mathrm{PAut}(A)$, we denote by $\mathscr{C}ov_{\gamma'}(E') \otimes_* \mathscr{C}ov_\gamma(E)$ the subspace of $E' \otimes_* E$ given by*

$$\overline{\mathrm{span}}\{\xi' \otimes_* \xi \mid \xi' \in \mathscr{C}ov_{\gamma'}(E'), \, \xi \in \mathscr{C}ov_\gamma(E)\}.$$

Furthermore, we denote by $\mathscr{C}ov(E') \otimes_* \mathscr{C}ov(E)$ the subfamily of $\mathscr{C}ov(E' \otimes_* E)$ given by

$$\big(\mathscr{C}ov(E') \otimes_* \mathscr{C}ov(E)\big)_{\gamma''} = \overline{\mathrm{span}}_{\gamma'\gamma = \gamma''} \big(\mathscr{C}ov_{\gamma'}(E') \otimes_* \mathscr{C}ov_\gamma(E)\big).$$

Next, we collect some easy properties of homogeneous elements. They can be proved by straight-forward calculations, but the identifications made in the previous proposition allows to shorten the proofs.

Lemma 1.51. *Let $\gamma, \gamma' \in \mathrm{PAut}(A)$.*

i) $\big\langle \mathscr{C}ov_\gamma(E) \big| \mathscr{C}ov_{\gamma'}(E) \big\rangle \subset \mathscr{C}ov_{\gamma^*\gamma'}(A)$.

ii) $\mathscr{C}ov_{\gamma'}(E') \otimes_* \mathscr{C}ov_\gamma(E) \subset \mathscr{C}ov_{\gamma'\gamma}(E' \otimes_* E)$.

iii) $\mathscr{C}ov_\gamma(E) \subset \mathscr{C}ov_{\gamma'}(E)$ *if* $\gamma' \geq \gamma$.

iv) $\mathscr{L}_\alpha^\beta(E, E')\big(\mathscr{C}ov_\gamma(E)\big) \subset \mathscr{C}ov_{\gamma'}(E')$ *where* $\gamma' = \beta\gamma\alpha^*$.

v) $\mathscr{C}ov_\gamma(E)$ *is a C^*-bimodule over* $\mathscr{C}ov_{\gamma^*\gamma}(A) \subset Z(A)$.

vi) If E is decomposable and full, A is decomposable.

vii) If E is decomposable, then $AE = E$.

Proof. For part ii), we only give a short proof under the assumption that $AE = E$; the general case can be covered by a straight-forward albeit tedious calculation.

i),ii) For all $\eta, \xi \in E$ and $\xi' \in E'$, we identify the operator $\theta_{\langle\eta|\xi\rangle}$ with $\theta_\eta^*\theta_\xi$ and the operator $\theta_{(\xi' \otimes_* \xi)}$ with the composition

$$(\theta_{\xi'} \otimes_* 1)\theta_\xi \colon A \to E \cong A \otimes_* E \to E' \otimes_* E.$$

The claims follow from the previous proposition and parts i,ii) of proposition 1.12.

iii) This follows from part ii) of proposition 1.49.

iv) Let $T \in \mathscr{L}_\alpha^\beta(E, E')$ and $\xi \in \mathscr{C}ov_\gamma(E)$. Then $(\theta_{T\xi})^*\zeta = \langle T\xi|\zeta\rangle = \alpha\big(\langle\xi|T^*\zeta\rangle\big)$ and hence $(\theta_{T\xi})^* = \alpha\theta_\xi^* T^* \in \mathscr{K}_{\alpha\alpha^*}^{\alpha\gamma^*\beta^*}(E, A)$. By part ii) of proposition 1.49, $T\xi$ belongs to $\mathscr{C}ov_{\gamma'}(E)$.

v,vi) This follows from i) and iv).

vii) By assumption, $E = \overline{\mathrm{span}}_\gamma \mathscr{C}ov_\gamma(E) \subset \overline{\mathrm{span}}_\gamma \mathrm{Im}(\gamma)E \subset AE$. ☐

Proposition 1.52. *i) For each $\xi \in \mathscr{C}ov_\gamma(E)$, $\gamma \in \mathrm{PAut}(A)$, the map $\theta^\xi \colon A \to E$ given by $a \mapsto a\xi$ is an (id, γ^*)-homogeneous compact operator. Its adjoint is given by $\theta^{\xi*}(\eta) = \gamma\big(\langle\xi|\eta\rangle\big)$ for all $\eta \in E$.*

ii) For each $\gamma \in \mathrm{PAut}(A)$, the map $\theta^-\colon \mathscr{C}ov_\gamma(E) \to \mathscr{K}^{\mathrm{id}}_{\gamma^*}(A,E)$ given by $\xi \mapsto \theta^\xi$ defines an isomorphism $\mathscr{C}ov_\gamma(E) \xrightarrow{\cong} \mathscr{K}^{\mathrm{id}}_{\gamma^*}(A,E)$.

Proof. Let $(u_\nu)_\nu$ be an approximate unit for $\mathrm{Im}(\gamma)$.

i) By assumption on ξ, the inner product $\langle \xi | E\rangle$ is contained in $\mathrm{Dom}(\gamma)$. For all $a \in A$ and $\eta \in E$, one has

$$\gamma\big(\langle \eta | a\xi\rangle\big) = \lim_\nu \gamma\big(\langle \eta | a u_\nu \xi\rangle\big) = \lim_\nu \gamma\big(\langle \eta|\xi\rangle\big) a u_\nu = \gamma\big(\langle \eta|\xi\rangle\big)a.$$

Therefore, the map θ^ξ is an (id,γ^*)-homogeneous operator. We show that it is elementary compact. By part v) of lemma 1.51, $\mathscr{C}ov_\gamma(E)$ is a C^*-module over the ideal $\mathscr{C}ov_{\gamma^*\gamma}(A)$. By the Cohen factorisation theorem, we can find elements $\xi' \in \mathscr{C}ov_\gamma(E)$ and $a' \in \mathscr{C}ov_{\gamma^*\gamma}(A)$ such that $\xi = \xi' a'$. Then

$$\theta^\xi a = a\xi = a\xi' a' = a\gamma(a')\xi' = \xi'\gamma^*(\gamma(a')a) = T^{\gamma^*}_{\xi',\gamma(a')^*}(a), \quad a \in A.$$

ii) We only need to show that the map θ^- is surjective. Let $T \in \mathscr{K}^{\mathrm{id}}_{\gamma^*}(A,E)$. By part viii) of proposition 1.2, the image of the operator T is contained in $E\,\mathrm{Dom}(\gamma)$. Consider the composition $T\gamma\colon \mathrm{Dom}(\gamma) \to \mathrm{Im}(\gamma) \to E\,\mathrm{Dom}(\gamma)$. It belongs to $\mathscr{K}^{\gamma}_{\gamma\gamma^*}(\mathrm{Dom}(\gamma),E)$. By part ii) of proposition 1.49, it is of the form $a \mapsto \xi a$ with some $\xi \in \mathscr{C}ov_\gamma(E)$. Then, by assumption on T, one has for all $a \in A$

$$\begin{aligned} Ta &= \lim_\nu T(au_\nu) = \lim_\nu T\gamma\big(\gamma^*(au_\nu)\big) \\ &= \lim_\nu \xi\gamma^*(au_\nu) = \lim_\nu au_\nu\xi = \lim_\nu a\xi\gamma^*(u_\nu) = a\xi = \theta^\xi a. \end{aligned}$$ ☐

1.2.2 Homogeneous elements of a C^*-algebra

In this subsection, we collect several characterisations of homogeneity in C^*-algebras and some properties of homogeneous elements of C^*-algebras. But first, we show that decomposability is not a very strong condition on a C^*-algebra.

Proposition 1.53. *A C^*-algebra A is decomposable if and only if the inclusion of the centre $Z(A) = \mathscr{C}ov_{\mathrm{id}}(A)$ in A is non-degenerate, i.e. if $Z(A)A = A$.*

Proof. Assume that A is decomposable. By lemma 1.51, for each $\gamma \in \mathrm{PAut}(A)$, the space $\mathscr{C}ov_\gamma(A)$ is a C^*-bimodule over the centre $Z(A)$. Hence, $Z(A)\mathscr{C}ov_\gamma(A) = \mathscr{C}ov_\gamma(A)$ and therefore $Z(A)A$ is dense in A.

Now, assume that $Z(A)A = A$. For each unitary $u \in M(A)$ and each $c \in Z(A)$, the product cu is contained in $\mathscr{C}ov_\gamma(A)$, where $\gamma = \mathrm{Ad}_u \in \mathrm{Aut}(A)$ is given by $\gamma(a) = uau^*$. By [34, remark 2.2.2], each element of A can be written as a sum of four unitaries in $M(A)$. Therefore, A is decomposable. ☐

Next, we collect results which will be needed in chapter 2. Let $\gamma \in \mathrm{PAut}(A)$.

Lemma 1.54. *An element $a \in A$ is γ-homogeneous if and only if one of the following conditions holds.*

i) $a \in \operatorname{Im}(\gamma)$ *and* $ab = \gamma(b\gamma^*(a))$ *for all* $b \in A$,

ii) $a \in \operatorname{Im}(\gamma)$ *and* $ab = \gamma(b)a$ *for all* $b \in \operatorname{Dom}(\gamma)$,

iii) $a \in \operatorname{Dom}(\gamma)$ *and* $a\gamma^*(b) = ba$ *for all* $b \in \operatorname{Im}(\gamma)$.

iv) $a \in \operatorname{Dom}(\gamma)$ *and* $ba = \gamma^*(\gamma(a)b)$ *for all* $b \in A$.

Proof. Let $(u_\nu)_\nu$ be an approximate unit for the ideal $\operatorname{Dom}(\gamma)$. Then $(v_\nu)_\nu := (\gamma(u_\nu))_\nu$ is an approximate unit for the ideal $\operatorname{Im}(\gamma)$. Each of the four conditions immediately implies that $\lim_\nu v_\nu a = a = \lim_\nu a u_\nu$ and hence that a is contained in $\operatorname{Dom}(\gamma) \cap \operatorname{Im}(\gamma)$. Now, the implications i) $\Rightarrow$ ii) $\Leftrightarrow$ iii) $\Leftarrow$ iv) are obvious. We show ii) $\Rightarrow$ i), and iii) $\Rightarrow$ iv) is analogous. For all $b \in A$, one has $ab = \lim_\nu a u_\nu b = \lim_\nu \gamma(u_\nu b)a = \lim_\nu \gamma(u_\nu b\gamma^*(a)) = \gamma(b\gamma^*(a))$. ☐

Proposition 1.55. *i)* $\alpha\big(\mathscr{C}ov_\gamma(A) \cap \operatorname{Dom}(\alpha)\big) \subset \mathscr{C}ov_{\alpha\gamma\alpha^*}(A)$ *for each* $\alpha \in \operatorname{PAut}(A)$,

ii) $\gamma\big(\mathscr{C}ov_\gamma(A)\big) = \mathscr{C}ov_\gamma(A) = \gamma^*\big(\mathscr{C}ov_\gamma(A)\big)$.

Proof. i) Let $a \in \mathscr{C}ov_\gamma(A) \cap \operatorname{Dom}(\alpha)$ and $b \in \operatorname{Dom}(\alpha\gamma\alpha^*)$. Then $b \in \operatorname{Dom}(\alpha^*) = \operatorname{Im}(\alpha)$, $\alpha^*(b) \in \operatorname{Dom}(\gamma)$ and $\gamma(\alpha^*(b)) \in \operatorname{Dom}(\alpha)$, hence

$$\alpha(a)b = \alpha\big(a\alpha^*(b)\big) = \alpha\big(\gamma(\alpha^*(b))a\big) = (\alpha\gamma\alpha^*)(b) \cdot \alpha(a).$$

ii) By lemma 1.54, $\mathscr{C}ov_\gamma(A)$ is contained in $\operatorname{Dom}(\gamma) \cap \operatorname{Dom}(\gamma^*)$. By part i), we have

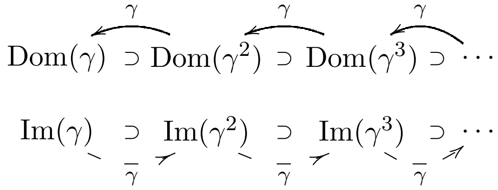
$$\gamma\big(\mathscr{C}ov_\gamma(A)\big) \subset \mathscr{C}ov_{\gamma\gamma\gamma^*}(A), \qquad \gamma^*\big(\mathscr{C}ov_\gamma(A)\big) \subset \mathscr{C}ov_{\gamma^*\gamma\gamma}(A).$$

Since $\gamma\gamma\gamma^*$ and $\gamma^*\gamma\gamma$ both are restrictions of γ, both of the sets above are contained in $\mathscr{C}ov_\gamma(A)$. Since γ and γ^* are mutually inverse, the claim follows. ☐

Part ii) of the preceding proposition exhibited a peculiar feature which we pursue in the next lemma and proposition. Neither of both is used in the rest of the thesis. Put $\operatorname{Dom}^\infty(\gamma) := \bigcap_{n>0} \operatorname{Dom}(\gamma^n)$ and $\operatorname{Im}^\infty(\gamma) := \bigcap_{n>0} \operatorname{Im}(\gamma^n)$.

Lemma 1.56. *The restrictions of* γ *and* γ^* *to the ideal* $J := \operatorname{Dom}^\infty(\gamma) \cap \operatorname{Im}^\infty(\gamma)$ *form a pair of inverse isomorphisms.*

Proof. The following chains of (partial) maps and inclusions show $\gamma(J) \subset J$.



Symmetrically, $\gamma^*(J) \subset J$. This proves the claim. ☐

We denote the restriction of γ to the ideal $\mathrm{Im}^\infty(\gamma) \cap \mathrm{Dom}^\infty(\gamma)$ by γ_∞.

Proposition 1.57. *One has* $\mathscr{C}ov_\gamma(A) = \mathscr{C}ov_{\gamma_\infty}(A)$.

Proof. This follows from lemma 1.54 and part ii) of proposition 1.55. □

1.2.3 C^*-families

Let A and B be C^*-algebras and let E be a C^*-B-A-bimodule. If both C^*-algebras are decomposable, the definition of C^*-pre-families on E can be refined so that one obtains a close analogue to the notion of a Fell bundle over the inverse semigroup $\mathrm{PAut}(B) \times \mathrm{PAut}(A)$ – the notion of an C^*-family. If A and B are commutative, this analogy can be made precise, see proposition 1.63. Basically, a C^*-family is a C^*-pre-family which is a module over a C^*-(pre-)family of coefficients $\mathscr{O}(E)$ naturally associated to E. This C^*-(pre-)family of coefficients is given by left and right multiplication by homogeneous elements of B and A, respectively, on E. In chapter 2, it will turn out that the Hopf C^*-pre-families associated to a decomposable pseudo-multiplicative unitary are in fact C^*-families.

First, we define the C^*-(pre-)family of coefficients $\mathscr{O}(E)$.

Proposition 1.58. *i) For each $a \in \mathscr{C}ov_\alpha(A)$, $\alpha \in \mathrm{PAut}(A)$, the map $\rho(a) \colon E \to E$ given by $\xi \mapsto \xi a$ is an (id, α^*)-homogeneous operator, and $\rho(a)^* = \rho(\alpha(a^*))$.*

ii) For each $b \in \mathscr{C}ov_\beta(B)$, $\beta \in \mathrm{PAut}(B)$, the map $\lambda(b) \colon E \to E$ given by $\xi \mapsto b\xi$ is a (β, id)-homogeneous operator, and $\lambda(b)^* = \lambda(b^*)$.

iii) The family $\mathscr{O}(E) \subset \mathscr{L}(E)$ given by $\mathscr{O}^\beta_\alpha(E) := \lambda\big(\mathscr{C}ov_\beta(B)\big)\rho\big(\mathscr{C}ov_{\alpha^*}(A)\big)$ for all $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$ is a C^*-pre-family.

Proof. Fort part ii), we only give a short proof under the assumption that $BE = E$; the general case can be covered by a straight-forward albeit tedious calculation. With respect to the identifications $E \cong B \otimes_* E \otimes_* A$, the maps $\lambda(b)$ and $\rho(a)$ correspond to $\theta_b \otimes_* 1 \otimes_* 1$ and $1 \otimes_* 1 \otimes_* \theta^a$, respectively. The claims follow from proposition 1.52, 1.49 and 1.24. Part iii) is immediate. □

Definition 1.59. *A C^*-pre-family $\mathscr{C} \subset \mathscr{L}(E)$ is a* C^*-family *if $\mathscr{C}\mathscr{O}(E) \subset \mathscr{C}$ and $\mathscr{C}^\beta_\alpha = \mathscr{C}^\beta_\alpha \cdot \mathscr{O}^{\beta^*\beta}_{\alpha^*\alpha}(E)$ for all $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$.*

Remark 1.60. i) The C^*-pre-family $\mathscr{O}(E)$ is a C^*-family.

ii) A C^*-pre-family $\mathscr{C} \subset \mathscr{L}(E)$ is a C^*-family if and only if $\mathscr{C}\mathscr{O}(E) \subset \mathscr{C}$ and $\mathscr{C}^{\epsilon'}_\epsilon = \mathscr{C}^{\epsilon'}_\epsilon \mathscr{O}^{\epsilon'}_\epsilon(E)$ for all partial identities $\epsilon \in \mathrm{PAut}(A)$ and $\epsilon' \in \mathrm{PAut}(B)$, because $\mathscr{C}^\beta_\alpha = \mathscr{C}^\beta_\alpha \cdot \mathscr{C}^{\beta^*\beta}_{\alpha^*\alpha}$ for all $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$.

Later, we will need the following lemma.

Lemma 1.61. *Assume that $BE = E$. Consider the space B as a C^*-$\mathbb{C}$-B-bimodule in the canonical way and denote this bimodule by $B^{\mathbb{C}}$.*

i) Let $\mathscr{C} \subset \mathscr{L}(E)$ be a C^*-pre-family such that $\mathscr{C}_\alpha^\beta = \mathscr{C}_\alpha^\beta \lambda(\mathscr{C}ov_{\beta^*\beta}(B))$ for all $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$. Then the identification $E \cong B^{\mathbb{C}} \otimes_* E$ as C^*-$\mathbb{C}$-A-bimodules induces an embedding $\overline{\mathrm{span}}_\beta \mathscr{C}_\alpha^\beta \subset \big(\mathscr{L}^{\mathrm{id}}(B^{\mathbb{C}}) \otimes_* \mathscr{C}\big)_\alpha$.

ii) Let E' be a C^*-B-A-bimodule such that $BE' = E'$ and let $\mathscr{D} \subset \mathscr{L}(E')$ be a C^*-pre-family such that $\mathscr{D}_\alpha^\beta = \mathscr{D}_\alpha^\beta \lambda(\mathscr{C}ov(B))_{\beta^*\beta}$ for all $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$. Let $\phi \colon \mathscr{C} \to \mathscr{D}$ be a morphism. If $\lambda(\mathscr{C}ov(B))\mathscr{C} \subset \mathscr{C}$, then $\phi(\lambda(b)c) = \lambda(b)\phi(c)$ for all $b \in \mathscr{C}ov(B)$ and $c \in \mathscr{C}$.

Proof. i) Denote by $\Phi \colon B^{\mathbb{C}} \otimes_* E \cong E$ the identification given by $\Phi(b' \otimes_* \xi) = b'\xi$, $b' \in B, \xi \in E$. Let $\beta \in \mathrm{PAut}(A), \alpha \in \mathrm{PAut}(A)$ and $c \in \mathscr{C}_\alpha^\beta$ and $b \in \mathscr{C}ov_{\beta^*\beta}(B)$. Then $\beta \circ \lambda(b) \colon b' \mapsto \beta(bb')$ defines a (id, β)-homogeneous operator on $B^{\mathbb{C}}$. For all $b' \in B$ and $\xi \in E$, one has

$$\begin{aligned}\Phi\big((\beta \circ \lambda(b) \otimes_* c)(b' \otimes_* \xi)\big) &= \Phi(\beta(bb') \otimes_* c\xi) \\ &= \beta(bb')c\xi = cbb'\xi = c\lambda(b)\Phi(b' \otimes_* \xi),\end{aligned}$$

whence $\mathrm{Ad}_{\Phi^{-1}}(c\lambda(b))$ is contained in $\mathscr{L}^{\mathrm{id}}(B^{\mathbb{C}}) \otimes_* \mathscr{C}$. The claim follows.

ii) By part i), for each $\alpha \in \mathrm{PAut}(A)$, the identification Φ induces an embedding $\iota_\alpha \colon \overline{\mathrm{span}}_\beta \mathscr{C}_\alpha^\beta \subset (\mathscr{L}^{\mathrm{id}}(B^{\mathbb{C}}) \otimes_* \mathscr{C})_\alpha$ and a similar embedding ι'_α for $\overline{\mathrm{span}}_\beta \mathscr{D}_\alpha^\beta$. Now for each $c \in \mathscr{C}_\alpha^\beta$ and $b \in \mathscr{C}ov_{\beta'}(B)$, $\beta, \beta' \in \mathrm{PAut}(B)$, $\alpha \in \mathrm{PAut}(A)$, one has

$$\begin{aligned}(\mathrm{id} \otimes_* \phi)\big(\iota_\alpha(\lambda(b)c)\big) &= (\mathrm{id} \otimes_* \phi)\big((\lambda(b) \otimes_* 1)\iota_\alpha(c)\big) \\ &= (\lambda(b) \otimes_* 1) \cdot (\mathrm{id} \otimes_* \phi)(\iota_\alpha(c)) = \iota'_\alpha\big(\lambda(b)\phi(c)\big).\end{aligned}$$

□

Remark 1.62. Using a similar technique like the one used in the proof of proposition 1.35, one can probably show that each morphism $\phi \colon \mathscr{C} \to \mathscr{D}$ of C^*-pre-families is $\rho(\mathscr{C}ov(A))$-$\lambda(\mathscr{C}ov(B))$-linear. We do not make this precise because of lack of time. Note that one should assume the inclusion $\rho(\mathscr{C}ov(A))\lambda(\mathscr{C}ov(B))\mathscr{C} \subset \mathscr{C}$.

Fell bundle picture for C^*-families

If the C^*-algebras A and B are commutative, C^*-families of E correspond to certain Fell bundles on groupoids. In the following, we describe this correspondence.

Proposition 1.63. *Let X and Y be locally compact Hausdorff spaces and let E be a C^*-$C_0(Y)$-$C_0(X)$-bimodule. Let $\mathscr{C} \subset \mathscr{L}(E)$ be a C^*-family. Then there exists an upper semi-continuous Fell bundle $\mathrm{Fell}(\mathscr{C})$ on the locally compact groupoid $\mathfrak{PHom}(Y) \times \mathfrak{PHom}(X)$ with natural isomorphisms*

$$\Gamma_0\big([\psi] \times [\phi], \mathrm{Fell}(\mathscr{C})\big) \cong \mathscr{C}_{\phi_*}^{\psi_*}, \quad \phi \in \mathrm{PHom}(X), \psi \in \mathrm{PHom}(Y),$$

such that with respect to these identifications, the multiplication and involution on $\mathscr{C}$ coincide with the product and involution on sections of $\mathrm{Fell}(\mathscr{C})$ induced by its Fell bundle structure.

Proof. For an open subset U of X, denote by id_U the identity map on $C_0(U)$.

i) For each pair of open subsets $V \subset Y$ and $U \subset X$, the $*$-homomorphism $C_0(V \times U) \to M\big(\mathscr{C}^{\mathrm{id}_V}_{\mathrm{id}_U}\big)$ given by $((f \otimes g)c)\xi := c(f\xi g)$ for all $f \in C_0(V)$, $g \in C_0(U)$, $c \in \mathscr{C}^{\mathrm{id}_V}_{\mathrm{id}_U}$ and $\xi \in E$ is non-degenerate because $\mathscr{C}$ is a C^*-family.

Let $\phi \in \mathrm{PHom}(X)$ and $\psi \in \mathrm{PHom}(Y)$. Put $V := \mathrm{Dom}(\psi)$ and $U := \mathrm{Dom}(\phi)$. Then $\mathscr{C}^{\psi_*}_{\phi_*}$ is a C^*-module over $\mathscr{C}^{\mathrm{id}_V}_{\mathrm{id}_U}$. By lemma A.2, it is a convex $C_0(V \times U)$-module. Denote by C^{ψ}_{ϕ} the pull-back of the corresponding Banach bundle over $V \times U$ to $[\psi] \times [\phi] \subset \mathfrak{PHom}\,(Y) \times \mathfrak{PHom}\,(X)$ via the source map $s \times s\colon [\psi] \times [\phi] \to V \times U$. Then $\Gamma_0\big(C^{\psi}_{\phi}\big) = \mathscr{C}^{\psi_*}_{\phi_*}$ by construction.

Let $\phi' \in \mathrm{PHom}(X)$ and $\psi' \in \mathrm{PHom}(Y)$ be restrictions of ϕ and ψ to subsets $U' \subset U$ and $V' \subset V$, respectively. We show that the restriction of C^{ψ}_{ϕ} to $[\psi'] \times [\phi']$ is equal to $C^{\psi'}_{\phi'}$. By the correspondence between modules and bundles, it is enough to show that $\mathscr{C}^{\psi_*}_{\phi_*} \cdot C_0(V' \times U')$ is equal to $\mathscr{C}^{\psi'_*}_{\phi'_*}$. By the considerations above, the definition of a C^*-pre-family and the definition of a C^*-family, we have

$$\mathscr{C}^{\psi'_*}_{\phi'_*} = \mathscr{C}^{\psi'_*}_{\phi'_*} \cdot C_0(V' \times U') \subset \mathscr{C}^{\psi_*}_{\phi_*} \cdot C_0(V' \times U') = \mathscr{C}^{\psi_*}_{\phi_*} \cdot \mathscr{O}^{\mathrm{id}_{V'}}_{\mathrm{id}_{U'}} \subset \mathscr{C}^{\psi_*\,\mathrm{id}_{V'}}_{\phi_*\,\mathrm{id}_{U'}} = \mathscr{C}^{\psi'_*}_{\phi'_*}.$$

Hence, there exists an upper semi-continuous Banach bundle C on the groupoid $\mathfrak{PHom}\,(Y) \times \mathfrak{PHom}\,(X)$ such that $C|_{[\psi]\times[\phi]} \cong C^{\psi}_{\phi}$ for all $\phi \in \mathrm{PHom}(X)$ and $\psi \in \mathrm{PHom}(Y)$. By functoriality of the correspondence between modules and bundles, the multiplication and involution on $\mathscr{C}$ induce families of multiplication and involution maps on the family $\big(C^{\psi}_{\phi}\big)_{\psi,\phi}$. These induce well-defined maps $C^2 \to C$ and $C \to C$ as in the definition of a Fell bundle because the multiplication and involution on $\mathscr{C}$ is coherent with respect to inclusions. It is easy to check that these maps endow C with the structure of an upper semi-continuous Fell bundle. □

1.2.4 Homogeneity of C^*-bimodules over commutative C^*-algebras

Decomposable C^*-bimodules over a commutative C^*-algebra are particularly nice behaved: The fibres of the corresponding Hilbert bundles decompose nicely over the effective groupoid of the underlying space. This decomposition property implies several factorisation results for compact homogeneous operators and internal tensor products which will be relevant to the chapter 2. After a discussion of the decomposition and factorisation results, we show that a decomposable C^*-bimodule can be disintegrated over the effective groupoid of the space underlying the corresponding Hilbert bundle.

Let $A = C_0(X)$ where X is a locally compact Hausdorff space, and let E and E' be decomposable C^*-bimodules over A.

Lemma 1.64. *Let $\phi \in \mathrm{PHom}(X)$ and let $\phi' \in \mathrm{PHom}(X)$ be the restriction of ϕ to an open subset $U \subset \mathrm{Dom}(\phi)$. Then $\mathscr{C}ov_{\phi_*}(E) \cdot C_0(U) = \mathscr{C}ov_{\phi'_*}(E)$.*

Proof. First, note that $\mathrm{Dom}(\phi'_*) = C_0(U)$. By parts iii) and iv) of lemma 1.51,

$$\begin{aligned}\mathscr{C}ov_{\phi'_*}(E) &= \mathscr{C}ov_{\phi'_*}(E)\, C_0(U)\\ &\subset C_0(X)\ \mathscr{C}ov_{\phi_*}(E)\, C_0(U) = \rho\big(\mathscr{O}^{\mathrm{id}}_{\mathrm{id}_{U*}}(E)\big)\, \mathscr{C}ov_{\phi_*}(E) \subset \mathscr{C}ov_{\phi''_*}(E),\end{aligned}$$

where $\phi''_* = \mathrm{id} \cdot \phi_* \cdot \mathrm{id}_{U*} = \phi'_*$. ☐

Notation 1.65. *Let ϕ be a partial homeomorphism of X. Given a point $x \in \mathrm{Dom}(\phi)$, we denote by $\mathscr{C}ov_{(\phi,x)}(E)$ the fibre of the Hilbert bundle corresponding to the C^*-submodule $\mathscr{C}ov_{\phi_*}(E) \subset E$ at x. It is, of course, a Hilbert space. The preceding lemma shows that if $\phi' \in \mathrm{PHom}(X)$ coincides with ϕ on a small neighbourhood of x, then $\mathscr{C}ov_{(\phi,x)}(E) = \mathscr{C}ov_{(\phi',x)}(E)$. Hence, the assignment $\mathscr{C}ov_{[\phi,x]}(E) := \mathscr{C}ov_{(\phi,x)}(E)$ is well-defined.*

Given partial homeomorphisms ϕ and ψ of X, we write $\phi \perp \psi$ if there exists no open subset $U \subset \mathrm{Dom}(\phi) \cap \mathrm{Dom}(\psi)$ such that $\phi|_U = \psi|_U$, or, equivalently, if the domain of $\phi \wedge \psi$ is empty.

Lemma 1.66. *For each pair of partial homeomorphisms $\phi, \psi \in \mathrm{PHom}(X)$ satisfying $\phi \perp \psi$, one has $\langle \mathscr{C}ov_{\phi_*}(E) | \mathscr{C}ov_{\psi_*}(E)\rangle = 0$. For each $x \in X$, one has $\mathscr{C}ov(E)_x = \bigoplus_{\mathfrak{x} \in \mathfrak{PHom}(X)_x} \mathscr{C}ov_{\mathfrak{x}}(E)$.*

Proof. By lemma 1.51, the inner product $\big\langle \mathscr{C}ov_{\phi_*}(E) \big| \mathscr{C}ov_{\psi_*}(E)\big\rangle$ belongs to the subspace $\mathscr{C}ov_\omega(C_0(X))$ of $C_0(X)$, where $\omega = \phi^*\psi_*$. If $\phi \perp \psi$, there exists no open set $U \subset X$ such that $\phi|_U = \psi|_U$, and since $C_0(X)$ is commutative, $\mathscr{C}ov_\omega(C_0(X)) = 0$. The second assertion follows immediately. ☐

Proposition 1.67. *$\mathscr{C}ov(E') \otimes_* \mathscr{C}ov(E) = \mathscr{C}ov(E' \otimes_* E)$.*

Proof. Let $x \in X$ and $\mathfrak{x} \in \mathfrak{PHom}\,(X)_x$. By lemma 1.66, one has a decomposition

$$\mathscr{C}ov_{\mathfrak{x}}(E' \otimes_* E) = \bigoplus_{\mathfrak{x}' \in \mathfrak{PHom}\,(X)_x} \mathscr{C}ov_{\mathfrak{x}'}(E') \otimes_* \mathscr{C}ov_{\mathfrak{x}'^{-1}\mathfrak{x}}(E).$$

Hence, for each $\phi'' \in \mathrm{PHom}(X)$, the fibre of the Hilbert bundle corresponding to the C^*-module $\mathscr{C}ov_{\phi''_*}(E' \otimes_* E)$ and the fibre of the sub-bundle corresponding to $\overline{\mathrm{span}}_{\phi'\phi=\phi''}\, \mathscr{C}ov_{\phi'_*}(E') \otimes_* \mathscr{C}ov_{\phi_*}(E)$ are equal. ☐

Proposition 1.68. *i) For each $\xi \in \mathscr{C}ov_\gamma(E\,\mathrm{Dom}(\alpha))$ and $\eta \in \mathscr{C}ov_{\gamma'}(E'\,\mathrm{Im}(\alpha))$, the elementary compact operator $T^\alpha_{\eta,\xi}$ is (β,α)-homogeneous with $\beta = \gamma'\alpha\gamma^*$.*

ii) For all $\alpha, \beta \in \mathrm{PAut}(A)$, one has $\mathscr{K}^\beta_\alpha(E, E') = \overline{\mathrm{span}}\{T^\alpha_{\eta,\xi}$ as above$\}$.

Proof. i) Writing $T^\alpha_{\eta,\xi} = \theta_\eta \alpha \theta^*_\xi$, the claim follows from propositions 1.49 and 1.12.

ii) Let $\phi, \psi \in \mathrm{PHom}(X)$. We want to compare the spaces given above fibre by fibre. It is easy to see that the right module structure on E turns $\mathscr{K}^{\psi^*\psi_*}_{\phi^*\phi_*}(E)$ into a $C_0(\mathrm{Dom}(\phi))$-algebra. By lemma A.2, the right module structure on E turns

$\mathscr{K}_{\phi_*}^{\psi_*}(E,E')$ into a non-degenerate convex $C_0(\mathrm{Dom}(\phi))$-Banach module. By the remark above and lemma 1.66, the fibres of both modules given in the statement at a point $x \in X$ are equal to

$$\bigoplus_{[\omega,x]\in\mathfrak{PHom}(X)_x} K\big(\mathscr{C}ov_{[\omega,x]}(E), \mathscr{C}ov_{[\psi\omega\phi^{-1},\phi(x)]}(E')\big).$$

Therefore, the Banach modules coincide. □

For each pair of Hilbert spaces H and H', one has a canonical isomorphism $K(H'\otimes H) = K(H')\otimes K(H)$. For the internal tensor product of C^*-modules, one has the following analogue.

Proposition 1.69. *Let E, E' and F, F' be decomposable C^*-bimodules over A. Then $\mathscr{K}(F,F')\otimes_* \mathscr{K}(E,E') = \mathscr{K}(F\otimes_* E, F'\otimes_* E')$.*

Proof. It is clear that the right hand side is contained in the left hand side. Let us prove the reverse inclusion. By proposition 1.68 and 1.67, for each pair of partial automorphisms $\alpha,\beta \in \mathrm{PAut}(A)$, the space $\mathscr{K}_\alpha^\beta(F\otimes_* E, F'\otimes_* E')$ is spanned by operators of the form $T_{\zeta',\zeta}^\alpha$, where

$$\zeta = \eta\otimes_*\xi \in \mathscr{C}ov_{\gamma_2}(F)\otimes_*\mathscr{C}ov_{\gamma_1}(E)\,\mathrm{Dom}(\alpha) \subset \mathscr{C}ov_{\gamma_2\gamma_1}(F\otimes_* E)\,\mathrm{Dom}(\alpha),$$
$$\zeta' = \eta'\otimes_*\xi' \in \mathscr{C}ov_{\gamma_2'}(F')\otimes_*\mathscr{C}ov_{\gamma_1'}(E')\,\mathrm{Im}(\alpha) \subset \mathscr{C}ov_{\gamma_2'\gamma_1'}(F'\otimes_* E')\,\mathrm{Im}(\alpha),$$

and $\gamma_2'\gamma_1'\alpha\gamma_1^*\gamma_2^* = \beta$. Then, in particular,

$$\zeta \in (F\otimes_* E)\big(\mathrm{Dom}(\gamma_1)\cap\mathrm{Dom}(\alpha)\big),\qquad \zeta' \in (F'\otimes_* E')\big(\mathrm{Dom}(\gamma_1')\cap\mathrm{Im}(\alpha)\big),$$

and the equation $T_{\zeta',\zeta a}^\alpha = T_{\zeta'\alpha(a)^*,\zeta}^\alpha$, $a\in\mathrm{Dom}(\alpha)$, shows that we may assume

$$\xi \in \mathscr{C}ov_{\gamma_1}(E)\alpha^*\big(\mathrm{Dom}(\gamma_1')\cap\mathrm{Im}(\alpha)\big) = \mathscr{C}ov_{\gamma_1}(E)\,\mathrm{Dom}(\gamma_1'\alpha) = \mathrm{Dom}(\delta)\,\mathscr{C}ov_{\gamma_1}(E),$$
$$\xi' \in \mathscr{C}ov_{\gamma_1'}(E')\alpha\big(\mathrm{Dom}(\gamma_1)\cap\mathrm{Dom}(\alpha)\big) = \mathscr{C}ov_{\gamma_1'}(E')\,\mathrm{Im}(\alpha\gamma_1^*) = \mathrm{Im}(\delta)\,\mathscr{C}ov_{\gamma_1'}(E'),$$

where $\delta = \gamma_1'\alpha\gamma_1^*$. Thus, without loss of generality, $\eta\in\mathscr{C}ov_{\gamma_2}(F)\,\mathrm{Dom}(\delta)$ and $\eta'\in\mathscr{C}ov_{\gamma_2'}(F')\,\mathrm{Im}(\delta)$. Consider the elementary compact operators $T_{\xi',\xi}^\alpha$ and $T_{\eta',\eta}^\delta$. By proposition 1.68, $T_{\xi',\xi}^\alpha \in \mathscr{K}_\alpha^\delta(E,E')$ and $T_{\eta',\eta}^\delta \in \mathscr{K}_\delta^\beta(F,F')$. We show that $T_{\zeta',\zeta}^\alpha = T_{\eta',\eta}^\delta \otimes_* T_{\xi',\xi}^\alpha$. Let $\eta''\otimes_*\xi'' \in F\otimes_* E$. Then

$$T_{\zeta',\zeta}^\alpha(\eta''\otimes_*\xi'') = \eta'\otimes_*\xi'\cdot\alpha\big(\langle\xi\big|\langle\eta|\eta''\rangle\xi''\rangle\big).$$

By covariance of ξ and ξ', we have

$$\big\langle\xi\big|\langle\eta|\eta''\rangle\xi''\big\rangle = \big\langle\langle\eta|\eta''\rangle^*\xi\big|\xi''\big\rangle = \big\langle\xi\gamma_1^*\big(\langle\eta|\eta''\rangle^*\big)\big|\xi''\big\rangle = \gamma_1^*\big(\langle\eta|\eta''\rangle\big)\,\langle\xi|\xi''\rangle,$$
$$\eta'\otimes_*\xi'(\alpha\gamma_1^*)\big(\langle\eta|\eta''\rangle\big) = \eta'\otimes_*\delta\big(\langle\eta|\eta''\rangle\big)\xi' = \eta'\delta\big(\langle\eta|\eta''\rangle\big)\otimes_*\xi'.$$

Therefore,

$$T_{\zeta',\zeta}^\alpha(\eta''\otimes_*\xi'') = \eta'\delta(\langle\eta|\eta''\rangle)\otimes_*\xi'\alpha(\langle\xi|\xi''\rangle) = \big(T_{\eta',\eta}^\delta\otimes_* T_{\xi',\xi}^\alpha\big)(\eta''\otimes_*\xi''). \qquad □$$

Now, we show that each decomposable C^*-bimodule over $C_0(X)$ can be disintegrated over the effective groupoid $\mathfrak{PHom}\,(X)$. The initial C^*-bimodule identifies with the push-forward of the Hilbert bundle obtained by the disintegration along the range map. If the support of the Hilbert bundle on the effective groupoid is Hausdorff, this push-forward construction can be described in the form of an internal tensor product with a canonical C^*-bimodule. In principle, this C^*-bimodule is nothing else but $L^2(\mathfrak{PHom}\,(X), \lambda)$, where λ denotes the family of counting measures. However, this can not be defined as a C^*-bimodule over $C_0(X)$ because in general, the topology on the groupoid $\mathfrak{PHom}\,(X)$ does not satisfy the Hausdorff condition. Therefore, we impose the support condition mentioned above and make the following definition. Let $\mathfrak{X} \subset \mathfrak{PHom}\,(X)$ be a closed Hausdorff subset. Then the space $C_c(\mathfrak{X})$ is a pre-C^*-module over $C_0(X)$ with respect to the operations

$$\langle \mathfrak{f}|\mathfrak{g}\rangle(x) := \sum_{\mathfrak{x}\in\mathfrak{PHom}\,(X)^x} \overline{\mathfrak{f}(\mathfrak{x})}\mathfrak{g}(\mathfrak{x}), \quad (\mathfrak{f}f)(\mathfrak{x}) := \mathfrak{f}(\mathfrak{x})f\big(r(x)\big), \quad \mathfrak{f},\mathfrak{g} \in C_c(\mathfrak{X}),\ f \in C_0(X).$$

We denote by $\Gamma^2(\mathfrak{X})$ the completion of this pre-C^*-module and by M the representation of $C_0(\mathfrak{X})$ on $\Gamma^2(\mathfrak{X})$ given by pointwise multiplication.

Proposition 1.70. *Let E be C^*-bimodule over $C_0(X)$.*

i) There exists a continuous Hilbert bundle $\mathrm{Cov}(E)$ *on* $\mathfrak{PHom}\,(X)$ *with natural isomorphisms*

$$\Gamma_0\big([\phi], \mathrm{Cov}(E)\big) \cong \mathscr{C}ov_{\phi_*}(E) \quad \textit{for all} \quad \phi \in \mathrm{PHom}(X),$$
$$\mathrm{Cov}(E)_{\mathfrak{x}} \cong \mathscr{C}ov_{\mathfrak{x}}(E) \quad \textit{for all} \quad \mathfrak{x} \in \mathfrak{PHom}\,(X).$$

ii) If E is decomposable and the subspace

$$\mathfrak{X} := \textit{closure of } \{\mathfrak{x} \in \mathfrak{PHom}\,(X) \mid \mathscr{C}ov_{\mathfrak{x}}(E) \neq 0\} \subset \mathfrak{PHom}\,(X)$$

is Hausdorff, then $E \cong \Gamma_0(\mathrm{Cov}(E)) \otimes_M \Gamma^2(\mathfrak{X})$.

Proof. i) For each $\phi \in \mathrm{PHom}(X)$, the space $\mathscr{C}ov_{\phi_*}(E)$, considered as a right C^*-module over $C_0(\mathrm{Dom}(\phi))$, corresponds to a continuous Hilbert bundle over $\mathrm{Dom}(\phi) \subset X$. Denote by $\mathrm{Cov}_\phi(E)$ the pull-back of this Hilbert bundle to the subset $[\phi] \subset \mathfrak{PHom}\,(X)$ via the restriction of the source map $s\colon [\phi] \to \mathrm{Dom}(\phi)$. Then $\Gamma_0(\mathrm{Cov}_\phi(E)) = \mathscr{C}ov_{\phi_*}(E)$, and the fibre of $\mathrm{Cov}_\phi(E)$ over a point $\mathfrak{x} \in [\phi]$ is equal to $\mathscr{C}ov_{\mathfrak{x}}(E)$.

Let $\phi' \in \mathrm{PHom}(X)$ be the restriction of ϕ to an open subset $U \subset \mathrm{Dom}(\phi)$. By lemma 1.64, we have $\mathscr{C}ov_{\phi_*}(E)\, C_0(U) = \mathscr{C}ov_{\phi'_*}(E)$. By the correspondence between modules and bundles, this implies that the restriction of $\mathrm{Cov}_\phi(E)$ to $[\phi']$ is equal to $\mathrm{Cov}_{\phi'}(E)$.

Hence, there exists a continuous Hilbert bundle $\mathrm{Cov}(E)$ on $\mathfrak{PHom}\,(X)$ such that $\mathrm{Cov}_\phi(E) = \mathrm{Cov}(E)|_{[\phi]}$ for all $\phi \in \mathrm{PHom}(X)$.

ii) The family of embeddings

$$j_\phi \colon \Gamma_0(\mathrm{Cov}_\phi(E)) \otimes_M C_0\big([\phi]\big) \cong \Gamma_0(\mathrm{Cov}_\phi(E)) \cong \mathscr{C}ov_{\phi_*}(E) \hookrightarrow E,$$

indexed by $\phi \in \mathrm{PHom}(X)$, define a map $j \colon \Gamma_0(\mathrm{Cov}(E)) \otimes_M \Gamma^2(\mathfrak{X}) \cong E$. The image of j is dense because E is decomposable. We show that for each $x \in X$, the map $j_x \colon \big(\mathrm{Cov}(E) \otimes_M \Gamma^2(\mathfrak{X})\big)_x \to E_x$ is isometric. With respect to the canonical identification

$$\begin{aligned}\Big(\Gamma_0(\mathrm{Cov}(E)) \otimes_M \Gamma^2(\mathfrak{X})\Big)_x &\cong \Gamma_0(\mathrm{Cov}(E)) \otimes_M l^2\big(\mathfrak{PHom}\,(X)^x\big) \\ &\cong \bigoplus_{\mathfrak{x} \in \mathfrak{PHom}\,(X)^x} \mathscr{C}ov_{\mathfrak{x}}(E),\end{aligned}$$

this map corresponds to the natural map $\bigoplus_{\mathfrak{x} \in \mathfrak{PHom}\,(X)^x} \mathscr{C}ov_{\mathfrak{x}}(E) \to E_x$ which is isometric by lemma 1.66. □

Remark 1.71. Using the techniques for the study of non-Hausdorff groupoids which will be developed in section 3.4, one can eliminate the support condition, replacing $\Gamma^2(\mathfrak{x})$ by a suitably defined C^*-module $L^2(\mathfrak{PHom}\,(X), \lambda)$. Then the internal tensor product $\mathrm{Cov}(E) \otimes_M L^2(\mathfrak{PHom}\,(X), \lambda)$ a-priori is a C^*-module over a C^*-algebra which properly contains $C_0(X)$, but one can easily show that in fact, the inner product takes values in $C_0(X)$ only.

Chapter 2

Pseudo-multiplicative unitaries and pseudo-Kac systems on C^*-modules

In this chapter, which is the main part of this thesis, we introduce and study pseudo-multiplicative unitaries and pseudo-Kac systems on C^*-modules. Building on the theory developed in the first chapter, we formulate decomposability conditions for such unitaries and pseudo-Kac systems and obtain the following main results: First, we define the legs of a decomposable pseudo-multiplicative unitary as families of homogeneous operators and show that they are Hopf C^*-families. Second, we construct crossed products in the framework of pseudo-Kac-systems and prove a duality theorem for iterated crossed products. Both results extend the work of Saad Baaj and Georges Skandalis [3].

Examples of pseudo-multiplicative unitaries and pseudo-Kac systems associated to groupoids are discussed in the third chapter. Up to now, these are the only examples satisfying the aforementioned decomposability condition for non-trivial reasons. For these examples, a one-line calculation shows that the legs form Hopf C^*-families. Thus, at the moment, the first main result lacks further illustrative examples. We discuss potential sources of interesting pseudo-multiplicative unitaries on C^*-modules – bicrossed product constructions and deformations – in a separate subsection. However, the constructions and results of the second part of this chapter are new and seem to be interesting already for r-discrete groupoids.

2.1 Preliminaries

The leg notation and the internal tensor product The common notation assigns a fixed rôle to the right and left factor of an internal tensor product of C^*-bimodules. For us, it will be useful to freely switch the order of the factors in internal tensor products. To do so, we introduce the following notation.

Notation 2.1. *Let A, B and C be C^*-algebras, let F be a C^*-C-B-bimodule and let E be a C^*-B-A-bimodule. We put $F \varogreaterthan E := F \otimes_* E$ and denote by $E \varolessthan F$ the completion of the pre-C^*-module $E \odot F$ over A with respect to the inner product and right module structure given by*

$$\langle \eta \odot \xi | \eta' \odot \xi' \rangle := \big\langle \eta \big| \langle \xi | \xi' \rangle_F \eta' \big\rangle_E, \quad (\eta \odot \xi) a := \eta a \odot \xi, \quad \eta, \eta' \in E,\, \xi, \xi' \in F,\, a \in A.$$

The formula $c(\eta \varolessthan \xi) := \eta \varolessthan c\xi$ for all $\eta \in E$, $\xi \in F$ and $c \in C$ defines a representation of C on $E \varolessthan F$ and turns it into a C^*-C-A-bimodule.

The flip $\Sigma\colon \eta \varolessthan \xi \leftrightarrow \xi \varogreaterthan \eta$ defines isomorphisms $\Sigma\colon E \varolessthan F \leftrightarrow F \varogreaterthan E$.

This notation also applies to internal tensor products of homogeneous operators, representations and C^*-pre-families in the obvious way.

As an example for the notation introduced above, let A, B, C and E, F be as above, let D be another C^*-algebra and let G be a C^*-D-C-bimodule. Then we can form e.g. the internal tensor products $G \varogreaterthan F \varogreaterthan E$, $G \varogreaterthan (E \varolessthan F)$ and $(F \varogreaterthan E) \varolessthan G$ with inner products given by

$$\begin{aligned} \Big\langle \xi \Big| \langle \eta | \langle \zeta | \zeta' \rangle \eta' \rangle \xi' \Big\rangle &= \langle \zeta \varogreaterthan \eta \varogreaterthan \xi | \zeta' \varogreaterthan \eta' \varogreaterthan \xi' \rangle_{(G \varogreaterthan F \varogreaterthan E)} \\ &= \langle \zeta \varogreaterthan (\xi \varolessthan \eta) | \zeta' \varogreaterthan (\xi' \varolessthan \eta') \rangle_{(G \varogreaterthan (E \varolessthan F))} \\ &= \langle (\eta \varogreaterthan \xi) \varolessthan \zeta | (\eta' \varogreaterthan \xi') \varolessthan \zeta' \rangle_{((F \varogreaterthan E) \varolessthan G)}, \end{aligned}$$

where $\zeta, \zeta' \in G$, $\eta, \eta' \in F$ and $\xi, \xi' \in E$.

Furthermore, we use an obvious generalisation of the leg notation [3] to denote maps between internal tensor product of C^*-bimodules which act on individual factors only, i.e. are of the form $1 \otimes_* W \otimes_* 1\colon F_2 \otimes_* E \otimes_* F_1 \to F_2 \otimes_* E' \otimes_* F_1$, $W \in \mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(E, E')$, up to a rearrangement of factors. We apply the same notation to C^*-pre-families and representations. For an illustration, look at lemma 2.5.

Hopf C^*-(pre-)families

Definition 2.2. *Let $(\mathscr{S}, \Delta)$ be a left Hopf C^*-pre-family on a C^*-bimodule E over a decomposable C^*-algebra A. If $\mathscr{S}$ is a C^*-family, $(\mathscr{S}, \Delta)$ is a* left Hopf C^*-family.

The asymmetry of the internal tensor product of C^*-bimodules and C^*-pre-families leads to the distinction between left and right Hopf C^*-pre-families. The former ones have already been introduced in subsection 1.1.5. Now that we have introduced the necessary notation, we can define the latter ones. We will see that the left leg of a decomposably regular pseudo-multiplicative unitary is a right Hopf C^*-pre-family and the right leg is a left Hopf C^*-pre-family.

Definition 2.3. *i) A* right Hopf C^*-(pre-)family *is a non-degenerate C^*-(pre)-family $\hat{\mathscr{S}}$ with a morphism $\hat{\Delta}\colon \hat{\mathscr{S}} \to \mathscr{M}(\hat{\mathscr{S}} \varolessthan \hat{\mathscr{S}})$ such that $(\hat{\mathscr{S}}, \hat{\Delta}^{op})$ is a left Hopf C^*-(pre-)family, where $\hat{\Delta}^{op} := \Sigma \circ \hat{\Delta}$ denotes the composition*

$\hat{\mathscr{S}} \to \mathscr{M}(\hat{\mathscr{S}} \otimes \hat{\mathscr{S}}) \xrightarrow{\Sigma} \mathscr{M}(\hat{\mathscr{S}} \otimes_* \hat{\mathscr{S}})$. *The Hopf C^*-(pre-)family $(\hat{\mathscr{S}}, \hat{\Delta}^{op})$ is the* opposite of $(\hat{\mathscr{S}}, \hat{\Delta})$.

Let A and B be C^*-algebras and let $(\hat{\mathscr{S}}, \hat{\Delta})$ be a right Hopf C^*-pre-family on a C^*-bimodule over A.

ii) Let $\mathscr{C}$ be a non-degenerate C^*-pre-family on a C^*-A-B-bimodule. A right coaction *of $(\hat{\mathscr{S}}, \hat{\Delta})$ on $\mathscr{C}$ is a non-degenerate morphism $\hat{\delta}\colon \mathscr{C} \to \mathscr{M}(\mathscr{C} \otimes \hat{\mathscr{S}})$ such that $\hat{\delta}(\mathscr{C})(1 \otimes \hat{\mathscr{S}}) \subset \mathscr{C} \otimes \hat{\mathscr{S}}$ and $(\hat{\delta} \otimes 1)\hat{\delta} = (1 \otimes \hat{\Delta})\hat{\delta}$.*
Let $\mathscr{C}$ and $\mathscr{C}'$ be non-degenerate C^*-pre-families on C^*-B-A-bimodules with right coactions $\hat{\delta}$ and $\hat{\delta}'$, respectively. A non-degenerate morphism $\phi\colon \mathscr{C} \to \mathscr{M}(\mathscr{C}')$ is equivariant *if $(\phi \otimes 1)\hat{\delta} = \hat{\delta}'\phi$.*

iii) Let $\mathscr{D}$ be a non-degenerate C^*-pre-family on a C^*-B-A-bimodule. A left coaction *of $(\hat{\mathscr{S}}, \hat{\Delta})$ on $\mathscr{D}$ is a non-degenerate morphism $\hat{\delta}\colon \mathscr{C} \to \mathscr{M}(\hat{\mathscr{S}} \otimes \mathscr{C})$ such that $\hat{\delta}(\mathscr{D})(\hat{\mathscr{S}} \otimes 1) \subset \hat{\mathscr{S}} \otimes \mathscr{D}$ and $(1 \otimes \hat{\delta})\hat{\delta} = (\hat{\Delta} \otimes 1)\hat{\delta}$.*
Let $\mathscr{D}$ and $\mathscr{D}'$ be non-degenerate C^*-pre-families on C^*-B-A-bimodules with left coactions $\hat{\delta}$ and $\hat{\delta}'$, respectively. A non-degenerate morphism $\psi\colon \mathscr{D} \to \mathscr{M}(\mathscr{D}')$ is equivariant *if $(1 \otimes \psi)\hat{\delta} = \hat{\delta}'\psi$.*

2.2 Pseudo-multiplicative unitaries on C^*-modules

In this section, we introduce pseudo-multiplicative unitaries on C^*-modules. This notion generalises several concepts introduced by Saad Baaj, Georges Skandalis [3], Étienne Blanchard [4] and Moto O'uchi [38]. It is a direct C^*-algebraic analogue of the von-Neumann algebraic notion of a pseudo-multiplicative unitary on a Hilbert space introduced by Jean-Michel Vallin [59]. We give a precise definition and discuss examples, standard constructions and the connection to related notions. In a separate subsection, we indicate one of the main motivations for the introduction of pseudo-multiplicative unitaries on Hilbert spaces and present a related example of pseudo-multiplicative unitaries on C^*-modules constructed by Moto O'uchi [39]. Then we turn to the main result of the section which is the construction of the legs of a pseudo-multiplicative unitary in the form of Hopf C^*-families. To do so, we need to assume certain regularity and decomposability conditions which are discussed in a separate subsection.

2.2.1 Definition, remarks and examples

Let E be a right C^*-module over a C^*-algebra A with commuting non-degenerate representations $\pi_r, \pi_s\colon A \to L_A(E)$.

Notation 2.4. *We use the notation E^r and E^s to distinguish between the two C^*-bimodule structures given by π_r and π_s, respectively. Thus, we put e.g.*

$$E^s \otimes E^s \otimes E := \big((E, \pi_s) \otimes E, 1 \otimes \pi_s\big) \otimes E,$$
$$E^s \otimes (E \otimes E^r) := (E, \pi_s) \otimes (E \otimes (E, \pi_r)).$$

Furthermore, we use multiple indexing and write

$$\begin{aligned} E \ogreaterthan E^{r,s} \olessthan E &:= \big(E \ogreaterthan (E, \pi_r), 1 \ogreaterthan \pi_s\big) \olessthan E = (E \ogreaterthan E^r, 1 \ogreaterthan \pi_s) \olessthan E \\ &\cong E \ogreaterthan \big(E^s \olessthan E, \pi_r \olessthan 1\big) = E \ogreaterthan (E^s \olessthan E, 1 \olessthan \pi_r). \end{aligned}$$

We write $E^{r,s}, E^{s,r}$ and the like if we consider E as a C^*-module with an ordered list of representations; when forming an internal tensor product with such a C^*-module, the first representation is used implicitly, i.e. we have

$$E \ogreaterthan E^{r,s} \olessthan E = (E \ogreaterthan E^{r,s}) \olessthan E \cong E \ogreaterthan (E^{s,r} \olessthan E).$$

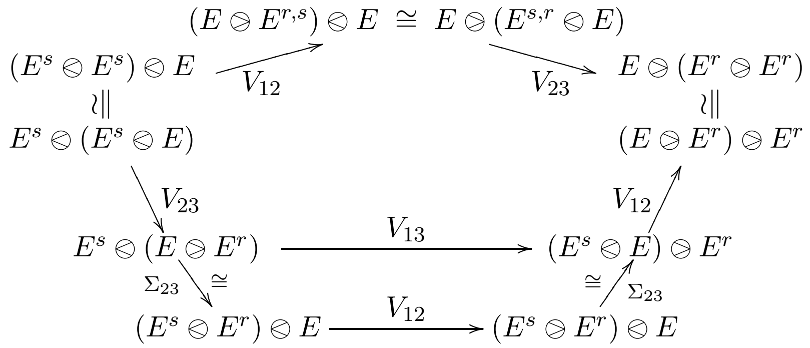
Lemma 2.5. *Assume that $V \in L_A(E^s \olessthan E, E \ogreaterthan E^r)$ intertwines the representations π_r and π_s as follows:*

$$\begin{aligned} V(1 \olessthan \pi_s(a)) &= (1 \ogreaterthan \pi_s(a))V, \\ V(\pi_r(a) \olessthan 1) &= (\pi_r(a) \ogreaterthan 1)V, \\ V(1 \olessthan \pi_r(a)) &= (\pi_s(a) \ogreaterthan 1)V, \quad a \in A. \end{aligned}$$

Then V, considered as an operator of the following C^*-bimodules,

$$E^s \olessthan E^s \to E \ogreaterthan E^{r,s}, \qquad E^{s,r} \olessthan E \to E^r \ogreaterthan E^r, \qquad E^s \olessthan E^r \to E^s \ogreaterthan E^r,$$

intertwines the representations given by left multiplication, and all operators in the following diagram are well-defined. Furthermore, one has isomorphisms of C^*-modules as shown below.



Proof. The existence of the isomorphisms follows immediately from the fact that π_s and π_r commute. The operators in the diagram are well-defined by proposition 1.24. $\square$

Using the leg notation and omitting explicit mentioning of the elements of A, the intertwining conditions in the lemma above can be shortly rewritten as follows:

$$V\pi_{s2} = \pi_{s2}V, \qquad V\pi_{r1} = \pi_{r1}V, \qquad V\pi_{r2} = \pi_{s1}V.$$

In the following, we will frequently use this short-hand notation.

Definition 2.6. *A unitary* $V\colon E^s \olessthan E \to E \ogreaterthan E^r$ *is* pseudo-multiplicative *if it satisfies the assumption of the previous lemma and the diagram above commutes.*

The condition $V_{12}V_{13}V_{23} = V_{23}V_{12}$ is called the *pentagon equation*.
This definition generalises the following concepts.

- If $A = \mathbb{C}$ and the representations π_r and π_s are given by scalar multiplication, V is a multiplicative unitary on a Hilbert space in the sense of Saad Baaj and Georges Skandalis [3].
- If A is commutative and the representations π_r and π_s both coincide with the module multiplication, V is a continuous field of multiplicative unitaries as defined by Étienne Blanchard [4].
- If A is commutative and the representation π_s coincides with right multiplication, V is a pseudo-multiplicative unitary in the sense of Moto O'uchi [38].

If one replaces

- E by a Hilbert space H,
- A by the opposite N^{op} of a von Neumann algebra N,
- the module structure of E over A by a non-degenerate normal representation α of N on H,
- the representations π_s and π_r by non-degenerate normal anti-representations β' and β of N on H,
- the internal tensor products $E^s \olessthan E$ and $E \ogreaterthan E^r$ with the relative tensor products $H\,{}_{\beta'}\!\otimes_{\alpha} H$ and $H\,{}_{\alpha}\!\otimes_{\beta} H$, taken with respect to some normal semi-finite faithful weight on N and its opposite on N^{op}, respectively,

and inserts the condition $V\alpha_2 = \alpha_1 V$, one obtains the definition of a pseudo-multiplicative unitary on a Hilbert space given by Jean-Michel Vallin [59, 15].

Isomorphisms of pseudo-multiplicative unitaries on C^*-modules are defined in the obvious way.

Definition 2.7. *For $i = 1, 2$, let E_i be a non-degenerate right C^*-module over a C^*-algebra A_i with commuting non-degenerate representations π_s^i and π_r^i of A_i. Two pseudo-multiplicative unitaries*

$$V_1\colon E_1^s \olessthan E_1^r \to E_1^s \ogreaterthan E_1^r \quad \text{and} \quad V_2\colon E_2^s \olessthan E_2^r \to E_2^s \ogreaterthan E_2^r$$

are isomorphic *if there exists an isomorphism $U\colon E_1 \to E_2$ such that*

$$\pi_s^2 = \mathrm{Ad}_U(\pi_s^1), \qquad \pi_r^2 = \mathrm{Ad}_U(\pi_r^1), \qquad (U \ogreaterthan U)V_1 = V_2(U \olessthan U).$$

Examples Multiplicative unitaries on Hilbert spaces, continuous fields of such unitaries and pseudo-multiplicative unitaries in the sense of Moto O'uchi all yield examples for pseudo-multiplicative unitaries on C^*-modules. Later in this thesis, we present in detail two examples belonging to the third class: Pseudo-multiplicative unitaries associated to locally compact groupoids – see subsection 3.2.1 – and examples associated to inclusions of finite-dimensional C^*-algebras [39] – see subsection 2.2.1. An example which does not belong to one of the first three classes is presented in subsection 2.4.5.

Let us also indicate two potential sources of further examples. First, we expect the bicrossed product construction for multiplicative unitaries [3] and locally compact quantum groups [56] to extend to pseudo-multiplicative unitaries on C^*-modules. Second, we sketch a vague idea connected to deformations of Hopf C^*-algebras. In [4], Blanchard introduces fields of multiplicative unitaries over parameter spaces and presents examples. It would be interesting to exhibit an example in which the fibres of such a deformation are linked not only "analytically" via the underlying parameter space, but also "algebraically" in the sense that one can identify additional deformations indexed by a groupoid over the parameter space. Then Hopf C^*-algebras indexed by different deformation parameters would be linked up by additional "Hopf C^*-bimodules" indexed by pairs of deformation parameters.

Standard constructions In the following, we collect several easy constructions. In subsection 3.2.1, the connections to corresponding constructions for groupoids are clarified.

Let $V\colon E^s \otimes E \to E \otimes E^r$ be a pseudo-multiplicative unitary.

- **The opposite** It is easy to see that the unitary

$$V^{op} := \Sigma V^* \Sigma \colon E^r \otimes E \xrightarrow{\Sigma} E \otimes E^r \xrightarrow{V^*} E^s \otimes E \xrightarrow{\Sigma} E \otimes E^s$$

  is pseudo-multiplicative. We call it the *opposite* of V.

- **Reduction** Let $I \subset A$ be an ideal and put $B := A/I$. We consider B as a left C^*-module over itself. Left multiplication defines a representation $A \to L_B(B)$. Put $F := E \otimes_* B$.

  Lemma 2.8. *Assume that $\pi_s(I)E$ and $\pi_r(I)E$ are contained in EI. Then the representations $\pi_s \otimes_* 1$ and $\pi_r \otimes_* 1$ of A on $L_B(E_I)$ factorise through the quotient map $A \to B$. Denote by π_s^I and π_r^I, respectively, the induced representations of B on E_I. Then one has isomorphisms*

$$E_I^s \otimes E_I \cong (E^s \otimes E) \otimes_* B, \qquad E_I \otimes E_I^r \cong (E \otimes E^r) \otimes_* B,$$

  and the operator

$$V_I \colon E_I^s \otimes E_I \cong (E^s \otimes E) \otimes_* B \xrightarrow{V \otimes_* 1} (E \otimes E^r) \otimes_* B \cong E_I \otimes E_I^r$$

  is a pseudo-multiplicative unitary.

Proof. The first statement follows from the inclusions $\pi_s(I)E\otimes_* B, \pi_r(I)E\otimes_* B \subset EI \otimes_* B$ and the equation $EI \otimes_* B = E \otimes_* IB = 0$. For the second statement, note first that one has isomorphisms

$$(E^s \oslash E) \otimes_* B \cong E^s_I \oslash E \cong E^s_I \oslash A \oslash E \cong E^s_I \oslash B \oslash E \cong E^s_I \oslash E_I$$

because the representation $\pi_s \otimes_* 1$ on E_I factorises through the quotient map $A \to B$. An analogous reasoning, applied to $E_I \ominus E^r_I$, proves the second statement.

The fact that V_I is a pseudo-multiplicative unitary follows immediately from the corresponding properties of V. □

We call V_I the *reduction* of V with respect to I.

The direct sum and external tensor product of two pseudo-multiplicative unitaries $V_1 \colon E^s_1 \oslash E_1 \to E_1 \ominus E^r_1$ and $V_2 \colon E^s_2 \oslash E_2 \to E_2 \ominus E^r_2$ is defined as follows.

- **Direct sum** The direct sum $E := E_1 \oplus E_2$ is a right C^*-module over the C^*-algebra $A := A_1 \oplus A_2$, and the direct sums $\pi_s := \pi^1_s \oplus \pi^2_s, \pi_r := \pi^1_r \oplus \pi^2_r$ define commuting representations of A on E. It is easy to see that the composition

$$E^s \oslash E \cong (E^s_1 \oslash E_1) \oplus (E^s_2 \oslash E_2) \xrightarrow{V_1 \oplus V_2} (E_1 \ominus E^r_1) \oplus (E_2 \ominus E^r_2) \cong E \ominus E^r$$

  is a pseudo-multiplicative unitary. We call it the *direct sum* of V_1 and V_2.

- **External tensor product** Let $A := A_1 \otimes A_2$ be the maximal C^*-tensor product and let $E := E_1 \otimes E_2$ be the corresponding external tensor product of C^*-modules. The tensor products $\pi_s := \pi^1_s \otimes \pi^2_s$ and $\pi_r := \pi^1_r \otimes \pi^2_r$ define commuting representations of A on E. Again, it is easy to see that the composition

$$E^s \oslash E \cong (E^s_1 \oslash E_1) \otimes (E^s_2 \oslash E_2) \xrightarrow{V_1 \otimes V_2} (E_1 \ominus E^r_1) \otimes (E_2 \ominus E^r_2) \cong E \ominus E^r$$

  is a pseudo-multiplicative unitary. We call it the *external tensor product* of V_1 and V_2.

Extended example: inclusion of C^*-algebras

In this subsection, we discuss one of the motivating examples for the development of the theory of quantum groupoids and pseudo-multiplicative unitaries on Hilbert spaces. It arises from the study of inclusions of factors [20] which are von Neumann algebras with trivial centre. Their study lies at the heart of von Neumann algebra theory because each von Neumann algebra can be written as a direct integral of factors over some measure space. Examples of *inclusions* of factors arise from group actions: given a factor M_1 with a suitable action of a group G, one can form the von Neumann-algebraic crossed product $M_2 := M_1 \rtimes G$. It contains M_1 and is a factor if the action satisfies certain assumptions. On the other hand, one can consider the fixed point algebra $M_0 := M_1^G$ of this action and ask whether this is a factor.

The study of more general inclusions of factors $M_0 \hookrightarrow M_1$ raised the question whether one can exhibit some (quantum) group(oid) G with an action on M_1 such that M_0 is isomorphic to the fixed point algebra M_1^G and the inclusion $M_1 \hookrightarrow M_2$ given by the basic construction [20] is isomorphic to the inclusion $M_1 \hookrightarrow M_1 \rtimes G$ [13, 15]. The ultimate goal is a Galois theory for inclusions of factors. The first step in this programme – to associate a pseudo-multiplicative unitary to an inclusion of factors – has been adapted by Moto O'uchi to certain inclusions of C^*-algebras [39]. We reproduce it here.

Let A_1 be a finite-dimensional C^*-algebra, let A_0 be a C^*-subalgebra of A_1 containing the identity of A_1 and let $P_1 \colon A_1 \to A_0$ be a faithful conditional expectation. Denote by E_1 the right C^*-module over A_0 obtained from P_1 by the Rieffel construction, and by $\phi_1 \colon A_1 \to L_{A_0}(E_1)$ the representation induced by left multiplication.

We assume that P_1 is of index-finite type [62, 1.2.2 and 2.1.6], i.e. there are elements $u_1, \ldots, u_n \in A_1$ such that $\mathrm{id}_{E_1} = \sum_{i=1}^n |u_i\rangle\langle u_i|$. Note that then $L_{A_0}(E_1) = K_{A_0}(E_1)$. The index of P_1 is the sum $\operatorname{Index} P_1 = \sum u_i u_i^*$. It is invertible and belongs to the centre of A_1.

The *reduced basic construction* is given by the C^*-algebra $A_2 := K_{A_0}(E_1)$, the inclusion $A_1 \subset A_2$ realised by ϕ_1, and the conditional expectation $P_2 \colon A_2 \to A_1$ defined by $P_2\big(|a\rangle\langle b|\big) := (\operatorname{Index} P_1)^{-1} ab^*$, $a, b \in A_1$.

We denote by E_2 the right C^*-module over A_0 obtained from $P_1 \circ P_2$ by the Rieffel construction, and by $\phi_2 \colon A_2 \to K_{A_0}(E_2)$ the representation induced by left multiplication. Then one has an isomorphism $\Phi \colon E_1 \otimes_{\phi_1} E_1 \xrightarrow{\cong} E_2$ given by

$$\Phi(a \otimes_{\phi_1} b) = |a\rangle\langle b^*|\phi_1(\operatorname{Index} P_1)^{1/2}, \quad \Phi^{-1}(x) = \sum_i x(u_i) \otimes_{\phi_1} (\operatorname{Index} P_1)^{-1/2} u_i^*,$$

where $a, b \in E_1$ and $x \in E_2$.

We make the following technical assumptions without further comment:

$$(\mathbf{A1}) \;\; A_0' \cap A_1 \subset Z(A_0' \cap A_2), \qquad\qquad (\mathbf{A2}) \;\; A_2 = A_1 \cdot (A_0' \cap A_2),$$

where A_0' denotes the centralisator of A_0 in A_2. The pseudo-multiplicative unitary associated to the conditional expectation is defined on the C^*-module

$$E := \{T \in K_{A_0}(E_1, E_2) \mid T\phi_1 = \phi_2 T\}$$

over the C^*-algebra $C := A_1' \cap A_2$. The map $q \colon A_0' \cap A_1 \to A_2$ given by $q(a)b = ba$ for $b \in E_1$, is a linear bijection of $A_0' \cap A_1$ onto C. Since $q(a)q(a') = q(a'a)$ for all $a, a' \in A_0' \cap A_1$, the C^*-algebra C is commutative if assumption (A1) is satisfied.

Denote by π_s and π_r the representations of C on E given by $\pi_s(c)e := ec$ and $\pi_r(c)e := ce$, respectively, where $c \in C, e \in E$ and $(ce)(e_1) := c \cdot (e(e_1))$ for all $e_1 \in E_1$.

Theorem 2.9 ([39], theorem 3.2). *Let C, E, π_s and π_r be as above and assume that the assumptions (A1) and (A2) hold. Then for all $x, y \in E$, there exist finitely many $x_i, y_i \in E$ such that $(\mathrm{id}_{E_1} \otimes_{\phi_1} y)x = \sum_i (x_i \otimes_{\phi_1} \mathrm{id}_{E_1}) y_i$ as operators $E_1 \to E_1 \otimes_{\phi_1} E_1 \otimes_{\phi_1} E_1$. Furthermore, the map $x \otimes y \mapsto \sum_i x_i \otimes y_i$ defines a pseudo-multiplicative unitary $V \colon E^s \otimes E \to E \otimes E^r$.*

2.2.2 Regularity and decomposability

In this subsection, we introduce regularity and decomposability conditions for pseudo-multiplicative unitaries on C^*-modules which allow us to construct their legs in the form of Hopf C^*-families. The notion of regularity of multiplicative unitaries on Hilbert spaces [3] is less general than that of manageability introduced by Woronowicz [63] but better suited to our purposes because it is simpler and allows us to focus on the new phenomena introduced by the C^*-bimodules. In contrast to regularity, decomposability is a condition on the two individual legs of a pseudo-multiplicative unitary – it may well happen that one is decomposable, while the other is not.

In the following, we will focus on the left leg. The corresponding definitions and results for the right leg can be obtained by first writing them down for the left leg of the *opposite* V^{op} of the initial pseudo-multiplicative unitary V and then rephrasing them in terms of V.

For the formulation of the regularity and decomposability condition, we employ the following notation. It will be used throughout the remainder of this chapter.

Let $V \colon E^s \otimes E \to E \otimes E^r$ be a pseudo-multiplicative unitary.

Notation 2.10. *We put $\mathscr{E}^r := \mathscr{C}ov(E^r)$, $\mathscr{E}^s := \mathscr{C}ov(E^s)$ and $\mathscr{A} := \mathscr{C}ov(A)$ as families, i.e. $\mathscr{E}^r_\gamma = \mathscr{C}ov_\gamma(E^r)$, $\mathscr{E}^s_\gamma = \mathscr{C}ov_\gamma(E^s)$ and $\mathscr{A}_\gamma = \mathscr{C}ov_\gamma(A)$ for all $\gamma \in \mathrm{PAut}(A)$. Furthermore, we put*

$$|\xi\rangle := \theta_\xi, \quad \langle\xi| := \theta_\xi^*, \quad |\mathscr{E}^r\rangle := \mathscr{K}_{\mathrm{id}}(A, E^r), \quad \langle\mathscr{E}^r| := \mathscr{K}_{\mathrm{id}}(E^r, A),$$
$$|\xi\} := \theta^\xi, \quad \{\xi| := \theta^{\xi*}, \quad |\mathscr{E}^r\} := \mathscr{K}^{\mathrm{id}}(A, E^r), \quad \{\mathscr{E}^r| := \mathscr{K}^{\mathrm{id}}(E^r, A), \quad \xi \in \mathscr{E}^r.$$

Notation 2.11. *In the following, we employ commutative diagrams in the category whose objects are C^*-bimodules over A and whose morphisms are families of closed subsets of homogeneous operators. The composition of morphisms is given by the product of such families as defined in 1.13.*

The definition of regularity involves the closed subspace

$$C(V) := \big(\langle E| \otimes 1\big)\, V \,\big(1 \otimes |E\rangle\big) \subset L_A(E^s \otimes A, A \otimes E^r).$$

In some more detail, it is equal to the composition

$$E^s \otimes A \xrightarrow{1 \otimes |E\rangle} E^s \otimes E \xrightarrow{V} E \otimes E^r \xrightarrow{\langle E| \otimes 1} A \otimes E^r.$$

In the following, we will frequently identify $E^s \otimes A$ and $A \otimes E^r$ with E without explicit mentioning of the canonical isomorphisms. Thus, we also identify the C^*-algebra $L_A(E^s \otimes A, A \otimes E^r)$ with $L_A(E)$ and consider $C(V)$ as a subspace of $L_A(E)$. In leg notation, we write $C(V) = \langle E|_1 V |E\rangle_2$. In the next definition, we will use the short-hand leg notation and the detailed diagrammatic notation in parallel.

Definition 2.12. *A pseudo-multiplicative unitary $V: E^s \otimes E \to E^s \otimes E^r$ is*

i) left (right) decomposable, *if the C^*-bimodule E^r (E^s) is decomposable,*

ii) regular *if the space $C(V) \subset L_A(E)$ defined above is equal to $K_A(E)$,*

iii) decomposably left (right) regular *if*

(a) V is left (right) decomposable,

(b) the sub-family $\mathscr{C}_r(V) := \langle \mathscr{E}^r|_1 V |E\rangle_2 \subset \mathscr{L}_{\mathrm{id}}(E^r)$, in detail given by the composition

$$E^r \cong E^{s,r} \otimes A \xrightarrow{|E\rangle_2} E^{s,r} \otimes E \xrightarrow{V} E^r \otimes E^r \xrightarrow{\langle \mathscr{E}^r|_1} A \otimes E^r \cong E^r,$$

is equal to $|\mathscr{E}^r\rangle\langle\mathscr{E}^r|$ (the sub-family of $\mathscr{L}_{\mathrm{id}}(E^s)$ given by

$$\mathscr{C}_s(V) := \langle E|_1 V |\mathscr{E}^s\rangle_2 : E^s \xrightarrow{|\mathscr{E}^s\rangle_2} E^s \otimes E^s \xrightarrow{V} E \otimes E^{r,s} \xrightarrow{\langle E|_1} E^s$$

is equal to $|\mathscr{E}^s\rangle\langle\mathscr{E}^s|$),

(c) The sub-family $\{\mathscr{E}^r|_2|\mathscr{E}^r\}_2|\mathscr{E}^r\} \subset \mathscr{K}^{\mathrm{id}}(A, E^r)$, in detail given by the composition

$$A \xrightarrow{|\mathscr{E}^r\}} E^r \cong E^r \otimes A \xrightarrow{|\mathscr{E}^r\}_2} E^r \otimes E^r \xrightarrow{\{\mathscr{E}^r|_2} E^r \otimes A \cong E^r,$$

is equal to $|\mathscr{E}^r\}$ $\big(\{\mathscr{E}^s|_2|\mathscr{E}^s\}_2|\mathscr{E}^s\} = |\mathscr{E}^s\}\big)$,

Let us discuss these conditions. If A is commutative and the representation π_r coincides with the right module structure of E, then, trivially, V is left decomposable. In particular, each pseudo-multiplicative unitary in the sense of [38] is left decomposable. This includes the examples associated to locally compact Hausdorff groupoids (3.2.1) and to inclusions of C^*-algebras (2.2.1). The former ones are also right decomposable provided the groupoid is decomposable, e.g. r-discrete, see 3.2.1.

The conditions i) and iiib) above are linked to each other as follows.

Lemma 2.13. *Let $V: E^s \otimes E \to E \otimes E^r$ be a pseudo-multiplicative unitary.*

i) If V is decomposably left (right) regular, then V is also regular.

ii) If V is left (right) decomposable, regular and if $\mathscr{K}_{\mathrm{id}}(E^r) = |\mathscr{E}^r\rangle\langle\mathscr{E}^r|$ (or $\mathscr{K}_{\mathrm{id}}(E^s) = |\mathscr{E}^s\rangle\langle\mathscr{E}^s|$, respectively), then condition iiib) of decomposable left (right) regularity is satisfied.

Proof. We prove both statements about decomposable left regularity.

i) By assumption, the C^*-bimodule E^r is decomposable. Therefore, $\langle E| = \overline{\mathrm{span}}_\beta \langle \mathscr{E}^r_\beta|$ and hence also $C(V) = \overline{\mathrm{span}}_\beta \mathscr{C}^\beta_r(V)$.

ii) By covariance of V and the assumptions, one has $\mathscr{C}^\beta_r(V) = C(V) \cap \mathscr{K}^\beta_{\mathrm{id}}(E^r) = \big(|\mathscr{E}^r\rangle\langle\mathscr{E}^r|\big)^\beta$ for each $\beta \in \mathrm{PAut}(A)$. ☐

Corollary 2.14. *A pseudo-multiplicative unitary over a commutative C^*-algebra which is is left (right) decomposable and regular satisfies condition iiib) of decomposable left (right) regularity.*

Proof. This follows from part ii) of proposition 1.68 and lemma 2.13. ☐

Condition iiic) is introduced because it is needed in order to show that the leg of a decomposably regular pseudo-multiplicative unitary is a Hopf C^*-family. We do not yet understand it conceptually.

Lemma 2.15. *If $\langle \mathscr{E}^r_{\mathrm{id}} | \mathscr{E}^r_{\mathrm{id}} \rangle \cdot A = A$, then condition iiic) of the definition above is satisfied.*

Proof. For all $\zeta, \zeta' \in \mathscr{E}^r_{\mathrm{id}}$, $\xi \in \mathscr{E}^r_\gamma$, $\gamma \in \mathrm{PAut}(A)$, and $a \in A$, one has

$$\{\zeta'|_2 \cdot |\zeta\}_2 \cdot |\xi\} a = \{\zeta'|_2 \cdot (a\xi \otimes \zeta) = a\xi\langle\zeta'|\zeta\rangle = |\xi'\}a, \quad \xi' := \xi\langle\zeta'|\zeta\rangle.$$

The assumption and lemma 1.51 imply that $\mathscr{E}^r_\gamma \cdot \langle \mathscr{E}^r_{\mathrm{id}} | \mathscr{E}^r_{\mathrm{id}} \rangle = \mathscr{E}^r_\gamma$ for all $\gamma \in \mathrm{PAut}(A)$, whence the claim follows. ☐

In the remainder of this paragraph, we collect several results which will be used in the next subsection. In the following lemma, we use the notation $\mathscr{E}^r \otimes \mathscr{E}^r$ which is defined in 1.50, and the notation $(\mathscr{E}^r, \pi_s) \otimes E$ for the family of closed subspaces of the C^*-bimodule $E^{s,r} \otimes E$ given by

$$\big((\mathscr{E}^r, \pi_s) \otimes E\big)^\alpha := \overline{\mathrm{span}}\,\{\xi \otimes \eta \in E^s \otimes E \mid \xi \in \mathscr{C}ov_\alpha(E^r), \eta \in E\}, \quad \alpha \in \mathrm{PAut}(A).$$

Lemma 2.16. *If V is decomposably left regular, then $V((\mathscr{E}^r, \pi_s) \otimes E) = \mathscr{E}^r \otimes \mathscr{E}^r$.*

Proof. Let $(T_\nu)_\nu$ be a bounded approximate unit for the C^*-algebra $\big(|\mathscr{E}^r\rangle\langle\mathscr{E}^r|\big)^{\mathrm{id}}$. We show that the net $(T_\nu \otimes 1)_\nu$ converges pointwise to the identity on $E \otimes E^r$.

Since $|\mathscr{E}^r_\alpha\rangle$ is a left C^*-module over $(|\mathscr{E}^r\rangle\langle\mathscr{E}^r|)^{\mathrm{id}}$ for each $\alpha \in \mathrm{PAut}(A)$, the product $(T_\nu\theta_\xi)_\nu$ converges to θ_ξ in norm for each fixed $\xi \in \mathscr{E}^r_\alpha$. But then also $T_\nu\xi \to \xi$ for each $\xi \in \mathscr{E}^r_\alpha$. By a standard argument, $\big((T_\nu \otimes 1)\zeta\big)_\nu$ converges to ζ for each element $\zeta \in E \otimes E^r$ which can be written as a finite linear combination of elementary tensors $\xi \otimes \eta$. Since vectors of this form are dense in $E \otimes E^r$ and the net $\big(\|T_\nu \otimes 1\|\big)_\nu$ is bounded, the assertion follows.

Combining this result with the assumption on V, we find for each $\alpha \in \mathrm{PAut}(A)$

$$\begin{aligned} V(\mathscr{E}^r_\alpha \otimes E) &\subset \big(|\mathscr{E}^r\rangle\langle\mathscr{E}^r|\big)_1^{\mathrm{id}}\, V\,|E\rangle_2\mathscr{E}^r_\alpha \\ &\subset \overline{\operatorname*{span}_{\beta\beta'^*\le \mathrm{id}}}\, |\mathscr{E}^r_\beta\rangle_1\langle\mathscr{E}^r_{\beta'}|_1\, V\,|E\rangle_2\mathscr{E}^r_\alpha \\ &= \overline{\operatorname*{span}_{\beta\beta'\beta''^*\le \mathrm{id}}}\, |\mathscr{E}^r_\beta\rangle_1\, |\mathscr{E}^r_{\beta'}\rangle\langle\mathscr{E}^r_{\beta''}|\,\mathscr{E}^r_\alpha \subset \overline{\operatorname*{span}_{\beta\beta'\le \mathrm{id}}}\, \mathscr{E}^r_\beta \otimes \mathscr{E}^r_{\beta'\alpha} \subset (\mathscr{E}^r \otimes \mathscr{E}^r)^\alpha. \end{aligned}$$

In the following, we will shorten the notation, considering entire families instead of individual closed subspaces. To prove the reverse inclusion, we use a bounded approximate unit $(S_\mu)_\mu$ for the C^*-algebra $K_A(E)$ and regularity of V to see that $V^*(\mathscr{E}^r \otimes \mathscr{E}^r)$ is contained in the family

$$|E\rangle_2\langle E|_2 V^*\,|\mathscr{E}^r\rangle_1\mathscr{E}^r = |E\rangle_2\,|\mathscr{E}^r\rangle\langle\mathscr{E}^r|\mathscr{E}^r = (\mathscr{E}^r, \pi_s) \otimes E. \qquad \square$$

Remark 2.17. If A is commutative, the result of the previous lemma holds irrespective of any regularity of V. To see this, observe that the intertwining properties of V imply that V induces a bijection between the families $\mathscr{C}ov(E^{s,r} \otimes E)$ and $\mathscr{C}ov(E^r \otimes E^r)$. If A is commutative, by proposition 1.67 the latter family decomposes as follows,

$$\mathscr{C}ov(E^r \otimes E^r) = \mathscr{C}ov(E^r) \otimes \mathscr{C}ov(E^r) = \mathscr{E}^r \otimes \mathscr{E}^r,$$

and a similar decomposition holds for the family $\mathscr{C}ov(E^{s,r} \otimes E)$.

Lemma 2.18. *If V is decomposably left regular, the following diagram commutes.*

$$\begin{array}{ccccccc} A & \xrightarrow{|\mathscr{E}^r\}} & E^r & \xrightarrow{|E\rangle_2} & E^{s,r}\otimes E & \xrightarrow{\qquad\langle E|_2\qquad} & E^r \\ \big\downarrow{\scriptstyle |\mathscr{E}^r\}} & & & & \big\downarrow{\scriptstyle V} & & \big\downarrow{\scriptstyle \{\mathscr{E}^r|_2} \\ E^r & & \xrightarrow{\qquad|\mathscr{E}^r\}_2\qquad} & & E^r\otimes E^r & \xrightarrow{\{\mathscr{E}^r|_2} E^r \xrightarrow{\{\mathscr{E}^r|} & A \end{array}$$

Proof. Consider the left hand side of the diagram. Since V commutes with π_{r1}, the family $V|\mathscr{E}^r\rangle_2|\mathscr{E}^r\}$ is generated by compositions of the form $a \mapsto \pi_r(a)\xi \otimes \eta \mapsto V(\pi_r(a)\xi \otimes \eta) = \pi_{r1}(a)V(\xi \otimes \eta)$ where $\xi \in \mathscr{E}^r$ and $\eta \in E$. The family $|\mathscr{E}^r\}_2|\mathscr{E}^r\}$ is generated by compositions of the form $a \mapsto \pi_r(a)\xi' \otimes \eta'$ where $\xi', \eta' \in \mathscr{E}^r$. By the previous lemma, both families coincide. Taking adjoints and multiplying both families by V on the right, one obtains commutativity of the square on the right hand side. $\square$

2.2.3 Construction of the legs

This subsection contains the main result of the first part of this chapter. We define the left leg of a decomposably left regular pseudo-multiplicative unitary and show that it is a Hopf C^*-family. The corresponding result for the right leg can be obtained by considering the opposite unitary and is summarised at the end of this subsection. For most proofs, we employ commutative diagrams as explained in the previous subsection.

Throughout this subsection, let $V\colon E^s \otimes E \to E \otimes E^r$ be a pseudo-multiplicative unitary.

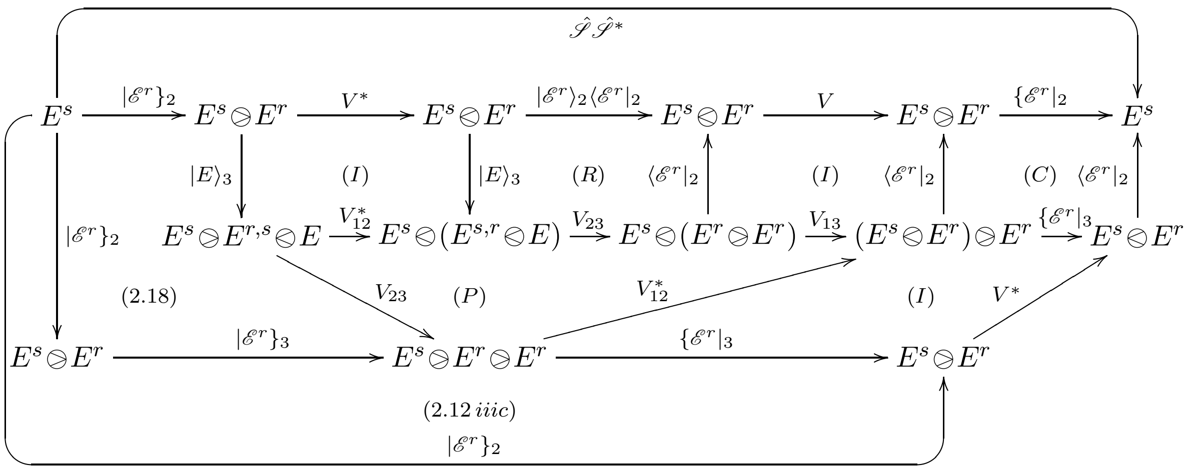
The left leg Denote by $\hat{\mathscr{S}} := \{\mathscr{E}^r|_2 V |\mathscr{E}^r\rangle_2 \subset \mathscr{L}(E^s)$ the composition

$$E^s \cong E^s \otimes A \xrightarrow{|\mathscr{E}^r\rangle_2} E^s \otimes E^r \xrightarrow{V} E^s \otimes E^r \xrightarrow{\{\mathscr{E}^r|_2} E^s \otimes A \cong E^s.$$

Since V commutes with the representation π_r, the family $\hat{\mathscr{S}}$ commutes with the representation π_r.

Proposition 2.19. *If V is decomposably left regular, $\hat{\mathscr{S}}$ is a non-degenerate C^*-pre-family.*

Proof. We prove $\hat{\mathscr{S}}\hat{\mathscr{S}}^* = \hat{\mathscr{S}}^*$ by means of the following diagram. The composition $\hat{\mathscr{S}}\hat{\mathscr{S}}^*$ is the arrow in the top line, and $\hat{\mathscr{S}}^*$ is equal to the composition obtained by first going down, then to the right and finally up again.



The individual labelled cells commute for the following reasons: (I) intertwining properties of V, (R) decomposable left regularity of V, (P) pentagon equation, (2.18) lemma 2.18, (C) by inspection, (2.12 iiic) by assumption iiic) of decomposable left regularity of V. Hence, the entire diagram commutes, whence $\hat{\mathscr{S}}\hat{\mathscr{S}}^* = \hat{\mathscr{S}}^*$. This also implies $\hat{\mathscr{S}} = \hat{\mathscr{S}}^*$.

For all $\alpha, \beta, \alpha', \beta' \in \mathrm{PAut}(A)$ satisfying $(\beta, \alpha) \leq (\beta', \alpha')$, one has $\hat{\mathscr{S}}_\alpha^\beta \subset \hat{\mathscr{S}}_{\alpha'}^{\beta'}$ by part iii) of lemma 1.51. Thus, $\hat{\mathscr{S}}$ is a C^*-pre-family.

The C^*-pre-family $\hat{\mathscr{S}}$ is non-degenerate since

$$\overline{\mathrm{span}}_{\alpha,\beta}\,(\hat{\mathscr{S}}_\alpha^\beta E) = \overline{\mathrm{span}}_{\alpha,\beta}\,(\{\mathscr{E}_\alpha^r|_2 V |\mathscr{E}_\beta^r\rangle_2 E) = \overline{\mathrm{span}}_\alpha\left(\{\mathscr{E}_\alpha^r|_2(E \otimes E^r)\right) = E$$

by assumption iiic) in definition 2.12. ∎

Proposition 2.20. *The C^*-pre-family $\hat{\mathscr{S}}$ is a C^*-family.*

Proof. Let $\alpha, \alpha', \beta, \beta' \in \mathrm{PAut}(A)$. By definition,

$$\begin{aligned} \hat{\mathscr{S}}_\alpha^\beta \cdot \mathscr{O}_{\alpha'}^{\beta'}(E^s) &= \{\mathscr{E}_\alpha^r|_2 \, V \, |\mathscr{E}_\beta^r\rangle_2 \cdot \pi_s(\mathscr{A}_{\beta'})\rho(\mathscr{A}_{\alpha'^*}) \\ &= \{\mathscr{E}_\alpha^r|_2 \, \rho(\mathscr{A}_{\alpha'^*}) \cdot V \cdot |\mathscr{E}_\beta^r\rangle_2 \, \pi_s(\mathscr{A}_{\beta'}). \end{aligned}$$

We show that $\{\mathscr{E}_\alpha^r|_2 \, \rho(\mathscr{A}_{\alpha'^*})$ and $|\mathscr{E}_\beta^r\rangle_2 \, \pi_s(\mathscr{A}_{\beta'})$ are contained in $\{\mathscr{E}_{\alpha\alpha'}^r|_2$ and $|\mathscr{E}_{\beta\beta'}^r\rangle_2$, respectively. By lemma 1.54, one has for all $\eta \in \mathscr{E}_\alpha^r$ and $a \in \mathscr{A}_{\alpha'^*}$

$$\begin{aligned} \{\eta|_2 \rho(a)(\zeta \ominus \zeta') &= \{\eta|_2(\zeta \ominus \zeta' a) \\ &= \zeta\alpha\big(\langle\eta|\zeta'\rangle a\big) = \zeta(\alpha \circ \alpha')\big(\alpha'^*(a)\langle\eta|\zeta'\rangle\big), \quad \zeta \ominus \zeta' \in E \ominus E^r. \end{aligned}$$

Now $a^* \in \mathscr{A}_{\alpha'}$ and by proposition 1.55, $\alpha'^*(a^*)$ belongs to $\mathscr{A}_{\alpha'}$. By lemma 1.51 and proposition 1.58 again, $\eta\alpha'^*(a^*) = \rho(\alpha'^*(a^*))\eta$ is contained in $\mathscr{E}_{\alpha\alpha'}^r$.
For all $\xi \in \mathscr{E}_\beta^r$ and $a \in \mathscr{A}_{\beta'}$, one has

$$|\xi\rangle_2 \pi_s(a)\zeta = a\zeta \ominus \xi = \zeta \ominus \xi a = |\xi a\rangle_2 \zeta, \quad \zeta \in E,$$

and $\xi a = \rho(a)\xi$ is contained in $\mathscr{E}_{\beta\beta'}^r$ by lemma 1.51 and proposition 1.58. Thus,

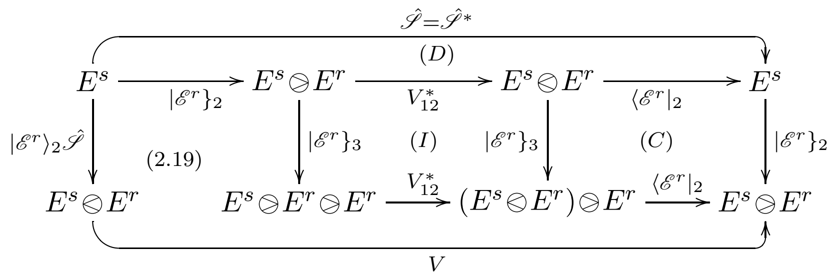
$$\hat{\mathscr{S}}_\alpha^\beta \cdot \mathscr{O}_{\alpha'}^{\beta'}(E^s) \subset \{\mathscr{E}_{\alpha\alpha'}^r|_2 \, V \, |\mathscr{E}_{\beta\beta'}^r\rangle_2 = \hat{\mathscr{S}}_{\alpha\alpha'}^{\beta\beta'}.$$

For $\beta' = \beta^*\beta$ and $\alpha' = \alpha^*\alpha$, we obtain $\hat{\mathscr{S}}_\alpha^\beta \cdot \mathscr{O}_{\alpha^*\alpha}^{\beta^*\beta}(E^s) = \hat{\mathscr{S}}_\alpha^\beta$ since $\mathscr{E}_\gamma^r = \mathscr{E}_\gamma^r \cdot \mathscr{A}_{\gamma^*\gamma}$ for all γ by part v) of lemma 1.51. ☐

Lemma 2.21. *The following diagram commutes.*

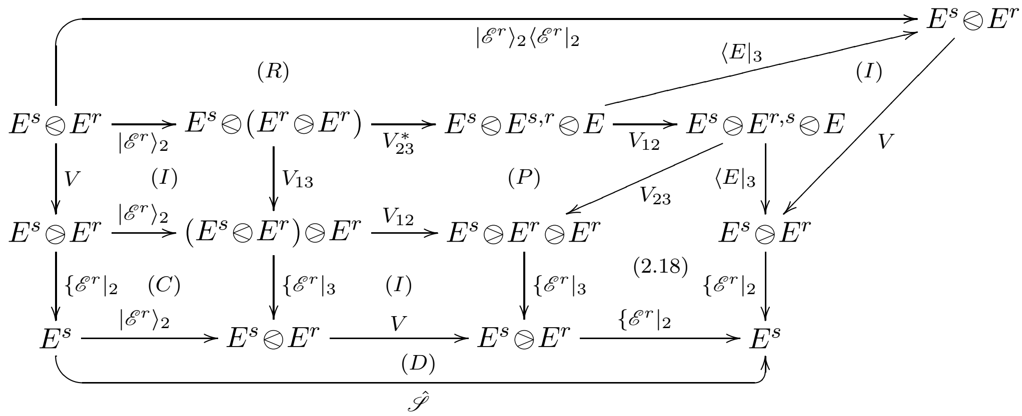
$$\begin{array}{ccccccc} E^s & \xrightarrow{\hat{\mathscr{S}}} & E^s & \xrightarrow{|\mathscr{E}^r\rangle_2} & E^s \ominus E^r & \xrightarrow{\langle\mathscr{E}^r|_2} & & & E^s \\ \downarrow{\scriptstyle \hat{\mathscr{S}}} & & & & \downarrow{\scriptstyle V} & & & & \downarrow{\scriptstyle \hat{\mathscr{S}}} \\ E^s & & \xrightarrow{|\mathscr{E}^r\}_2} & & E^s \ominus E^r & \xrightarrow{\{\mathscr{E}^r|_2} & E^s & \xrightarrow{\hat{\mathscr{S}}} & E^s \end{array}$$

Proof. For the left square, consider the following diagram.



The individual cells commute for the following reasons: (I) intertwining property of V, (C) by inspection, (D) definition, (2.19) by the commutative diagram in the proof of the previous proposition – the composition $V|\mathscr{E}^r\rangle_2\hat{\mathscr{S}}$ is equal to the composition of the first four arrows going from the left corner E^s to the right, and the composition $\langle\mathscr{E}^r|_2 V_{12}^*|\mathscr{E}^r\}_3|\mathscr{E}^r\}_2$ is equal to the composition of the arrows going down, to the right, slanted right upwards and finally up in that diagram.

For the right square, consider the following diagram.



The individual cells commute for the following reasons: (I) intertwining properties of V, (C) by inspection, (D) definition, (P) pentagon equation, (R) decomposable left regularity of V, (2.18) lemma 2.18. By definition, the composition $\{\mathscr{E}^r|_2 V|\mathscr{E}^r\rangle_2\langle\mathscr{E}^r|_2$ is equal to $\hat{\mathscr{S}}\langle\mathscr{E}^r|_2$, whence the claim follows. ☐

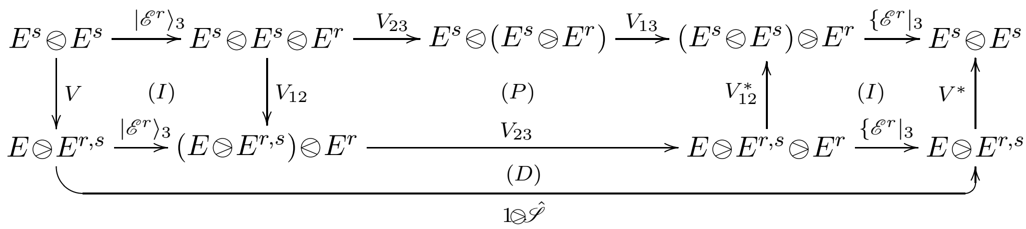
Proposition 2.22. *i) For each $x \in \hat{\mathscr{S}}$, the operator $1 \otimes x \in \mathscr{L}(E \otimes E^{r,s})$ is well-defined. Furthermore, one has a morphism $\hat{\Delta}\colon \hat{\mathscr{S}} \to \mathscr{L}(E^s \otimes E^s)$ given by $\hat{\Delta}(x) = V^*(1 \otimes x)V$.*

ii) In $\mathscr{L}(E^s \otimes E^s)$, one has $\hat{\Delta}(\hat{\mathscr{S}})(1 \otimes \hat{\mathscr{S}}) = \hat{\mathscr{S}} \otimes \hat{\mathscr{S}} = \hat{\Delta}(\hat{\mathscr{S}})(\hat{\mathscr{S}} \otimes 1)$. In particular, $\hat{\Delta}(\hat{\mathscr{S}})$ is a non-degenerate C^*-sub-family of $\mathscr{M}(\hat{\mathscr{S}} \otimes \hat{\mathscr{S}})$.

iii) The map $\hat{\Delta}\colon \hat{\mathscr{S}} \to \mathscr{M}(\hat{\mathscr{S}} \otimes \hat{\mathscr{S}})$ is coassociative in the sense that $(\hat{\Delta} \otimes 1)\hat{\Delta} = (1 \otimes \hat{\Delta})\hat{\Delta}$.

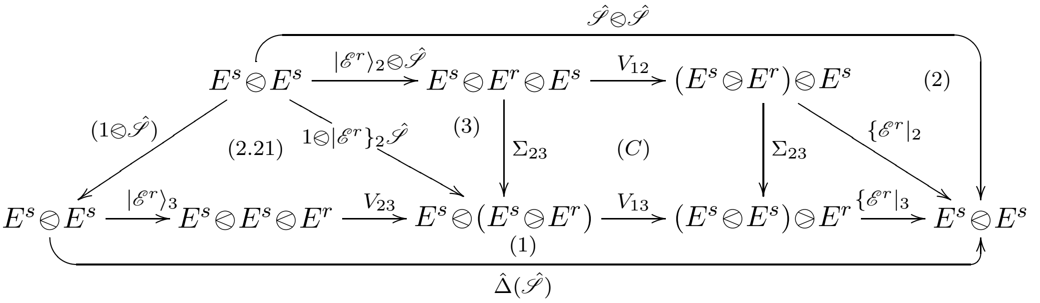
Proof. i) The first assertion follows from the fact that the family $\hat{\mathscr{S}}$ commutes with the representation π_r. The existence of the morphism follows from example 1.42 ii).

ii) We first prove $\hat{\Delta}(\hat{\mathscr{S}}) = \{\mathscr{E}^r|_3 V_{13} V_{23}|\mathscr{E}^r\rangle_3$ by means of the following diagram.



The individual cells commute for the following reasons: (I) intertwining properties of V, (P) pentagon equation, (D) definition. Therefore, the composition on the top line is equal to $\hat{\Delta}(\mathscr{S})$.

We prove $\hat{\Delta}(\mathscr{S})(1 \otimes \hat{\mathscr{S}}) = \hat{\mathscr{S}} \otimes \hat{\mathscr{S}}$ by means of the following diagram.



The labelled cells commute for the following reasons: (1) by the diagram above, (2.21) lemma 2.21, (2) by definition of $\hat{\mathscr{S}}$ and because $|\mathscr{E}^r\}_2 V \subset \mathscr{L}^{\mathrm{id}}(E^s \otimes E^r, E^s)$, (C) by inspection. We show that cell (3) commutes. The families $1 \otimes |\mathscr{E}^r\}_2 \hat{\mathscr{S}}$ and $\Sigma_{23} \circ \big(|\mathscr{E}^r\rangle_2 \otimes \hat{\mathscr{S}}\big)$ are generated by maps of the form

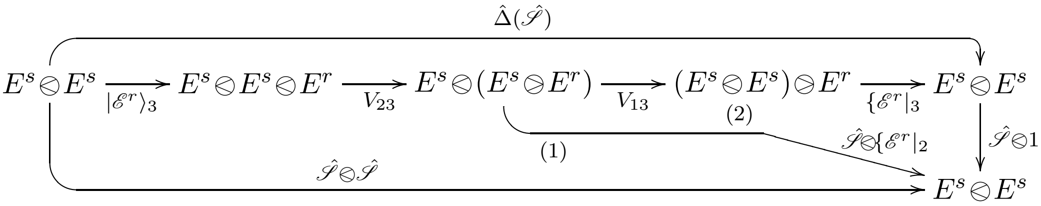
$$\zeta \otimes \zeta' \mapsto \zeta \otimes (\hat{s}\zeta' \otimes \xi), \quad \hat{s} \in \hat{\mathscr{S}}^{\beta}_{\alpha},\ \xi \in \mathscr{E}^r_{\alpha'},\ \alpha, \alpha', \beta \in \mathrm{PAut}(A),\ \alpha'^*\alpha \leq \mathrm{id}$$

and

$$\zeta \otimes \zeta' \mapsto (\zeta \otimes \xi') \otimes \hat{s}'\zeta' \mapsto \zeta \otimes (\hat{s}'\zeta' \otimes \xi'),$$
$$\hat{s}' \in \hat{\mathscr{S}}^{\beta}_{\alpha},\ \xi' \in \mathscr{E}^r_{\alpha'},\ \alpha, \alpha', \beta \in \mathrm{PAut}(A),\ \alpha \curlyvee \alpha',$$

respectively. Hence, $\Sigma_{23} \circ \big(|\mathscr{E}^r\rangle_2 \otimes \hat{\mathscr{S}}\big)$ is contained in $1 \otimes |\mathscr{E}^r\}_2 \hat{\mathscr{S}}$. We prove the reverse inclusion. We can write $\xi = \xi' a = \alpha'(a)\xi'$ with $\xi' \in \mathscr{E}^r_{\alpha'}$ and $a \in \mathscr{A}_{\alpha'^*\alpha'}$. Then $\hat{s}\zeta' \otimes \xi = \hat{s}\zeta' \otimes \alpha'(a)\xi' = \rho(\alpha'(a))\hat{s}\zeta' \otimes \xi'$. By part i) of proposition 1.55, $\alpha'(a)$ is contained in $\mathscr{A}_{\alpha'\alpha'^*}$. By lemma 1.51 and proposition 1.58, the product $\hat{s}' := \rho(\alpha'(a))\hat{s}$ belongs to $\hat{\mathscr{S}}^{\beta}_{\alpha'\alpha'^*\alpha}$. Since $\alpha'^*\alpha \leq \mathrm{id}$, we have $\alpha'\alpha'^*\alpha \leq \alpha'$. Therefore, $\hat{s}'$ belongs to $\hat{\mathscr{S}}^{\beta}_{\alpha'}$, and the map $1 \otimes (|\xi\}_2 \hat{s})$ is equal to $\Sigma_{23} \circ (|\xi'\rangle_2 \otimes \hat{s}')$.

We prove $(\hat{\mathscr{S}} \otimes 1)\hat{\Delta}(\mathscr{S}) = \hat{\mathscr{S}} \otimes \hat{\mathscr{S}}$ by means of the following diagram.



Cell (1) commutes by definition of $\hat{\mathscr{S}}$ and because $V|\mathscr{E}^r\rangle_2 \in \mathscr{L}_{\mathrm{id}}(E^s, E^s \otimes E^r)$. We show that cell (2) commutes. By lemma 2.21,

$$(\hat{\mathscr{S}} \otimes 1)\{\mathscr{E}^r|_3 V_{13} = \big(\big(\hat{\mathscr{S}}\{\mathscr{E}^r|_2 V_{12}\big) \otimes 1\big)\Sigma_{23} = \big(\hat{\mathscr{S}}\langle\mathscr{E}^r|_2 \otimes 1\big) \circ \Sigma_{23}.$$

The families $\big(\hat{\mathscr{S}}\langle\mathscr{E}^r|_2 \otimes 1\big) \circ \Sigma_{23}$ and $\hat{\mathscr{S}} \otimes \{\mathscr{E}^r|_2$ are generated by maps of the form

$$\zeta \otimes (\zeta' \otimes \zeta'') \mapsto \zeta \otimes \zeta'' \otimes \zeta' \mapsto \hat{s}\langle\xi|\zeta''\rangle\zeta \otimes \zeta',$$
$$\hat{s} \in \hat{\mathscr{S}}^{\beta}_{\alpha},\ \xi \in \mathscr{E}^r_{\beta'},\ \alpha, \beta, \beta' \in \mathrm{PAut}(A),\ \beta\beta'^* \leq \mathrm{id},$$

and

$$\zeta \otimes (\zeta' \otimes \zeta'') \mapsto \hat{s}'\zeta \otimes \zeta'\beta'\big(\langle\xi'|\zeta''\rangle\big) = \beta'\big(\langle\xi'|\zeta''\rangle\big)\hat{s}'\zeta \otimes \zeta' = \hat{s}'\langle\xi'|\zeta''\rangle\zeta \otimes \zeta',$$
$$\hat{s}' \in \hat{\mathscr{S}}^{\beta}_{\alpha},\ \xi' \in \mathscr{E}^r_{\beta'},\ \alpha, \beta, \beta' \in \mathrm{PAut}(A),\ \beta \curlyvee \beta',$$

respectively. Hence, $\hat{\mathscr{S}} \otimes \{\mathscr{E}^r|_2$ is contained in $(\hat{\mathscr{S}}\langle\mathscr{E}^r|_2 \otimes 1) \circ \Sigma_{23}$. We prove the reverse inclusion. We can write $\xi = \xi' a'$ with $\xi' \in \mathscr{E}^r_{\beta'}$ and $a' \in \mathscr{A}_{\beta'^*\beta'}$. Then $\hat{s}\langle\xi|\zeta''\rangle = \hat{s}a'^*\langle\xi'|\zeta''\rangle$, and the element $\hat{s}' := \hat{s}a'^*$ belongs to $\hat{\mathscr{S}}^{\beta}_{\alpha}\lambda(\mathscr{A}_{\beta'^*\beta'}) \subset \hat{\mathscr{S}}^{\beta\beta'^*\beta'}_{\alpha}$. Since $\beta\beta'^* \leq \mathrm{id}$, we have $\beta\beta'^*\beta' \leq \beta'$. Therefore, $\hat{s}'$ belongs to $\hat{\mathscr{S}}^{\beta'}_{\alpha}$, and the map $(\hat{s}\langle\xi|_2 \otimes 1) \circ \Sigma_{23}$ is equal to $\hat{s}' \otimes \{\xi'|_2$.

iii) By the pentagon equation, one has for each $\hat{s} \in \hat{\mathscr{S}}$

$$(1 \otimes \hat{\Delta})\hat{\Delta}(\hat{s}) = V^*_{23}V^*_{13} \cdot \underbrace{\hat{s}_3}_{V^*_{12}\hat{s}_3V_{12}} \cdot V_{13}V_{23}$$
$$= V^*_{12}V^*_{23}\hat{s}_3V_{23}V_{12} = (\hat{\Delta} \otimes 1)\hat{\Delta}(\hat{s}). \qquad \square$$

Theorem 2.23. *Let $V\colon E^s \otimes E \to E \otimes E^r$ be a decomposably left regular pseudo-multiplicative unitary. Then the pair $(\hat{\mathscr{S}}, \hat{\Delta})$ defined above is a right Hopf C^*-family.* □

The right leg The *right leg of V* is the sub-family $\mathscr{S} := \langle\mathscr{E}^s|_1\, V\, |\mathscr{E}^s\}_1 \subset \mathscr{L}(E^r)$ given by the composition

$$E^r \cong A \otimes E^r \xrightarrow{|\mathscr{E}^s\}_1} E^s \otimes E^r \xrightarrow{V} E^s \otimes E^r \xrightarrow{\langle\mathscr{E}^s|_1} A \otimes E^r \cong E^r.$$

Since V commutes with the representation π_{s2}, the family $\mathscr{S}$ commutes with the representation π_s.

Corollary 2.24. *Let $V\colon E^s \otimes E \to E \otimes E^r$ be a decomposably right regular pseudo-multiplicative unitary and let $\mathscr{S} \subset \mathscr{L}(E^r)$ be defined as above.*

i) $\mathscr{S}$ is a non-degenerate C^*-family.

ii) For each $y \in \mathscr{S}$, the operator $y \otimes 1 \in \mathscr{L}(E^{s,r} \otimes E)$ is well-defined, and the formula $y \mapsto V(y \otimes 1)V^*$ defines a non-degenerate morphism $\Delta\colon \mathscr{S} \to \mathscr{M}(\mathscr{S} \otimes \mathscr{S})$.

iii) The pair $(\mathscr{S}, \Delta)$ is a left Hopf C^*-family.

Proof. Consider the opposite $V^{op} = \Sigma V^* \Sigma$. It is decomposably left regular, and its left leg coincides with $\mathscr{S}$. This proves i). The remaining parts follow from theorem 2.23, applied to V^{op}, in a similar fashion. □

2.3 Coactions of the associated Hopf C^*-families

Our primary interest in Hopf C^*-families lies in their coactions on C^*-algebras, associated crossed products and duality theorems. The first of these notions has yet to be defined. This is done in the first subsection where we also discuss the relation to coactions on C^*-pre-families. Next, we study a regularity condition for coactions of the legs of a pseudo-multiplicative unitary which is relevant to the proof of the duality theorem 2.74. The section ends with a discussion of coaction unitaries. They give rise to coactions and will be fundamental constituents of pseudo-Kac systems.

Again, we focus on the left leg of a pseudo-multiplicative unitary and on right Hopf C^*-families.

2.3.1 Coactions on C^*-algebras

In order to define coactions of Hopf C^*-families on C^*-algebras, we first have to construct an internal tensor product of C^*-algebras with C^*-families: by analogy to Hopf C^*-algebras, a coaction of a Hopf C^*-family $(\hat{\mathscr{S}}, \hat{\Delta})$ on a C^*-algebra C should be a $*$-homomorphism $\hat{\delta}\colon C \to M(C \otimes \hat{\mathscr{S}})$ subject to several conditions, where the C^*-algebra $C \otimes \hat{\mathscr{S}}$ yet has to be defined. The necessary definitions are straight-forward. However, they raise several questions which suggest that for Hopf C^*-families, it is more natural to consider coactions on C^*-pre-families than coactions on C^*-algebras. This problem is discussed at the end of this subsection.

Internal tensor product of C^*-families with C^*-algebras

Let A and B be C^*-algebras and let E be a C^*-B-A-bimodule such that $BE = E$. We denote by σ the representation of B on E given by left multiplication.

Definition 2.25. *A C^*-A-*algebra (C, π) *consists of a C^*-algebra C and a non-degenerate $*$-homomorphism $\pi\colon A \to M(C)$. A* morphism *between C^*-A-algebras (C, π) and (C', π') is a non-degenerate $*$-homomorphism $\phi\colon C \to M(C')$ intertwining π and π', i.e. satisfying $\phi\big(\pi(a)c\big) = \pi'(a)\phi(c)$ for all $a \in A$ and $c \in C$. We denote the category of all C^*-A-algebras by $\mathbf{C^*_A}$.*

Let (C, π) be a C^*-A-algebra. We denote by $\mathscr{C}ov(C, \pi)$ the family given by

$$\mathscr{C}ov_\alpha(C, \pi) := \{c \in C\pi(\mathrm{Dom}(\alpha)) \mid \forall a \in \mathrm{Dom}(\alpha) : c\pi(a) = \pi(\alpha(a))c\}, \qquad \alpha \in \mathrm{PAut}(A).$$

Let $\mathscr{D}$ be a C^*-pre-family on E. The internal tensor product *$(C, \pi) \otimes \mathscr{D}$ is the C^*-subalgebra of $L_C(C \otimes E)$ given by*

$$(C, \pi) \otimes \mathscr{D} := \overline{\mathrm{span}}_\beta \big(\mathscr{C}ov(C, \pi) \otimes \mathscr{D}\big)^\beta, \quad \textit{where}$$
$$\big(\mathscr{C}ov(C, \pi) \otimes \mathscr{D}\big)^\beta := \overline{\mathrm{span}}_{\alpha \curlyvee \alpha'} \mathscr{C}ov_\alpha(C, \pi) \otimes \mathscr{D}^\beta_{\alpha'}, \quad \beta \in \mathrm{PAut}(B).$$

Remark 2.26. Let (C,π) be a C^*-A-algebra. Then, adopting the convention of definition 1.13, $\mathscr{C}ov(C,\pi)$ is a pre-C^*-family. Consider C as a C^*-A-C-bimodule via π. Then $K_C(C) \subset L_C(C)$ can be identified with C, acting on itself by left multiplication, and for each $\gamma \in \mathrm{PAut}(A)$, the space $\mathscr{C}ov_\gamma(C,\pi)$ identifies with $\mathscr{K}^\gamma_{\mathrm{id}}(C)$. This follows easily from the equality $\mathscr{K}^\gamma_{\mathrm{id}}(C) = K_C(C) \cap \mathscr{L}^\gamma_{\mathrm{id}}(C)$.

To prove functoriality of the internal tensor product constructed above, we need to make the following preliminary remarks. Let $\phi\colon (C,\pi) \to (C',\pi')$ be a morphism of C^*-A-algebras. Then ϕ induces a non-degenerate $*$-homomorphism $\mathrm{ind}_E\,\phi\colon L_C(C \otimes E) \to L_{C'}(C' \otimes E)$ via the map $L_C(C \otimes E) \to L_{C'}(C' \otimes C \otimes E)$ given by $T \mapsto 1 \otimes T$, and the identification $C' \otimes C \otimes E \cong C' \otimes E$. If ϕ is injective, so is $\mathrm{ind}_E\,\phi$. We omit the subscript E if it is clear from the context.

Proposition 2.27. *Let $\mathscr{D}$ be a C^*-pre-family on E and let (C,π) be a C^*-A-algebra.*

i) The space $(C,\pi) \otimes \mathscr{D}$ is a C^*-algebra.

ii) Let $\phi\colon (C,\pi) \to (C',\pi')$ be a morphism of C^*-A-algebras. Then the induction homomorphism $\mathrm{ind}_E\,\phi$ restricts to a non-degenerate $*$-homomorphism $\mathrm{ind}_{\mathscr{D}}\,\phi\colon (C,\pi) \otimes \mathscr{D} \to M\big((C',\pi') \otimes \mathscr{D}\big)$.

iii) If $\mathscr{D}$ is a C^*-family, the representation $1 \otimes \sigma$ of B on $C \otimes E$ turns $(C,\pi) \otimes \mathscr{D}$ into a C^*-B-algebra. The maps $(C,\pi) \mapsto \big((C,\pi)\otimes\mathscr{D}, 1\otimes\sigma\big)$ and $\phi \mapsto \mathrm{ind}_{\mathscr{D}}\,\phi$ define a functor $\mathbf{ind}_{\mathscr{D}}\colon \boldsymbol{C^*_A} \to \boldsymbol{C^*_B}$.

iv) Let $\psi\colon \mathscr{D} \to \mathscr{M}(\mathscr{D}')$ be a non-degenerate morphism of non-degenerate C^*-families. Then the family of maps

$$\begin{aligned} 1 \times \psi^\beta_{\alpha'}\colon \mathscr{C}ov_\alpha(C,\pi) \times \mathscr{D}^\beta_{\alpha'} &\to \mathscr{C}ov_\alpha(C,\pi) \otimes \mathscr{M}(\mathscr{D})^\beta_{\alpha'}, \\ (c,d) &\mapsto c \otimes d, \end{aligned}$$

where $\alpha \curlyvee \alpha' \in \mathrm{PAut}(A)$, $\beta \in \mathrm{PAut}(B)$, defines a non-degenerate $*$-homomorphism $1 \otimes \psi\colon (C,\pi) \otimes \mathscr{D} \to M\big((C,\pi) \otimes \mathscr{D}'\big)$. Furthermore, $1 \otimes \psi$ is a morphism of C^*-B-algebras. The family $\big(1 \otimes \psi\big)_{(C,\pi)}$ defines a natural transformation $\boldsymbol{\psi}\colon \mathbf{ind}_{\mathscr{D}} \to \mathbf{ind}_{\mathscr{D}'}$.

v) Let E' be a C^*-module over B with a non-degenerate representation σ' of a C^*-algebra B' and let $\mathscr{D}'$ be a non-degenerate C^*-family on E'. Then one has a natural transformation $\mathbf{ind}_{\mathscr{D}\otimes\mathscr{D}'} \to \mathbf{ind}_{\mathscr{D}'}\mathbf{ind}_{\mathscr{D}}$ given by inclusions of C^*-algebras.

Proof. i,ii) This follows from proposition 1.28 and its proof, respectively.

iii) The assumption on $\mathscr{D}$ implies that $\lambda(\mathscr{C}ov(B))\mathscr{D} = \mathscr{D}$, whence the first claim follows. The existence of the functor is immediate.

iv) Denote by $B^{\mathbb{C}}$ the space B, considered as a C^*-$\mathbb{C}$-B-module in the canonical way. By definition 1.30, the morphism ψ induces a $*$-homomorphism

$$\mathrm{id} \otimes \psi \otimes \mathrm{id} \colon \mathscr{L}_{\mathrm{id}}(C) \otimes \mathscr{D} \otimes \mathscr{L}^{\mathrm{id}}(B^{\mathbb{C}}) \to M\big(\mathscr{L}_{\mathrm{id}}(C) \otimes \mathscr{D}' \otimes \mathscr{L}^{\mathrm{id}}(B^{\mathbb{C}})\big).$$

Since $\mathscr{D}$ is a non-degenerate C^*-family on E, one has $BE = E$. By part i) of lemma 1.61, the identification $E \cong E \otimes B^{\mathbb{C}}$ induces an embedding

$$(C, \pi) \otimes \mathscr{D} \subset \mathscr{L}_{\mathrm{id}}(C) \otimes \mathscr{D} \otimes \mathscr{L}^{\mathrm{id}}(B^{\mathbb{C}}),$$

and a similar inclusion holds for $(C, \pi) \otimes \mathscr{D}'$. Now, the $*$-homomorphism $\mathrm{id} \otimes \psi$ is obtained as the restriction of $\mathrm{id} \otimes \psi \otimes \mathrm{id}$. It is non-degenerate because ψ is non-degenerate, and it intertwines the representations of B on E and E' by part ii) of lemma 1.61. The last statement in the claim is obvious.

v) For each C^*-A-algebra (C, π), the C^*-pre-family $\mathscr{C}ov(C, \pi) \otimes \mathscr{D}$ is contained in $\mathscr{C}ov\left((C, \pi) \otimes \mathscr{D}, 1 \otimes \sigma\right)$. By associativity of the internal tensor product of C^*-pre-families, one obtains

$$\begin{aligned} \mathscr{C}ov(C, \pi) \otimes (\mathscr{D} \otimes \mathscr{D}') &= \big(\mathscr{C}ov(C, \pi) \otimes \mathscr{D}\big) \otimes \mathscr{D}' \\ &\subset \mathscr{C}ov\left((C, \pi) \otimes \mathscr{D}, 1 \otimes \sigma\right) \otimes \mathscr{D}'. \end{aligned}$$ □

Hooptedoodle 2.28. *Denote by* $[\boldsymbol{C^*_A}, \boldsymbol{C^*_B}]$ *the (quasi-)category*[1] *consisting of all functors* $\boldsymbol{C^*_A} \to \boldsymbol{C^*_B}$ *as objects and natural transformations as morphisms. Then the map* $\mathscr{C} \mapsto \mathbf{ind}_{\mathscr{C}}$ *and* $\psi \mapsto \boldsymbol{\psi}$ *defines a (quasi-)functor from the category of* C^**-families on* C^**-*B*-*A*-bimodules with non-degenerate morphisms* $\mathscr{C} \to \mathscr{M}(\mathscr{D})$ *to the (quasi-)category* $[\boldsymbol{C^*_A}, \boldsymbol{C^*_B}]$.

Definition of coactions

Let $(\hat{\mathscr{S}}, \hat{\Delta})$ be a right Hopf C^*-family on a C^*-bimodule E over A. By the previous proposition, it defines

- a functor $\hat{\boldsymbol{S}}$ on $\boldsymbol{C^*_A}$ given by $\hat{\boldsymbol{S}}(C, \pi) := ((C, \pi) \otimes \hat{\mathscr{S}}, 1 \otimes \sigma)$ and $\hat{\boldsymbol{S}}(\phi) := \mathrm{ind}_{\hat{\mathscr{S}}} \phi$, where $\sigma \colon A \to L_A(E)$ denotes the representation defining the bimodule structure, and
- a natural transformation $\hat{\boldsymbol{\Delta}} \colon \hat{\boldsymbol{S}} = \mathbf{ind}_{\hat{\mathscr{S}}} \to \mathbf{ind}_{\hat{\mathscr{S}} \otimes \hat{\mathscr{S}}} \to \hat{\boldsymbol{S}}^2$,

such that the following diagram commutes.

$$\begin{array}{ccc} \hat{\boldsymbol{S}} & \xrightarrow{\hat{\boldsymbol{\Delta}}} & \hat{\boldsymbol{S}} \circ \hat{\boldsymbol{S}} \\ {\scriptstyle \hat{\boldsymbol{\Delta}}} \downarrow & & \downarrow {\scriptstyle \hat{\boldsymbol{\Delta}}_{\hat{\boldsymbol{S}}}} \\ \hat{\boldsymbol{S}} \circ \hat{\boldsymbol{S}} & \xrightarrow[\hat{\boldsymbol{S}}\hat{\boldsymbol{\Delta}}]{} & \hat{\boldsymbol{S}} \circ \hat{\boldsymbol{S}} \circ \hat{\boldsymbol{S}} \end{array}$$

[1] Depending on the Universe you choose to live in.

Hooptedoodle 2.29. *Consider the (quasi-)category whose objects are endofunctors of the category $\boldsymbol{C^*_A}$ and whose morphisms are natural transformations. Composition of functors defines a (quasi-)monoidal structure on this (quasi-)category, and the pair $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$ is a (quasi-)cosemigroup in this (quasi-)category.*

Definition 2.30. *i) A* (right) coaction *of $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$ on a C^*-A-algebra (C, π) is a morphism $\hat{\delta}\colon (C, \pi) \to \hat{\boldsymbol{S}}(C, \pi)$ of C^*-A-algebras which satisfies $\hat{\delta}(C)(1 \otimes \hat{\mathscr{S}}) \subset (C, \pi) \otimes \hat{\mathscr{S}}$ and is* coassociative *in the sense that the following diagram commutes.*

$$\begin{array}{ccc} (C,\pi) & \xrightarrow{\hat{\delta}} & \hat{\boldsymbol{S}}(C,\pi) \\ {\scriptstyle \hat{\delta}}\downarrow & & \downarrow{\scriptstyle \hat{\boldsymbol{\Delta}}_{(C,\pi)}} \\ \hat{\boldsymbol{S}}(C,\pi) & \xrightarrow[\hat{\boldsymbol{S}}\hat{\delta}]{} & \hat{\boldsymbol{S}}^2(C,\pi) \end{array}$$

ii) Let $\hat{\delta}'\colon (C', \pi') \to \boldsymbol{S}(C', \pi')$ be another coaction of $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$. A morphism $\phi\colon (C, \pi) \to (C', \pi')$ is equivariant with respect to the coactions $\hat{\delta}$ and $\hat{\delta}'$ *if the following diagram commutes.*

$$\begin{array}{ccc} (C,\pi) & \xrightarrow{\hat{\delta}} & \hat{\boldsymbol{S}}(C,\pi) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \hat{\boldsymbol{S}}\phi} \\ (C',\pi') & \xrightarrow{\hat{\delta}'} & \hat{\boldsymbol{S}}(C',\pi') \end{array}$$

Remark 2.31. Coactions of a Hopf C^*-family $(\hat{\mathscr{S}}, \hat{\Delta})$ on C^*-A-algebras form a category:

- its objects are all triples $(C, \pi, \hat{\delta})$ where $\hat{\delta}$ is a coaction of $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$ on a C^*-A-algebra (C, π),
- morphisms $(C, \pi, \hat{\delta}) \to (C', \pi', \hat{\delta}')$ are all morphisms $(C, \pi) \to (C', \pi')$ of C^*-A-algebras which are equivariant with respect to the coactions $\hat{\delta}$ and $\hat{\delta}'$.

We loosely use the term coaction for objects of this category. For examples of coactions in the context of groupoids, see subsection 3.2.3 and section 3.3.

Coactions on C^*-algebras versus coactions on C^*-families

Let (C, π) be a C^*-A-algebra. It is natural to ask whether coactions on (C, π) and coactions on the associated C^*-pre-family $\mathscr{C}ov(C, \pi)$ correspond to each other via a restriction and an extension procedure. The extension process is easy, but the restriction is obstructed by the following problem: given a coaction $\hat{\delta}$ on (C, π), we do not know whether $\hat{\delta}(\mathscr{C}ov(C, \pi))$ is contained in $\mathscr{M}(\mathscr{C}ov(C, \pi) \otimes \hat{\mathscr{S}})$. After proving the extension result, we discuss several related open questions.

Proposition 2.32. *Let F be a C^*-module over a C^*-algebra B with a non-degenerate representation $\pi\colon A \to L_B(F)$. Let $\mathscr{C} \subset \mathscr{L}_{\mathrm{id}}(F)$ be a non-degenerate C^*-pre-family such that $\mathscr{C}\lambda(\mathscr{A}) = \mathscr{C}$ and $\mathscr{C}^\alpha = \mathscr{C}^\alpha\lambda(\mathscr{A}_{\alpha^*\alpha})$ for all $\alpha \in \mathrm{PAut}(A)$. Let $\hat{\delta}$ be a right coaction of $(\hat{\mathscr{S}}, \hat{\Delta})$ on $\mathscr{C}$. Then the C^*-algebra $C := \overline{\mathrm{span}}_\alpha \mathscr{C}^\alpha$ satisfies $\pi(A)C = C$, and $\hat{\delta}$ extends to a coaction of $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$ on (C,π). If $\hat{\delta}$ is injective, so is the extension.*

Proof. The first claim follows from the equation $\pi(A)C = \overline{\mathrm{span}}_\alpha \big(\lambda(\mathscr{A})\mathscr{C}\big)^\alpha = \overline{\mathrm{span}}_\alpha \mathscr{C}^\alpha = C$. Observe that by construction, $\mathscr{C}$ is a subfamily of $\mathscr{C}ov(C,\pi)$. The extension of the morphism $\hat{\delta}$ is constructed along the lines of the proof of part iv) of proposition 2.27. Denote by $A^{\mathbb{C}}$ the space A, considered as a C^*-$\mathbb{C}$-A-bimodule in the canonical way. By definition, the morphism $\hat{\delta}$ induces a $*$-homomorphism

$$\mathrm{id} \otimes_* \hat{\delta} \otimes_* \mathrm{id}\colon \mathscr{L}_{\mathrm{id}}(B) \otimes \mathscr{C} \otimes \mathscr{L}^{\mathrm{id}}(A^{\mathbb{C}}) \to M\big(\mathscr{L}_{\mathrm{id}}(B) \otimes \mathscr{C} \otimes \hat{\mathscr{S}} \otimes \mathscr{L}^{\mathrm{id}}(A^{\mathbb{C}})\big).$$

By assumption on $\mathscr{C}$, one has $AF = F$. By lemma 1.61, the identification $F \cong B \otimes F \otimes A^{\mathbb{C}}$ induces an embedding $C \hookrightarrow 1 \otimes \mathscr{C} \otimes \mathscr{L}^{\mathrm{id}}(A^{\mathbb{C}})$. Hence, the $*$-homomorphism $\mathrm{id} \otimes_* \hat{\delta} \otimes_* \mathrm{id}$ restricts to a $*$-homomorphism $\hat{\delta}'\colon C \to L_B(F \otimes E)$. By construction, $\hat{\delta}'$ extends $\hat{\delta}$, whence

$$\hat{\delta}'(C)(1 \otimes \hat{\mathscr{S}}) \subset \overline{\mathrm{span}}_\alpha \big(\hat{\delta}(\mathscr{C})(1 \otimes \hat{\mathscr{S}})\big)^\alpha \subset \overline{\mathrm{span}}_\alpha \big(\mathscr{C} \otimes \hat{\mathscr{S}}\big)^\alpha \subset (C,\pi) \otimes \hat{\mathscr{S}}.$$

By construction, $\hat{\delta}'$ intertwines the representation π with $1 \otimes \sigma$. Coassociativity of $\hat{\delta}'$ follows from the coassociativity of $\hat{\delta}$. If $\hat{\delta}$ is injective, it is isometric, and hence $\hat{\delta}'$ is isometric, too. □

As indicated above, the definition of the internal tensor product of C^*-pre-families with C^*-algebras and the definition of coactions on C^*-algebras raise several questions which we can not yet answer. Most of these questions are related to the following problem:

Question 1 Let B be a C^*-algebra, F a C^*-A-B-bimodule and $\mathscr{C} \subset \mathscr{L}_{\mathrm{id}}(F)$ a C^*-pre-family. Put $C := \overline{\mathrm{span}}_\alpha \mathscr{C}^\alpha$ and denote the representation of A on F by σ. Then one always has an inclusion $\mathscr{C} \subset \mathscr{C}ov(C,\sigma)$. Under which conditions is this inclusion an equality?

Several related open questions follow.

Question 2 In the situation of part v) of proposition 2.27, under which conditions is the inclusion $\mathbf{ind}_{\mathscr{D} \otimes \mathscr{D}'}(C,\pi) \subset \mathbf{ind}_{\mathscr{D}'}\mathbf{ind}_{\mathscr{D}}(C,\pi)$ an equality?

Let $(C,\pi,\hat{\delta})$ be a coaction of $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$.

Question 3 Is $\mathscr{C}ov\left((C,\pi) \otimes \hat{\mathscr{S}}, 1 \otimes \sigma\right)$ equal to $(C,\pi) \otimes \hat{\mathscr{S}}$?

Question 4 Is $\hat{\delta}(\mathscr{C}ov(C,\pi))$ contained in $\mathscr{M}\big(\mathscr{C}ov(C,\pi) \otimes \hat{\mathscr{S}}\big)$?

Question 5 Does injectivity of $\hat{\delta}$ imply decomposability of C?

We expect that in general, the answer to all questions is "no". Apparently, the best behaved notion of a coaction of a Hopf C^*-family on a C^*-algebra is the following.

Definition 2.33. *A coaction (C, π, δ) of $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$ is a $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-*algebra *if it is* induced *from a $(\hat{\mathscr{S}}, \hat{\Delta})$-pre-family as in proposition 2.32.*

This point of view is supported by the fact that in the context of decomposable pseudo-Kac systems, dual coactions on crossed products will turn out to be of this kind.

2.3.2 Regular coactions on C^*-pre-families

In this subsection, we introduce a regularity property for coactions of the legs of a pseudo-multiplicative unitary on C^*-pre-families which occurs in the proof of the duality theorem for iterated crossed products 2.74. Regarding C^*-pre-families as sections of generalised Fell bundles, regularity can be thought of as a support condition for those sections.

We give the definition of regularity, note several examples and establish an easy criterion. Then, we prove a property of regular coactions which is crucial for the duality theorem. Finally, we show that every coaction restricts to a maximal regular coaction.

Throughout this subsection, let $V\colon E^s \otimes E \to E \otimes E^r$ be a decomposably left regular pseudo-multiplicative unitary. For each $\alpha \in \mathrm{PAut}(A)$, put $J_\alpha := \langle \mathscr{E}^r_\alpha | \mathscr{E}^r_\alpha \rangle$. By lemma 1.51, J_α is an ideal in $\mathscr{A}_{\alpha^*\alpha} \subset Z(A)$.

Definition 2.34. *A right coaction $\hat{\delta}$ of $(\hat{\mathscr{S}}, \hat{\Delta})$ on a C^*-pre-family $\mathscr{C}$ is* regular *if $\mathscr{C}^\alpha_\beta = \mathscr{C}^\alpha_\beta \lambda(J_\alpha)$ for all $\alpha \in \mathrm{PAut}(A)$ and all β. A $(\hat{\mathscr{S}}, \hat{\Delta})$-family is* regular *if its coaction is regular. A $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebra is* regular *if it is induced from a regular $(\hat{\mathscr{S}}, \hat{\Delta})$-family.*

Remark 2.35. From the proof of proposition 2.20, it follows that

$$\hat{\mathscr{S}}^\beta_\alpha = \hat{\mathscr{S}}^\beta_\alpha \, \lambda(J_\beta) \, \rho(J_\alpha) \quad \text{for all } \alpha, \beta \in \mathrm{PAut}(A).$$

We will see that coactions constructed out of coaction unitaries and dual coactions on crossed products are always regular. Additionally, we have the following criterion. It will imply that coactions of r-discrete groupoids are always regular, see 3.3.

Proposition 2.36. *Let $(\mathscr{C}, \hat{\delta})$ be a right coaction of $(\hat{\mathscr{S}}, \hat{\Delta})$ on a C^*-pre-family. If $\hat{\delta}$ is injective and $\hat{\delta}(\mathscr{C}) \subset \mathscr{C} \otimes \hat{\mathscr{S}}$, the coaction $\hat{\delta}$ is regular.*

Proof. Let $\beta \in \mathrm{PAut}(B)$ and $\alpha \in \mathrm{PAut}(A)$. Then $\hat{\delta}(\mathscr{C}^\alpha_\beta)$ is contained in $(\mathscr{C} \otimes \hat{\mathscr{S}})^\alpha_\beta$. By remark 2.35, $(\mathscr{C} \otimes \hat{\mathscr{S}})^\alpha_\beta = (\mathscr{C} \otimes \hat{\mathscr{S}})^\alpha_\beta \lambda(J_\alpha)$. Furthermore, $\hat{\delta}(c\lambda(a)) = \hat{\delta}(c)\lambda(a)$ for all $a \in \mathscr{A}$. Let (u_ν) be an approximate unit of the ideal J_α. Since $\hat{\delta}$ is injective, the net $\big(\hat{\delta}(c\lambda(u_\nu))\big)_\nu$ converges to $\hat{\delta}(c)$ for each $c \in \mathscr{C}^\alpha_\beta$. Hence, $\mathscr{C}^\alpha_\beta = \mathscr{C}^\alpha_\beta \lambda(J_\alpha)$. $\square$

Remark 2.37. It seems natural to call the Hopf C^*-family $(\hat{\mathscr{S}}, \hat{\Delta})$ *A-discrete* if $\lambda(\mathscr{A}) \subset \mathscr{M}(\hat{\mathscr{S}})$ is actually contained in $\hat{\mathscr{S}}$. If this condition holds, then each coaction $(\mathscr{C}, \hat{\delta})$ which satisfies $\lambda(\mathscr{A})\mathscr{C} = \mathscr{C}$ is regular, because

$$\hat{\delta}(\mathscr{C}) = \hat{\delta}(\lambda(\mathscr{A})\mathscr{C}) = (1 \otimes \lambda(\mathscr{A}))\hat{\delta}(\mathscr{C}) \subset (1 \otimes \hat{\mathscr{S}})\hat{\delta}(\mathscr{C}) \subset \mathscr{C} \otimes \hat{\mathscr{S}}.$$

The regularity condition is relevant to the duality theorem 2.74 for the following reason.

Proposition 2.38. *Let $(C, \pi, \hat{\delta})$ be a coaction $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebra which is induced from a regular coaction $(\mathscr{C}, \hat{\delta})$ of $(\hat{\mathscr{S}}, \hat{\Delta})$. Then*

$$\overline{\operatorname{span}}_\beta\, (\mathscr{C} \otimes \hat{\mathscr{S}})^\beta (1 \otimes K_A(E)) = K_C(C \otimes E).$$

Proof. Let $c \otimes \hat{s} \in \mathscr{C} \otimes \hat{\mathscr{S}}$, $c_1, c_2 \in \mathscr{C}$ and $\xi, \xi', \eta \in E$. Then

$$\begin{aligned} (c \otimes \hat{s})\big(1 \otimes |\xi\rangle\langle\eta|\big)d \otimes \zeta &= (c \otimes \hat{s})\big(d \otimes \xi\langle\eta|\zeta\rangle\big) = c\langle\eta|\zeta\rangle d \otimes \hat{s}\xi, \\ |c_1 \otimes \xi'\rangle\langle c_2 \otimes \eta| d \otimes \zeta &= c_1 \cdot c_2^*\langle\eta|\zeta\rangle d \otimes \xi' \end{aligned}$$

for all $d \in C$ and $\zeta \in E$. This shows that $\overline{\operatorname{span}}_\beta\, (\mathscr{C} \otimes \hat{\mathscr{S}})^\beta (1 \otimes K_A(E))$ is contained in $K_C(C \otimes E)$. Let us prove the reverse inclusion. For all $\alpha \in \mathrm{PAut}(A)$, one has

$$\overline{\operatorname{span}}_\beta\, \hat{\mathscr{S}}^\beta_\alpha E = \{\mathscr{E}^r_\alpha|_2\, V\, |E\rangle_2\, E = \{\mathscr{E}^r_\alpha|_2\, (E \otimes E^r) = E\,\alpha\big(\langle\mathscr{E}^r_\alpha|E\rangle\big)$$

and therefore, in $L_C(C, C \otimes E)$,

$$\overline{\operatorname{span}}_\beta\big(\mathscr{C}^\alpha \otimes \hat{\mathscr{S}}^\beta_\alpha|E\rangle\big) \supset \mathscr{C}^\alpha \otimes |EJ_\alpha\rangle = \mathscr{C}^\alpha\pi(J_\alpha) \otimes |E\rangle.$$

Since $C = \overline{\operatorname{span}}_\alpha\, \mathscr{C}^\alpha = \overline{\operatorname{span}}_\alpha \mathscr{C}^\alpha J_\alpha$, the reverse inclusion follows. ☐

In the remaining part of the subsection we show that every coaction of $(\hat{\mathscr{S}}, \hat{\Delta})$ restricts to a maximal regular coaction. Given a coaction $(\mathscr{C}, \hat{\delta})$, a natural candidate for the "regularisation" is the family $\tilde{\mathscr{C}}$ given by $\tilde{\mathscr{C}}^\alpha_\beta := \mathscr{C}^\alpha_\beta \lambda(J_\alpha)$ for all $\alpha \in \mathrm{PAut}(A)$ and all β. However, neither it is clear whether $\tilde{\mathscr{C}}$ is a C^*-pre-family nor whether $\hat{\delta}$ restricts to a coaction on $\tilde{\mathscr{C}}$. So, we need some preparations. They could be simplified if we could show the following equation:

$$(\dagger) \qquad J_\alpha \stackrel{?}{=} \alpha^*(J_{\alpha^*}) \text{ for all } \alpha \in \mathrm{PAut}(A).$$

Lemma 2.39. *For all $\alpha \in \mathrm{PAut}(A)$, put $I_\alpha := J_\alpha \cap \alpha^*(J_{\alpha^*})$.*

i) For all $\alpha, \gamma \in \mathrm{PAut}(A)$, one has $\hat{\mathscr{S}}^\gamma_\alpha = \hat{\mathscr{S}}^\gamma_\alpha \rho(I_\alpha)$.

ii) The C^*-subalgebra $I_{\mathrm{id}} \subset A$ is non-degenerate.

iii) For all $\alpha, \alpha' \in \mathrm{PAut}(A)$, one has $\alpha^*(J_{\alpha'}\alpha(J_\alpha)) \subset J_{\alpha'\alpha}$ and $\alpha^*(I_{\alpha'}\alpha(I_\alpha)) \subset I_{\alpha'\alpha}$.

Proof. i) By remark 2.35 and proposition 1.2, one has

$$\hat{\mathscr{S}}_{\alpha}^{\gamma} = \big(\hat{\mathscr{S}}_{\alpha^*}^{\gamma^*}\big)^* = \rho(J_{\alpha^*})^* \hat{\mathscr{S}}_{\alpha}^{\gamma} = \hat{\mathscr{S}}_{\alpha}^{\gamma} \rho(\alpha^*(J_{\alpha^*})),$$

and by remark 2.35 again, we obtain $\hat{\mathscr{S}}_{\alpha}^{\gamma} = \hat{\mathscr{S}}_{\alpha}^{\gamma} \rho(I_{\alpha})$.

ii) Since E is full, one has by part i) and proposition 2.19

$$A = \langle E|E\rangle = \langle E|\hat{\mathscr{S}}_{\mathrm{id}}^{\mathrm{id}} E\rangle \subset \langle E|EI_{\mathrm{id}}\rangle = AI_{\mathrm{id}}.$$

iii) Let us prove the first inclusion. Since E is full and π_s is non-degenerate, we have $J_{\alpha'\alpha} = \langle(\mathscr{E}^r_{\alpha'\alpha}, \pi_s) \circledcirc E\big|(\mathscr{E}^r_{\alpha'\alpha}, \pi_s) \circledcirc E\rangle$. By lemma 2.16, this space contains the following inner product.

$$\begin{aligned}\big\langle V^*(\mathscr{E}^r_{\alpha'} \circledcirc \mathscr{E}^r_{\alpha})\big|V^*(\mathscr{E}^r_{\alpha'} \circledcirc \mathscr{E}^r_{\alpha})\big\rangle &= \big\langle \mathscr{E}^r_{\alpha'} \circledcirc \mathscr{E}^r_{\alpha}\big|\mathscr{E}^r_{\alpha'} \circledcirc \mathscr{E}^r_{\alpha}\big\rangle \\ &= \big\langle \mathscr{E}^r_{\alpha}\big|\pi_r(J_{\alpha'})\mathscr{E}^r_{\alpha}\big\rangle \\ &= \big\langle \mathscr{E}^r_{\alpha}\big|\mathscr{E}^r_{\alpha}\alpha^*(J_{\alpha'}\alpha(J_{\alpha}))\big\rangle = \alpha^*\big(J_{\alpha'}\alpha(J_{\alpha})\big)\end{aligned}$$

Next, let us prove the second inclusion. Replacing α and α' by α'^* and α^*, respectively, we find that $\alpha'\big(\alpha'^*(J_{\alpha'^*})J_{\alpha^*}\big)$ is contained in $J_{\alpha'}$. Therefore,

$$\begin{aligned}\alpha^*(I_{\alpha'}\alpha(I_{\alpha})) &= \alpha^*\big(J_{\alpha'}\alpha(J_{\alpha})\big) \cdot \alpha^*\big(\alpha'^*(J_{\alpha'^*})\alpha\big(\alpha^*(J_{\alpha^*})\big)\big) \\ &\subset J_{\alpha'\alpha} \cdot \alpha^*\big(\alpha'^*(J_{\alpha'\alpha})\big) = I_{\alpha'\alpha}.\end{aligned}$$ □

Let B be a C^*-algebra, let $\mathscr{C}$ be a non-degenerate C^*-pre-family on a C^*-A-B-bimodule F and let $\hat{\delta}$ be a coaction of $(\hat{\mathscr{S}}, \hat{\Delta})$ on $\mathscr{C}$. Consider the sub-family $\tilde{\mathscr{C}} \subset \mathscr{C}$ given by $\tilde{\mathscr{C}}_{\beta}^{\alpha} := \mathscr{C}_{\beta}^{\alpha}\lambda(I_{\alpha})$ for all $\alpha \in \mathrm{PAut}(A)$ and $\beta \in \mathrm{PAut}(B)$.

Lemma 2.40. *The sub-family $\tilde{\mathscr{C}} \subset \mathscr{C}$ is a non-degenerate C^*-pre-family.*

Proof. By the previous lemma, one has for all $\alpha, \alpha' \in \mathrm{PAut}(A)$ and all $\beta, \beta' \in \mathrm{PAut}(B)$

$$\begin{aligned}\tilde{\mathscr{C}}_{\beta'}^{\alpha'}\tilde{\mathscr{C}}_{\beta}^{\alpha} = \mathscr{C}_{\beta'}^{\alpha'}\lambda(I_{\alpha'})\mathscr{C}_{\beta}^{\alpha}\lambda(I_{\alpha}) &= \mathscr{C}_{\beta'}^{\alpha'}\lambda\big(I_{\alpha'}\alpha^*(I_{\alpha})\big)\mathscr{C}_{\beta}^{\alpha} \\ &= \mathscr{C}_{\beta'}^{\alpha'}\mathscr{C}_{\beta}^{\alpha}\lambda\big(\alpha\big(I_{\alpha'}\alpha^*(I_{\alpha})\big)\big) \subset \mathscr{C}_{\beta'\beta}^{\alpha'\alpha}\lambda(I_{\alpha'\alpha}) = \tilde{\mathscr{C}}_{\beta'\beta}^{\alpha'\alpha}.\end{aligned}$$

Furthermore, by proposition 1.2, one has

$$\begin{aligned}\big(\tilde{\mathscr{C}}_{\beta}^{\alpha}\big)^* = \lambda(I_{\alpha}^*)\mathscr{C}_{\beta^*}^{\alpha^*} &= \lambda(J_{\alpha}\alpha^*(J_{\alpha^*}))\mathscr{C}_{\beta^*}^{\alpha^*} \\ &= \mathscr{C}_{\beta^*}^{\alpha^*}\lambda\big(\alpha(J_{\alpha}\alpha^*(J_{\alpha^*}))\big) = \mathscr{C}_{\beta^*}^{\alpha^*}\lambda(I_{\alpha^*}) = \tilde{\mathscr{C}}_{\beta^*}^{\alpha^*}.\end{aligned}$$

If $(\beta', \alpha') \geq (\beta, \alpha)$, then $\tilde{\mathscr{C}}_{\beta}^{\alpha} = \mathscr{C}_{\beta}^{\alpha}\lambda(I_{\alpha})$ is contained in $\mathscr{C}_{\beta'}^{\alpha'}\lambda(I_{\alpha'}) = \tilde{\mathscr{C}}_{\beta'}^{\alpha'}$. Therefore, $\tilde{\mathscr{C}}$ is a C^*-pre-family. Finally, by part ii) of the previous lemma, one has $F = AF = J_{\mathrm{id}}F$. By remark 1.17 ii), we obtain $\tilde{\mathscr{C}}_{\mathrm{id}}^{\mathrm{id}}F = \mathscr{C}_{\mathrm{id}}^{\mathrm{id}}\lambda(I_{\mathrm{id}})F = F$. □

Lemma 2.41. *If $\mathscr{C}\lambda(\mathscr{A}) \subset \mathscr{C}$, then $\mathscr{C} \circledcirc \hat{\mathscr{S}} = \tilde{\mathscr{C}} \circledcirc \hat{\mathscr{S}}$.*

Proof. Let $\beta \in \mathrm{PAut}(B)$ and $\alpha, \alpha', \gamma \in \mathrm{PAut}(A)$ such that $\alpha \curlyvee \alpha'$. By part i) of lemma 2.39, one has

$$\mathscr{C}^{\alpha}_{\beta} \otimes \hat{\mathscr{S}}^{\gamma}_{\alpha'} = \mathscr{C}^{\alpha}_{\beta} \otimes \hat{\mathscr{S}}^{\gamma}_{\alpha'}\, \rho(I_{\alpha'}) = \mathscr{C}^{\alpha}_{\beta}\, \lambda(I_{\alpha'}) \otimes \hat{\mathscr{S}}^{\gamma}_{\alpha'}.$$

The inclusion $I_{\alpha'} \subset \mathscr{A}_{\alpha'^*\alpha'}$ and the assumption on $\mathscr{C}$ imply that the product $\mathscr{C}^{\beta}_{\alpha}\, \lambda(I_{\alpha'})$ is contained in $\mathscr{C}^{\beta}_{\alpha\alpha'^*\alpha'}$. Since $\alpha \curlyvee \alpha'$, one has $\alpha\alpha'^*\alpha' \leq \alpha'$. Thus, $\mathscr{C}^{\beta}_{\alpha}\, \lambda(I_{\alpha'}) \subset \mathscr{C}^{\beta}_{\alpha'}$. Since $\lambda(I_{\alpha'}) = \lambda(I_{\alpha'})^2$, we obtain $\mathscr{C}^{\alpha}_{\beta} \otimes \hat{\mathscr{S}}^{\gamma}_{\alpha'} \subset \tilde{\mathscr{C}}^{\beta}_{\alpha'} \otimes \hat{\mathscr{S}}^{\gamma}_{\alpha'}$ which implies the claim. ☐

Proposition 2.42. *i) Let $(\mathscr{C}, \hat{\delta})$ be a right coaction of $(\hat{\mathscr{S}}, \hat{\Delta})$ on a C^*-pre-family. Then the sub-family $\tilde{\mathscr{C}} \subset \mathscr{C}$ given by $\tilde{\mathscr{C}}^{\alpha}_{\beta} := \mathscr{C}^{\alpha}_{\beta}\lambda(I_{\alpha})$ is a non-degenerate C^*-pre-family. If $\mathscr{C} = \mathscr{C}\lambda(\mathscr{A})$, the coaction $\hat{\delta}$ restricts to a regular coaction on $\tilde{\mathscr{C}}$.*

ii) Let $(C, \pi, \hat{\delta})$ be a coaction of $(\hat{\mathscr{S}}, \hat{\Delta})$ such that $C = \overline{\mathrm{span}}_{\alpha}\, \mathscr{C}ov_{\alpha}(C, \pi)$. Then the subspace $\tilde{C} := \overline{\mathrm{span}}_{\alpha}\big(\mathscr{C}ov_{\alpha}(C, \pi)\pi(I_{\alpha})\big)$ of C is a non-degenerate C^*-subalgebra, and the restrictions of π and $\hat{\delta}$ define a coaction of $(\hat{\mathscr{S}}, \hat{\Delta})$ on $\tilde{C}$.

Proof. Part i) and ii) follow from lemmas 2.40 and 2.41, respectively. ☐

2.3.3 Coaction unitaries

Coaction unitaries are closely related to pseudo-multiplicative unitaries on C^*-modules. The construction of the legs of a decomposably regular pseudo-multiplicative unitary carries over to coaction unitaries, giving rise to C^*-pre-families, C^*-algebras and coactions thereon. In the context of pseudo-Kac systems, coaction unitaries will give rise to dual coactions on reduced crossed products.

There are two variants of coaction unitaries – right and left ones. The former ones give rise to right coactions of the left leg of a pseudo-multiplicative unitary, the latter give rise to left coactions of the right leg. We focus on right coaction unitaries and summarise the corresponding results for left ones at the end of the subsection.

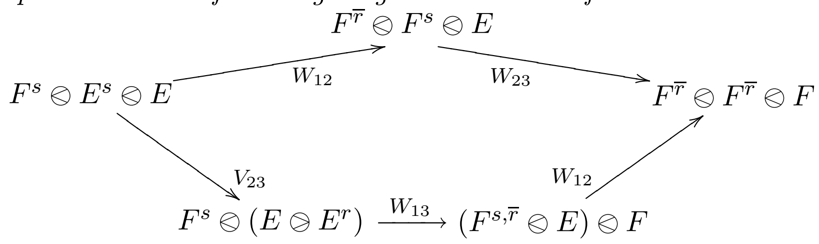
Right coaction unitaries Let E be a C^*-module over a C^*-algebra A with commuting non-degenerate representations π_r and π_s of A and let $V\colon E^s \otimes E \to E \otimes E^r$ be a pseudo-multiplicative unitary.

Let F be a right C^*-module over A^{op} and let $\pi_{\overline{r}}\colon A^{op} \to L_{A^{op}}(F)$ and $\pi_s\colon A \to L_{A^{op}}(F)$ be two commuting non-degenerate representations. Then one has the following analogue of lemma 2.5.

Lemma 2.43. *If $W \in L_{A^{op}}(F^s \otimes E, F^{\overline{r}} \otimes F)$ satisfies*

$$W\pi_{s2} = \pi_{s2}W, \qquad W\pi_{r2} = \pi_{s1}W, \qquad W\pi_{\overline{r}1} = \pi_{\overline{r}2}W,$$

then all operators in the following diagram are well-defined.



□

Definition 2.44. *A unitary* $W: F^s \otimes E \to F^{\overline{r}} \otimes F$ *is a* right coaction unitary *for the pseudo-multiplicative unitary* V *if it satisfies the assumption of the previous lemma and the diagram above commutes.*

Proposition 2.45. *Let* W *be a right coaction unitary for* V *and assume that* V *is left decomposable and decomposably left regular.*

i) The sub-family $\hat{\mathscr{S}}(W) := \langle F|_2 \, W \, |\mathscr{E}^r\rangle_2 \subset \mathscr{L}_{\mathrm{id}}(F^s)$ *given by the composition*

$$F^s \cong F^s \otimes A \xrightarrow{|\mathscr{E}^r\rangle_2} F^s \otimes E^r \xrightarrow{W} F^{\overline{r},s} \otimes F \xrightarrow{\langle F|_2} F^{\overline{r},s} \otimes A \cong F^s$$

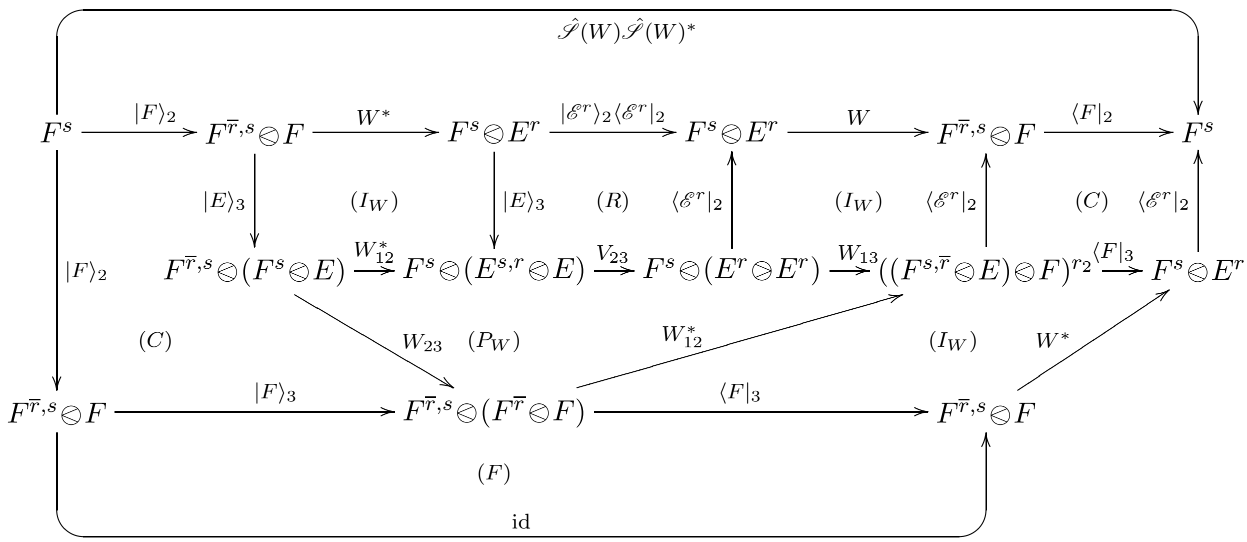
is a non-degenerate C^**-pre-family.*

ii) One has $\pi_s(\mathscr{C}ov(A))\hat{\mathscr{S}}(W) = \hat{\mathscr{S}}(W)$.

iii) The map $T \mapsto W^*(1_{F^{\overline{r}}} \otimes T)W$ *defines a regular coaction* $\hat{\delta}$ *of* $(\hat{\mathscr{S}}, \hat{\Delta})$ *on* $\hat{\mathscr{S}}(W)$.

iv) If the representation $\pi_{\overline{r}}$ *is faithful, then* $\big(\hat{\mathscr{S}}(W), \hat{\delta}\big)$ *is a regular* $(\hat{\mathscr{S}}, \hat{\Delta})$*-family.*

Proof. i) The proof is completely analogous to the one of proposition 2.19. We prove $\hat{\mathscr{S}}(W)\hat{\mathscr{S}}(W)^* = \hat{\mathscr{S}}(W)^*$ by means of the following diagram.



The individual cells commute for the following reasons: (C) by inspection, (I_W) intertwining properties of W, (R) regularity of V, (P_W) pentagon equation for W, (F) F is full and $\pi_{\overline{r}}$ non-degenerate.

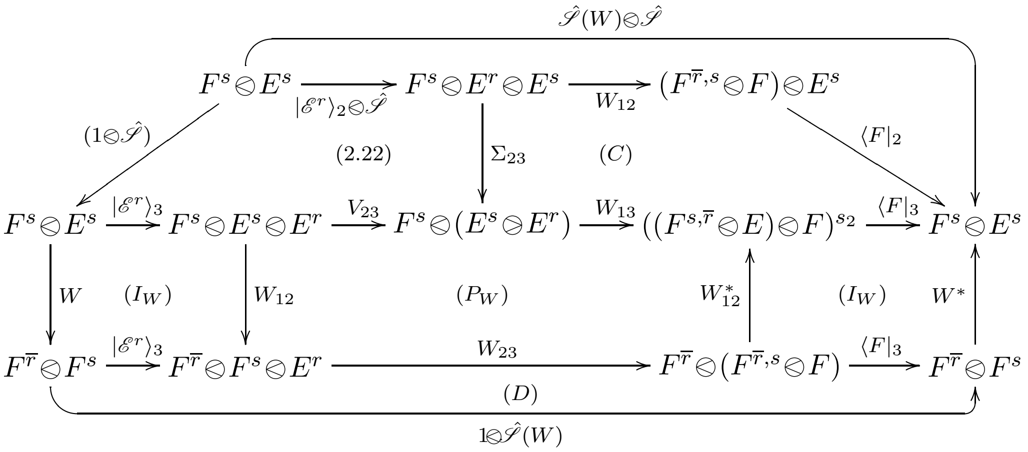
Furthermore, one has $\overline{\operatorname{span}}_\alpha \hat{\mathscr{S}}(W)^\alpha F = \langle F|_2 W |E\rangle_2 F = \pi_{\overline{r}}(\langle F|F\rangle)F = F$.

ii) Since $\pi_{s1} W = W \pi_{r2}$, the product $\lambda(\mathscr{C}ov(A))\hat{\mathscr{S}}(W)$ is equal to

$$\pi_{s1}(\mathscr{C}ov(A))\,\langle F|_2\, W\, |\mathscr{E}^r\rangle_2 = \langle F|_2\, W \cdot \pi_{r2}(\mathscr{C}ov(A))|\mathscr{E}^r\rangle_2.$$

By lemma 1.51, this equals $\langle F|_2\, W\, |\mathscr{E}^r\rangle_2 = \hat{\mathscr{S}}(W)$.

iii) We prove $W^*(1 \otimes \hat{\mathscr{S}}(W))W(1 \otimes \hat{\mathscr{S}}) = \hat{\mathscr{S}}(W) \otimes \hat{\mathscr{S}}$ by means of the following diagram.



The individual cells commute for the following reasons: (I_W) intertwining properties of W, (P_W) pentagon equation for W, (D) definition of $\hat{\mathscr{S}}(W)$, (C) by inspection, (2.22) by the second diagram in the proof of proposition 2.22.

Therefore, the map $T \mapsto W^*(1 \otimes T)W$ defines a non-degenerate morphism $\hat{\delta}\colon \hat{\mathscr{S}}(W) \to \mathscr{M}(\hat{\mathscr{S}}(W) \otimes \hat{\mathscr{S}})$. The fact that $\hat{\delta}$ is coassociative is proved as part iii) of proposition 2.22. Finally, the coaction $\hat{\delta}$ is regular because

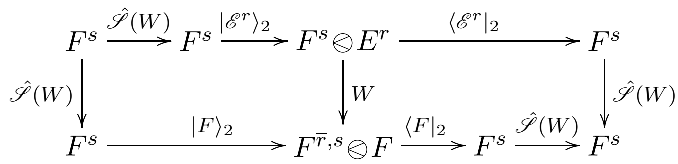
$$\hat{\mathscr{S}}(W)^\beta = \langle F|_2\, W\, |\mathscr{E}^r_\beta\rangle_2 = \langle F|_2\, W\, |\mathscr{E}^r_\beta J_\beta\rangle_2 = \hat{\mathscr{S}}(W)^\beta \pi_s(J_\beta), \quad \beta \in \mathrm{PAut}(A),$$

where $J_\beta = \langle \mathscr{E}^r_\beta | \mathscr{E}^r_\beta \rangle$.

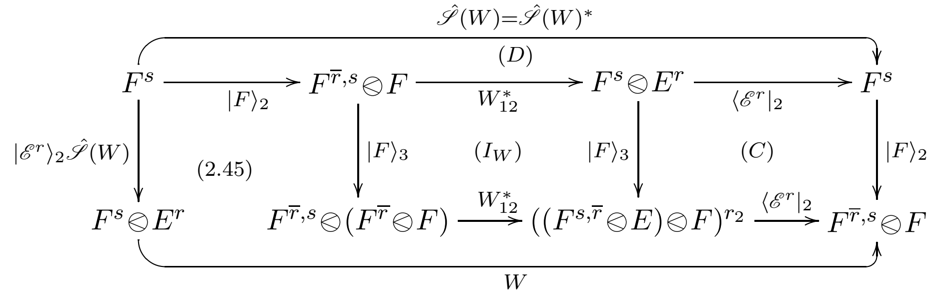
iv) If the representation $\pi_{\overline{r}}$ on F is faithful, the map $T \mapsto 1_{F^{\overline{r}}} \otimes T$ is injective. Therefore, the coaction $\hat{\delta}$ is injective as well. □

For later use we note the following analogue of lemma 2.21.

Lemma 2.46. *The following diagram commutes.*

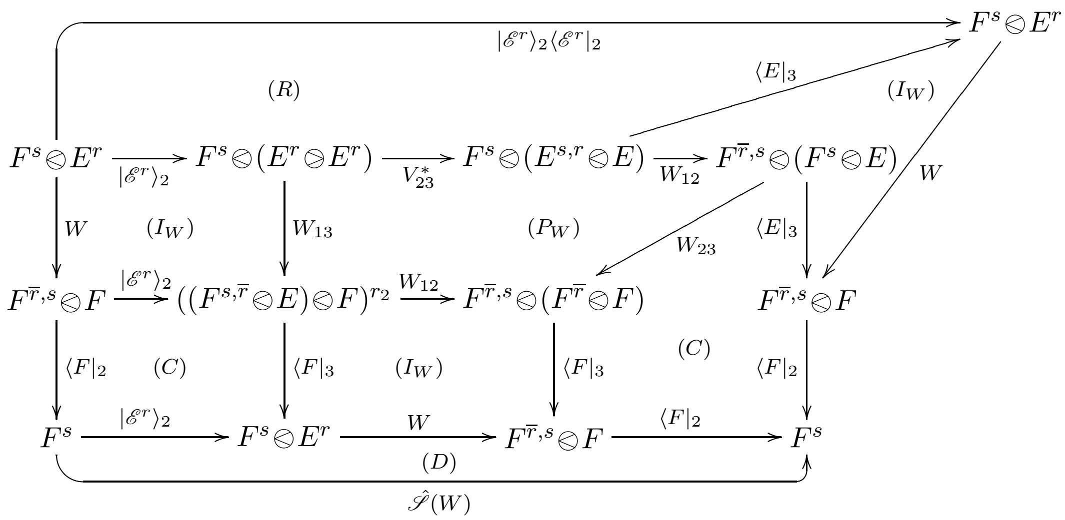


Proof. For the left square, consider the following diagram.



The individual cells commute for the following reasons: (I_W) intertwining property of W, (C) by inspection, (D) definition, (2.45) by the commutative diagram in the proof of the previous proposition.

For the right square, consider the following diagram.



The individual cells commute for the following reasons: (I_W) intertwining properties of W, (C) by inspection, (D) definition, (P_W) pentagon equation for W, (R) decomposable left regularity of V. $\square$

If V is not left decomposable, one still has the following results.

Proposition 2.47. *Let W be a right coaction unitary for V. Then the subspace $\widehat{S}(W) := \langle F|_2 \, W \, |E\rangle_2 \subset L_{A^{op}}(F)$ is a non-degenerate C^*-algebra, and the representation $\pi_s\colon A \to L_{A^{op}}(F)$ restricts to a non-degenerate $*$-homomorphism $\pi_s \to M\big(\widehat{S}(W)\big)$. One has $W\,|E\rangle_2\,\widehat{S}(W) = |F\rangle_2\,\widehat{S}(W)$ and $\widehat{S}(W)\,\langle F|_2\,W = \widehat{S}(W)\,\langle E|_2$.* $\square$

Left coaction unitaries Let F be a right C^*-module over A^{op} and let $\pi_{\overline{s}} \colon A^{op} \to L_{A^{op}}(F)$ and $\pi_r \colon A \to L_{A^{op}}(F)$ be two commuting non-degenerate representations. Then one has the following analogue of lemma 2.5.

Lemma 2.48. *If $W \in L_{A^{op}}(F \ogreaterthan F^{\overline{s}}, E \ogreaterthan F^r)$ satisfies*

$$W\pi_{r1} = \pi_{r1}W, \qquad W\pi_{r2} = \pi_{s1}W, \qquad W\pi_{\overline{s}1} = \pi_{\overline{s}2}W,$$

then all operators in the following diagram are well-defined.

$$\begin{array}{ccccc} & & E \ogreaterthan F^r \ogreaterthan F^{\overline{s}} & & \\ & \overset{W_{12}}{\nearrow} & & \overset{W_{23}}{\searrow} & \\ F \ogreaterthan F^{\overline{s}} \ogreaterthan F^{\overline{s}} & & & & E \ogreaterthan E^r \ogreaterthan F^r \\ \quad\searrow^{W_{23}} & & & & \nearrow^{V_{12}}\quad \\ & F \ogreaterthan (E \ogreaterthan F^{r,\overline{s}}) & \xrightarrow{W_{13}} & (E^s \olessthan E) \ogreaterthan F^r & \end{array}$$

□

Definition 2.49. *A unitary $W \colon F \ogreaterthan F^{\overline{s}} \to E \ogreaterthan F^r$ is a* left coaction unitary *for V if it satisfies the assumptions of the previous lemma.*

So, W is a left coaction unitary for V if and only if $W^{op} := \Sigma W^* \Sigma$ is a right coaction unitary for $V^{op} = \Sigma V^* \Sigma$.

Proposition 2.50. *Let V be a decomposably right regular pseudo-multiplicative unitary and let W be a left coaction unitary for V.*

i) The sub-family $\mathscr{S}(W) := \langle \mathscr{E}^r|_1 W |F\rangle_1 \subset \mathscr{L}_{\mathrm{id}}(F^r)$ given by the composition

$$F^r \cong A \ogreaterthan F^{\overline{s},r} \xrightarrow{|F\rangle_1} F \ogreaterthan F^{\overline{s},r} \xrightarrow{W} E^s \ogreaterthan F^r \xrightarrow{\langle \mathscr{E}^r|_1} A \ogreaterthan F^r \cong F^r$$

is a non-degenerate C^*-pre-family.

ii) One has $\pi_r(\mathscr{C}ov(A))\mathscr{S}(W) = \mathscr{S}(W)$.

iii) The map $T \mapsto W(T \ogreaterthan 1_{F^{\overline{s}}})W^*$ defines a coaction δ of $(\mathscr{S}, \Delta)$ on $\mathscr{S}(W)$.

iv) If the representation $\pi_{\overline{s}}$ is faithful, $\big(\mathscr{S}(W), \delta\big)$ is a regular $(\mathscr{S}, \Delta)$-family.

□

Proposition 2.51. *Let W be a left coaction unitary for V. The subspace $S(W) := \langle E|_1 W |F\rangle_1 \subset L_{A^{op}}(F)$ is a non-degenerate C^*-pre-family, and the representation $\pi_r \colon A \to L_A(F)$ restricts to a non-degenerate $*$-homomorphism $\pi_r \to M(\widehat{S}(W))$.*

□

2.4 Pseudo-Kac systems on C^*-modules

In this section, we introduce the notion of a pseudo-Kac system on C^*-modules and use it as a framework to construct reduced crossed products for coactions of Hopf C^*-pre-families and to generalise the Baaj-Skandalis duality theorem.

The definition of a pseudo-Kac system builds on the notion of a Kac system on a Hilbert space introduced by Saad Baaj and Georges Skandalis [3], but brings in a new level of complexity: the underlying Hilbert space, the multiplicative unitary and the symmetry comprising a Kac system are now replaced by families of four C^*-bimodules, six unitaries and four symmetries. The involved system of axioms calls for a careful motivation. In various examples, the families of C^*-modules and operators arise from one single module and two additional maps after completion with respect to several different inner products. The algebraic essence of the underlying triples and the transition to the C^*-algebraic setting are discussed in the first subsection. The definition of a pseudo-Kac system and several remarks follow.

Like a decomposable pseudo-multiplicative unitary, a decomposable pseudo-Kac system has two legs which form a "dual pair" of Hopf C^*-families. The reduced crossed product constructions given by Saad Baaj and Georges Skandalis carry over to coactions of these Hopf C^*-families, as does the duality theorem. In the adaptation to the present situation, much attention has to be paid to the choice of the right member of the family of C^*-bimodules and of the family of unitaries in each equation – each one plays its own rôle. This phenomenon seems to be inherent to the problem and not to our axiomatisation: an ad-hoc proof of the duality theorem adapted to locally compact groupoids already reaches a comparable complexity.

The main example for the theory developed here is provided by locally compact groupoids. At the end of this section, we present another example of a pseudo-Kac system which is not decomposable.

In the discussion of pseudo-multiplicative unitaries in the previous section, we considered both left and right Hopf C^*-families and left and right coactions. Following [3], we restrict to right coactions and hence also to right Hopf C^*-families. At the moment, we do not see whether the simultaneous consideration of left and right coactions would make the treatment more transparent.

It would be nice if the axiom system could be simplified. We suspect that in the context of measured quantum groupoids and von Neumann algebras, the corresponding notion of a pseudo-Kac system should consist of just one pseudo-multiplicative unitary and one additional symmetry – there, the problems which force us to introduce families of operators should be amenable to the Tomita-Takesaki theory and the theory of operator-valued weights.

2.4.1 Motivation

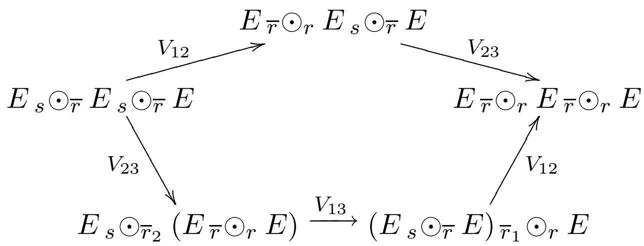
As a motivation for the axiom system, we first present its algebraic essence as a "toy definition".

The algebraic setting

An algebraic pseudo-Kac system consists of a pseudo-multiplicative isomorphism and an additional "antipode" subject to several conditions. This isomorphism is the algebraic counterpart of the notion of a pseudo-multiplicative unitary on a C^*-module.

> **Preliminary definition** *A* pseudo-multiplicative isomorphism *consists of an algebra A, a vector space E with pairwise commuting representations $\pi_{\overline{r}}\colon A^{op} \to L(E)$ and $\pi_r, \pi_s\colon A \to L(E)$, and an isomorphism $V\colon E\,{}_s{\odot}_{\overline{r}}\,E \to E\,{}_{\overline{r}}{\odot}_r\,E$ which intertwines the representations $\pi_{\overline{r}}, \pi_r, \pi_s$ as follows*
>
> $$V\pi_{r1} = \pi_{r1}V, \quad V\pi_{s2} = \pi_{s2}V, \quad V\pi_{\overline{r}1} = \pi_{\overline{r}2}V, \quad V\pi_{r2} = \pi_{s1}V,$$
>
> *and satisfies the pentagon equation $V_{12}V_{13}V_{23} = V_{23}V_{12}$.*

One easily checks that the intertwining conditions imply that the maps in the following diagram are well-defined.



The pentagon equation holds exactly if this diagram commutes.

> **Preliminary definition** *An* algebraic pseudo-Kac system (A, E, U, V) *consists of*
>
> *i) an algebra A and a vector space E with representations $\pi_{\overline{r}}, \pi_{\overline{s}}\colon A^{op} \to L(E)$ and $\pi_r, \pi_s\colon A \to L(E)$ all of which commute pairwise,*
>
> *ii) an involutive isomorphism $U\colon E \to E$ such that $U\pi_{\overline{r}} = \pi_{\overline{s}}U$ and $U\pi_r = \pi_s U$,*
>
> *iii) a pseudo-multiplicative isomorphism $V : E\,{}_s{\odot}_{\overline{r}}\,E \to E\,{}_{\overline{r}}{\odot}_r\,E$*
>
> *such that $\pi_{\overline{s}1}V = V\pi_{\overline{s}1}$, $\pi_{\overline{s}2}V = V\pi_{\overline{s}2}$, $(\Sigma(1 \odot U)V)^3 = 1$ and the map*
>
> $$\hat{V} := \Sigma(U \odot 1)V(U \odot 1)\Sigma\colon E\,{}_{\overline{r}}{\odot}_r\,E \to E\,{}_r{\odot}_{\overline{s}}\,E$$
>
> *is a pseudo-multiplicative isomorphism.*

This notion of an algebraic pseudo-Kac system does not include any regularity condition for the multiplicative isomorphism. Note that a useful regularity condition depends on the identification of a natural replacement for the space of maps $K_A(E, A)$, where E is a C^*-module over A.

Examples of algebraic pseudo-Kac systems associated to étale groupoids and to pairs of algebras are presented in subsection 3.2.1 and 2.4.5.

The transition to the C^*-algebraic setting

Formal problems Let us now shift from algebraic pseudo-Kac system to the C^*-algebraic setting. The appropriate counterpart of the notion of a pseudo-multiplicative isomorphism has already been introduced in 2.6. There, one replaced

- the algebra A by a C^*-algebra A and the vector space E by a C^*-module over A,
- the representations π_r, π_s by $*$-representations of A on the C^*-module E,
- the isomorphism $V\colon E\,{}_s{\odot}_{\overline{r}}\,E \to E\,{}_{\overline{r}}{\odot}_r\,E$ by a unitary $V\colon E^s \oslash E \to E \ominus E^r$.

The representation $\pi_{\overline{r}}$ present in the algebraic definition is implicit in the C^*-algebraic notion in form of the right module structure of E over A. Also, the counterpart of the equation $V\pi_{\overline{r}1} = \pi_{\overline{r}2}V$ is implicit in the fact that V is a map of C^*-modules. The remaining intertwining conditions for V with respect to the representations π_r and π_s and the pentagon equation imposed on V are exactly the same in both settings.

To extend a pseudo-multiplicative unitary to a pseudo-Kac system, one should introduce a fourth representation $\pi_{\overline{s}}$ of A^{op} on E and a counterpart of the involutive isomorphism U acting on E. The theory of Kac systems suggests that U should be a unitary map on E. Now, several problems arise:

- Since $\pi_{\overline{r}}$ corresponds to the right module structure of E and U intertwines $\pi_{\overline{r}}$ and $\pi_{\overline{s}}$ in the algebraic setting, one must either have $\pi_{\overline{r}} = \pi_{\overline{s}}$ or introduce another C^*-module F which gets identified with E via U.
- In this case, however, $\hat{V}$ would be a map $E \ominus F \to E \oslash F$ and no longer fit the definition of a pseudo-multiplicative unitary. Similar problems arise in the equation $(\Sigma(1 \otimes U)V)^3 = 1$.
- Let us consider the domain and range of $\hat{V}$ with the definition of pseudo-multiplicative unitaries of C^*-modules. Since the algebraic tensor product gets replaced by the internal tensor product on C^*-modules, π_r should get replaced by a right C^*-module. Thus, an additional C^*-modules appears.

Conceptual problems The purpose of pseudo-Kac systems will be the construction of crossed products for coactions of a dual pair of right Hopf C^*-families, $(\hat{\mathscr{S}}, \hat{\Delta})$ and $(\mathscr{S}, \Delta)$, say. Let δ be a right coaction of $(\mathscr{S}, \Delta)$ on a C^*-algebra C. By analogy to the setting of Hilbert spaces, the corresponding reduced crossed product should be equal to the product $\overline{\operatorname{span}}\big(\delta(C)(1 \otimes \hat{\mathscr{S}})\big)$. However, the internal tensor product $1 \otimes \hat{\mathscr{S}} = (1 \otimes \hat{\mathscr{S}}^{\beta}_{\mathrm{id}})_{\beta}$ is formed of the strictly adjointable operators of $\hat{\mathscr{S}}$ only – for $\alpha \not\leq \mathrm{id}$, the subspace $\hat{\mathscr{S}}^{\beta}_{\alpha}$ does not enter the definition of the reduced crossed product. Thus, in the definition of the reduced crossed product, the factor $1 \otimes \overline{\operatorname{span}}_{\beta}\hat{\mathscr{S}}^{\beta}_{\mathrm{id}}$ is to small – the C^*-algebra $\overline{\operatorname{span}}_{\beta}\hat{\mathscr{S}}^{\beta}_{\mathrm{id}}$ should be replaced by a larger C^*-algebra $\hat{S}_0$ which carries a canonical coaction of $\hat{\mathscr{S}}$. Arguably, this coaction should be the left leg of a coaction unitary which should enter the definition of a pseudo-Kac system. Symmetrically, the consideration of crossed products for coactions of the Hopf C^*-family $(\hat{\mathscr{S}}, \hat{\Delta})$ suggests to introduce an additional coaction unitary.

The example of locally compact groupoids Next, let us consider the fundamental example which shall be covered by the theory of pseudo-Kac systems on C^*-modules. Let G be a locally compact Hausdorff groupoid with left Haar system λ. In subsection 3.2.1, we associate to G a pseudo-multiplicative unitary $V\colon E^s \otimes E \to E \otimes E^r$, where

- E is the C^*-module $L^2(G,\lambda)$ over $A = C_0(G^0)$,
- π_r and π_s are the representations induced by the range and the source map, respectively,
- V is the operator induced by the map $V_0\colon C_c\big(G\,{}_s{\times}_r\,G\big) \to C_c\big(G\,{}_r{\times}_r\,G\big)$ given by $(V_0 f)(x,y) := f(x, x^{-1}y)$ for all $(x,y) \in G\,{}_r{\times}_r\,G$ and $f \in C_c\big(G\,{}_s{\times}_r\,G\big)$.

In analogy to the situation of the Kac system associated to a locally compact group, the additional operator U should be induced by the inversion $\nu : G \to G$ given by $x \mapsto x^{-1}$. This inversion defines a unitary map between the C^*-modules $E = L^2(G,\lambda)$ and $L^2(G,\lambda^{-1}) =: F$. Here, λ^{-1} denotes the right Haar system associated to λ, i.e. the family of measures $\lambda^{-1}_v := \nu^*(\lambda^v)$ on the fibres $G_v, v \in G^0$, of the source map. Put $\pi_{\overline{r}} := \pi_r$ and $\pi_{\overline{s}} := \pi_s$ and use the same notation for the representations on F induced by the range and source map of G, respectively. Now, the crucial point is the following observation:

> *The map $V_0\colon C_c\big(G\,{}_s{\times}_r\,G\big) \to C_c\big(G\,{}_r{\times}_r\,G\big)$ defined above extends to unitaries $F^s \otimes E \to F^r \otimes E$ and $F \otimes F^r \to E \otimes F^r$.*

This result will be proved in section 3.2.

Summary In the definition of C^*-algebraic pseudo-Kac system, one needs to introduce additional C^*-modules and unitaries. This seems to open Pandora's box – if one C^*-algebraic pseudo-Kac systems consists of a collection of C^*-modules and unitaries, the formation of direct sums and external tensor products of such systems might lead to an explosion of the number of C^*-modules and unitaries involved.

2.4.2 Definition and first properties

In this subsection, we introduce the notion of a pseudo-Kac system. As already indicated, it consists of families of C^*-bimodules and families of operators. The defining properties again are families of equations, for which we employ the following notation.

Notation 2.52. *Given two sets $\mathcal{M}, \mathcal{N}$ of homogeneous operators of C^*-bimodules with different domains and different ranges, we denote by $\mathcal{M}\mathcal{N}, \mathcal{M}^*, \mathcal{M} \otimes \mathcal{N}$ and $\mathcal{M} \ominus \mathcal{N}$ the set of all possible compositions, adjoints and internal tensor products that can be formed with elements of $\mathcal{M}$ and $\mathcal{N}$. We say that two families commute if $\mathcal{M}\mathcal{N} = \mathcal{N}\mathcal{M}$. Furthermore, we apply the leg notation to such families in the obvious way. Finally, an equation of the form $\mathcal{M} = 1$, in this context, means that all elements of $\mathcal{M}$ are identity morphisms, but of potentially different objects.*

This notation is used in the following definition. In the remarks ensuing the definition, the corresponding equations are written out in full detail.

Definition 2.53. *A C^*-pseudo-Kac system (A, I, E, π, U, V) consists of*

- *a C^*-algebra A,*
- *a family I of four indices, say $(r, s, \overline{r}, \overline{s})$,*
- *a family E of right C^*-modules E_r, E_s over A and $E_{\overline{r}}, E_{\overline{s}}$ over A^{op},*
- *families of pairwise commuting non-degenerate and faithful $*$-representations π_r^x, π_s^x of A and $\pi_{\overline{r}}^x, \pi_{\overline{s}}^x$ of A^{op} on E_x defined for all $x \in I$, except for $\pi_x^x, x \in I$,*
- *a family U of two pairs of mutually inverse unitaries*

  $$U^{\overline{r}}\colon E_{\overline{r}} \to E_{\overline{s}}, \quad U^{\overline{s}}\colon E_{\overline{s}} \to E_{\overline{r}}, \quad U^r\colon E_r \to E_s, \quad U^s\colon E_s \to E_r,$$

  where $U^{\overline{s}} = (U^{\overline{r}})^*$ and $U^s = (U^r)^*$,
- *a family V of unitaries*

  $$\begin{aligned} V^{ss}&\colon E_s \ominus E_s^{\overline{r}} \to E_{\overline{r}} \ominus E_s^r, & V^{\overline{s}\overline{r}}&\colon E_{\overline{s}}^s \otimes E_{\overline{r}} \to E_{\overline{s}}^{\overline{r}} \otimes E_r, \\ V^{\overline{r}\overline{r}}&\colon E_{\overline{r}}^s \otimes E_{\overline{r}} \to E_{\overline{r}} \ominus E_{\overline{r}}^r, & V^{sr}&\colon E_s \ominus E_r^{\overline{r}} \to E_s^{\overline{r}} \otimes E_r, \\ V^{r\overline{r}}&\colon E_r^s \otimes E_{\overline{r}} \to E_r^{\overline{r}} \otimes E_r, & V^{s\overline{s}}&\colon E_s \ominus E_{\overline{s}}^{\overline{r}} \to E_{\overline{r}} \ominus E_{\overline{s}}^r, \end{aligned}$$

subject to the following conditions:

i) *the family U satisfies $U\pi_r U = \pi_s$ and $U\pi_{\overline{r}}U = \pi_{\overline{s}}$,*

ii) *V and U intertwine the families of representations $\pi_{\overline{r}}, \pi_r, \pi_{\overline{s}}, \pi_s$ as follows:*

$$\begin{aligned} V\pi_{\overline{s}1} &= \pi_{\overline{s}1}V, & V\pi_{\overline{s}2} &= \pi_{\overline{s}2}V, & V\pi_{r1} &= \pi_{r1}V, \\ V\pi_{s2} &= \pi_{s2}V, & V\pi_{\overline{r}1} &= \pi_{\overline{r}2}V, & V\pi_{r2} &= \pi_{s1}V, \end{aligned}$$

iii) *the families V and $\hat{V} := \Sigma U_1 V U_1 \Sigma$ both satisfy the* pentagon equations

$$V_{12}V_{13}V_{23} = V_{23}V_{12}, \qquad \hat{V}_{12}\hat{V}_{13}\hat{V}_{23} = \hat{V}_{23}\hat{V}_{12},$$

iv) *the families U and V satisfy the* irreducibility equation $(\Sigma U_2 V)^3 = 1$*;*

v) *The pseudo-multiplicative unitaries $V^{\overline{r}\overline{r}}$ and $\hat{V}^{rr}$, the latter being given by the composition*

$$\hat{V}^{rr} \colon E_r^{\overline{r}} \olessthan E_r \xrightarrow{U_1^r\Sigma} E_s \ogreaterthan E_r^{\overline{r}} \xrightarrow{V^{sr}} E_s^{\overline{r}} \olessthan E_r \xrightarrow{\Sigma U_1^s} E_r \ogreaterthan E_r^{\overline{s}},$$

are regular.

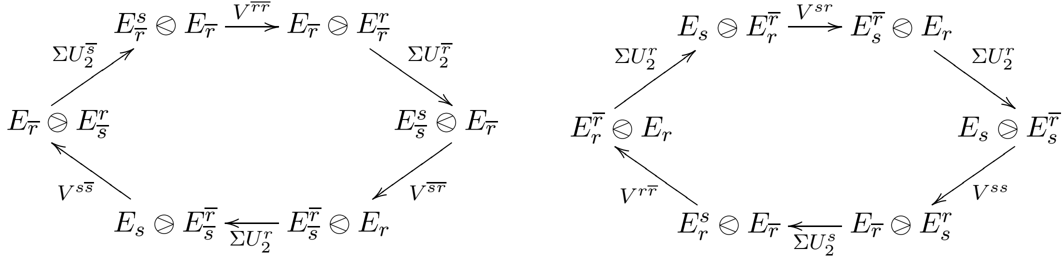
Of particular interest to us are decomposable pseudo-Kac systems. To these, the techniques of the previous chapter and the previous section can be applied. In particular, we can associate Hopf C^*-families, consider coactions of these Hopf C^*-families on C^*-algebras and – in the following subsections – turn to crossed product constructions and duality theorems.

Definition 2.54. *A pseudo-Kac system (A, I, π, E, U, V) is* decomposable *if the C^*-bimodules $E_{\overline{r}}^r, E_{\overline{r}}^s, E_r^{\overline{r}}$ and $E_r^{\overline{s}}$ are decomposable, and if the pseudo-multiplicative unitaries $V^{\overline{r}\overline{r}}$ and $\hat{V}^{rr}$ are decomposably left and right regular.*

Remarks 2.55. i) One has $UU = 1$.

ii) The pentagonal equation for the family V is equivalent to the fact that each of the eight diagrams in figure 2 commutes. The fact that each map in these diagrams is well-defined follows from the intertwining properties imposed on the family V by condition ii).

iii) The last condition is equivalent to the fact that the following two diagrams commute.



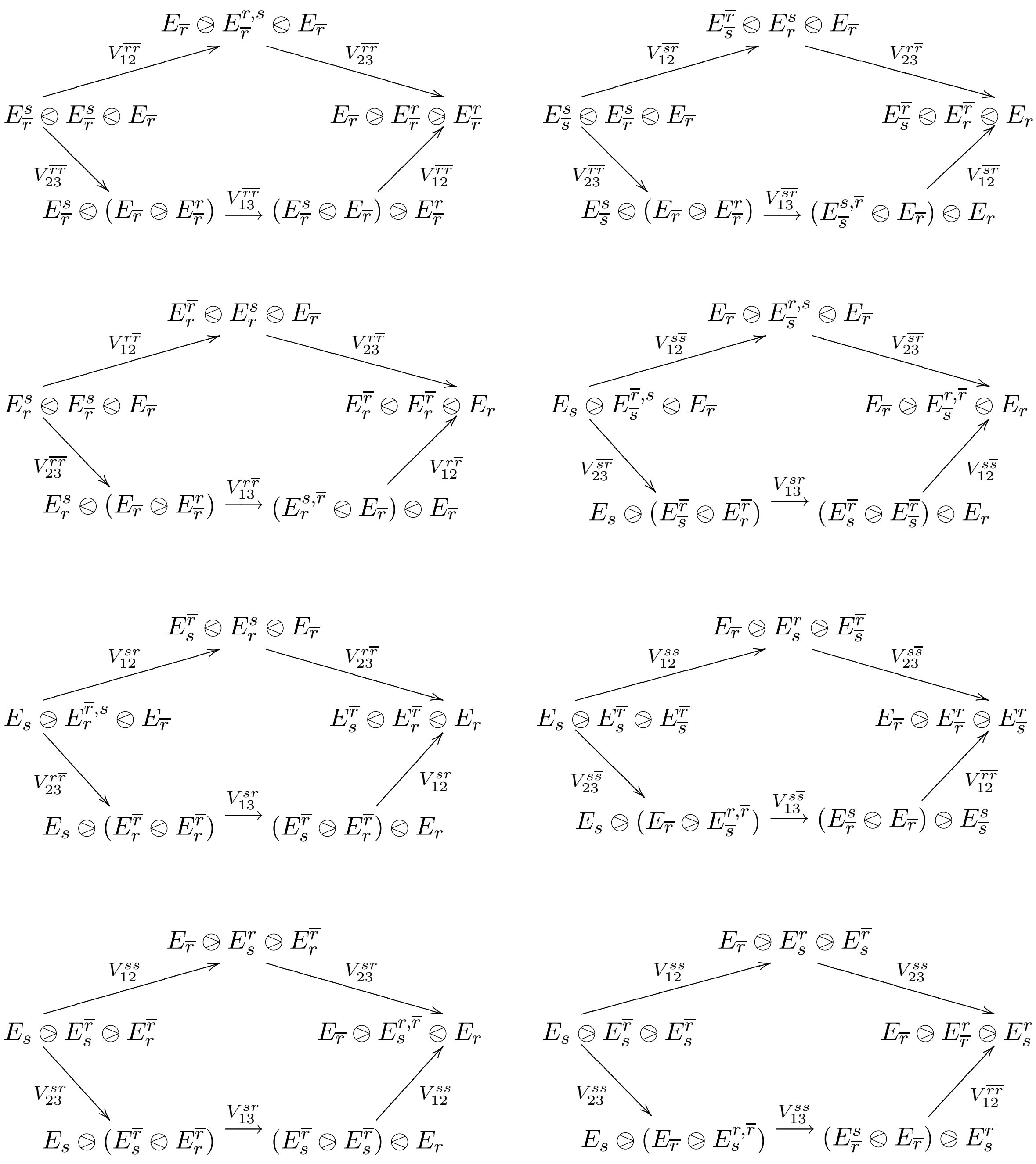


Figure 2.1: The eight pentagon diagrams of a C^*-pseudo-Kac system

iv) The operator $V^{\overline{r}\overline{r}}$ is a pseudo-multiplicative unitary, and V^{ss} and $V^{r\overline{r}}$ are a left and a right coaction unitary for it.

v) The definition of the family $\hat{V}$ raises some ambiguities concerning the notation of the indices. The indexing does not interfere with the formulation of the axioms, but is very important in the remainder of the section. It will be fixed in the next paragraph.

The dual pseudo-Kac system

The notion of a pseudo-Kac system has a rich symmetry: out of one pseudo-Kac system, several related ones can be constructed. As in the case of pseudo-multiplicative unitaries, this symmetry will be used to transfer results about one leg of a decomposable pseudo-Kac system to the other leg.

The most important symmetry is the duality of pseudo-Kac systems which is explained in the following.

Proposition 2.56. *Let $\mathcal{K} = (A, I, \pi, E, U, V)$ be a pseudo-Kac system on C^*-modules. Put $\hat{I} = (\hat{r}, \hat{s}, \hat{\overline{r}}, \hat{\overline{s}})$ where $\hat{r} = \overline{s}$, $\hat{s} = \overline{r}$, $\hat{\overline{r}} = r$, $\hat{\overline{s}} = s$. We index the family $\hat{V} = \Sigma U_1 V U_1 \Sigma$ as follows:*

$$\begin{aligned}
\hat{V}^{sr} &:= \widehat{V^{ss}} \colon E^{\overline{r}}_{s} \olessthan E_r \xrightarrow{U_1^r \Sigma} E_s \ogreaterthan E^{\overline{r}}_{s} \xrightarrow{V^{ss}} E_{\overline{r}} \ogreaterthan E^{r}_{s} \xrightarrow{\Sigma U_1^{\overline{r}}} E^{r}_{s} \olessthan E_{\overline{s}},\\
\hat{V}^{\overline{r}\overline{s}} &:= \widehat{V^{\overline{r}\overline{r}}} \colon E_{\overline{r}} \ogreaterthan E^{r}_{\overline{s}} \xrightarrow{U_1^{\overline{s}} \Sigma} E^{s}_{\overline{r}} \olessthan E_{\overline{r}} \xrightarrow{V^{\overline{r}\overline{r}}} E_{\overline{r}} \ogreaterthan E^{r}_{\overline{r}} \xrightarrow{\Sigma U_1^{\overline{r}}} E^{r}_{\overline{r}} \olessthan E_{\overline{s}},\\
\hat{V}^{\overline{r}s} &:= \widehat{V^{r\overline{r}}} \colon E_{\overline{r}} \ogreaterthan E^{r}_{s} \xrightarrow{U_1^{s} \Sigma} E^{s}_{r} \olessthan E_{\overline{r}} \xrightarrow{V^{r\overline{r}}} E^{\overline{r}}_{r} \olessthan E_{r} \xrightarrow{\Sigma U_1^{r}} E_{r} \ogreaterthan E^{\overline{s}}_{s},\\
\hat{V}^{\overline{r}\overline{r}} &:= \widehat{V^{\overline{s}\overline{r}}} \colon E_{\overline{r}} \ogreaterthan E^{r}_{\overline{r}} \xrightarrow{U_1^{\overline{r}} \Sigma} E^{s}_{\overline{s}} \olessthan E_{\overline{r}} \xrightarrow{V^{\overline{s}\overline{r}}} E^{\overline{r}}_{\overline{s}} \olessthan E_{r} \xrightarrow{\Sigma U_1^{\overline{s}}} E_{r} \ogreaterthan E^{\overline{s}}_{\overline{r}},\\
\hat{V}^{rr} &:= \widehat{V^{sr}} \colon E^{\overline{r}}_{r} \olessthan E_r \xrightarrow{U_1^r \Sigma} E_s \ogreaterthan E^{\overline{r}}_{r} \xrightarrow{V^{sr}} E^{\overline{r}}_{s} \olessthan E_{r} \xrightarrow{\Sigma U_1^{s}} E_{r} \ogreaterthan E^{\overline{s}}_{r},\\
\hat{V}^{\overline{s}r} &:= \widehat{V^{s\overline{s}}} \colon E^{\overline{r}}_{\overline{s}} \olessthan E_r \xrightarrow{U_1^r \Sigma} E_s \ogreaterthan E^{\overline{r}}_{\overline{s}} \xrightarrow{V^{s\overline{s}}} E_{\overline{r}} \ogreaterthan E^{r}_{\overline{s}} \xrightarrow{\Sigma U_1^{\overline{r}}} E^{r}_{\overline{s}} \olessthan E_{\overline{s}}.
\end{aligned}$$

Then $\hat{\mathcal{K}} := (A^{op}, \hat{I}, \pi, E, U, \hat{V})$ is a C^*-pseudo-Kac-system as well.

Proof. We check the axioms one after the other. Axiom i) is satisfied because

$$U\pi_{\hat{r}}U = U\pi_{\overline{s}}U = \pi_{\overline{r}} = \pi_{\hat{s}}, \qquad U\pi_{\hat{\overline{r}}}U = U\pi_{\overline{r}}U = \pi_{\overline{s}} = \pi_{\hat{\overline{s}}}.$$

To check axiom ii), we use a self-explanatory column-notation:

$$\hat{V} \cdot \begin{Bmatrix} \pi_{\hat{\overline{s}}1} \\ \pi_{\hat{\overline{s}}2} \\ \pi_{\hat{r}1} \\ \pi_{\hat{s}2} \\ \pi_{\hat{\overline{r}}1} \\ \pi_{\hat{r}2} \end{Bmatrix} = \Sigma U_1 V U_1 \Sigma \cdot \begin{Bmatrix} \pi_{s1} \\ \pi_{s2} \\ \pi_{\overline{s}1} \\ \pi_{\overline{r}2} \\ \pi_{r1} \\ \pi_{\overline{s}2} \end{Bmatrix} = \Sigma U_1 V \cdot \begin{Bmatrix} \pi_{s2} \\ \pi_{r1} \\ \pi_{\overline{s}2} \\ \pi_{\overline{s}1} \\ \pi_{r2} \\ \pi_{\overline{r}1} \end{Bmatrix} \cdot U_1 \Sigma =$$

$$= \Sigma U_1 \cdot \begin{Bmatrix} \pi_{s2} \\ \pi_{r1} \\ \pi_{\overline{s}2} \\ \pi_{\overline{s}1} \\ \pi_{s1} \\ \pi_{\overline{r}2} \end{Bmatrix} \cdot V U_1 \Sigma = \begin{Bmatrix} \pi_{s1} \\ \pi_{s2} \\ \pi_{\overline{s}1} \\ \pi_{\overline{r}2} \\ \pi_{r2} \\ \pi_{\overline{r}1} \end{Bmatrix} \cdot \Sigma U_1 V U_1 \Sigma = \begin{Bmatrix} \pi_{\hat{\overline{s}}1} \\ \pi_{\hat{\overline{s}}2} \\ \pi_{\hat{r}1} \\ \pi_{\hat{s}2} \\ \pi_{\overline{r}2} \\ \pi_{s1} \end{Bmatrix} \cdot \hat{V}.$$

For axiom iii), note that the bidual family $\hat{\hat{V}}$ is given by

$$\Sigma U_1 \hat{V} U_1 \Sigma = \Sigma U_1 \Sigma U_1 V U_1 \Sigma U_1 \Sigma = U_2 U_1 V U_1 U_2.$$

It is easy to see that the family $U_2 U_1 V U_1 U_2$ satisfies the pentagon equation if V does. The irreducibility condition iv) holds because

$$(\Sigma U_2 \hat{V})^3 = (\Sigma U_2 \Sigma U_1 V U_1 \Sigma)^3 = (V \Sigma U_2)^3 = U_2 \Sigma (\Sigma U_2 V)^3 \Sigma U_2 = U_2 \Sigma \cdot \Sigma U_2 = 1.$$

Finally, the pseudo-multiplicative unitaries

$$\hat{V}^{\overline{\hat{r}\hat{r}}} = \hat{V}^{rr} = \widehat{V^{sr}} \quad \text{and} \quad \widehat{\hat{V}^{\hat{s}\hat{r}}} = \widehat{\hat{V}^{\overline{rs}}} = \widehat{\widehat{V^{\overline{rr}}}} = U_2 U_1 V^{\overline{rr}} U_1 U_2$$

are regular by assumption on $\mathcal{K}$. ☐

Definition 2.57. *The pseudo-Kac system $\hat{\mathcal{K}}$ is the* dual *of $\mathcal{K}$.*

Observe that the operator $\hat{V}^{rr}$ is a pseudo-multiplicative unitary, and $\hat{V}^{\overline{rr}}$ and $\hat{V}^{\overline{s}r}$ are a left and a right coaction unitary for it.

The bidual pseudo-Kac system $\hat{\hat{\mathcal{K}}}$ is not equal to the initial pseudo-Kac $\mathcal{K}$, but isomorphic in a suitable sense. It is given by $\hat{\hat{\mathcal{K}}} = (A, \hat{\hat{I}}, \pi, E, U, \hat{\hat{V}})$, where the index set $\hat{\hat{I}} = (\hat{\hat{r}}, \hat{\hat{s}}, \hat{\hat{\overline{r}}}, \hat{\hat{\overline{s}}})$ is given by $\hat{\hat{r}} = s$, $\hat{\hat{r}} = s$, $\hat{\hat{\overline{r}}} = \overline{s}$, $\hat{\hat{\overline{s}}} = \overline{r}$, and the family $\hat{\hat{V}}$ is given by $\hat{\hat{V}} = U_1 U_2 V U_2 U_1$.

It is easy to see that the bidual of the bidual is strictly equal to the initial pseudo-Kac system. The *predual pseudo-Kac system* $\tilde{\mathcal{K}} = \hat{\hat{\hat{\mathcal{K}}}}$ of $\mathcal{K}$ is given by $\tilde{\mathcal{K}} = (A^{op}, \tilde{I}, \pi, E, U, \tilde{V})$, where the index set $\tilde{I} = (\tilde{r}, \tilde{s}, \tilde{\overline{r}}, \tilde{\overline{s}})$ is given by $\tilde{r} = \overline{r}$, $\tilde{s} = \overline{s}$, $\tilde{\overline{r}} = s$, $\tilde{\overline{s}} = r$, and the family $\tilde{V}$ is given by $\tilde{V} = \Sigma U_2 V U_2 \Sigma$.

Examples and standard constructions

In this thesis, we give two examples of pseudo-Kac systems. In section 3.2, we associate to each locally compact groupoid a pseudo-Kac system which is decomposable if the groupoid is decomposable. In section 2.4.5, we construct pseudo-Kac systems out of center-valued tracial conditional expectations. The following constructions can be used to obtain further examples of pseudo-Kac systems.

- **Completion** Let (A_0, E_0, U_0, V_0) be an algebraic pseudo-Kac system, put $I = (r, s, \overline{r}, \overline{s})$ and let A, π, E, U and V be a C^*-algebra, families of C^*-modules with representations and unitaries as in definition 2.53, respectively, such
  - that A_0 is dense in A, the family E is obtained by completing E_0 with respect to various pre-C^*-module structures,
  - the representations on E are induced by the corresponding representations on E and
  - the families U and V are induced by the unitaries U_0 and V_0, respectively.

  Then E, π, U and V satisfy the conditions v)-viii) of definition 2.53. If (A, I, π, E, U, V) is a pseudo-Kac system, we say that it is *induced* by the algebraic Kac system.

Given two pseudo-Kac systems, one can form their direct sum and their external tensor product; we do not write down the details.

The left and the right leg of a decomposable pseudo-Kac system

Let $\mathcal{K} := (A, I, \pi, E, U, V)$ be a decomposable pseudo-Kac system.

Notation 2.58. *We denote by $(\hat{\mathscr{S}}, \hat{\Delta})$ the right Hopf C^*-family of the left leg of the pseudo-multiplicative unitary $V^{\overline{r}\overline{r}}$,*

$$\hat{\mathscr{S}} = \{\mathscr{E}^{r}_{\overline{r}}|_2\, V^{\overline{r}\overline{r}}\, |\mathscr{E}^{r}_{\overline{r}}\rangle_2 \subset \mathscr{L}(E^{s}_{\overline{r}}), \qquad \hat{\Delta}(T) = V^{\overline{r}\overline{r}*}(1 \otimes T)V^{\overline{r}\overline{r}},$$

where $\mathscr{E}^{r}_{\overline{r}} := \mathscr{C}ov(E^{r}_{\overline{r}})$. We denote by $(\hat{\mathscr{S}}_0, \hat{\delta}_0)$ the right coaction of $(\hat{\mathscr{S}}, \hat{\Delta})$ associated to the left coaction unitary $V^{r\overline{r}}$ in subsection 2.3.3,

$$\hat{\mathscr{S}}_0 = \langle E_r|_2\, V^{r\overline{r}}|\, \mathscr{E}^{r}_{\overline{r}}\rangle_2 \subset \mathscr{L}_{\mathrm{id}}(E^{s}_{r}), \qquad \hat{\delta}_0(T) = T \mapsto V^{r\overline{r}*}(1 \otimes T)V^{r\overline{r}}, \tag{2.1}$$

and call it the initial coaction *of $(\hat{\mathscr{S}}, \hat{\Delta})$. The extension of this coaction to the C^*-algebra $\hat{S}_0 := \overline{\mathrm{span}}_\alpha\, \hat{\mathscr{S}}_0^\alpha$ in the sense of proposition 2.32 is denoted by $\hat{\delta}_0$ again. The* left leg *of $\mathcal{K}$ is the pair $\big((\hat{\mathscr{S}}, \hat{\Delta}), (\hat{\mathscr{S}}_0, \hat{\delta}_0)\big)$. The* right leg *$\big((\mathscr{S}, \Delta), (\mathscr{S}_0, \delta_0)\big)$ of $\mathcal{K}$ is the left leg of the dual pseudo-Kac system $\hat{\mathcal{K}}$, i.e. $(\mathscr{S}, \Delta)$ is the right Hopf C^*-family of the left leg of the pseudo-multiplicative unitary $\hat{V}^{rr}$,*

$$\mathscr{S} = \{\mathscr{E}^{\overline{s}}_{r}|_2\, \hat{V}^{rr}\, |\mathscr{E}^{\overline{s}}_{r}\rangle_2 \subset \mathscr{L}(E^{\overline{r}}_{r}), \qquad \Delta(T) = \hat{V}^{rr*}(1 \otimes T)\hat{V}^{rr},$$

where $\mathscr{E}^{\overline{s}}_r := \mathscr{C}ov(E^{\overline{s}}_r)$*, and* $(\mathscr{S}_0, \delta_0)$ *is the right coaction of* $(\mathscr{S}, \Delta)$ *associated to the left coaction unitary* $\hat{V}^{\overline{s}r}$*,*

$$\mathscr{S}_0 = \langle E_{\overline{s}}|_2 \hat{V}^{\overline{s}r} |\mathscr{E}^{\overline{s}}_r\rangle_2 \subset \mathscr{L}_{\mathrm{id}}(E^{\overline{r}}_{\overline{s}}), \qquad \delta_0(T) = \hat{V}^{\overline{s}r*}(1 \otimes T)\hat{V}^{\overline{s}r}.$$

The extension of this coaction to $S_0 := \overline{\operatorname{span}}_\alpha \mathscr{S}^\alpha_0$ *is denoted by* δ_0 *again.*

Recall that associated to the Hopf C^*-families $(\hat{\mathscr{S}}, \hat{\Delta})$ and $(\mathscr{S}, \Delta)$, we have the pairs of functors and natural transformations $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$ and $(\boldsymbol{S}, \boldsymbol{\Delta})$ given by

$$\begin{aligned} &\hat{\boldsymbol{S}}\colon \boldsymbol{C^*_A} \to \boldsymbol{C^*_A}, && \hat{\boldsymbol{S}}(C,\pi) = \big((C,\pi) \otimes \hat{\mathscr{S}}, 1 \otimes \pi_s\big), \quad \hat{\boldsymbol{S}}\phi = \mathrm{ind}_\phi, \\ &\hat{\boldsymbol{\Delta}}\colon \hat{\boldsymbol{S}} \to \hat{\boldsymbol{S}}^2, && \hat{\boldsymbol{\Delta}}_{(C,\pi)} = \mathrm{id}_C \otimes \hat{\Delta} \end{aligned}$$

on objects (C, π) and morphisms ϕ in the category $\boldsymbol{C^*_A}$, and

$$\begin{aligned} &\boldsymbol{S}\colon \boldsymbol{C^*_{A^{op}}} \to \boldsymbol{C^*_{A^{op}}}, && \boldsymbol{S}(C,\pi) = \big((C,\pi) \otimes \mathscr{S}, 1 \otimes \pi_{\overline{r}}\big), \quad \boldsymbol{S}\phi = \mathrm{ind}_\phi, \\ &\boldsymbol{\Delta}\colon \boldsymbol{S} \to \boldsymbol{S}^2, && \boldsymbol{\Delta}_{(C,\pi)} = \mathrm{id}_C \otimes \Delta, \end{aligned}$$

on objects (C, π) and morphisms ϕ in the category $\boldsymbol{C^*_{A^{op}}}$.

Next, we recall the notion of regularity for coactions on C^*-pre-families and C^*-algebras introduced in subsection 2.3.2. The reduced crossed product construction will give rise to such coactions, and the formulation of the duality theorem depends on them.

Notation 2.59. *A right coaction of* $(\hat{\mathscr{S}}, \hat{\Delta})$ *on a* C^**-pre-family* $\mathscr{C}$ *is* regular *if* $\mathscr{C}^\alpha_\beta = \mathscr{C}^\alpha_\beta \lambda(\hat{J}_\alpha)$ *for all* $\alpha \in \mathrm{PAut}(A)$ *and all* β*, where* $\hat{J}_\alpha = \big\langle (\mathscr{E}^r_{\overline{r}})_\alpha \big| (\mathscr{E}^r_{\overline{r}})_\alpha \big\rangle$*. Likewise, a right coaction of* $(\mathscr{S}, \Delta)$ *on a* C^**-pre-family* $\mathscr{C}$ *is* regular *if* $\mathscr{C}^{\alpha'}_\beta = \mathscr{C}^{\alpha'}_\beta \lambda(J_{\alpha'})$ *for all* $\alpha' \in \mathrm{PAut}(A^{op})$ *and all* β*, where* $J_{\alpha'} = \big\langle (\mathscr{E}^{\overline{s}}_r)_{\alpha'} \big| (\mathscr{E}^{\overline{s}}_r)_{\alpha'} \big\rangle$*.*

A right coaction (C, π, δ) *of* $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$ *is a* regular $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebra *if it is induced from a regular right coaction of* $(\hat{\mathscr{S}}, \hat{\Delta})$ *on a non-degenerate* C^**-pre-family* $\mathscr{C} \subset \mathscr{C}ov(C, \pi)$ *in the sense of proposition 2.32. Likewise, the notion of an* $(\boldsymbol{S}, \boldsymbol{\Delta})$*-algebra is defined.*

Observe that the initial coactions belong to this class by proposition 2.45.

2.4.3 Crossed products and dual coactions

In this section, we introduce reduced crossed products for coactions of the Hopf C^*-families comprising the left and the right leg of a pseudo-Kac system. These are "twisted tensor products" formed of the C^*-algebra or C^*-pre-family underlying the coaction under consideration and of the C^*-pre-family underlying the initial coaction of the respective dual Hopf C^*-family. The initial coaction induces a dual coaction of the dual Hopf C^*-family on the reduced crossed product.

The constructions are similar to the case of Kac systems [3], but much more book-keeping is needed in order to keep track of the multitude of C^*-bimodules and unitaries involved.

First, we collect some standard relations which are satisfied by each pseudo-Kac system. They are used in the construction of dual coactions and in the proof of the duality theorem. Then we discuss coactions of the right leg of a pseudo-Kac system and associated reduced crossed products. The corresponding constructions and results for coactions of the left leg are summarised at the end.

Throughout this subsection, let $\mathcal{K} = (A, I, E, \pi, V, U)$ be a pseudo-Kac system.

Standard relations

The following relations are straight-forward reformulations of the corresponding results for Kac systems introduced and proved in [3, section 6]. We adopt the notation 2.52 for sets of morphisms to collect families of related equalities in one equation.

Proposition 2.60. *One has the following families of relations.*

i) $\hat{V}V\widetilde{V} = U_1\Sigma$,

ii) $V_{12}U_2V_{23}U_2 = U_2V_{23}U_2V_{13}V_{12}$,

iii) $\hat{V}_{23}V_{12}V_{13} = V_{13}\hat{V}_{23}$ *and* $\widetilde{V}_{12}V_{13} = V_{13}V_{23}\widetilde{V}_{12}$,

iv) $V_{23}V_{12}V_{23}^* = \hat{V}_{23}^*V_{13}\hat{V}_{23}$ *and* $\widetilde{V}_{12}V_{13}\widetilde{V}_{12}^* = V_{12}^*V_{23}V_{12}$,

v) each of the following pairs of families commute:

$$\Sigma_{23}\hat{V}_{23}V_{23} \text{ with } V_{12}, \qquad V_{12}\widetilde{V}_{12}\Sigma_{12} \text{ with } V_{23},$$
$$V_{12} \text{ with } \widetilde{V}_{23}, \qquad V_{23} \text{ with } \hat{V}_{12}.$$

Proof. By condition v) of definition 2.53, one has

$$\hat{V}V\widetilde{V} = \underbrace{U_1U_2 \cdot \Sigma U_2 V \Sigma U_2}_{\hat{V}} V \underbrace{\Sigma U_2 V \cdot U_2 \Sigma}_{\widetilde{V}} = U_1U_2 \cdot U_2\Sigma = U_1\Sigma.$$

The assertions ii),iii) and v) are direct generalisations of proposition 6.1 (1)-(5) and of proposition 6.5 (b),(c) in [3]. The proofs carry over verbatim. For relation iv), combine the pentagon equation and iii). □

In the situation of Kac systems acting on Hilbert spaces, these relations imply the equations

$$V(s \otimes 1)V^* = \Delta(s), \ s \in S, \quad \text{and} \quad V(U\hat{s}U \otimes 1)V^* = U\hat{s}U \otimes 1, \ \hat{s} \in \hat{S},$$

which are used to construct dual coactions on reduced crossed products and to prove of the duality theorem. In the setting of pseudo-Kac systems, these two equations split up into several related but formally different ones.

Proposition 2.61. *Let E_0 and V^0 be a C^*-module and a unitary belonging to the family E and V, respectively, and assume that $\phi_2\colon E_0 \to \operatorname{Dom} V^0$ and $\psi_2\colon \operatorname{Im} V^0 \to E_0$ are maps of the form $\zeta \mapsto \zeta \otimes \eta$ and $\zeta \mapsto \zeta \otimes \xi$, respectively. Then*

$$V(\phi_1^* V^0 \psi_1)_1 V^* = \hat{V}^* (\phi_1^* V^0 \psi_1)_2 \hat{V}, \qquad \hat{V}(\phi_2^* V^0 \psi_2)_2 \hat{V}^* = (\phi_2^* V^0 \psi_2)_2,$$
$$\widetilde{V}(\phi_2^* V^0 \psi_2)_1 \widetilde{V}^* = V^* (\phi_2^* V^0 \psi_2)_2 V.$$

Proof. By parts iv) and v) of the previous proposition, one has

$$\begin{aligned} V(\phi_1^* V \psi_1)_1 V^* &= \phi_1^* V_{23} V_{12} V_{23}^* \psi_1 = \phi_1^* \hat{V}_{23}^* V_{13} \hat{V}_{23} \psi_1 = \hat{V}^* (\phi_1^* V \psi_2)_1 \hat{V}, \\ \hat{V}(\phi_2^* V \psi_2)_2 \hat{V}^* &= \phi_3^* \hat{V}_{12} V_{23} \hat{V}_{12}^* \psi_3^* = \phi_3^* V_{23} \psi_3 = (\phi_2^* V \psi_2)_2, \\ \widetilde{V}(\phi_2^* V \psi_2)_1 \widetilde{V}^* &= \phi_3^* \widetilde{V}_{12} V_{13} \widetilde{V}_{12}^* \psi_3 = \phi_3^* V_{12}^* V_{23} V_{12} \psi_3 = V^* (\phi_2^* V \psi_2)_2 V. \end{aligned}$$ □

Coactions of the right leg

First, we discuss reduced crossed products for coaction of the right leg on C^*-pre-families. The corresponding constructions and results for coactions on C^*-algebras will be summarised at the end.

Let B be a C^*-algebra and let $\mathscr{C}$ be a non-degenerate C^*-pre-family on a C^*-A^{op}-B-bimodule E with a right coaction δ of $(\mathscr{S}, \Delta)$.

Proposition 2.62. *The image $\delta(\mathscr{C})$ is contained in $\mathscr{L}^{\mathrm{id}}(E \otimes E_r^s)$, and $\delta(\mathscr{C})(1 \otimes \hat{\mathscr{S}}_0) \subset \mathscr{L}(E \otimes E_r^s)$ is a C^*-pre-family.*

Proof. Since $V^{r\overline{r}}$ commutes with π_{s2}, the C^*-pre-family $\mathscr{S}$ commutes with π_s. Therefore, $\delta(\mathscr{C})$ commutes with the representation π_s on $E \otimes E_r$.

Since $\delta(\mathscr{C})$ and $1 \otimes \hat{\mathscr{S}}_0$ are C^*-pre-families, it is enough to show that the family $X := (1 \otimes \hat{\mathscr{S}}_0)\delta(\mathscr{C})$ is contained in $\delta(\mathscr{C})(1 \otimes \hat{\mathscr{S}}_0)$. By equation (2.1), one has

$$X = \langle E_r|_3 \, V_{23}^{r\overline{r}} \, |\mathscr{E}_{\overline{r}}^r\rangle_3 \, \delta(\mathscr{C}) = \langle E_r|_3 \, V_{23}^{r\overline{r}} \left(\delta(\mathscr{C}) \otimes 1\right) |\mathscr{E}_{\overline{r}}^r\rangle_3.$$

By proposition 2.61, one has $V^{r\overline{r}}(s \otimes 1)V^{r\overline{r}*} = \Delta(s)$ for all $s \in \mathscr{S}$. By uniqueness of the extension of morphisms to multiplier families, we can rewrite in the expression above $V_{23}^{r\overline{r}} \left(\delta(\mathscr{C}) \otimes 1\right) = \left((\mathrm{id} \otimes \Delta)(\delta(\mathscr{C}))\right) V_{23}^{r\overline{r}}$. Since $\mathscr{S}_{\mathrm{id}}^{\mathrm{id}}$ is non-degenerate and $(\mathrm{id} \otimes \Delta)\delta = (\delta \otimes 1)\delta$, we can rewrite

$$\begin{aligned} \langle E_r|_3 \left((\mathrm{id} \otimes \Delta)(\delta(\mathscr{C}))\right) &= \langle \mathscr{S}_{\mathrm{id}}^{\mathrm{id}} E_r|_3 \left((\delta \otimes 1)(\delta(\mathscr{C}))\right) \\ &= \langle E_r|_3 \left((\delta \otimes 1)\left((1 \otimes \mathscr{S}_{\mathrm{id}}^{\mathrm{id}})\delta(\mathscr{C})\right)\right) \subset \langle E_r|_3 \left(\delta(\mathscr{C}) \otimes \mathscr{S}\right). \end{aligned}$$

Inserting this in the expression for X again, we obtain

$$X \subset \langle E_r|_3 \left(\delta(\mathscr{C}) \otimes \mathscr{S}\right) V_{23}^{r\overline{r}} \, |\mathscr{E}_{\overline{r}}^r\rangle_3 \subset \delta(\mathscr{C})(1 \otimes \hat{\mathscr{S}}_0).$$ □

Definition 2.63. *The* reduced crossed product *of the coaction $(\mathscr{C}, \delta)$ is the C^*-pre-family $\mathscr{C} \rtimes \hat{\mathscr{S}}_0 := \delta(\mathscr{C})(1 \otimes \hat{\mathscr{S}}_0) \subset \mathscr{L}(E \otimes E_r^s)$.*

A similar argument as the one in the proof above shows the following result.

Corollary 2.64. *The product $\mathscr{S}\hat{\mathscr{S}}_0 \subset \mathscr{L}(E^s_r)$ is a C^*-pre-family.* □

The reduced crossed product carries a dual coaction which is constructed using the standard relations collected in the previous subsection.

Proposition 2.65. *The formula*

$$\delta(c)(1 \otimes \hat{s}) \mapsto \big(\delta(c) \otimes 1\big)\big(1 \otimes \hat{\delta}_0(\hat{s})\big), \quad c \in \mathscr{C},\ \hat{s} \in \hat{\mathscr{S}}_0,$$

defines a regular coaction $\hat{\delta}$ on the reduced crossed product $\mathscr{C} \rtimes \hat{\mathscr{S}}_0$. With this coaction, it becomes a regular $(\hat{\mathscr{S}}, \hat{\Delta})$-pre-family.

Proof. Consider the operator $\tilde{V}^{rs}$ is given by the composition

$$E^{\overline{s}}_r \otimes E_s \xrightarrow{U^r_2\Sigma} E_s \otimes E^{\overline{r}}_s \xrightarrow{V^{ss}} E_{\overline{r}} \otimes E^r_s \xrightarrow{\Sigma U^s_2} E^s_r \otimes E_{\overline{r}}.$$

It intertwines the representation π_s as follows:

$$\tilde{V}^{rs}\pi_{s1} = \Sigma U^s_2 V^{ss} U^r_2 \Sigma \pi_{s1} = \Sigma U^s_2 V^{ss} \pi_{r2} U^r_2 \Sigma = \Sigma U^s_2 \pi_{s1} V^{ss} U^r_2 \Sigma = \pi_{s2}\tilde{V}^{rs}.$$

We show that for each $T \in \mathscr{C} \rtimes \hat{\mathscr{S}}_0$, the operator $T \otimes 1_{E_s}$ on $E \otimes E^{\overline{s},s}_r \otimes E_s$ is well-defined, and that the map $\Phi\colon \mathscr{L}^{\mathrm{id}}(E \otimes E^{\overline{s}}_r) \to \mathscr{L}(E \otimes E^s_r \otimes E^s_{\overline{r}})$ given by $T \mapsto \tilde{V}^{rs}_{23}(T \otimes 1_{E_s})\tilde{V}^{rs*}_{23}$ implements the formula given above.

The C^*-pre-families $\mathscr{S}$ and $\hat{\mathscr{S}}_0$ commute with the representation $\pi_{\overline{s}}$ because they are the left legs of the operators $\hat{V}^{rr}$ and $V^{r\overline{r}}$, respectively, which commute with the representation $\pi_{\overline{s}1}$ by definition and proposition 2.56. Since the family $\mathscr{S} \subset \mathscr{L}(E^{\overline{r}}_r)$ is non-degenerate, it follows that the family $\delta(\mathscr{C}) \subset \mathscr{M}(\mathscr{C} \otimes \mathscr{S})$ commutes with the representation $\pi_{\overline{s}2}$. Therefore, $\mathscr{C} \rtimes \hat{\mathscr{S}}_0$ commutes with $\pi_{\overline{s}2}$.

By proposition 2.61 and part v) of proposition 2.60, one has

$$\begin{aligned}
\Phi(1 \otimes \hat{s}) &= \tilde{V}^{rs}_{23}(1 \otimes \hat{s} \otimes 1)\tilde{V}^{rs*}_{23} = 1 \otimes \hat{\delta}_0(\hat{s}), \quad \hat{s} \in \hat{\mathscr{S}}_0,\\
\Phi(1 \otimes s) &= \tilde{V}^{rs}_{23}(1 \otimes s \otimes 1)\tilde{V}^{rs*}_{23} = (1 \otimes s \otimes 1), \quad s \in \mathscr{S}.
\end{aligned}$$

Since $\mathscr{S} \subset \mathscr{L}(E^{\overline{r}}_r)$ is non-degenerate, the second equation implies $\Phi(\delta(c)) = \delta(c) \otimes 1$. Therefore, the map Φ implements the formula above.

The formula shows that the map $\hat{\delta}$ is coassociative. Furthermore,

$$\begin{aligned}
\hat{\delta}(\mathscr{C} \rtimes \hat{\mathscr{S}}_0)(1 \otimes 1 \otimes \hat{\mathscr{S}}) &= (\delta(\mathscr{C}) \otimes 1)(1 \otimes \hat{\delta}_0(\hat{\mathscr{S}}_0))(1 \otimes 1 \otimes \hat{\mathscr{S}})\\
&= (\delta(\mathscr{C}) \otimes 1)(1 \otimes \hat{\mathscr{S}}_0 \otimes \hat{\mathscr{S}})\\
&\overset{(*)}{=} \big(\delta(\mathscr{C})(1 \otimes \hat{\mathscr{S}}_0)\big) \otimes \hat{\mathscr{S}} = (\mathscr{C} \rtimes \hat{\mathscr{S}}_0) \otimes \hat{\mathscr{S}},
\end{aligned}$$

where we used the fact that $\delta(\mathscr{C})$ commutes with the representation π_{s2} in the equation $(*)$. The coaction $\hat{\delta}_0$ on $\hat{\mathscr{S}}_0$ is regular by proposition 2.45. Therefore, $\hat{\delta}$ is regular. □

Definition 2.66. *The* dual coaction *on the reduced crossed product $\mathscr{C} \rtimes \hat{\mathscr{S}}_0$ is the coaction $\hat{\delta}$ constructed in the previous proposition.*

The reduced crossed product construction extends to morphisms.

Proposition 2.67. *Let $\mathscr{C}'$ be a non-degenerate C^*-pre-family with a coaction δ' of $(\mathscr{S}, \Delta)$ and let $\phi\colon \mathscr{C} \to \mathscr{M}(\mathscr{C}')$ be a non-degenerate equivariant morphism. Then the morphism $\phi \otimes 1$ extends to an equivariant non-degenerate morphism $\phi \rtimes 1\colon \mathscr{C} \rtimes \hat{\mathscr{S}}_0 \to \mathscr{M}(\mathscr{C}' \rtimes \hat{\mathscr{S}}_0)$.*

Proof. By proposition 1.43, the morphism ϕ induces a non-degenerate morphism $\phi \otimes 1\colon \mathscr{C} \otimes \mathscr{S}\hat{\mathscr{S}}_0 \to \mathscr{M}(\mathscr{C}' \otimes \mathscr{S}\hat{\mathscr{S}}_0)$. By 1.41, the latter extends to the multiplier C^*-pre-family $\mathscr{M}(\mathscr{C} \otimes \mathscr{S}\hat{\mathscr{S}}_0)$. Since $\delta(\mathscr{C})$ is contained in $\mathscr{M}(\mathscr{C} \otimes \mathscr{S})$, the C^*-pre-family $\mathscr{C} \rtimes \hat{\mathscr{S}}_0$ is contained in $\mathscr{M}(\mathscr{C} \otimes \mathscr{S}\hat{\mathscr{S}}_0)$. Since ϕ is equivariant, $(\phi \otimes 1)(\delta(\mathscr{C}))$ is contained $\delta(\phi(\mathscr{C}))$. Hence, the image $(\phi \otimes 1)(\mathscr{C} \rtimes \hat{\mathscr{S}}_0)$ is contained in $\mathscr{M}(\mathscr{C}' \rtimes \hat{\mathscr{S}}_0)$. The formula for the dual coactions shows that $\phi \otimes 1$ is equivariant. ☐

Corollary 2.68. *The reduced crossed product construction defines a functor from the category of right coactions of the Hopf C^*-family $(\mathscr{S}, \Delta)$ to the category of regular right coactions of the Hopf C^*-family $(\hat{\mathscr{S}}, \hat{\Delta})$.*

Proof. Functoriality follows immediately from the construction given in the proof of the previous proposition. ☐

The reduced crossed product construction applies to coactions on C^*-algebras as well. Straight-forward modifications yield the following definition and results.

Proposition/Definition 2.69. *Let (C, π, δ) be a coaction of $(\boldsymbol{S}, \boldsymbol{\Delta})$. Then the family*

$$C \rtimes \hat{\mathscr{S}}_0 := \delta(C)(1 \otimes \hat{\mathscr{S}}_0) \subset \mathscr{L}_{\mathrm{id}}(C \otimes E^s_r)$$

is a non-degenerate C^*-pre-family. The formula

$$\delta(c)(1 \otimes \hat{s}) \mapsto (\delta(c) \otimes 1)(1 \otimes \hat{\delta}_0(\hat{s})), \quad c \in C,\ \hat{s} \in \hat{\mathscr{S}}_0,$$

defines a regular coaction $\hat{\delta}$ of $(\hat{\mathscr{S}}, \hat{\Delta})$ on $C \rtimes \hat{\mathscr{S}}_0$. The pair $\big(C \rtimes \hat{\mathscr{S}}_0, \hat{\delta}\big)$ is a regular $(\hat{\mathscr{S}}, \hat{\Delta})$-pre-family, called the reduced crossed product C^*-pre-family *associated to (C, π, δ).*

The reduced crossed product $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebra *$C \rtimes \hat{S}_0$ is the extension of this $(\hat{\mathscr{S}}, \hat{\Delta})$-pre-family according to proposition 2.32; it is also regular.*

Let (C', π', δ') be another coaction of $(\boldsymbol{S}, \boldsymbol{\Delta})$ and let $\phi\colon (C, \pi) \to (C', \pi')$ be a non-degenerate morphism. Then ϕ defines equivariant non-degenerate morphisms $\phi \rtimes 1\colon C \rtimes \hat{\mathscr{S}}_0 \to C' \rtimes \hat{\mathscr{S}}_0$ and $\phi \rtimes 1\colon (C \rtimes \hat{S}_0, \pi_{s2}) \to (C' \rtimes \hat{S}_0, \pi_{s2})$.

The reduced crossed product construction defines a functor from the category of right coactions of $(\boldsymbol{S}, \boldsymbol{\Delta})$ on C^*-algebras with non-degenerate equivariant morphisms to the category of regular $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebras. ☐

Coactions of the left leg

Reduced crossed products for coactions of the left leg $(\hat{\mathscr{S}}, \hat{\Delta})$ are defined similarly as for the right leg. For completeness, we reproduce the corresponding constructions and results.

Proposition/Definition 2.70. *Let $\mathscr{C} \subset \mathscr{L}(E)$ be a non-degenerate C^*-pre-family on a C^*-A-B-bimodule E with a right coaction $\hat{\delta}$ of $(\hat{\mathscr{S}}, \hat{\Delta})$. The image $\mathrm{Ad}_{U_2^{\overline{r}}}\big(\hat{\delta}(\mathscr{C})\big)$ is contained in $\mathscr{L}^{\mathrm{id}}(E \otimes E^{\overline{r}}_{\overline{s}})$, and the product*

$$\mathscr{C} \rtimes \mathscr{S}_0 := \mathrm{Ad}_{U_2^{\overline{r}}}\big(\hat{\delta}(\mathscr{C})\big)(1 \otimes \mathscr{S}_0) \subset \mathscr{L}(E \otimes E^{\overline{r}}_{\overline{s}})$$

is a C^*-pre-family. The formula

$$\mathrm{Ad}_{U_2^{\overline{r}}}\big(\hat{\delta}(c)\big)(1 \otimes s) \mapsto \big(\mathrm{Ad}_{U_2^{\overline{r}}}\big(\hat{\delta}(c)\big) \otimes 1\big)(1 \otimes \delta_0(s)), \quad c \in \mathscr{C},\, s \in \mathscr{S}_0,$$

defines a regular coaction δ of $(\mathscr{S}, \Delta)$ on $\mathscr{C} \rtimes \mathscr{S}_0$. With respect to this coaction, it is a regular $(\mathscr{S}, \Delta)$-pre-family, called the reduced crossed product C^*-pre-family *associated to $(\mathscr{C}, \hat{\delta})$.*

Let $\mathscr{C}'$ be a non-degenerate C^*-pre-family with a coaction $\hat{\delta}'$ of $(\hat{\mathscr{S}}, \hat{\Delta})$ and let $\psi \colon \mathscr{C} \to \mathscr{M}(\mathscr{C}')$ be a non-degenerate equivariant morphism. Then the morphism $\psi \otimes 1$ extends to an equivariant non-degenerate morphism $\psi \rtimes 1 \colon \mathscr{C} \rtimes \mathscr{S}_0 \to \mathscr{M}(\mathscr{C}' \rtimes \mathscr{S}_0)$.

The reduced crossed product construction defines a functor from the category of right coactions of the Hopf C^*-family $(\hat{\mathscr{S}}, \hat{\Delta})$ to the category of regular right coactions of the Hopf C^*-family $(\mathscr{S}, \Delta)$.

Proof. The proof proceeds along the same lines as the proofs of propositions 2.62 and 2.65. For the construction of the coaction, one uses the map $T \mapsto V^{\overline{sr}}_{23}(T \otimes 1_{E_{\overline{s}}})V^{\overline{sr}*}_{23}$. Let us show that $V^{\overline{sr}}(s_0 \otimes 1)V^{\overline{sr}*} = \delta_0(s_0)$ for all $s_0 \in S_0$, because we will need this relation later on anyway. By definition of S_0 and $\hat{V}$, one has

$$S_0 = \langle E_{\overline{s}}|_2 \, \hat{V}^{\overline{s}r} \, |E_r\rangle_2 = \langle E_{\overline{r}}|_1 \, V^{s\overline{s}} |E_s\rangle_1.$$

By definition of δ_0 and by proposition 2.61, we obtain

$$\delta_0(s_0) = \hat{V}^{\overline{s}r*}(1 \otimes s_0)\hat{V}^{\overline{s}r} = V^{\overline{sr}}(s_0 \otimes 1)V^{\overline{sr}*}, \quad s_0 \in S_0. \qquad \square$$

Finally, we state the corresponding result for coactions on C^*-algebras.

Proposition/Definition 2.71. *Let $(C, \pi, \hat{\delta})$ be a coaction of $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$. The family*

$$C \rtimes \mathscr{S}_0 := \mathrm{Ad}_{U_2^{\overline{r}}}\big(\hat{\delta}(C)\big)(1 \otimes \mathscr{S}_0) \subset \mathscr{L}_{\mathrm{id}}(C \otimes E^{\overline{r}}_{\overline{s}})$$

is a C^*-pre-family. The formula

$$\mathrm{Ad}_{U_2^{\overline{r}}}\big(\hat{\delta}(c)\big)(1 \otimes s) \mapsto \big(\mathrm{Ad}_{U_2^{\overline{r}}}\big(\hat{\delta}(c)\big) \otimes 1\big)(1 \otimes \delta_0(s)), \quad c \in C,\, s \in \mathscr{S}_0,$$

defines a regular coaction δ *of* $(\mathscr{S}, \Delta)$ *on* $C \rtimes \mathscr{S}_0$. *With respect to this coaction, it is a regular* $(\mathscr{S}, \Delta)$*-pre-family, called the* reduced crossed product C^*-pre-family *associated to* $(C, \hat{\delta})$.

The reduced crossed product $(\boldsymbol{S}, \boldsymbol{\Delta})$-algebra $C \rtimes S_0$ *is the extension of this* $(\mathscr{S}, \Delta)$*-pre-family according to proposition 2.32; it is regular.*

Let $(C', \pi', \hat{\delta}')$ *be another coaction of* $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$ *and let* $\psi \colon (C, \pi) \to (C', \pi')$ *be a non-degenerate morphism. Then* ψ *defines equivariant non-degenerate morphisms* $\psi \rtimes 1 \colon C \rtimes \mathscr{S}_0 \to \mathscr{M}(C' \rtimes \mathscr{S}_0)$ *and* $\psi \rtimes 1 \colon (C \rtimes S_0, \pi_{\overline{r}2}) \to (C' \rtimes S_0, \pi_{\overline{r}2})$.

The reduced crossed product construction defines a functor from the category of right coactions of $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$ *on* C^**-algebras with non-degenerate equivariant morphisms to the category of regular* $(\boldsymbol{S}, \boldsymbol{\Delta})$*-algebras.* □

2.4.4 The duality theorem

Now, we come to the second main result of this thesis – the generalisation of the Takesaki-Takai-Baaj-Skandalis duality theorem to coactions of Hopf C^*-families. Let us first put it into a categorical perspective. Denote by $\boldsymbol{Coact}_{(\boldsymbol{S},\boldsymbol{\Delta})}$ and $\boldsymbol{Coact}_{(\hat{\boldsymbol{S}},\hat{\boldsymbol{\Delta}})}$ the category of right coactions of $(\boldsymbol{S}, \boldsymbol{\Delta})$ and $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$, respectively, on C^*-algebras with non-degenerate equivariant morphisms. The reduced crossed product construction defines two functors

$$\boldsymbol{Coact}_{(\boldsymbol{S},\boldsymbol{\Delta})} \underset{-\rtimes \boldsymbol{S}_0}{\overset{-\rtimes \hat{\boldsymbol{S}}_0}{\rightleftarrows}} \boldsymbol{Coact}_{(\hat{\boldsymbol{S}},\hat{\boldsymbol{\Delta}})}.$$

The images of these functors are contained in the full sub-categories of all regular $(\boldsymbol{S}, \boldsymbol{\Delta})$-algebras and all regular $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebras, respectively. Denoting these sub-categories by $\boldsymbol{C}^{*reg}_{(\boldsymbol{S},\boldsymbol{\Delta})}$ and $\boldsymbol{C}^{*reg}_{(\hat{\boldsymbol{S}},\hat{\boldsymbol{\Delta}})}$, respectively, we can refine the previous diagram as follows.

$$\begin{array}{ccc} \boldsymbol{Coact}_{(\boldsymbol{S},\boldsymbol{\Delta})} & \xrightarrow{-\rtimes \hat{\boldsymbol{S}}_0} & \boldsymbol{C}^{*reg}_{(\hat{\boldsymbol{S}},\hat{\boldsymbol{\Delta}})} \\ \cup & & \cap \\ \boldsymbol{C}^{*reg}_{(\boldsymbol{S},\boldsymbol{\Delta})} & \xleftarrow[-\rtimes \boldsymbol{S}_0]{} & \boldsymbol{Coact}_{(\hat{\boldsymbol{S}},\hat{\boldsymbol{\Delta}})} \end{array}$$

This diagram does not commute, but the following duality theorem calculates the compositions of the restrictions of the reduced crossed product functors

$$\boldsymbol{C}^{*reg}_{(\boldsymbol{S},\boldsymbol{\Delta})} \underset{-\rtimes \boldsymbol{S}_0}{\overset{-\rtimes \hat{\boldsymbol{S}}_0}{\rightleftarrows}} \boldsymbol{C}^{*reg}_{(\hat{\boldsymbol{S}},\hat{\boldsymbol{\Delta}})}.$$

Its content will be interpreted after the proof at the end of this subsection. The proof of the duality theorem given in [3] carries over with increased book-keeping, but regularity of coactions enters as a new phenomenon. First, we collect some preliminary results.

Proposition 2.72. *i) The space $S'_0 := \langle E_{\overline{s}}|_2 \hat{V}^{sr} |E_r\rangle_2 \subset L_A(E_s)$ is a C^*-algebra, and the representation $\pi_{\overline{r}}$ restricts to a non-degenerate $*$-homomorphism $A \to M(S'_0)$.*

ii) The map $s'_0 \mapsto V^{sr}(s'_0 \otimes 1_{E^{\overline{r}}_r})V^{sr*}$ defines a coaction δ'_0 of $(\boldsymbol{S}, \boldsymbol{\Delta})$ on $(S'_0, \pi_{\overline{r}})$.

iii) The map $\langle \eta|_2 \hat{V}^{\overline{s}r} |\xi\rangle_2 \mapsto \langle \eta|_2 \hat{V}^{sr} |\xi\rangle_2$ defines an isomorphism

$$\phi\colon (S_0, \pi_{\overline{r}}, \delta_0) \xrightarrow{\cong} (S'_0, \pi_{\overline{r}}, \delta'_0)$$

.

iv) Conjugation by the unitary $\tilde{V}^{\overline{s}r}$, which is given by the composition

$$E_{\overline{s}} \otimes E^s_r \xrightarrow{U^{\overline{s}}_2\Sigma} E^s_r \otimes E_{\overline{r}} \xrightarrow{V^{r\overline{r}}} E^{\overline{r}}_r \otimes E_r \xrightarrow{\Sigma U^r_2} E_s \otimes E^{\overline{r}}_r,$$

maps $s_0 \otimes 1_{E^s_r}$ to $\phi(s_0) \otimes 1_{E^{\overline{r}}_r}$ for each $s_0 \in S_0$.

Proof. Let $\eta \in E_{\overline{s}}, \xi \in E_r$ and put $s_0 := \langle \eta|_2 \hat{V}^{\overline{s}r} |\xi\rangle_2$, $s'_0 := \langle \eta|_2 \hat{V}^{sr} |\xi\rangle_2$. By definition of $\hat{V}$, one has $s_0 = \langle \eta'|_1 V^{s\overline{s}} |\xi'\rangle_1$ and $s'_0 = \langle \eta'|_1 V^{ss} |\xi'\rangle_1$ where $\eta' = U^{\overline{s}}\eta$ and $\xi' = U^r\xi$. Now

$$\tilde{V}^{\overline{s}r}(s_0 \otimes 1_{E^s_r})\tilde{V}^{\overline{s}r*} = \langle \eta'|_2 \tilde{V}^{\overline{s}r}_{23} V^{s\overline{s}}_{12} \tilde{V}^{\overline{s}r*}_{23} |\xi'\rangle_2 = \langle \eta'|_2 V^{ss} |\xi'\rangle_2 = s'_0 \otimes 1_{E^{\overline{r}}_r}$$

by part v) of proposition 2.60. Since $\tilde{V}^{\overline{s}r}$ commutes with the representation $\pi_{\tilde{r}1} = \pi_{\overline{r}1}$, this equation implies that $(S'_0, \pi_{\overline{r}})$ is a C^*-A-algebra and that the map ϕ defines an isomorphism $(S_0, \pi_{\overline{r}}) \xrightarrow{\cong} (S'_0, \pi_{\overline{r}})$.

Denote by $\iota\colon S_0 \hookrightarrow L_A(E_{\overline{s}} \otimes E^s_r)$ and $\iota'\colon S'_0 \hookrightarrow L_A(E_s \otimes E^{\overline{r}}_r)$ the injections given by $x \mapsto x \otimes 1$. Then $\iota' \circ \phi = \mathrm{Ad}_{\tilde{V}^{\overline{s}r}} \circ \iota$. Combining this observation with the formula for the initial coaction δ_0 given in the proof of 2.70, we find that for each $s_0 \in S_0$, we have

$$(\iota' \otimes 1)\big((\phi \otimes 1)\delta_0(s_0)\big) = \tilde{V}^{\overline{s}r}_{12}\, (\delta_0(s_0))_{13}\, \tilde{V}^{\overline{s}r*}_{12} = \tilde{V}^{\overline{s}r}_{12}\, V^{\overline{s}\overline{r}}_{13}\, (s_0)_1\, V^{\overline{s}\overline{r}*}_{13}\, \tilde{V}^{\overline{s}r*}_{12}.$$

By part iii) of proposition 2.60, we have a commutative diagram

$$\begin{array}{ccccc}
 & & (E^{\overline{r}}_{\overline{s}} \otimes E^s_r) \otimes E_r & & \\
 & \overset{V^{\overline{s}\overline{r}}_{13}}{\nearrow} & & \overset{\tilde{V}^{\overline{s}r}_{12}}{\searrow} & \\
(E^s_{\overline{s}} \otimes E^s_r) \otimes E_{\overline{r}} & & & & (E^{\overline{r}}_s \otimes E^{\overline{r}}_r) \otimes E_r \\
 & \underset{\tilde{V}^{\overline{s}r}_{12}}{\searrow} & & \underset{V^{sr}_{13}}{\nearrow} & \\
 & E_s \otimes E^{\overline{r},s}_r \otimes E_{\overline{r}} & \xrightarrow{V^{r\overline{r}}_{23}} & E_s \otimes (E^{\overline{r}}_r \otimes E^{\overline{r}}_r) &
\end{array}$$

Therefore, we have

$$\begin{aligned}(\iota' \otimes 1)\big((\phi \otimes 1)\delta_0(s_0)\big) &= V_{13}^{sr}\, V_{23}^{r\overline{r}}\, \tilde{V}_{12}^{\overline{s}r}\, (s_0)_1\, \tilde{V}_{12}^{\overline{s}r*}\, V_{23}^{r\overline{r}*}\, V_{13}^{sr*} \\ &= V_{13}^{sr}\, V_{23}^{r\overline{r}}\, \big(\phi(s_0) \otimes 1 \otimes 1\big)\, V_{23}^{r\overline{r}*}\, V_{13}^{sr*} \\ &= V_{13}^{sr}\, \big(\phi(s_0) \otimes (1 \otimes 1)\big)\, V_{13}^{sr*} \\ &= (\iota' \otimes 1)\big(V^{sr}(\phi(s_0) \otimes 1_{E_r^{\overline{r}}})V^{sr*}\big),\end{aligned}$$

which shows that under the isomorphism ϕ, the coaction δ_0 on S_0 corresponds to the map on S_0' given in part ii). In particular, this map is a coaction. ☐

An analogous result holds for the space $\hat{S}_0' := \langle E_r|_2\, V^{\overline{sr}}\, |E_{\overline{r}}\rangle_2 \subset L_A(E_{\overline{s}})$.

Proposition 2.73. *One has $\hat{S}_0 \cdot U^{r*} S_0' U^r = K_A(E_r)$.*

Proof. Since $\hat{V}^{rr}$ is regular, one has

$$K_A(E_r) = \langle E_r|_2\, \hat{V}^{rr*}\, |E_r\rangle_1 = \langle E_r|_2\, \hat{V}^{rr*} U_1^s \Sigma\, |E_s\rangle_2.$$

By part i) of proposition 2.60 and part iii) of remark 2.55, $\hat{V}^{rr} V^{r\overline{r}} \tilde{V}^{rs} = U_1^s \Sigma$. Thus, we can replace $\hat{V}^{rr*} U_1^s \Sigma$ by $V^{r\overline{r}} \tilde{V}^{rs}$. Since $\hat{S}_0$ is non-degenerate, we have $\hat{S}_0 K_A(E_r) = K_A(E_r)$. Multiplying the equation above on the left by $\hat{S}_0$ and applying lemma 2.46, we find

$$K_A(E_r) = \hat{S}_0 \langle E_r|_2\, V^{r\overline{r}} \tilde{V}^{rs}\, |E_s\rangle_2 = \hat{S}_0 \langle E_{\overline{r}}|_2\, \tilde{V}^{rs}\, |E_s\rangle_2 = \hat{S}_0 U^{r*} S_0' U^r. \qquad ☐$$

Now we can state and proof the generalisation of the Takesaki-Takai-Baaj-Skandalis duality theorem.

Theorem 2.74. *Let (C, π, δ) be a regular $(\boldsymbol{S}, \boldsymbol{\Delta})$-algebra and let $\hat{\delta}$ denote the dual coaction of $(\hat{\mathscr{S}}, \hat{\Delta})$ on the crossed product $C \rtimes \hat{S}_0$. Then the reduced crossed product $C \rtimes \hat{S}_0 \rtimes S_0$ is isomorphic to $K_C(C \otimes E_r)$. Under this isomorphism, the bidual coaction is given by the composition $\mathrm{Ad}_{(1 \otimes W')} \circ (\delta \otimes 1)$, where the unitary W' is given by $W' = \Sigma \circ \hat{V}^{rr} \colon E_r^{\overline{r}} \otimes E_r \to E_r^{\overline{s}} \otimes E_r$.*

Proof. The C^*-algebra $C \rtimes \hat{S}_0 \rtimes S_0$ is generated by elements of the form

$$\big(\delta(c) \otimes 1\big)\, \big(1 \otimes \mathrm{Ad}_{U_2^{\overline{r}}}\, \hat{\delta}_0(\hat{s}_0)\big)\, (1 \otimes 1 \otimes s_0) \;\in\; L_C(C \otimes E_r^s \otimes E_{\overline{s}}),$$
$$c \in C,\ \hat{s}_0 \in \hat{S}_0,\ s_0 \in S_0.$$

Conjugation by $U_3^r V_{23}^{r\overline{r}} U_3^{\overline{r}*} \colon C \otimes E_r^s \otimes E_{\overline{s}} \to C \otimes E_r^{\overline{r}} \otimes E_s$ acts on a product of the form above as follows:

i) $\delta(c) \otimes 1 \mapsto \mathrm{Ad}_{U_3^r}((\delta \otimes 1)\delta(c))$ by coassociativity, definition of Δ and proposition 2.61,

ii) $\mathrm{Ad}_{U_2^{\overline{r}}}\, \hat{\delta}_0(\hat{s}_0) \mapsto 1 \otimes U^r \hat{s}_0 U^{r*}$ by definition of $\hat{\delta}_0$,

iii) $1 \otimes s_0 \mapsto 1 \otimes \phi(s_0)$ by part iv) of proposition 2.72.

Therefore, the image of the element above is

$$\big(\mathrm{Ad}_{U_3^r}\left((\delta \otimes 1)\delta(c)\right)\big)\left(1 \otimes 1 \otimes U^r \hat{s}_0 U^{r*} \cdot \phi(s_0)\right) \in L_C(C \otimes E_r^{\overline{r}} \otimes E_s).$$

This is equal to the image of the element

$$\delta(c)\left(1 \otimes \hat{s}_0 \cdot U^{r*}\phi(s_0)U^r\right) \in L_C(C \otimes E_r)$$

under the composition of the induction homomorphism

$$L_C(C \otimes E_r) \to L_C\big((C \otimes E_r, \delta) \otimes C \otimes E_r\big)$$

followed by the identification

$$((C \otimes E_r), \delta) \otimes C \otimes E_r \cong C \otimes E_r^{\overline{r}} \otimes E_r,$$

with conjugation by U_3^r. By proposition 2.73, one has $\hat{S}_0 \cdot U^{r*}S_0' U^r = K_A(E_r)$. Thus, the iterated crossed product $C \rtimes \hat{S}_0 \rtimes S_0$ is isomorphic to $\delta(C)(1_C \otimes K_A(E_r))$. Let $(\mathscr{C}, \delta)$ be a regular coaction of $(\mathscr{S}, \Delta)$ which (C, π, δ) is an extension of. Then the formula $\delta(\mathscr{C})(1 \otimes \mathscr{S}) = \mathscr{C} \otimes \mathscr{S}$ and proposition 2.38 imply

$$\begin{aligned} C \rtimes \hat{S}_0 \rtimes S_0 &\cong \delta(C)(1_C \otimes K_A(E_r)) \\ &= \overline{\mathrm{span}}_{\beta,\beta'}\ \delta(\mathscr{C})^{\beta}\big(1_C \otimes \mathscr{S}_{\mathrm{id}}^{\beta'} K_A(E_r)\big) \\ &= \overline{\mathrm{span}}_{\beta''}\ \big((\mathscr{C} \otimes \mathscr{S})^{\beta''}(1_C \otimes K_A(E_r))\big) = K_C(C \otimes E_r). \end{aligned}$$

Now, let us identify the bidual coaction. It maps the initial element to

$$\big(\delta(c) \otimes 1 \otimes 1\big)\left(1 \otimes \mathrm{Ad}_{U_2^{\overline{r}}}\left(\hat{\delta}_0(\hat{s}_0)\right) \otimes 1\right)\left(1 \otimes 1 \otimes \delta_0(s_0)\right)$$

and therefore the element above to

$$(\dagger) \qquad \big(\delta(c) \otimes 1\big)\left(1 \otimes \hat{s}_0 \otimes 1\right)\big(1 \otimes \left((\mathrm{Ad}_{U^s} \circ \phi \otimes 1)\delta_0(s_0)\right)\big).$$

On the other hand, by coassociativity, the map $\delta \otimes 1$ sends the element above to

$$\big((1 \otimes \Delta)\delta(c)\big)\left(1 \otimes 1 \otimes \hat{s}_0 \cdot U^{r*}\phi(s_0)U^r\right).$$

Conjugation by $W' = 1 \otimes \Sigma\hat{V}^{rr}$ acts on this element as follows:

i) $(1 \otimes \Delta)\delta(c) \mapsto \delta(c) \otimes 1 \otimes 1$ because $\Sigma\hat{V}^{rr}(\Delta(s))\hat{V}^{rr*}\Sigma = s \otimes 1$ for each $s \in \mathscr{S}$ by definition of Δ,

ii) $1 \otimes \hat{s}_0 \mapsto \hat{s}_0$ because $\hat{V}^{rr}$ commutes with $1 \otimes \hat{S}_0$ by proposition 2.61,

iii) $1 \otimes U^{r*}\phi(s_0)U^r \mapsto (\mathrm{Ad}_{U^r} \circ \phi \otimes 1)\delta_0(s_0)$ because

$$\begin{aligned}\Sigma\hat{V}^{rr}(1 \otimes U^{r*}\phi(s_0)U^r)\hat{V}^{rr*}\Sigma &= U_1^s V^{sr}(\phi(s_0) \otimes 1)V^{sr*}U_1^{s*} \\ &= U_1^s\big((\phi \otimes 1)(\delta_0(s_0))\big)U_1^{s*}\end{aligned}$$

by part iii) of proposition 2.72.

A comparison with the expression (†) completes the proof. □

Remark 2.75. A similar result holds for coactions of regular $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebras.

Roughly speaking, the preceding theorem shows that the restricted reduced crossed product functors

$$\boldsymbol{C}^{*reg}_{(\boldsymbol{S},\boldsymbol{\Delta})} \underset{-\rtimes \boldsymbol{S}_0}{\overset{-\rtimes \hat{\boldsymbol{S}}_0}{\rightleftarrows}} \boldsymbol{C}^{*reg}_{(\hat{\boldsymbol{S}},\hat{\boldsymbol{\Delta}})}.$$

are inverse equivalences of categories up to "Morita equivalence". To make this precise, one would have to define the notion of Morita equivalence for $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebras and $(\boldsymbol{S}, \boldsymbol{\Delta})$-algebras which involves coactions of Hopf C^*-families on C^*-modules. This could certainly be carried out, building on the theory developed in the first chapter and the theory of coactions of Hopf C^*-algebras on C^*-modules laid out in [2], but will not be pursued in this thesis. In the context of Kac systems associated to locally compact groups, a precise formulation of the slogan above is given in [2].

A second look at the definition

At the end it seems appropriate to have a second look at the definition of a pseudo-Kac system. Each member of the family V has been used to define an important ingredient of a pseudo-Kac system:

$V^{\overline{rr}}$ – for the Hopf C^*-family $(\hat{\mathscr{S}}, \hat{\Delta})$,

$\hat{V}^{rr}$ and hence V^{sr} – for the Hopf C^*-family $(\mathscr{S}, \Delta)$,

$V^{r\overline{r}}$ – for the canonical coaction $(\hat{\mathscr{S}}_0, \hat{\delta}_0)$,

$\hat{V}^{\overline{s}r}$ – for the canonical coaction $(\mathscr{S}_0, \delta_0)$,

$\tilde{V}^{rs}$ and hence V^{ss} – for dual coactions on reduced crossed products of the form $\mathscr{C} \rtimes \hat{\mathscr{S}}_0$,

$V^{\overline{sr}}$ – for dual coactions on reduced crossed products of the form $\mathscr{C} \rtimes \mathscr{S}_0$.

2.4.5 Extended example of a pseudo-Kac system

In this subsection, we present an example of a pseudo-Kac system which is not so much interesting in itself but rather serves as an illustration of the axiom system. It generalises the groupoid associated to the full equivalence relation $X \times X$ on a space X, but in contrast to pseudo-Kac systems associated to groupoids, here the underlying C^*-algebra may be non-commutative. Furthermore, of the families of C^*-modules and unitaries comprising the system, no two members need to coincide. The construction is inspired by examples in Franck Lesieur's thesis [32, section 7.6,7.7]. We subsequently discuss the pseudo-multiplicative unitary, an algebraic pseudo-Kac system and finally the C^*-pseudo-Kac system which is obtained as a completion of the former one.

The pseudo-multiplicative unitary

Let τ be a conditional expectation from a C^*-algebra A to a C^*-subalgebra B of $Z(A)$ such that $\tau(A)A = A$. We associate to τ a pseudo-multiplicative unitary as follows. The underlying C^*-module E and the representations π_s and π_r of A on E will be of the form

$$E = A_\tau \otimes A, \qquad \pi_s = 1 \otimes \varrho, \qquad \pi_r = \varrho_\tau \otimes 1,$$

where the individual components are defined as follows:

- A_τ is the C^*-A-B-module obtained via the Rieffel construction from τ, which is the completion of A with respect to the inner product $\langle a|a'\rangle := \tau(a^*a')$, and A and B act via left and right multiplication, respectively,
- $\varrho_\tau \colon A \to L_B(A_\tau)$ the implicit representation,
- $\varrho \colon A \to L_A(A)$ is the representation given by left multiplication,
- $E := A_\tau \otimes A$ is the internal tensor product over B, taken with respect to the representation $\varrho|_B \colon B \to L_A(A)$.

Denote by $a \mapsto a_\tau$ the canonical map $A \to A_\tau$. Since $BA = A$, the notation $1_\tau \otimes a$ is well-defined for each $a \in A$ even if A is not unital.

Proposition 2.76. *The formula*

$$V((a_\tau \otimes b) \otimes (c_\tau \otimes d)) := (a_\tau \otimes c) \otimes (1_\tau \otimes db), \qquad a, b, c, d \in A,$$

defines a regular pseudo-multiplicative unitary $V \colon E^s \otimes E \to E \otimes E^r$. If A is decomposable, V is right decomposable.

Proof. Throughout the proof, let a, b, c, d, e, f and a', b', c', d' denote arbitrary elements of A. Let us first show that V is isometric and hence also well-defined. One has

$$\begin{aligned}\big\langle (a_\tau \otimes c) \otimes (1_\tau \otimes db) \,\big|\, (a' \otimes c') \otimes (1_\tau \otimes d'b')\big\rangle_{(E \otimes E^r)} \\ &= \big\langle 1_\tau \otimes db \,\big|\, \varrho_\tau(\langle a_\tau \otimes c | a'_\tau \otimes c'\rangle_E) 1_\tau \otimes d'b'\big\rangle_E \\ &= \big\langle 1_\tau \otimes db \,\big|\, (c^* \tau(a^* a') c')_\tau \otimes d'b'\big\rangle_E \\ &= b^* d^* \cdot \tau(c^* c') \tau(a^* a') \cdot d'b',\end{aligned}$$

where we used the equation $\tau(c^* b c') = \tau(c^* c' b) = \tau(c^* c') b$ for $b := \tau(a^* a')$. On the other hand, one has

$$\begin{aligned}\big\langle (a_\tau \otimes b) \otimes (c_\tau \otimes d) \,\big|\, (a'_\tau \otimes b') \otimes (c'_\tau \otimes d')\big\rangle_{(E^s \otimes E)} \\ &= \big\langle a_\tau \otimes b \,\big|\, a'_\tau \otimes \varrho\big(\langle c_\tau \otimes d | c'_\tau \otimes d'\rangle_E\big) b'\big\rangle_E \\ &= \big\langle a_\tau \otimes b \,\big|\, a'_\tau \otimes d^* \tau(c^* c') d' b'\big\rangle_E \\ &= b^* \cdot \tau(a^* a') \cdot d^* \tau(c^* c') d' b'.\end{aligned}$$

Since B is contained in $Z(A)$, both expressions coincide. This proves that the formula given in the proposition defines an isometric operator V. Note that it is enough to consider elementary tensor products instead of finite sums in this calculation. In $E \otimes E^r$, one has for all $a, b, c, d \in A$ the equality

$$(a_\tau \otimes b) \otimes (c_\tau \otimes d) = (a_\tau \otimes b) \otimes \pi_r(c)(1_\tau \otimes d) = (a_\tau \otimes bc) \otimes (1_\tau \otimes d).$$

Therefore, the image of V is dense and V is a unitary.

Next, we show that V is pseudo-multiplicative. A short glance shows that V commutes with the representations π_{r1} and π_{s2}, and that $V\pi_{r2} = \pi_{s1}V$. The composition $V_{12}V_{13}V_{23} \colon E^s \otimes E^s \otimes E \to E \otimes E^r \otimes E^r$ is given by

$$\begin{aligned}(a_\tau \otimes b) \otimes (c_\tau \otimes d) \otimes (e_\tau \otimes f) &\xrightarrow{V_{23}} (a_\tau \otimes b) \otimes \big((c_\tau \otimes e) \otimes (1_\tau \otimes fd)\big) \\ &\xrightarrow{V_{13}} \big((a_\tau \otimes 1) \otimes (c_\tau \otimes e)\big) \otimes (1_\tau \otimes fdb) \\ &\xrightarrow{V_{12}} (a_\tau \otimes c) \otimes (1_\tau \otimes e) \otimes (1_\tau \otimes fdb),\end{aligned}$$

and the composition $V_{23}V_{12}$ is given by

$$\begin{aligned}(a_\tau \otimes b) \otimes (c_\tau \otimes d) \otimes (e_\tau \otimes f) &\xrightarrow{V_{12}} (a_\tau \otimes c) \otimes (1_\tau \otimes db) \otimes (e_\tau \otimes f) \\ &\xrightarrow{V_{23}} (a_\tau \otimes c) \otimes (1_\tau \otimes e) \otimes (1_\tau \otimes fdb).\end{aligned}$$

Therefore, V satisfies the pentagon equation.

Next, we show that V is regular. One has

$$\begin{aligned}\big(\langle a_\tau \otimes b|_1 V|c'_\tau \otimes d'\rangle_2\big)a'_\tau \otimes b' &= \langle a_\tau \otimes b|_1((a'_\tau \otimes c') \otimes (1_\tau \otimes d'b')) \\ &= \pi_r\big(\langle a_\tau \otimes b|a'_\tau \otimes c'\rangle\big)(1_\tau \otimes d'b') \\ &= (b^*\tau(a^*a')c')_\tau \otimes d'b' \\ &= (b^*c')_\tau \otimes \tau(a^*a')d'b' \\ &= |(b^*c')_\tau \otimes d'\rangle\langle a_\tau \otimes 1| \cdot (a'_\tau \otimes b'),\end{aligned}$$

where we used the fact that $\tau(a^*a')$ belongs to $Z(A)$.

Let us prove the last statement. For each $a \in A$ and $b\,\mathscr{C}ov_\gamma(A)$, $\gamma \in \mathrm{PAut}(A)$, the internal tensor product $a_\tau \otimes b$ is contained in $(A_\tau \otimes A)\,\mathrm{Dom}(\gamma)$, and the equation

$$(a_\tau \otimes b)a' = a_\tau \otimes ba' = a_\tau \otimes \gamma(a')b = \pi_s(\gamma(a'))(a_\tau \otimes b), \quad a' \in \mathrm{Dom}(\gamma),$$

shows that $a_\tau \otimes b$ belongs to $\mathscr{C}ov_\gamma(E^s)$. If A is decomposable, the closed linear span of such elements is equal to E, whence the C^*-bimodule E^s is decomposable. □

It would be interesting to check whether V is right decomposably regular; we leave this question unanswered because of lack of time.

To illustrate the construction, let us discuss a particular case. Let X be a compact space and μ a probability measure on X with support $\operatorname{supp}\mu = X$. Denote also by μ the state on the C^*-algebra $A := C(X)$ corresponding to the measure μ, i.e. $\mu(f) = \int_X f\,d\mu, f \in C(X)$. Put $B = \mathbb{C}$ and denote by V_μ the pseudo-multiplicative unitary associated to the state μ in the previous proposition. Denote by $G = X \times X$ the compact groupoid associated to the equivalence relation on X which identifies all points. For each $x \in X$, one has $G^x = \{x\} \times X$, and the family $\lambda^x := \delta_x \times \mu$, $x \in X$, is a left Haar system on X. Denote by V_G^{op} the opposite of the pseudo-multiplicative unitary associated to the groupoid G.

Proposition 2.77. *One has $V_\mu \cong V_\mu^{op}$.*

Proof. To distinguish between the different representations occurring in the definition of the operators V_μ and V_G^{op}, let us introduce indices and write $\pi_s^G, \pi_r^G, \pi_s^\mu, \pi_r^\mu$ and E_G, E_μ. Observe that $L^2(G,\lambda) = C(X, L^2(X,\mu))$. The representations π_s^G and π_r^G are given by

$$\big(\pi_s^G(f)\xi\big)(x,y) = f(y)\xi(x,y), \qquad \big(\pi_r^G(f)\xi\big)(x,y) = f(x)\xi(x,y),$$

where $f \in C(X)$, $\xi \in C(X, L^2(X,\mu))$ and $x, y \in X$. We identify E_μ and E_G via the map

$$\begin{gathered}\Upsilon\colon E_\mu = L^2(X,\mu) \otimes C(X) \xrightarrow{\cong} C(X, L^2(X,\mu)) = \overline{C(X) \odot C(X)} = E_G, \\ f_1 \otimes f_2 \mapsto f_2 \odot f_1, \quad f_1, f_2 \in C(X).\end{gathered}$$

Then one has $\Upsilon(\pi_s^\mu)\Upsilon^* = \pi_r^G$, $\Upsilon(\pi_r^\mu)\Upsilon^* = \pi_s^G$ and the map

$$W\colon E_G^r \otimes E_G \xrightarrow{\Upsilon^* \otimes \Upsilon^*} E_\mu^s \otimes E_\mu \xrightarrow{V_\mu} E_\mu \otimes E_\mu^r \xrightarrow{\Upsilon \otimes \Upsilon} E_G \otimes E_G^s$$

is given by

$$\begin{aligned}((f_1 \odot f_2) \otimes (g_1 \odot g_2)) &\xrightarrow{\Upsilon^* \otimes \Upsilon^*} (f_2 \otimes f_1) \otimes (g_2 \otimes g_1) \\ &\xrightarrow{V_\mu} (f_2 \otimes g_2) \otimes (1 \otimes g_1 f_1) \xrightarrow{\Upsilon \otimes \Upsilon} (g_2 \odot f_2) \otimes (g_1 f_1 \odot 1),\end{aligned}$$

for all $f_1, f_2, g_1, g_2 \in C(X)$. Rewriting this, we obtain

$$\begin{aligned}\big(W((f_1 \odot f_2) \otimes (g_1 \odot g_2))\big)((x,y),(z,x)) &= g_2(x) f_2(y) g_1(z) f_1(z) \\ &= f_1(z) f_2(y) g_1(z) g_2(x).\end{aligned}$$

On the other hand, a short calculation shows that the unitary V_G^{op} is given by

$$(V_G^{op} h)((x,y),(z,x)) = h\big((z,x)\cdot(x,y),(z,x)\big) = h((z,y),(z,x)).$$

for all $x, y, z \in X$ and $h \in C(X \times X)$. This proves $W = V_G^{op}$. □

The algebraic pseudo-Kac system

Let A be an algebra and let $B \subset Z(A)$ be a subalgebra such that $BA = A$. The space $E := A \odot_B A$, the representations

$$\begin{aligned}&\pi_{\overline{r}}, \pi_{\overline{s}}\colon A^{op} \to L(E), && \pi_{\overline{r}}(a^{op})(b \odot c) := b \odot ca, && \pi_{\overline{s}}(a^{op})(b \odot c) := ba \odot c, \\ &\pi_r, \pi_s\colon A \to L(E), && \pi_r(a)(b \odot c) := ab \odot c, && \pi_s(a)(b \odot c) := b \odot ac\end{aligned}$$

and the linear maps

$$\begin{aligned}&V\colon E \,{}_s\!\odot_{\overline{r}}\, E \to E \,{}_{\overline{r}}\!\odot_r\, E, && V(a \odot b) \odot (c \odot d) := (a \odot c) \odot (1 \odot db), \\ &U\colon E \to E, && U(a \odot b) := b \odot a\end{aligned}$$

form an algebraic pseudo-Kac system.

Proof. First, observe that the assumption on A and B implies that for each $a \in A$, the element $a \odot 1 \in A \odot_B A$ is well-defined. Second, observe that the assumption furthermore implies that the multiplication map $A \odot_A A \to A$ is an isomorphism. Therefore, the maps

$$\begin{aligned}&\Phi\colon E \,{}_s\!\odot_{\overline{r}}\, E \to A \odot_B A \odot_B A, && (a \odot b) \odot (c \odot d) \mapsto a \odot c \odot db, \\ &\Psi\colon E \,{}_{\overline{r}}\!\odot_r\, E \to A \odot_B A \odot_B A, && (a' \odot b') \odot (c' \odot d') \mapsto a' \odot b'c' \odot d',\end{aligned}$$

induce isomorphisms. A short calculation shows that $V = \Psi^{-1}\Phi$. This proves that V is a well-defined isomorphism. A short glance shows that U and V intertwine the representations $\pi_{\overline{r}}, \pi_{\overline{s}}, \pi_r$ and π_s as desired.

The calculation in proposition 2.76 shows that V satisfies the pentagon equation. The map $\Sigma(1 \odot U)V \colon E \,{}_s\!\odot_{\overline{r}} E \to E \,{}_s\!\odot_{\overline{r}} E$ and its powers are given by

$$
\begin{aligned}
(a \odot b) \odot (c \odot d) &\mapsto (\underbrace{db}_{a'} \odot \underbrace{1}_{b'}) \odot (\underbrace{a}_{c'} \odot \underbrace{c}_{d'}) \\
&\mapsto (\underbrace{c}_{a''} \odot \underbrace{1}_{b''}) \odot (\underbrace{db}_{c''} \odot \underbrace{a}_{d''}) \mapsto (a \odot 1) \odot (c \odot db).
\end{aligned}
$$

In $E \,{}_s\!\odot_{\overline{r}} E$, the last term is equal to $(a \odot b) \odot (c \odot d)$. Thus, $(\Sigma(1 \odot U)V)^3 = \mathrm{id}$. The map $\hat{V}$ is given by

$$
\begin{aligned}
\hat{V} \colon (a \odot b) \odot (c \odot d) &\xrightarrow{(U \odot \mathrm{id})\Sigma} (d \odot c) \odot (a \odot b) \\
&\xrightarrow{V} (d \odot a) \odot (1 \odot bc) \xrightarrow{\Sigma(U \odot \mathrm{id})} (1 \odot bc) \odot (a \odot d)
\end{aligned}
$$

for all $a, b, c, d \in A$. A short calculation shows that $\hat{V}$ satisfies the pentagon equation:

$$
\begin{array}{ccc}
(a \odot b) \odot (c \odot d) \odot (e \odot f) & \xrightarrow{\hat{V}_{12}} & (1 \odot bc) \odot (a \odot d) \odot (e \odot f) \\
 & & \downarrow \hat{V}_{23} \\
\hat{V}_{23} \downarrow & & (1 \odot bc) \odot (1 \odot de) \odot (a \odot f) \\
 & & \uparrow \hat{V}_{12} \\
(a \odot b) \odot (1 \odot de) \odot (c \odot f) & \xrightarrow{\hat{V}_{13}} & (1 \odot bc) \odot (1 \odot de) \odot (a \odot f)
\end{array}
$$

□

The pseudo-Kac system

Let τ be a conditional expectation from a C^*-algebra A to a C^*-subalgebra B of $Z(A)$ such that $\tau(A)A = A$. Additionally, let us assume that τ satisfies $\tau(aa') = \tau(a'a)$ for all $a, a' \in A$. We associate to τ a pseudo-Kac system, building on the algebraic pseudo-Kac system introduced above.

Denote by τ^{op} the opposite trace on A^{op} and let (A_τ, ϱ_τ) and $(A^{op}_{\tau^{op}}, \varrho_{\tau^{op}})$ denote the corresponding Rieffel constructions. Furthermore, denote by $a \mapsto a^{op}$ the canonical anti-isomorphism $A \xrightarrow{\cong} A^{op}$. Consider the following family $\mathcal{E}$ of C^*-modules.

$$
E_{\overline{r}} := A_\tau \otimes A, \qquad E_{\overline{s}} := A \otimes A_\tau, \qquad E_r := A^{op} \otimes A^{op}_{\tau^{op}}, \qquad E_s := A^{op}_{\tau^{op}} \otimes A^{op}.
$$

These C^*-modules identify with the completions of the space $A \odot_B A$ with respect to the following inner products.

$$
\begin{aligned}
\langle a \odot b | a' \odot b' \rangle_{E_{\overline{r}}} &:= \tau(a^* a') b^* b', & \langle a \odot b | a' \odot b' \rangle_{E_{\overline{s}}} &:= a^* a' \tau(b^* b'), \\
\langle a \odot b | a' \odot b' \rangle_{E_r} &:= [a' a^* \tau(b' b^*)]^{op}, & \langle a \odot b | a' \odot b' \rangle_{E_s} &:= [\tau(a' a^*) b' b^*]^{op}.
\end{aligned}
$$

Under this identification, the right module structures of E_r and E_s over A^{op} are induced by $(a \odot b) \cdot c^{op} = ca \odot b$ and $(a \odot b) \cdot c^{op} = a \odot cb$, respectively.

Proposition 2.78. *The algebraic pseudo-Kac system associated to A and B in the previous paragraph induces a pseudo-Kac system.*

Proof. It is immediate that the flip $a \odot b \mapsto b \odot a$ extends to a family of unitaries as in parts iii),v) of definition 2.53.

Let us check that the map $(a \odot b) \odot (c \odot d) \mapsto (a \odot c) \odot (1 \odot db)$ induces a family of unitaries V as claimed. For all $a, b, c, d, a', b', c', d' \in A$, one has

$$\langle (a \odot b) \otimes (c \odot d) | (a' \odot b') \otimes (c' \odot d') \rangle_{(E_x^s \otimes E_{\overline{r}})} =$$
$$= \langle a \odot b | a' \odot \tau(c^*c')d^*d'b' \rangle_{E_x} = \begin{cases} [a'a^*\tau(c^*c')\tau(d^*d'b'b^*)]^{op}, & x = r, \\ a^*a'\tau(c^*c')\tau(b^*d^*d'b'), & x = \overline{s}, \\ \tau(a^*a')\tau(c^*c')b^*d^*d'b', & x = \overline{r}, \end{cases}$$

$$\langle (a \odot c) \otimes (1 \odot db) | (a' \odot c') \otimes (1 \odot d'b') \rangle_{(E_x^{\overline{r}} \otimes E_r)} =$$
$$= \langle a \odot c | a' \odot c'\tau(d'b'b^*d^*) \rangle_{E_x} = \begin{cases} [a'a^*\tau(d'b'b^*d^*)\tau(c'c^*)]^{op}, & x = r, \\ a^*a'\tau(c^*c')\tau(d'b'b^*d^*), & x = \overline{s}, \\ [\tau(a'a^*)c'\tau(d'b'b^*d^*)c^*]^{op}, & x = s. \end{cases}$$

Here, we used the fact that the image of τ lies in the centre of $M(A)$. Since τ is tracial, we can deduce the existence of the unitaries $V^{r\overline{r}}\colon E_r^s \otimes E_{\overline{r}} \to E_r^{\overline{r}} \otimes E_r$ and $V^{\overline{s}r}\colon E_{\overline{s}}^s \otimes E_{\overline{r}} \to E_{\overline{s}}^{\overline{r}} \otimes E_r$. Furthermore, we have

$$\langle (a \odot b) \otimes (c \odot d) | (a' \odot b') \otimes (c' \odot d') \rangle_{(E_s \otimes E_x^{\overline{r}})} =$$
$$= \langle (c \odot d) | (c' \odot d'\tau(a'a^*)b'b^*) \rangle_{E_x} = \begin{cases} c^*c'\tau(d^*d'b'b^*)\tau(a'a^*), & x = \overline{s} \\ [\tau(c'c^*)d'\tau(a'a^*)b'b^*d^*]^{op}, & x = s, \\ [c'c^*\tau(d'b'b^*d^*)\tau(a'a^*)]^{op}, & x = r, \end{cases}$$

$$\langle (a \odot c) \otimes (1 \odot db) | (a' \odot c') \otimes (1 \odot d'b') \rangle_{(E_{\overline{r}} \otimes E_x^r)} =$$
$$= \langle 1 \odot db | \tau(a^*a')c^*c' \odot d'b' \rangle_{E_x} = \begin{cases} \tau(a^*a')c^*c'\tau(b^*d^*d'b'), & x = \overline{s}, \\ [\tau(a^*a')\tau(c^*c')d'b'b^*d^*]^{op}, & x = s, \\ \tau(a^*a')\tau(c^*c')b^*d^*d'b', & x = \overline{r}. \end{cases}$$

Again, since τ is tracial, we can deduce the existence of the unitaries $V^{s\overline{s}}\colon E_s \otimes E_{\overline{s}}^{\overline{r}} \to E_{\overline{r}} \otimes E_{\overline{s}}^r$ and $V^{ss}\colon E_s \otimes E_s^{\overline{r}} \to E_{\overline{r}} \otimes E_s^r$. Finally, comparing the last two lines in each block, we obtain the existence of the unitaries $V^{\overline{r}\overline{r}}\colon E_{\overline{r}}^{\hat{r}} \otimes E_{\overline{r}} \to E_{\overline{r}} \otimes E_{\overline{r}}^r$ and $V^{sr}\colon E_s \otimes E_r^{\overline{r}} \to E_s^{\overline{r}} \otimes E_r$.

The pseudo-multiplicative unitary $V^{\overline{r}\overline{r}}$ is regular by proposition 2.76. A similar calculation shows that the pseudo-multiplicative unitary $\hat{V}^{rr}\colon E_r^{\overline{r}} \otimes E_r \to E_r \otimes E_r^{\overline{s}}$ is regular. □

Chapter 3

Applications to locally compact groupoids

Fundamental examples for pseudo-multiplicative unitaries and pseudo-Kac systems are those associated locally compact Hausdorff groupoids. The pseudo-multiplicative unitary and the pseudo-Kac system are decomposable if the underlying groupoid satisfies a certain decomposability condition which is close to being r-discrete. This property – which is the topic of the first section – provides a control over the discrepancy between the range and the source map of the groupoid and entails a number of consequences which may be interesting in their own right. It is a spacial analogue to the decomposability property for C^*-bimodules introduced in the first chapter.

For pseudo-Kac systems associated to decomposable groupoids, the constructions and notions introduced in the previous chapter can be made very transparent. The legs are easily computed, and coactions of these legs can be related to classical notions such as groupoid actions and Fell bundles. This is done in the second and the third section.

In applications of groupoids, the Hausdorff condition is frequently not satisfied. The study of non-Hausdorff groupoids requires its own methods: one can approach the situation via topology and construct a Hausdorff "resolution" of such a groupoid [55], or via algebra and construct auxiliary algebras or modules out of special functions on such a groupoid [8, 23]. In the last section, we present a new approach along the first line which is related to a Hausdorff compactification studied by James Fell [16]. The results obtained elucidate several other constructions considered by Mahmood Khoshkam, Georges Skandalis [23] and Jean-Louis Tu [55].

3.1 Decomposable groupoids

In this section, we introduce a decomposability condition for groupoids which will render the associated C^*-bimodules and hence also the pseudo-multiplicative unitaries and the associated pseudo-Kac systems decomposable. The precise definition, an equivalent characterisation, first implications and the decomposability result for the associated C^*-bimodule are presented in the first subsection.

The following two subsections are included for their own interest. First, we study Haar systems on decomposable groupoids. Then the techniques and constructions introduced in the first chapter are illustrated by representations of decomposable groupoids. Partial integration of a continuous representation yields a C^*-family, and several constructions on the level of representations have counterparts on the level of C^*-families.

Throughout this section, let G be a locally compact groupoid with open range and source maps.

3.1.1 Definition and first properties

Generally, the range and the source map of a groupoid are mutually independent in the following sense: given a point of the groupoid, its image under the range map can not be computed from its image under the source map. We call the groupoid decomposable if, locally on the groupoid, the range and the source map differ by a partial homeomorphism of the unit space.

Definition 3.1. *An open subset $U \subset G$ is* covariant *if it satisfies the following assumption:*

$$\forall x, y \in U \colon r(x) = r(y) \Leftrightarrow s(x) = s(y).$$

We denote by $\mathscr{C}ov(G)$ the family of all open covariant subsets of G. The groupoid G is decomposable *if the family $\mathscr{C}ov(G)$ forms a basis for the topology of G.*

As a first example, note that every r-discrete groupoid is decomposable. Of particular interest to us will be the dynamics on the unit space introduced by the covariant subsets of G or, more precisely, by the adjoint action of G.

Proposition 3.2. *i) Let $U \subset G$ be an open covariant subset. Then the map $s(x) \mapsto r(x), x \in U$, defines a homeomorphism $q_U \colon s(U) \to r(U)$.*

ii) Let G be decomposable. For each covariant open subset $U \subset G$, denote by $\mathfrak{q}_U \colon U \to \mathfrak{PHom}(G^0)$ the map $x \mapsto [q_U, s(x)], x \in U$. Then the family $\left(\mathfrak{q}_U\right)_U$ defines a continuous groupoid homomorphism $\mathfrak{q} \colon G \to \mathfrak{PHom}(G^0)$.

iii) For each $\phi \in \mathrm{PHom}(G^0)$, the subset $\mathrm{Cov}_\phi(G) := \mathfrak{q}^{-1}([\phi])$ of G is open and covariant.

iv) For each germ $[\phi, v] \in \mathfrak{PHom}(G^0)$, the subset $\mathrm{Cov}_{[\phi,v]}(G) := \mathfrak{q}^{-1}([\phi, v])$ is relatively closed and open in $G^{\phi(v)}$ and in G_v.

v) The subsets of $\mathrm{PHom}(G^0)$ *given by*

$$\{q_U \mid U \in \mathscr{C}ov(G)\}, \quad \{\phi \in \mathrm{PHom}(G^0) \mid [\phi] \subset \mathfrak{q}(G)\}$$

are equal and form an inverse semigroup.

Proof. i) By assumption on U, the map q_U is a bijection. Since the range and source maps are continuous and open, it is a homeomorphism.

ii) It is clear that for each pair of open covariant subsets $U, U' \in \mathscr{C}ov(G)$, the restrictions $\mathfrak{q}_U|_{U\cap U'}$ and $\mathfrak{q}_{U'}|_{U\cap U'}$ coincide. Since $\mathscr{C}ov(G)$ covers G, one obtains a well-defined map $\mathfrak{q}\colon G \to \mathfrak{PHom}(G^0)$. It is also clear that this map is a homomorphism. Let us show that it is continuous. Let ϕ be a partial homeomorphism on G^0 and let $U \in \mathscr{C}ov(G)$. Then the intersection $\mathfrak{q}^{-1}([\phi]) \cap U$ is equal to the set of points $x \in U$ such that $q_U|_V = \phi|_V$ for some neighbourhood V of $s(x)$. Therefore, this intersection is open, and hence $\mathfrak{q}^{-1}([\phi])$ is open in G. Since sets of the form $[\phi], \phi \in \mathrm{PHom}(G^0)$, constitute a basis for the topology on $\mathfrak{PHom}(G^0)$, the map $\mathfrak{q}$ is continuous.

iii) The set $\mathrm{Cov}_\phi(G)$ is open because $\mathfrak{q}$ is continuous, and covariant because $r(x) = \phi(s(x))$ for each $x \in \mathrm{Cov}_\phi(G)$.

iv) The set $\mathrm{Cov}_{[\phi,v]}(G)$ is relatively open in G_v because it is equal to the intersection of the open set $\mathrm{Cov}_\phi(G)$ with G_v. It is relatively closed because G_v is the disjoint union of the sets $\mathrm{Cov}_{\mathfrak{x}}(G)$ where $\mathfrak{x} \in \mathfrak{PHom}(G^0)_v$, each of which is open. The statement concerning $G^{\phi(v)}$ follows similarly.

v) The left hand side is contained in the right hand side because $\mathfrak{q}(U) = [q_U]$ for each $U \in \mathscr{C}ov(G)$. Let $\phi \in \mathrm{PHom}(G^0)$. If $[\phi] \subset \mathfrak{q}(G)$, there exists a family $(U_\nu)_\nu$ of covariant open subsets of G such that $[\phi] = \bigcup_\nu \mathfrak{q}(U_\nu)$. Then the union $U := \bigcup_\nu U_\nu$ is open and covariant since $r(x) = \phi(s(x))$ for all $x \in U$, and $q_U = \phi$. Therefore, the right hand side is contained in the left hand side.

The last claim follows from the equations $q_U \circ q_{U'} = q_{UU'}$ and $q_U^{-1} = q_{(U^{-1})}$, $U, U' \in \mathscr{C}ov(G)$. □

The homomorphism $\mathfrak{q}\colon G \to \mathfrak{PHom}(G^0)$ constructed in the previous proposition yields another characterisation of decomposable groupoids – they are exactly the extensions of r-discrete groupoids by group bundles.

Definition 3.3. *The* stationary subgroupoid *of G is the subspace* $\mathrm{Stat}(G) := \mathrm{Cov}_{\mathrm{id}}(G)$ *of G, equipped with the operations inherited from G.*

Note that by proposition 3.2 iii), the stationary subgroupoid is open in G. Recall that a sequence of groupoid homomorphisms $N \xrightarrow{\sigma} G \xrightarrow{\rho} H$ is *exact* if $\sigma(N) = \ker\rho$, where $\ker\rho = \{x \in G \mid \rho(x) \in H^0\}$, and an *extension* if furthermore σ is injective and ρ is surjective.

Proposition 3.4. *G is decomposable if and only if there exists an extension $N \to G \to H$ where N is stationary in the sense that $r(n) = s(n)$ for all $n \in N$, and H is r-discrete.*

Proof. Assume that G is decomposable. Then the sequence $\mathrm{Stat}(G) \hookrightarrow G \xrightarrow{\mathfrak{q}} \mathfrak{q}(G)$ is an extension of the form desired.

Conversely, let $N \xrightarrow{\sigma} G \xrightarrow{\rho} H$ be an extension of the form above. Then the inverse image of each open covariant subset $U \subset H$ under ρ is open and covariant in G, and the family $\{\rho^{-1}(U) \mid U \in \mathscr{C}ov(H)\}$ covers G. □

The next result is the primary reason for our interest in decomposable groupoids – it implies that the associated pseudo-Kac system is decomposable. Assume that the groupoid G is Hausdorff and let λ be a left Haar system for G. Denote by π_r and π_s the representations of $C_0(G^0)$ on the C^*-module $L^2(G,\lambda)$ given by

$$\big(\pi_s(f)\xi\big)(x) = f(s(x))\xi(x), \qquad \big(\pi_r(f)\xi\big)(x) = f(r(x))\xi(x),$$

for all $x \in G$, $\xi \in L^2(G,\lambda)$ and $f \in C_0(G^0)$.

Proposition 3.5. *Let G be a decomposable Hausdorff groupoid with left Haar system λ. Then the C^*-bimodule $\big(L^2(G,\lambda),\pi_s\big)$ is decomposable, and one has*

$$\mathscr{C}ov_{\phi^*}\big(L^2(G,\lambda),\pi_s\big) = \overline{C_c(\mathrm{Cov}_\phi(G))}, \qquad \phi \in \mathrm{PHom}(G^0).$$

Proof. Since G is decomposable, the support of each function $f \in C_c(G)$ can be covered by a finite collection of covariant open sets. By a partition of unity argument, we can write f as a finite sum of functions $f_i \in C_c(U_i)$ where $U_i \in \mathscr{C}ov(G)$. Therefore, the linear span of the family of subspaces $C_c(U), U \in \mathscr{C}ov(G)$, is dense in $L^2(G,\lambda)$.

Let $\phi \in \mathrm{PHom}(G^0)$. Then $\mathrm{Dom}(\phi^*) = C_0(\mathrm{Im}(\phi))$, and for each $f \in C_0(\mathrm{Im}(\phi))$, $\xi \in C_c(\mathrm{Cov}_\phi(G))$ and $x \in \mathrm{supp}(\xi)$, one has

$$(\xi f)(x) = \xi(x)f(r(x)) = \xi(x)f(\phi(s(x))) = (\phi^* f)(s(x))\xi(x) = \big(\pi_s(\phi^* f)\xi\big)(x).$$

Clearly, $C_c(\mathrm{Cov}_\phi(G))\, C_0(\mathrm{Im}(\phi)) = C_c(\mathrm{Cov}_\phi(G))$. Therefore, $C_c(\mathrm{Cov}_\phi(G))$ is contained in $\mathscr{C}ov_{\phi^*}(L^2(G,\lambda))$. By part iv) of the previous proposition, the fibre of the module $L^2(G,\lambda)$ over a point $v \in G^0$ is equal to

$$\big(L^2(G,\lambda)\big)_v = L^2(G^v,\lambda^v) = \bigoplus_{\mathfrak{x}\in\mathfrak{PHom}(G^0)^v} \overline{C_c(\mathrm{Cov}_{\mathfrak{x}}(G))}, \quad v \in G^0.$$

Therefore, the fibres of the two sub-modules of $L^2(G,\lambda)$ given in the first equation of the proposition coincide, and hence, the two sub-modules coincide as well. By the argument given above, this also implies that the C^*-bimodule $\big(L^2(G,\lambda),\pi_s\big)$ is decomposable. □

Remark 3.6. The C^*-bimodule $\big(L^2(G,\lambda),\pi_r\big)$ satisfies $\mathscr{C}ov_{\mathrm{id}}\big(L^2(G,\lambda),\pi_r\big) = L^2(G,\lambda)$ because π_r coincides with the right module structure, and therefore is decomposable independent of any assumption on G.

3.1.2 Haar systems

Haar systems on locally compact groupoids differ from Haar measures on locally compact groups in two decisive points – neither do they need to exist nor do they need to be unique. For an r-discrete groupoid, however, the fibres of the range map are discrete, and the family of counting measures is a canonical choice for a Haar system. For a decomposable groupoid, the situation is similar: the fibres of the range map are disjoint unions of translates of locally compact groups. This implies that Haar systems on a decomposable groupoid correspond bijectively with Haar systems on the stationary subgroupoid, as will be shown in the next proposition. Then we introduce unimodular Haar systems and study the interrelation of Haar systems with groupoid homomorphisms. Apart from the next subsection, none of these results is used anywhere else in this thesis.

Proposition 3.7. *Let G be a decomposable Hausdorff groupoid and let λ_0 be a Haar system for* $\mathrm{Stat}(G)$*.*

i) For each $x \in G$, the map $y \mapsto xy$ defines a homeomorphism $m_x\colon \mathrm{Stat}(G)^{s(x)} \to \mathrm{Cov}_{\mathfrak{q}(x)}(G)$.

ii) Let $\mathfrak{x} \in \mathfrak{q}(G)$. Then for each pair of points $x, x' \in \mathfrak{q}^{-1}(\mathfrak{x})$, the measures $m_{x*}\big(\lambda_0^{s(x)}\big)$ and $m_{x'*}\big(\lambda_0^{s(x')}\big)$ on $\mathrm{Cov}_{\mathfrak{x}}(G)$ coincide. Denote this measure by $\lambda^{\mathfrak{x}}$.

iii) For each $v \in G^0$, denote by λ^v the unique Borel measure on G^v whose restriction to $\mathrm{Cov}_{\mathfrak{x}}(G)$ is equal to $\lambda^{\mathfrak{x}}$ for each germ $\mathfrak{x} \in \mathfrak{PHom}\,(G^0)^v$. Then the family $(\lambda^v)_{v\in G^0}$ is a Haar system for G.

iv) If λ' is a left Haar system on G satisfying $\lambda'^v|_{\mathrm{Stat}(G)^v} = \lambda_0^v$ for all $v \in G^0$, then $\lambda' = \lambda$. The assignments

$$\lambda_0 \mapsto \lambda \quad \text{and} \quad \lambda' \mapsto \lambda'|_{\mathrm{Stat}(G)} := \big(\lambda'^v|_{\mathrm{Stat}(G)^v}\big)_{v\in G^0}$$

define a bijection between the set of left Haar systems on G and on $\mathrm{Stat}(G)$*, respectively.*

Proof. i) Obvious.

ii) One has $m_{x'*}\big(\lambda_0^{s(x)}\big) = m_{x*}\big(m_{x^{-1}x'*}\big(\lambda_0^{s(x)}\big)\big)$. Since the product $x^{-1}x'$ is contained in $\mathrm{Stat}(G)^{s(x)}$ and the measure λ_0 is left-invariant, one has $m_{x^{-1}x'*}\big(\lambda_0^{s(x)}\big) = \lambda_0^{s(x)}$. The claim follows.

iii) We check conditions i)-iii) in the definition of a left Haar system. Let $v \in G^0$. Since λ_0^v is a regular Borel measure with support $\mathrm{Stat}(G)^v$ and the map m_x is a homeomorphism for each $x \in G$, each of the measures $\lambda^{\mathfrak{x}}$, $\mathfrak{x} \in \mathfrak{PHom}\,(G^0)^v$, is regular. By proposition 3.2 iv), λ^v is regular as well.

Next, let $f \in C_c(\mathrm{Cov}_\phi(G))$ for some $\phi \in \mathrm{PHom}(G^0)$ and let $v \in \mathrm{Im}(\phi)$. We need to show that the function F on $\mathrm{Im}(\phi)$ defined by

$$F(v') := \int_{G^{v'}} f(y) d\lambda^{v'}(y)$$

is continuous at v. Let $(v_\nu)_\nu$ be a net in $\mathrm{Im}(\phi)$ converging to v. We can choose a point x and a net $(x_\nu)_\nu$ converging to x in $\mathrm{Cov}_\phi(G)$ such that $r(x) = v$ and $r(x_\nu) = v_\nu$ for all ν. Put $g := (m_x)^*\big(f|_{G^{\phi^{-1}(v)}}\big)$ and $g_\nu := (m_{x_\nu})^*\big(f|_{G^{\phi^{-1}(v_\nu)}}\big)$. Then

$$F(v_\nu) = \int_{\mathrm{Stat}(G)^{\phi^{-1}(v)}} g_\nu(y) d\lambda_0^{\phi^{-1}(v)}(y),$$

and a similar equation holds for $F(v)$. By the Tietze extension theorem, we can find a continuous function $h \in C_c(\mathrm{Stat}(G))$ extending g. Denote by g'_ν the restriction of h to $\mathrm{Stat}(G)^{v_\nu}$. By assumption on λ_0, the net

$$F'_\nu := \int_{\mathrm{Stat}(G)^{\phi^{-1}(v')}} g'_\nu(y) d\lambda_0^{\phi^{-1}(v)}(y)$$

converges to $F(v)$. Since f and h are continuous and have compact support and since x_ν converges to x, we obtain $\|g_\nu - g'_\nu\| \to 0$ as $\nu \to \infty$. Furthermore, by construction, the union $\bigcup_\nu \mathrm{supp}(g_\nu - g'_\nu)$ is contained in a compact set. Therefore,

$$F(v_\nu) - F'_\nu = \int_{\mathrm{Stat}(G)^{\phi^{-1}(v_\nu)}} \left(g_\nu - g'_\nu\right)(y) d\lambda_0^{\phi^{-1}(v_\nu)}(y) \to 0 \quad \text{as} \quad \nu \to \infty.$$

This proves that $F(v_\nu) \to F(v)$ as $\nu \to \infty$.

The fact that the family $(\lambda^v)_v$ is left-invariant follows from the decompositions

$$\begin{aligned}
\left(G^{s(x)}, \lambda^{s(x)}\right) &= \coprod_{\mathfrak{q}(x') \in \mathfrak{q}(G^{s(x)})} \Big(\mathrm{Cov}_{\mathfrak{q}(x')}(G), m_{x'*}\big(\lambda_0^{s(x')}\big)\Big), \\
\left(G^{r(x)}, \lambda^{r(x)}\right) &= \coprod_{\mathfrak{q}(x') \in \mathfrak{q}(G^{s(x)})} \Big(\mathrm{Cov}_{\mathfrak{q}(xx')}(G), m_{xx'*}\big(\lambda_0^{s(x')}\big)\Big), \quad x \in G.
\end{aligned}$$

iv) The first statement follows from left-invariance of the Haar system λ', and the second statement is an immediate corollary. □

Remark 3.8. If locally, the map r has continuous sections, the proof of the continuity condition in part iii) substantially simplifies: If σ is a section of the range map $r\colon \mathrm{Cov}_\phi(G) \to \mathrm{Im}(\phi)$ defined on a neighbourhood $V \subset G^0$ of v, then

$$\int_{G^{v'}} f(x) d\lambda^{v'}(x) = \int_{\mathrm{Stat}(G)^{\phi^{-1}(v')}} f(\sigma(v')y) d\lambda_0^{\phi^{-1}(v')}(y), \quad v' \in V.$$

The function assigning this integral to v' is continuous on V by assumption on λ_0.

The following definition is motivated by the application in the next subsection.

Definition 3.9. *Let G be a decomposable Hausdorff groupoid. A Haar system λ for G is* unimodular, *if for each germ $\mathfrak{x} \in \mathfrak{PHom}(G^0)$, the restrictions of the measures $\lambda^{r(\mathfrak{x})}$ and $\lambda^{-1}_{s(\mathfrak{x})}$ to the set $\mathrm{Cov}_{\mathfrak{x}}(G)$ coincide.*

Corollary 3.10. *Let G be a decomposable Hausdorff groupoid. If its stationary groupoid $\mathrm{Stat}(G)$ admits a unimodular left Haar system which is invariant under the adjoint action of G, i.e. which satisfies $\lambda_0^{r(x)} = \big(\mathrm{Ad}_x\big)_*\big(\lambda_0^{s(x)}\big)$ for all $x \in G$, where $\mathrm{Ad}_x \colon G^{s(x)} \to G^{r(x)}$ is given by $y \mapsto xyx^{-1}$, then G has a unimodular left Haar system, too.*

Proof. Let λ_0 be a left Haar system on $\mathrm{Stat}(G)$ as above and denote by λ the associated left Haar system on G, see proposition 3.7. Let $x \in G$ and put $\mathfrak{x} := \mathfrak{q}(x)$. Denote by $i \colon G \to G$ the inversion map $y \mapsto y^{-1}$. Then $\lambda^{-1}_{s(x)} = i_*(\lambda^{s(x)})$. By assumption on λ_0, in the notation of proposition 3.7 one has

$$\begin{aligned}
\big(\lambda^{-1}_{s(x)}\big)\big|_{\mathrm{Cov}_{\mathfrak{x}}(G)} &= i_*\Big(\lambda^{s(x)}\big|_{i(\mathrm{Cov}_{\mathfrak{x}}(G))}\Big)\\
&= i_*\Big(\lambda^{r(x^{-1})}\big|_{\mathrm{Cov}_{\mathfrak{x}^{-1}}(G)}\Big)\\
&= (i \circ m_{x^{-1}})_*\big(\lambda_0^{s(x^{-1})}\big)\\
&= (m_x \circ \mathrm{Ad}_{x^{-1}} \circ i)_*\big(\lambda_0^{s(x^{-1})}\big) = m_{x*}\big(\lambda_0^{s(x)}\big).
\end{aligned}$$

Here, we used unimodularity of the Haar system λ_0 and the equation $i \circ m_{x^{-1}} = m_x \circ \mathrm{Ad}_{x^{-1}} \circ i$,

$$\begin{aligned}
(i \circ m_{x^{-1}})(y) = i(x^{-1}y) &= y^{-1}x\\
&= x(x^{-1}y^{-1}x) = (m_x \circ \mathrm{Ad}_{x^{-1}} \circ i)(y),\ y \in G^{s(x^{-1})}. \quad \square
\end{aligned}$$

In the next subsection, we will need to compare Haar systems on the domain and the range of a groupoid homomorphism. The following proposition shows that under certain surjectivity assumptions, the push-forward of the Haar system of the domain is related to the Haar system on the range by a continuous “modular” function on the domain.

Proposition 3.11. *Let $\sigma \colon G' \to G$ be a proper continuous homomorphism of decomposable Hausdorff groupoids with left Haar systems λ' and λ, respectively. If the restriction $\sigma^{v'} \colon G'^{v'} \to G^{\sigma(v')}$ is surjective for each $v' \in G'^0$, there exists a continuous strictly positive function D_σ on G' such that $\lambda^{\sigma(v')} = \sigma_*\big(D_\sigma \lambda'^{v'}\big)$ for each $v' \in G'^0$.*

Proof. Let $v' \in G'^0$ and put $v := \sigma(v')$. The measure $\mu^{v'} := \sigma_*\big(\lambda'^{v'}\big)|_{\mathrm{Stat}(G)^v}$ on the group $\mathrm{Stat}(G)^v$ is non-zero. The assumption on $\sigma^{v'}$ and the left-invariance of the measure $\lambda'^{v'}$ imply that $\mu^{v'}$ is left-invariant also. Since $\lambda^v|_{\mathrm{Stat}(G)^v}$ is a non-zero left-invariant measure on the locally compact group $\mathrm{Stat}(G)^v$ as well, there exists a number $D^0_\sigma(v') > 0$ such that $\lambda^v|_{\mathrm{Stat}(G)^v} = D^0_\sigma(v')\mu^{v'}$.

We show that the function D^0_σ thus defined is continuous at v'. Choose a non-negative function $f \in C_c(\mathrm{Stat}(G))$ such that $f(v) = 1$. Then the functions F' on G'^0 and F on G^0 given by

$$F(u) := \int_{\mathrm{Stat}(G)^{\sigma(u)}} f(x) d\lambda^{\sigma(u)}(x), \qquad F'(u') := \int_{G'^{u'}} f(\sigma(x'))d\lambda'^{u'}(x'),$$

are continuous and related by the formula

$$F(\sigma(u'))D^0_\sigma(u') = \int_{\mathrm{Stat}(G)^{\sigma(u')}} f(x)d\underbrace{\big(D^0_\sigma(u')\lambda^{\sigma(u')}\big)}_{\sigma_*\left(\lambda'^{u'}\right)}(x) = F'(u').$$

Since the functions σ^*F and F' are strictly positive on a neighbourhood of v', the function D^0_σ must be continuous at v' as well.

Put $D_\sigma := s^*(D^0_\sigma)$. We show that $\lambda^v = \sigma_*\big(D_\sigma\lambda'^{v'}\big)$. Let $x \in G^v$ and choose $x' \in G^{v'}$ such that $\sigma(x') = x$. Then, by proposition 3.7, one has

$$\begin{aligned}
\lambda^v|_{\mathrm{Cov}_{\mathfrak{q}(x)}(G)} &= m_{x*}\Big(\lambda^{s(x)}|_{\mathrm{Stat}(G)^{s(x)}}\Big) \\
&= \underbrace{(m_x \circ \sigma)_*}_{(\sigma\circ m_{x'})_*}\Big(D^0_\sigma(s(x'))\lambda'^{s(x')}|_{\mathrm{Stat}(G')^{s(x')}}\Big) \\
&= \sigma_*\Big(D^0_\sigma(s(x'))\lambda'^{s(x')}|_{\mathrm{Cov}_{\mathfrak{q}(x')}(G')}\Big) = \sigma_*\big(D_\sigma\lambda'^{s(x')}\big)|_{\mathrm{Cov}_{\mathfrak{q}(x)}(G)}. \qquad \square
\end{aligned}$$

3.1.3 C^*-families associated to representations

This subsection is included as an extended example for the notions and constructions introduced in the first chapter and will be used nowhere else in this thesis.

To each continuous representation of a decomposable groupoid, we associate a C^*-family which encodes that representation. This C^*-family is obtained by partial integration along the fibres of the range map. The relation to the classical integration procedure of measurable representations which involves an additional integration over the unit space is discussed in the next paragraph. The functoriality of this association is clarified in the last paragraph.

Throughout this section, let G be a decomposable Hausdorff groupoid with a unimodular left Haar system λ. A *continuous representation* of G consists of a continuous Hilbert bundle H on G^0 and a family of unitaries $U_x\colon H_{s(x)} \to H_{r(x)}, x \in G$, such that

i) $U_v = \mathrm{id}_{H_v}$ for all $v \in G^0$,

ii) $U_xU_y = U_{xy}$ for all $x, y \in G$,

iii) $(U_x)^{-1} = (U_{x^{-1}})$ for all $x \in G$,

iv) for every pair of continuous sections η, ξ of H, the function on G given by $x \mapsto \langle \eta(r(x)) \big| U_x \xi(s(x)) \rangle$ is continuous.

The *trivial representation* of G consists of the trivial Hilbert bundle $\mathbb{C} \times G^0$ and the family of identity operators $\mathrm{id}_x = \mathrm{id}_{\mathbb{C}}, x \in G$.
The *left regular representation* consists of the Hilbert bundle $L^2(G, \lambda)$ and the family of unitaries $U_x \colon L^2\big(G^{s(x)}, \lambda^{s(x)}\big) \rightarrow L^2\big(G^{r(x)}, \lambda^{r(x)}\big)$ given by $(U_x f) = f(x^{-1} \cdot -)$, $x \in G$.

Throughout this section, let (H, U) be a continuous representation of G.

The C^*-family associated to a continuous representation

Consider the C^*-module $\Gamma_0(H)$ as a C^*-bimodule over $C_0(G^0)$ via the representation M of $C_0(G^0)$ given by pointwise multiplication.

Proposition 3.12. *Let (H, U) be a continuous representation of a decomposable Hausdorff groupoid G.*

i) Let $\phi \in \mathrm{PHom}(G^0)$ and $f \in C_c(\mathrm{Cov}_\phi(G))$. Then for each $\xi \in \Gamma_0(H)$, the section $U_f \xi$ of H over $\mathrm{Im}(\phi)$ given by

$$(U_f \xi)(v) := \int_{G^v} f(x) U_x \xi(\phi^{-1}(v)) d\lambda^v(x), \quad v \in \mathrm{Im}(\phi),$$

is continuous and vanishes at infinity.

ii) The map $U_f \colon \xi \mapsto U_f \xi$ defines a (ϕ_*, ϕ_*)-homogeneous operator on the C^*-bimodule $\Gamma_0(H)$. Its adjoint is given by $(U_f)^* = U_g$, where the function $g \in C_c(\mathrm{Cov}_{\phi^{-1}}(G))$ is defined by $g(x) = \overline{f(x^{-1})}$.

iii) One has $\mathrm{Cov}_\phi(G) \cap \mathrm{Cov}_\psi(G) = \mathrm{Cov}_{\phi \wedge \psi}(G)$ for all $\phi, \psi \in \mathrm{PHom}(G^0)$.

iv) The family $\mathscr{C}^*(H, U) \subset \mathscr{L}(\Gamma_0(H))$ given by

$$\mathscr{C}^*(U, H)_{\phi_*}^{\psi_*} := \overline{\{U_f \mid f \in C_c(\mathrm{Cov}_{\phi \wedge \psi}(G))\}}, \quad \phi, \psi \in \mathrm{PHom}(G^0),$$

is a C^*-family.

v) The C^*-family associated to the trivial representation is given by

$$\mathscr{C}^*(\mathbb{C} \times G^0, \mathrm{id})_{\phi_*}^{\psi_*} = (\phi \wedge \psi)_* \circ M\big(C_0(s(\mathrm{Cov}_{\phi \wedge \psi}(G)))\big), \quad \phi, \psi \in \mathrm{PHom}(G^0).$$

Proof. i) This follows from the continuity of the representation and condition ii) on the Haar system.

ii) Let $\xi, \eta \in \Gamma_0(H)$. Then the functions $\langle \eta | U_f \xi \rangle$ and $\langle U_g \eta | \xi \rangle$ belong to $C_0(\mathrm{Im}(\phi))$ and $C_0(\mathrm{Dom}(\phi))$, respectively. For each $v \in \mathrm{Im}(\phi)$, one has

$$\langle \eta | U_f \xi \rangle(v) = \big\langle \eta(v) \big| (U_f \xi)(v) \big\rangle_{H_v} = \int_{G^v} \big\langle \eta(v) \big| f(x) U_x \xi(\phi^{-1}(v)) \big\rangle_{H_v} d\lambda^v(x).$$

Since $U_x^* = U_{x^{-1}}$ and $\overline{f(x)} = g(x^{-1})$, the integrand is equal to

$$\big\langle \overline{f(x)} U_x^* \eta(v) \big| \xi\big(\phi^{-1}(v)\big) \big\rangle_{H_v} = \big\langle g(x^{-1}) U_{x^{-1}} \eta(v) \big| \xi\big(\phi^{-1}(v)\big) \big\rangle_{H_v}.$$

Substituting $x' = x^{-1}$ and using unimodularity of the Haar system, we obtain

$$\begin{aligned}
\langle \eta | U_f \xi \rangle(v) &= \int_{G_v} \big\langle g(x') U_{x'} \eta(v) \big| \xi(\phi^{-1}(v)) \big\rangle_{H_v} d\lambda_v(x') \\
&= \int_{G^{\phi(v)}} \big\langle g(x') U_{x'} \eta(v) \big| \xi(\phi^{-1}(v)) \big\rangle_{H_v} d\lambda^{\phi^{-1}(v)}(x') \\
&= \big\langle (U_g \eta)(\phi^{-1}(v)) \big| \xi(\phi^{-1}(v)) \big\rangle_{H_v} \\
&= \big\langle U_g \eta \big| \xi \big\rangle (\phi^{-1}(v)).
\end{aligned}$$

Since the representation M coincides with the right module structure, we can deduce that the operator U_f is (ϕ_*, ϕ_*)-homogeneous.

iii) Evident.

iv) Let $\psi \in \mathrm{PHom}(G^0)$ and $g \in C_c(\mathrm{Cov}_\psi(G))$. A short calculation shows that $U_f U_g = U_{(f \star g)}$ where

$$(f \star g)(x) = \int_{G^{r(x)}} f(y) g(y^{-1} x) d\lambda^{r(x)}(y), \quad x \in G.$$

By part iii) and lemma A.1, this implies that the family $\mathscr{C}^*(H, U)$ is closed under multiplication. It is closed under involution by part ii) and therefore a C^*-pre-family. The fact that it is a C^*-family is easy to check.

v) Let $f \in C_c(\mathrm{Cov}_\phi(G))$, $\phi \in \mathrm{PHom}(G^0)$. Then

$$(\mathrm{id}_f\, \xi)(v) = \int_{G^v} f(x) d\lambda^v(x) \cdot \xi(\phi^{-1}(v)), \quad v \in G^0, \, \xi \in C_c(G^0).$$

Therefore, $\mathrm{id}_f = \phi_* \circ M(F)$ where $F \in C_0(\mathrm{Dom}(\phi))$ is given by $F(\phi^{-1}(v)) = \int_{G^v} f(x) d\lambda^v(x)$, $v \in \mathrm{Im}(\phi)$. Thus, $\|\, \mathrm{id}_f \,\| = \|F\|_\infty$. This implies the claim. □

We call $\mathscr{C}^*(H, U)$ *the C^*-family associated to the continuous representation* (H, U).

Integration with respect to a measure

The C^*-family constructed above can be related to the C^*-algebra classically associated to a continuous representation (H,U). Before we explain this relation, let us first recall the definition of this C^*-algebra [45]. It depends on the choice of a quasi-invariant measure μ on G^0 and acts on the direct integral $L^2(H,\mu) = \int H d\mu$. A positive regular Borel measure μ on G^0 is *quasi-invariant* if the induced measures $\nu := \int_{G^0} \lambda^u d\mu(u)$ and $\nu^{-1} = \int_{G^0} \lambda_u^{-1} d\mu(u)$ are equivalent [45, 41]. The *direct integral* $L^2(H,\mu)$ of H with respect to a measure μ on G^0 is the Hilbert space completion of the space $\Gamma_c(H)$ with respect to the inner product $\langle \eta|\xi\rangle := \int_{G^0} \langle \eta(v)|\xi(v)\rangle d\mu(v)$. Let μ be a quasi-invariant measure and denote by D the Radon-Nikodym derivative $D = d\nu/d\nu^{-1}$. Then, for each function $f \in C_c(G)$, the formula

$$(U_f^\mu \xi)(v) := \int_{G^v} f(x) D^{-1/2}(x) U_x \xi(s(x)) d\lambda^v(x), \quad v \in G^0,$$

defines an operator $U_f^\mu \in \mathcal{B}(L^2(H,\mu))$, and the map $f \mapsto U_f^\mu$ defines a $*$-representation of the convolution algebra $C_c(G)$. The C^*-algebra generated by the image $U^\mu(C_c(G))$ is called *the C^*-algebra associated to* (H,U,μ) and denoted by $C^*(H,U,\mu)$. This C^*-algebra can be obtained from the C^*-family $\mathscr{C}^*(H,U)$ by tensoring with a C^*-family which corresponds to the adjoint representation of G on the measure space (G^0,μ) – in fact $C^*(H,U,\mu) = \overline{\operatorname{span}}\,\mathscr{C}^*(H,U,\mu)$ where $\mathscr{C}^*(H,U,\mu) = \mathscr{C}^*(H,U) \otimes_M \mathscr{C}^*(\mathbb{C} \times G^0, \mathrm{id}, \mu)$, see the following proposition.

Proposition 3.13. *Let (H,U) be a continuous representation of a decomposable Hausdorff groupoid G with a unimodular Haar system λ and let μ be a quasi-invariant measure on G^0. Consider $L^2(H,\mu)$ as a C^*-$C_0(G^0)$-$\mathbb{C}$-bimodule via the representation M of $C_0(G^0)$ given by pointwise multiplication.*

i) The family $\mathscr{C}^*(H,U,\mu) \subset \mathscr{L}_{\mathrm{id}}(L^2(H,\mu))$ given by

$$\mathscr{C}^*(U,H,\mu)^{\phi_*} := \{U_f^\mu \mid f \in C_c(\mathrm{Cov}_\phi(G))\}^{-\!\!-}, \quad \phi \in \mathrm{PHom}(G^0),$$

is a C^*-family, and $\overline{\operatorname{span}}\,\mathscr{C}^*(H,U,\mu) = C^*(H,U,\mu)$.

ii) For each $\phi \in \mathrm{PHom}(G^0)$, put $D_\phi := d\mu|_{\mathrm{Im}(\phi)}/d\phi_*\big(\mu|_{\mathrm{Dom}(\phi)}\big)$. Then the functions $r^*\big(D_\phi\big)$ and $D = d\nu/d\nu^{-1}$ coincide on $\mathrm{Cov}_\phi(G)$.

iii) The C^*-family associated to the trivial representation and μ is given by

$$\mathscr{C}^*(\mathbb{C} \times G^0, \mathrm{id}, \mu)^{\phi_*} = \phi_*^\mu \circ M\big(C_0(s(\mathrm{Cov}_\phi(G)))\big), \quad \phi \in \mathrm{PHom}(G^0),$$

where $\phi_*^\mu \in \mathcal{B}(L^2(H,\mu))$ is the operator given by

$$(\phi_*^\mu \xi)(v) = \begin{cases} D_\phi^{-1/2}(v)\xi(\phi^{-1}(v)), & v \in \mathrm{Im}(\phi), \\ 0, & v \notin \mathrm{Im}(\phi), \end{cases} \qquad v \in G^0,\ \xi \in L^2(H,\mu).$$

iv) There exists an isomorphism $V\colon \Gamma^0(H)\otimes_M L^2(G^0,\mu)\to L^2(H,\mu)$*, given by*

$$(V(f\otimes_M \xi))(v) = f(v)\xi(v), \quad v\in G^0.$$

v) For all $g\in C_c(\mathrm{Cov}_{\phi\wedge\psi}(G))$, $f\in C_0(s(\mathrm{Cov}_\phi(G)))$, $\phi,\psi\in \mathrm{PHom}(G^0)$*, one has*

$$\mathrm{Ad}_V(U_g\otimes_M \phi^\mu_*\cdot M(f)) = U^\mu_h \quad \textit{where} \quad h := gs^*(f)\in C_c(\mathrm{Cov}_{\psi\wedge\phi}(G)).$$

The map Ad_V *defines an isomorphism*

$$\mathscr{C}^*(H,U)\otimes_M \mathscr{C}^*(\mathbb{C}\times G^0,\mathrm{id},\mu)\xrightarrow{\cong}\mathscr{C}^*(H,U,\mu).$$

Proof. i) The first statement is an analogue of proposition 3.12. The second statement follows from the fact that G is decomposable.

ii) By unimodularity of λ, one has for all $f\in C_c(\mathrm{Cov}_\phi(G))$

$$\begin{aligned}\int_{\mathrm{Cov}_\phi(G)} f(x)D_\phi(r(x))d\nu^{-1}(x) &= \int_{\mathrm{Dom}(\phi)}\int_{G_v} f(x)d\lambda_v(x)D_\phi(\phi(v))d\mu(v)\\ &= \int_{\mathrm{Dom}(\phi)}\int_{G^{\phi(v)}} f(x)d\lambda^{\phi(v)}(x)D_\phi(\phi(v))d\mu|_{\mathrm{Dom}(\phi)}(v)\\ &= \int_{\mathrm{Im}(\phi)}\int_{G^{v'}} f(x)d\lambda^{v'}(x)\underbrace{D_\phi(v')d\phi_*\big(\mu|_{\mathrm{Dom}(\phi)}\big)(v')}_{d\mu|_{\mathrm{Im}(\phi)}(v')}\\ &= \int_{\mathrm{Cov}_\phi(G)} f(x)d\nu(x).\end{aligned}$$

iii) Let $f\in C_c(\mathrm{Cov}_\phi(G))$, $\phi\in\mathrm{PHom}(G^0)$. A similar calculation as in proposition 3.12 shows that $\mathrm{id}^\mu_f = \phi^\mu_*\circ M(F)$, where $F\in C_0(s(\mathrm{Cov}_\phi(G)))$ is given by $F(\phi^{-1}(v)) = \int_{G^v} f(x)d\lambda^v(x)$, $v\in\mathrm{Im}(\phi)$. The operator $\phi^\mu_*\colon L^2\big(H|_{\mathrm{Dom}(\phi)},\mu|_{\mathrm{Dom}(\phi)}\big)\to L^2\big(H|_{\mathrm{Im}(\phi)},\mu|_{\mathrm{Im}(\phi)}\big)$ is isometric,

$$\begin{aligned}\|\phi^\mu_*\xi\|^2 &= \int_{\mathrm{Im}(\phi)}\overline{\xi(\phi^{-1}(v))}\xi(\phi^{-1}(v))\underbrace{D^{-1}_\phi(v)d\mu|_{\mathrm{Im}(\phi)}(v)}_{d\phi_*(\mu|_{\mathrm{Dom}(\phi)})}\\ &= \int_{\mathrm{Dom}(\phi)}\overline{\xi(v)}\xi(v)d\mu|_{\mathrm{Dom}(\phi)}(v) = \|\xi\|^2, \quad \xi\in L^2(H,\mu),\end{aligned}$$

and therefore, $\|\mathrm{id}^\mu_f\| = \|F\|_{\infty,\mathrm{supp}(\mu)}$. The claim follows.

iii) Evident.

iv) The operators id^μ_f and U_g satisfy the conditions of proposition 1.24 by proposition 3.12 and part i). The action of the operator $V(U_g\otimes_M\phi^\mu_*\cdot M(f))$ on an element $\eta \otimes_M \xi \in \Gamma_0(H) \otimes_M L^2(G^0, \mu)$ is given by

$$\begin{aligned}
V\big(U_g\eta \otimes_M \phi_*^\mu \cdot M(f)\xi\big)(v) &= \\
&= \int_{G^v} g(x) U_x \eta(\phi^{-1}(v)) d\lambda^v(x) D_\phi^{-1/2}(v) f(\phi^{-1}(v)) \xi(\phi^{-1}(v)) \\
&= \int_{G^v} g(x) f(s(x)) D^{-1/2}(x) U_x \eta(\phi^{-1}(v)) \xi(\phi^{-1}(v)) d\lambda^v(x) \\
&= \big(U_h^\mu V(\eta \otimes_M \xi)\big)(v), \quad v \in \mathrm{Im}(\phi).
\end{aligned}$$

The last statement follows from the formula proved in the first part. □

Functoriality of the association

We conclude the extended example by a discussion of the relation between groupoid homomorphisms and the partial integration of continuous representations. Let G' be a decomposable Hausdorff groupoid and let $\sigma\colon G' \to G$ be a continuous homomorphism. Denote by σ_0 the restriction of σ to G'^0. The representation (H, U) pulls back to the representation $\sigma^*(H, U) = (\sigma_0^* H, \sigma^* U)$ of G', where $(\sigma^* U)_{x'} = U_{\sigma(x')}$ for all $x' \in G'$.

Proposition 3.14. *Let $\sigma\colon G' \to G$ be a continuous homomorphism of decomposable groupoids.*

i) Consider $C_0(G'^0)$ as a C^*-$C_0(G^0)$-$C_0(G'^0)$-bimodule via the representation $\sigma_0^*\colon C_0(G^0) \to M\big(C_0(G'^0)\big)$. Then the family $\mathscr{C}^*(\sigma_0) \subset \mathscr{L}\big(C_0(G'^0)\big)$ given by

$$\mathscr{C}^*(\sigma_0)^{\phi_*}_{\psi_*} := \psi_* \circ M\big(C_0(s(V^\phi_\psi))\big), \quad V^\phi_\psi := \mathrm{Cov}_\psi(G') \cap \sigma^{-1}(\mathrm{Cov}_\phi(G)),$$

$\phi \in \mathrm{PHom}(G^0)$, $\psi \in \mathrm{PHom}(G'^0)$, is a C^*-family.[1]

ii) One has an isomorphism $V\colon \Gamma_0(H) \otimes_{\sigma_0^*} C_0(G'^0) \to \Gamma_0(\sigma_0^* H)$ given by

$$\big(V(\eta \otimes_{\sigma_0^*} \xi)\big)(v') = \eta(\sigma(v'))\xi(v'), \quad \eta \in \Gamma_0(H),\ \xi \in C_0(G'^0).$$

Assume that σ is proper and that the restriction $\sigma^{v'}\colon G'^{v'} \to G^{\sigma(v')}$ of σ is surjective for each $v' \in G'^0$. Denote by D_σ the function on G' associated to σ in proposition 3.11.

iii) For each $g \in C_c\big(\mathrm{Cov}_\phi(G)\big), \phi \in \mathrm{PHom}(G^0)$ and each $f' \in C_0(s(V^\phi_\psi))$, $\psi \in \mathrm{PHom}(G'^0)$, one has

$$\mathrm{Ad}_V\big(U_g \otimes_{\sigma_0^*} \psi_* \cdot M(f')\big) = (\sigma^* U)_{h'} \ \text{ where } \ h' := D_\sigma \cdot \sigma^*(g) \cdot s^*(f').$$

The map Ad_V induces an embedding

$$\mathscr{C}^*(H, U) \otimes_{\sigma_0^*} \mathscr{C}^*(\sigma_0) \hookrightarrow \mathscr{C}^*(\sigma^*(H, U)).$$

[1] Here, we cavalierly neglect the partial automorphism on the C^*-algebra $\mathbb{C}$ defined on the 0-ideal.

iv) If G' and G are r-discrete, this embedding is an isomorphism.

Proof. i) Let ϕ, ψ be as above and $f' \in C_0(s(V_\psi^\phi))$. For all $g \in C_0(\mathrm{Dom}(\phi))$, one has

$$(\psi_* \circ M(f'))\sigma_0^*(g) = \psi_*(\sigma_0^*(g))(\psi_* \circ M(f')) = \sigma_0^*(\phi_*(g))(\psi_* \circ M(f'))$$

as operators on $C_0(G'^0)$, because $\sigma(\psi^{-1}(v')) = \phi^{-1}(\sigma(v'))$, $v' \in V_\psi^\phi$. Together with proposition 3.12 ii), this shows that the operator $\psi_* \circ M(f')$ is (ϕ_*, ψ_*)-homogeneous.

Let us show that $\mathscr{C}^*(\sigma_0)$ is a C^*-family. Since $\|\psi_* \circ M(f')\| = \|f'\|_\infty$, the space $\psi_* \circ M(C_0(V_\psi^\phi))$ is closed. It is easy to see that the family $\mathscr{C}^*(\sigma_0)$ is closed under involution and multiplication. The fact that it is a C^*-family is easy to check.

ii) Evident.

iii) The operators U_g and $\psi_* \circ M(f')$ satisfy the conditions of proposition 1.24 by part ii) of proposition 3.12 and by part i). We compute the action of the operator $V(U_g \otimes_* \psi_* \circ M(f'))$ on an element $\eta \otimes_* \xi \in \Gamma_0(H) \otimes_{\sigma_0^*} C_0(G'^0)$. By proposition 3.11, we have

$$\begin{aligned}(U_g\eta)(\sigma(v')) &= \int_{G^{\sigma(v')}} g(x)U_x\eta(s(x)) \underbrace{d\lambda^{\sigma(v')}(x)}_{d\sigma_*\left(D_\sigma \lambda'^{v'}\right)(x)} \\ &= \int_{G'^{v'}} g(\sigma(x'))U_{\sigma(x')}\eta(\sigma(s(x')))D_\sigma(x')d\lambda'^{v'}(x').\end{aligned}$$

Therefore, we find

$$\begin{aligned}\big(V(U_g\eta \otimes_* \psi_* M(f')\xi)\big)(v') &= (U_g\eta)(\sigma(v')) \cdot f'(\psi^{-1}(v'))\xi(\psi^{-1}(v')) \\ &= \int_{G^{v'}} D_\sigma(x')g(\sigma(x'))f'(s(x'))U_{\sigma(x')} \\ &\qquad\qquad \cdot \eta(\sigma(s(x')))\xi(s(x'))d\lambda'^{v'}(x') \\ &= \int_{G^{v'}} h'(x')U_{\sigma(x')}\big((V(\eta \otimes_{\sigma_0^*} \xi))(s(x'))\big)d\lambda'^{v'}(x') \\ &= \big((\sigma^*U)_{h'}V(\eta \otimes_* \xi)\big)(v').\end{aligned}$$

iv) Since the open G-sets and G'-sets cover G and G', respectively, each compactly supported continuous function on G' can be written as a sum of functions $h' \in C_0(V')$ where V' is a G'-set whose image $\sigma(V')$ is contained in some G-set V. Given such a function, choose $g \in C_c(V)$ such that $\sigma^*(g)|_{\mathrm{supp}(h')} = 1$. This is possible because the image $\sigma(\mathrm{supp}(h')) \subset V$ is compact. Finally, put $f' := s_{V'*}(h')$ where $s_{V'} \colon V' \to s(V')$ denotes the restriction of the source map, which is a homeomorphism by assumption on V'. Then the internal tensor product $U_g \otimes q_{V'*} \circ M(f')$ is well-defined. For all $\eta \in \Gamma_0(H), \xi \in C_0(G'^0)$ and $x' \in V'$, we have

$$\begin{aligned}\big(V(U_g\eta \otimes_* q_{V'*}M(f')\xi)\big)(r(x')) &= g(\sigma(x'))U_{\sigma(x')}\eta(\sigma(s(x')))f'(s(x'))\xi(s(x')) \\ &= \big((\sigma^*U)_{h'}V(\eta \otimes_* \xi)\big)(r(x')).\end{aligned}$$

Therefore, the embedding is surjective. □

3.2 The pseudo-Kac system of a locally compact groupoid

In this section, we study the main examples for the theory developed in the previous chapter: pseudo-multiplicative unitaries and pseudo-Kac systems associated to locally compact groupoids. The pseudo-multiplicative unitaries have already been considered by Moto O'uchi [38], but a satisfactory definition of their legs remained an open question. The results of the previous section imply that the unitary associated to a decomposable groupoid is itself decomposable. In this case, the generalised Baaj-Skandalis construction introduced in the second chapter can be applied.

After a review of the pseudo-multiplicative unitaries, we exemplify the "toy definition" of an algebraic pseudo-Kac system and associate a C^*-pseudo-Kac system to each locally compact groupoid. The latter is less complex than the general definition allows because several members of the respective families of C^*-modules, representations and unitaries comprising the pseudo-Kac system coincide. Again, this system is decomposable if the underlying groupoid is decomposable.

Next, we compute the legs of this pseudo-Kac system. They are related to the function algebra and to the left regular representation of the groupoid, respectively. Then we determine the canonical coactions of the legs. The underlying C^*-algebras are the function algebra on one side and the reduced groupoid C^*-algebra on the other side. Finally, we show that a certain class of coactions of the left leg corresponds bijectively to groupoid actions. The study of coactions of the right leg is left to a separate section.

3.2.1 Construction

The pseudo-multiplicative unitary

We construct the pseudo-multiplicative unitary associated to a groupoid and relate several standard operations on unitaries obtained this way to the corresponding operations on the level of the underlying groupoids. Except for the regularity and decomposability considerations, the following result already appeared in [38]. For completeness, we reproduce the entire proof.

Proposition 3.15. *Let G be a locally compact Hausdorff groupoid with a left Haar system λ. Put*

$$A := C_0(G^0), \qquad\qquad E := L^2(G,\lambda)$$

and denote by π_s, π_r the representations of $C_0(G^0)$ on $L^2(G,\lambda)$ induced by the range and source map of G, respectively, i.e.

$$(\pi_s(f)\xi)(x) := f(s(x))\xi(x), \qquad\qquad (\pi_r(f)\xi)(x) := f(r(x))\xi(x),$$

for all $x \in G$, $f \in C_0(G^0)$ and $\xi \in L^2(G,\lambda)$. Clearly, these two representations commute. The map $V_0\colon C_c(G\,{}_s\times_r G) \to C_c(G\,{}_r\times_r G)$ given by

$$(V_0 f)(x,y) := f(x, x^{-1}y), \quad (x,y) \in G\,{}_r\times_r G,\ f \in C_c(G\,{}_s\times_r G),$$

extends to a regular pseudo-multiplicative unitary $V\colon E^s \oslash E^r \to E^s \oslash E^r$. This pseudo-multiplicative unitary is always left decomposable and left decomposably regular. If G is decomposable, it is right decomposable and right decomposably regular.

Proof. First, note that the spaces $C_c(G\,{}_s\times_r G)$ and $C_c(G\,{}_r\times_r G)$ are dense subsets of $E^s \oslash E$ and $E \oslash E^r$, respectively. The map V_0 is a linear isomorphism because it is the transpose of the homeomorphism $G\,{}_r\times_r G \to G\,{}_s\times_r G$ given by $(x,y) \mapsto (x, x^{-1}y)$. It extends to a unitary because

$$\begin{aligned}\langle V_0 f | V_0 g\rangle_{(E\oslash E^r)}(u) &= \int \overline{f(x,x^{-1}y)}g(x,x^{-1}y)d\lambda^{r(x)}(y)d\lambda^u(x)\\ &= \int \overline{f(x,y')}g(x,y')d\lambda^{s(x)}(y')d\lambda^u(x)\\ &= \langle f|g\rangle_{(E^s\oslash E)}(u), \quad u \in G^0,\end{aligned}$$

for all $f, g \in C_c(G\,{}_s\times_r G)$.

For all $f \in C_c(G\,{}_s\times_r G\,{}_s\times_r G)$ and all $(x,y,z) \in G\,{}_s\times_r G\,{}_s\times_r G$, one has

$$\begin{aligned}(V_{12}V_{13}V_{23}f)(x,y,z) &= (V_{13}V_{23}f)(x,x^{-1}y,z)\\ &= (V_{23}f)(x,x^{-1}y,x^{-1}z) = f(x,x^{-1}y,y^{-1}z),\\ (V_{23}V_{12}f)(x,y,z) &= (V_{12}f)(x,y,y^{-1}z) = f(x,x^{-1}y,y^{-1}z).\end{aligned}$$

Since $C_c(G\,{}_s\times_r G\,{}_s\times_r G)$ is dense in $E^s \oslash E^s \oslash E$, the unitary V satisfies the pentagonal equation.

For every f, g and $f', g' \in C_c(G)$, one has

$$\begin{aligned}\big((\langle g| \oslash 1)V(1 \oslash |f\rangle)h\big)(y) &= \int_{G^{r(y)}} \overline{g(x)}h(x)f(x^{-1}y)d\lambda^{r(y)}(x),\\ \big(|f'\rangle\langle g'|h\big)(y) &= \int_{G^{r(y)}} \overline{g'(x)}h(x)f'(y)d\lambda^{r(y)}(x), \quad h \in C_c(G), y \in G.\end{aligned}$$

Therefore, V is regular.

The unitary V is left decomposable by remark 3.6. If G is decomposable, V is right decomposable by proposition 3.5. The remaining regularity statements follow from corollary 2.14, proposition 2.36 and lemma 2.15. □

Examples 3.16. Let (G, λ) and (G', λ') be two locally compact Hausdorff groupoids with left Haar systems and denote by V and V', respectively, the associated pseudo-multiplicative unitaries.

- Let $C \subset G^0$ be a closed subset such that $(r \circ s^{-1})(C) \subset C$. Then the ideal $I := \{f \in C_0(G^0) \mid f|_C = 0\}$ of $C_0(G^0)$ satisfies the assumptions of lemma 2.8. The reduction of V with respect to I is isomorphic to the pseudo-multiplicative unitary associated to the subgroupoid $G_C \subset G$ and the left Haar system $\lambda_C \subset \lambda$, where $G_C := s^{-1}(C)$ and $\lambda_C := (\lambda^v)_{v \in C}$.
- The direct sum $V \oplus V'$ is isomorphic to the pseudo-multiplicative unitary associated to the coproduct $(G, \lambda) \coprod (G', \lambda')$.
- The external tensor product $V \otimes V'$ is isomorphic to the pseudo-multiplicate unitary associated to the product $(G, \lambda) \times (G', \lambda')$.

The algebraic pseudo-Kac system

Combining the pseudo-multiplicative isomorphism of the previous paragraph with the inversion map of the underlying groupoid, we obtain an algebraic pseudo-Kac system.

Proposition 3.17. *Let G be an étale groupoid. Put $A = C_c(G^0)$. Then the space $E := C_c(G)$ of compactly supported functions on G with the representations $\pi_{\overline{r}} := \pi_r, \pi_{\overline{s}} := \pi_s$ induced from the range and source maps, and the operators induced from the involution and multiplication on G,*

$$V\colon E\,{}_s\!\odot_{\overline{r}} E \to E\,{}_{\overline{r}}\!\odot_r E, \qquad (Vf)(x,y) := f(x, x^{-1}y),$$
$$U\colon E \to E, \qquad (Uf)(x) = f(x^{-1}),$$

form an algebraic pseudo-Kac system.

Proof. A straight-forward variation of the proof of proposition 3.15 shows that V is a pseudo-multiplicative isomorphism. One easily checks that V commutes with the representations $\pi_{\overline{s}1}$ and $\pi_{\overline{s}2}$. The map $\Sigma(1 \odot U)V$ is the transpose of the homeomorphism $\phi\colon G\,{}_s\!\times_r G \to G\,{}_s\!\times_r G$ given by the composition $(x,y) \mapsto (y,x) \mapsto (y, x^{-1}) \mapsto (y, y^{-1}x^{-1})$. A short calculation shows that ϕ^3 is the identity map:

$$(\underbrace{y}_{x'}, \underbrace{y^{-1}x^{-1}}_{y'}) \overset{\phi}{\mapsto} (\underbrace{y^{-1}x^{-1}}_{y'}, \underbrace{xy}_{y'^{-1}}\underbrace{y^{-1}}_{x'^{-1}}) = (\underbrace{y^{-1}x^{-1}}_{x''}, \underbrace{x}_{y''}) \overset{\phi}{\mapsto} (\underbrace{x}_{y''}, \underbrace{x^{-1}}_{y''^{-1}}\underbrace{xy}_{x''^{-1}}) = (x,y).$$

The map $\hat{V}$ is the transpose of the homeomorphism

$$G\,{}_r\!\times_r G \to G\,{}_r\!\times_s G, \qquad (x,y) \mapsto (y^{-1},x) \mapsto (y^{-1},yx) \mapsto (yx,y).$$

A straight-forward calculation shows that it satisfies the pentagon equation. □

The C^*-pseudo-Kac system

The construction of the Kac system associated to a locally compact group carries over to locally compact Hausdorff groupoids with minor modifications.

Proposition 3.18. *Let G be a locally compact Hausdorff groupoid with left Haar system λ. Put*

- $A := C_0(G^0)$,
- $I := (r, s, \overline{r}, \overline{s})$,
- $E_r := E_{\overline{r}} := L^2(G,\lambda)$ *and* $E_s := E_{\overline{s}} := L^2(G,\lambda^{-1})$,

and denote by

- $\pi_r := \pi_{\overline{r}}$ *the family of representations of* $C_0(G^0)$ *on* $L^2(G,\lambda)$ *and* $L^2(G,\lambda^{-1})$ *induced by the range map of* G,
- $\pi_s := \pi_{\overline{s}}$ *the family of representations induced by the source map.*

Then the map V_0 introduced in proposition 3.15 and the map $U_0 \colon C_c(G) \to C_c(G)$ given by $(U_0 f)(x) := f(x^{-1})$ induce families of unitaries U and V as in definition 2.53, and the tuple (A, I, E, π, U, V) forms a pseudo-Kac system. This pseudo-Kac system is decomposable if G is decomposable.

Proof. It is clear that U_0 induces unitaries as in definition 2.53. We show that V_0 extends to unitaries

$$\begin{aligned}
&(1) \qquad V^{\overline{r}\overline{r}}, V^{r\overline{r}} \colon E^s_r \otimes E_r \to E_r \otimes E^r_r \cong E^r_r \otimes E_r,\\
&(2) \qquad V^{\overline{s}\overline{r}}, V^{sr} \colon E^s_s \otimes E_r \cong E_s \otimes E^r_r \to E^r_s \otimes E_r,\\
&(3) \qquad V^{ss}, V^{s\overline{s}} \colon E_s \otimes E^r_s \to E_r \otimes E^r_s.
\end{aligned}$$

Part (1) follows from proposition 3.15. For parts (2) and (3), let $f \in C_c(G\,{}_r\!\times_r G)), g \in C_c(G\,{}_s\!\times_r G)$ and $u \in G^0$. Then

$$\begin{aligned}
\langle f|V_0 g\rangle_{(E^r_s \otimes E_r)}(u) &= \int_{G_u}\int_{G^{r(x)}} \overline{f(x,y)}g(x,x^{-1}y)\,d\lambda^{r(x)}(y)\,d\lambda^{-1}_u(x)\\
&= \int_{G_u}\int_{G^u} \overline{f(x,xy')}g(x,y')\,d\lambda^u(y')\,d\lambda^{-1}_u(x)\\
&= \left\langle V_0^{-1} f|g\right\rangle_{(E^s_s \otimes E_r)}(u),\\
\left\langle f|V^0 g\right\rangle_{(E_r \otimes E^r_s)}(u) &= \int_{G_u}\int_{G^{r(y)}} \overline{f(x,y)}g(x,x^{-1}y)\,d\lambda^{r(y)}(x)\,d\lambda^{-1}_u(y)\\
&= \int_{G_u}\int_{G^u} \overline{f(yx',y)}g(yx',x'^{-1})\,d\lambda^u(x')\,d\lambda^{-1}_u(y).
\end{aligned}$$

Upon substituting $y' := yx'$ and $x'' := x'^{-1}$, this expression becomes

$$\int_{G_u}\int_{G_{r(x'')}} \overline{f(y', y'x'')} g(y', x'') d\lambda^{-1}_{r(x'')}(y') d\lambda^{-1}_u(x'') = \left\langle V_0^{-1} f | g \right\rangle_{(E_s \otimes E_s^r)}(u).$$

The fact that these families satisfy the conditions vi)-viii) of the definition follows from proposition 3.15 and calculations similar to those carried out in the previous subsection. □

3.2.2 The left and right leg

Let G be a decomposable Hausdorff groupoid with left Haar system λ. We determine the legs of the pseudo-Kac system associated to G in the previous subsection. Let us fix the following notation. Denote by $m\colon C_b(G) \to L_{C_0(G^0)}(L^2(G,\lambda))$ and

$$m'\colon C_b(G\,{}_s\!\times_r G) \to L_{C_0(G^0)}\big(L^2(G,\lambda)^{\pi_s} \otimes L^2(G,\lambda)\big)$$

the representations given by pointwise multiplication. Note that the internal tensor product $L^2(G,\lambda)^{\pi_s} \otimes L^2(G,\lambda)$ can be identified with the completion of the space $C_c(G\,{}_s\!\times_r G)$ with respect to the inner product

$$\langle f|g\rangle(v) := \int_{G^v}\int_{G^{s(x)}} \overline{f(x,y)} g(x,y) d\lambda^{s(x)}(y) d\lambda^v(x), \quad v \in G^0,\ f, g \in C_c(G\,{}_s\!\times_r G),$$

and that the representations m and m' are injective.

Proposition 3.19. *Let G be a decomposable Hausdorff groupoid with left Haar system λ.*

i) For each $\phi, \psi \in \mathrm{PHom}(G^0)$, the space $\hat{\mathscr{S}}^{\psi_*}_{\phi_*}$ is contained in $\hat{\mathscr{S}}^{\mathrm{id}}_{\mathrm{id}}$, and the map $\hat{\Delta}^{\psi_*}_{\phi_*}$ is extended by the $*$-homomorphism $\hat{\Delta}^{\mathrm{id}}_{\mathrm{id}}$. One has

$$\hat{\mathscr{S}}^{\mathrm{id}}_{\mathrm{id}} = m\big(C_0(G)\big) \subset \mathscr{L}^{\mathrm{id}}_{\mathrm{id}}(L^2(G,\lambda), \pi_s) \subset L_{C_0(G^0)}\big(L^2(G,\lambda)\big)$$

and $\hat{\Delta}^{\mathrm{id}}_{\mathrm{id}}(m(f)) = m'(f')$, $f \in C_0(G)$, where $f' \in C_b(G\,{}_s\!\times_r G)$ is defined by $f'(x,y) = f(xy)$ for all $(x,y) \in G\,{}_s\!\times_r G$.
The canonical coaction $(\hat{S}_0, \pi_s, \hat{\delta}_0)$ is given by

$$\hat{S}_0 = m(C_0(G^0)) = \hat{\mathscr{S}}^{\mathrm{id}}_{\mathrm{id}}, \qquad \hat{\delta}_0 = \hat{\Delta}^{\mathrm{id}}_{\mathrm{id}}.$$

ii) The C^*-family $\mathscr{S} \subset \mathscr{L}(L^2(G,\lambda), \pi_r)$ is given by

$$\mathscr{S}^{\psi_*}_{\phi_*} = \overline{\lambda(C_c(\mathrm{Cov}_{(\phi\wedge\psi)}(G)))} \subset \mathscr{L}\big(L^2(G,\lambda), \pi_r\big), \quad \phi, \psi \in \mathrm{PHom}(G^0),$$

where for each $f \in C_c(G)$, the map $\lambda(f)$ on $L^2(G,\lambda)$ is defined by

$$(\lambda(f)\xi)(x) := \int_{G^{r(x)}} f(y)\xi(y^{-1}x) d\lambda^{r(x)}(y), \quad x \in G,\ \xi \in C_c(G).$$

The coproduct Δ is given by the formula

$$\big(\Delta(\lambda(f))(\xi \otimes \eta)\big)(x,y) = \int_{G^{r(x)}} f(z)\xi(z^{-1}x)\eta(z^{-1}y)d\lambda^{r(x)}(z),$$

where $(x,y) \in G\,{}_r\!\times_r G$, $\xi,\eta \in L^2(G,\lambda)$ and $\lambda(f) \in \hat{\mathscr{S}}$.
The C^*-algebra of the canonical coaction is given by

$$S_0 = C^*_r(G) \subset L_{C_0(G^0)}\big(L^2(G,\lambda^{-1})\big),$$

and the coaction δ_0 is given by the same formula as Δ.

Proof. i) The first claim follows from remark 3.6. For all $\eta,\xi,\zeta \in C_c(G)$, one has

$$\begin{aligned}\big((1 \otimes \{\eta|)V^{\overline{rr}}(1 \otimes |\xi\rangle)\,\zeta\big)(x) &= \int_{G^{r(x)}} \overline{\eta(y)}\big(V^{\overline{rr}}(\zeta \otimes \xi))(x,y)\big)d\lambda^{r(x)}(y)\\ &= \int_{G^{r(x)}} \overline{\eta(y)}\zeta(x)\xi(x^{-1}y)d\lambda^{r(x)}(y) = (m(f)\zeta)(x),\end{aligned}$$

where $f \in C_b(G)$ is given by

$$f(x) = \int_{G^{r(x)}} \overline{\eta(y)}\xi(x^{-1}y)d\lambda^{r(x)}(y).$$

The support of f is compact because it is contained in $(\operatorname{supp}\eta)\cdot(\operatorname{supp}\xi)^{-1}$. Hence, $\hat{\mathscr{S}}^{\mathrm{id}}_{id}$ is contained in $m(C_0(G))$. An easy application of the Stone-Weierstrass theorem shows that this inclusion is an equality.

Let us prove the formula for the coproduct. Let $f \in C_0(G)$, $\xi,\eta \in C_c(G)$ and $(x,y) \in G\,{}_s\!\times_r G$. Since $\big((V^{\overline{rr}})^{-1}(\xi \otimes \eta)\big)(x,y) = \xi(x)\eta(xy)$, one has

$$\begin{aligned}\big(V^{\overline{rr}*}(1 \otimes m(f))V^{\overline{rr}}(\eta \otimes \xi)\big)(x,y) &= f(xy)\big(V^{\overline{rr}}(\eta \otimes \xi)(x,xy)\big)\\ &= f(xy)\eta(x)\xi(y) = (m'(f')(\eta \otimes \xi))(x,y).\end{aligned}$$

ii) By definition, the C^*-family $\mathscr{S}$ is the left leg of the pseudo-multiplicative unitary

$$\hat{V}^{rr} \colon (L^2(G,\lambda),\pi_r) \otimes L^2(G,\lambda) \to L^2(G,\lambda) \otimes (L^2(G,\lambda),\pi_s),$$

which by the calculation in the proof of proposition 3.17 is given by $(\hat{V}^{rr}f)(x,y) = f(yx,y)$, $f \in C_c(G\,{}_r\!\times_r G)$, $(x,y) \in G\,{}_r\!\times_s G$. Let $\phi,\psi \in \mathrm{PHom}(G^0)$. By proposition 3.5, 1.52 and 1.49, one has

$$\mathscr{S}^{\psi_*}_{\phi_*} = \overline{\operatorname{span}}\big\{C_c(\mathrm{Cov}_{\psi^{-1}}(G))\big|_2 \cdot \hat{V}^{rr} \cdot \big|C_c(\mathrm{Cov}_{\phi^{-1}}(G))\big\rangle_2.$$

For all $\eta \in C_c(\mathrm{Cov}_{\psi^{-1}}(G))$ and $\xi \in C_c(\mathrm{Cov}_{\phi^{-1}}(G))$, one has

$$\begin{aligned}\big((1 \otimes \{\eta|)\hat{V}^{rr}(1 \otimes |\xi\rangle)\,\zeta\big)(x) &= \int_{G^{r(x)}} \overline{\eta(y)}\big(\hat{V}^{rr}(\zeta \otimes \xi)(x,y)\big)d\lambda^{r(x)}(x)\\ &= \int_{G^{r(x)}} \overline{\eta(y)}\zeta(yx)\xi(y)d\lambda^{r(x)}(x)\\ &= \big(\lambda(f)\xi\big)(x), \quad x \in G,\ \zeta \in C_c(G),\end{aligned}$$

where the function $f \in C_c(G)$ is given by $f(y) = \xi(y^{-1})\overline{\eta(y^{-1})}$ for all $y \in G$. The support of f is contained in

$$\big(\mathrm{Cov}_{\phi^{-1}}(G) \cap \mathrm{Cov}_{\psi^{-1}}(G)\big)^{-1} = \mathrm{Cov}_{(\phi\wedge\psi)}(G),$$

whence the formula for the C^*-family $\mathscr{S}$ follows. A similar calculation with $\hat{V}^{rr}$ replaced by $\hat{V}^{\bar{s}r}$ proves the equality $S_0 = C^*_r(G)$. Next, let us check the formula for the coproduct. Let $f \in C_c(\mathrm{Cov}_{\phi\wedge\psi}(G))$. Then the operator $\Delta(\lambda(f))$ is given by

$$\begin{aligned}\big(\hat{V}^{\bar{s}r*}(1 \otimes \lambda(f))\hat{V}^{\bar{s}r}(\eta \otimes \xi)\big)(x,y) &= \big((1 \otimes \lambda(f))\hat{V}^{\bar{s}r}(\eta \otimes \xi)\big)(y^{-1}x, y) \\ &= \int_{G^{r(x)}} f(z)(\hat{V}^{\bar{s}r}(\eta \otimes \xi))(y^{-1}x, z^{-1}y)d\lambda^{r(x)}(z) \\ &= \int_{G^{r(x)}} f(z)\eta(\underbrace{z^{-1}y \cdot y^{-1}x}_{z^{-1}x}, z^{-1}y)d\lambda^{r(x)}(z)\end{aligned}$$

for all $(x,y) \in G\,{}_r\!\times_r G$ and $\eta, \xi \in C_c(G)$. The formula for the coaction follows from a similar calculation. □

3.2.3 Coactions of the left leg and actions of the groupoid

Let G be a locally compact Hausdorff groupoid. We show that $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebras correspond bijectively with actions of the groupoid G on C^*-algebras. Let us first recall the definition of groupoid actions on C^*-algebras [31]. An *action of* G on a $C_0(G^0)$-algebra C is an isomorphism of $C_0(G)$-algebras $d\colon s^*C \to r^*C$ such that $d_x \circ d_y = d_{xy}$ for all $(x,y) \in G\,{}_s\!\times_r G$.

Given two C^*-algebras C and D, we denote by $C \overset{max}{\otimes} D$ the maximal C^*-tensor product of C and D. The following lemma relates the $C_0(G^0)$-tensor product to the internal tensor product.

Lemma 3.20. *Let C be a $C_0(X)$-algebra and let E be a C^*-module over $C_0(X)$. Let $D \subset L_{C_0(X)}(E)$ be a nuclear C^*-subalgebra which is a $C_0(X)$-algebra with respect to the representation of $C_0(X)$ on E given by the right module structure. Then the map $\phi\colon C \overset{max}{\otimes} D \to L_C(C \otimes E)$ given by $c \otimes d \mapsto c \otimes d$ induces an isomorphism $C \otimes_{C_0(X)} D \cong C \otimes D$.*

Proof. The existence and surjectivity of ϕ follow from the universal property of the maximal tensor product. In the notation used in [4], by [4, proposition 3.21] the map ϕ induces an isomorphism $C \overset{m}{\otimes}_{C_0(X)} D \cong C \otimes D$. Since D is nuclear, $C \overset{m}{\otimes}_{C_0(X)} D \cong C \otimes_{C_0(X)} D$ by [4, proposition 3.24]. □

Now, we show how to construct a groupoid action out of an $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebra. Put $X := G^0$. For two $C_0(X)$-algebras C and D, put

$$\tilde{M}(C \otimes_{C_0(X)} D) := \big\{T \in M(C \otimes_{C_0(X)} D) \mid T(1 \otimes D) \subset C \otimes_{C_0(X)} D\big\}.$$

Proposition 3.21. *Let $\hat{\delta}$ be an injective coaction of $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$ on an object $(C, \pi) \in \boldsymbol{C^*_A}$. Then*

i) $\pi(C_0(X)) \subset ZM(C)$, *i.e.* C *is a* $C_0(X)$*-algebra,*

ii) $\hat{\boldsymbol{S}}(C, \pi) = \big((C \otimes_{C_0(X)} (C_0(G), \pi_r)), \pi_{s2}\big) = (r^*C, \pi_s)$,

iii) $\hat{\delta}(C)$ *is contained in the multiplier algebra* $\tilde{M}(r^*C) \subset M(r^*C)$,

iv) for each $x \in G$, *the composition of* $\hat{\delta}$ *with the map* $\tilde{M}(r^*C) \to C_{r(x)}$ *given by evaluation at* x *factorises to a* $*$*-homomorphism* $\hat{\delta}_x \colon C_{s(x)} \to C_{r(x)}$,

v) $\hat{\delta}_x \hat{\delta}_y = \hat{\delta}_{xy}$ *for all* $(x, y) \in G\,{}_s{\times}_r\, G$,

vi) the family $(\hat{\delta}_x)_{x \in G}$ *defines a map* $d \colon s^*C \to r^*C$ *of* $C_0(G)$*-algebras,*

vii) if (C, π) *is an* $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$*-algebra, then* d *is an action of* G *on* C.

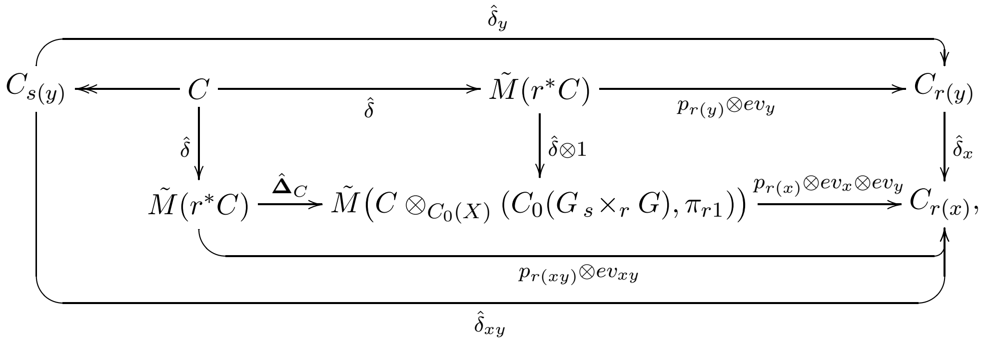
Proof. i) By covariance of $\hat{\delta}$, one has $\hat{\delta}\big([\pi(C_0(X)), C]\big) = [\pi_{s2}(C_0(X)), \hat{\delta}(C)]$. Since $\hat{\mathscr{S}} = m(C_0(G))$ commutes with the representation π_s and $\hat{\delta}(C)$ is contained in $M(C \otimes \hat{\mathscr{S}})$, this is 0. By injectivity of $\hat{\delta}$, we obtain $[\pi(C_0(X)), C] = 0$.

ii) $C \otimes_{C_0(X)} (C_0(G), \pi_r) \cong C \otimes m(C_0(G)) \subset L_C(C \otimes L^2(G, \lambda))$ by lemma 3.20.

iii) This follows from the condition $\hat{\delta}(C)(1 \otimes \hat{\mathscr{S}}) \subset C \otimes \hat{\mathscr{S}}$.

iv) Denote by $p_{r(x)} \colon C \to C_{r(x)}$ the quotient map and by $ev_x \colon C_0(G) \to \mathbb{C}$ the evaluation map. It is easy to see that the tensor product $p_{r(x)} \otimes ev_x$ extends to a quotient map $\tilde{M}(r^*C) \to C_{r(x)}$ as desired. If $f \in C_0(G^0)$ vanishes at $s(x)$, then $(p_{r(x)} \otimes ev_x) \circ \hat{\delta}\big(\pi(f)c\big) = (p_{r(x)} \otimes ev_x)\big(\pi_{s2}(f)\hat{\delta}(c)\big) = 0$ for all $c \in C$. Hence, the composition factorises to a $*$-homomorphism $\hat{\delta}_x \colon C_{s(x)} \to C_{r(x)}$.

v) Let $(x, y) \in G\,{}_s{\times}_r\, G$. Then $\hat{\delta}_x \hat{\delta}_y = \hat{\delta}_{xy}$ follows from commutativity of the diagram



where we used the identification $C_0(G\,{}_s{\times}_r\, G) = \big(C_0(G), \pi_s\big) \otimes_{C_0(X)} \big(C_0(G), \pi_r\big)$ and the formula $\big(\hat{\Delta}(f)\big)(x, y) = f(xy)$.

vi) Consider the multiplier algebra $M(r^*C)$ as a $C_0(X)$-algebra via the representation π_s. Since $\hat{\delta}(C)$ and $1 \otimes_{C_0(X)} C_0(G)$ commute in $M(r^*C)$, one has a morphism

$$d\colon s^*C = C \otimes_{C_0(X)} \big(C_0(G), \pi_s\big) \to C \otimes_{C_0(X)} \big(C_0(G), \pi_r\big) = M(r^*C),$$
$$c \otimes f \mapsto \hat{\delta}(c)(1 \otimes f).$$

By assumption on $\hat{\delta}$, the image of d is contained in r^*C. Let us show that $d_x = \hat{\delta}_x$ for each $x \in G$. Choose $f \in C_0(G)$ such that $f(x) = 1$. Then

$$\begin{aligned} d_x\big(p_{s(x)}(c)\big) &= (p_{r(x)} \otimes ev_x)(d(c \otimes f)) = (p_{r(x)} \otimes ev_x)(\hat{\delta}(c)(1 \otimes f)) \\ &= (p_{r(x)} \otimes ev_x)\hat{\delta}(c) = \hat{\delta}_x(p_{s(x)}(c)), \quad c \in C. \end{aligned}$$

vii) Using parts i) and ii), it is easy to see that the assumption implies $\hat{\delta}(C)(1 \otimes \hat{\mathscr{S}}) = C \otimes \hat{\mathscr{S}}$. Therefore, d is surjective. Then d_v is surjective for each $v \in G^0$ by [31, prop. 3.1]. Since $d_v = d_v^2$, it must be the identity. Therefore, $d_{x^{-1}}$ is the inverse of d_x for each $x \in G$. Using [31, prop 3.1] again, we find that d is an isomorphism. □

The previous proposition has the following converse.

Proposition 3.22. *Let C be a $C_0(X)$-algebra and let $d\colon s^*C \to r^*C$ be an action of G on C. Then the composition of the map $i_1\colon C \mapsto M(s^*C), c \mapsto c \otimes 1$, with the extension $M(s^*C) \to M(r^*C)$ of d defines a coaction $\hat{\delta}\colon C \to \tilde{M}(r^*C)$ which turns C into an $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebra.*

Proof. Since d is a morphism of $C_0(G)$-algebras, we have for all $c \in C$ and $f \in C_0(X)$

$$\hat{\delta}(fc) = d(fc \otimes 1) = d(c \otimes \pi_s(f)) = d(c \otimes 1)\pi_{s2}(f) = \hat{\delta}(c)\pi_{s2}(f).$$

Hence, $\hat{\delta}$ is a morphism from C to $\hat{\boldsymbol{S}}(C)$. Again, since d is a morphism of $C_0(G)$-algebras, one has

$$\hat{\delta}(C)(1 \otimes C_0(G)) = d(C \otimes 1)(1 \otimes C_0(G)) = d(C \otimes C_0(G)) = r^*C.$$

For each $x \in X$, denote by $\hat{\delta}_x$ the map associated to $\hat{\delta}$ in part ii) of the previous proposition. The same calculation as shown there proves that $\hat{\delta}_x = d_x$. Since maps of the form $p_{r(xy)} \otimes ev_x \otimes ev_y, (x, y) \in G\,{}_s{\times}_r\, G$, separate the points of $\tilde{M}\big(C \otimes_{C_0(X)} (C(G\,{}_s{\times}_r\, G), \pi_{r1})\big)$, the same diagram as shown there proves coassociativity of $\hat{\delta}$. Therefore, $\hat{\delta}$ is a coaction. Finally, the map i_1 is injective by [4, cor. 3.16], and since d is injective, so is $\hat{\delta}$. Considering C as a C^*-pre-family on the C^*-C-C-bimodule C via

$$C_\alpha^\beta := \begin{cases} C, & \alpha = \mathrm{id}, \beta = \mathrm{id}, \\ \{0\}, & \text{otherwise}, \end{cases} \qquad \alpha, \beta \in \mathrm{PAut}(C_0(G^0)),$$

and δ as a morphism of C^*-pre-families in the obvious way, (C, δ) becomes a $(C_0(G), \hat{\Delta})$-pre-family, and the coaction of $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$ is induced from this $(C_0(G), \hat{\Delta})$-family by tautology. □

Alternatively, in the last part of the proof above, one could also consider the C^*-pre-family $\mathscr{C}ov(C)$ in order to produce a $(C_0(G), \hat{\Delta})$-pre-family.

Theorem 3.23. *There exists a bijective correspondence between actions of G on $C_0(X)$-algebras and $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{\Delta}})$-algebras.*

Proof. Combine the previous two propositions and observe that both times one has $\hat{\delta}_x = d_x$ for all $x \in G$, whence the constructions given there are mutually inverse. □

Remark 3.24. This paragraph has shown that the semigroup grading techniques introduced in the first chapter are not needed for the treatment of the left leg of the pseudo-Kac system associated to a groupoid. It is convenient to focus on the grading degree $(\mathrm{id}, \mathrm{id})$ and – roughly speaking – to neglect components of other degrees if they are given or put them equal to 0.

3.3 Coactions of the right leg and Fell bundles in the r-discrete case

Let G be an r-discrete Hausdorff groupoid. In the previous section, we associated to G a pseudo-Kac system, computed its legs and investigated coactions of the left leg. The study of coactions of the right leg $(\boldsymbol{S}, \boldsymbol{\Delta})$ requires a separate section. We show that injective coactions of $(\boldsymbol{S}, \boldsymbol{\Delta})$ correspond bijectively with upper semi-continuous Fell bundles over G. This generalises an analogous result for discrete groups [35, 44, 2]. In the first subsection, we show that the reduced C^*-algebra of a Fell bundle on the groupoid G carries a canonical injective coaction. This does not depend on the assumption that G is r-discrete. The reverse process – the construction of a Fell bundle out of an injective coaction – proceeds in several steps. Following the proof of the corresponding result in the group case [2], we construct a Haar mean and show that injective coactions of $(\boldsymbol{S}, \boldsymbol{\Delta})$ are automatically non-degenerate in a strong sense. Next, we associate to each injective coaction a Fell bundle, using the Haar mean again, and show that the reduced C^*-algebra of this Fell bundle coincides with the C^*-algebra underlying the coaction. Then, the correspondence follows easily. As a by-product, it will follow that each injective coaction of $(\boldsymbol{S}, \boldsymbol{\Delta})$ is a regular $(\boldsymbol{S}, \boldsymbol{\Delta})$-algebra.

Throughout this section, let G be an r-discrete Hausdorff groupoid and denote by $(\boldsymbol{S}, \boldsymbol{\Delta})$ the functor and the natural transformation of the right leg of the pseudo-Kac system associated to the groupoid G.

3.3.1 From Fell bundles to coactions

Let F be an upper semi-continuous Fell bundle on the groupoid G. The *reduced C^*-algebra* of F is defined as follows. Denote by F^0 the restriction of F to G^0 and note that $\Gamma_0(F^0)$ is a C^*-algebra. The space $\Gamma_c(F)$, equipped with the operations

$$\langle\eta|\xi\rangle(v) := \int_{G_v} \eta(x)^*\xi(x)d\lambda_v^{-1}(x), \qquad (\xi f)(x) := \xi(x)f(s(x))$$

where $\xi, \eta \in \Gamma_c(F), f \in \Gamma_0(F^0)$ and $x \in G, v \in G^0$, is a pre-C^*-module over $\Gamma_0(F^0)$. Its completion is denoted by $\Gamma^2(F)$. For each section $f \in \Gamma_c(F)$, one has a left convolution operator λ_f on $\Gamma^2(F)$ given by

$$(\lambda_f\xi)(x) := \int_{G^{r(x)}} f(y)\xi(y^{-1}x)d\lambda^{r(x)}(y), \quad \xi \in \Gamma_c(F),\, x \in G.$$

The *reduced C^*-algebra of the Fell bundle F* is the C^*-subalgebra of $L_{\Gamma_0(F^0)}\big(\Gamma^2(F)\big)$ generated the convolution operators $\lambda_f, f \in \Gamma_c(F)$. It is denoted by $C^*_r(F)$. The map $\lambda\colon \Gamma_c(F) \to C^*_r(F)$ is injective. Denote by π_r and π_s the representations of $C_0(G^0)$ on the C^*-module $\Gamma^2(F)$ induced by the range and source map of G, respectively.

Proposition 3.25. *Let F be an upper semi-continuous Fell bundle on a locally compact Hausdorff groupoid G. The formulas*

$$\big(\pi(b)f\big)(x) := b\big(r(x)\big)f(x), \qquad \big(f\pi(b)\big)(x) := f(x)b\big(s(x)\big),$$
$$x \in G,\, b \in C_0(G^0),\, f \in \Gamma_c(F),$$

define a non-degenerate $*$-homomorphism $\pi\colon C_0(G^0) \to M(C^*_r(F))$. The C^*-algebra $\mathscr{S}(C^*_r(F), \pi) = C^*_r(F) \circledast \mathscr{S}$ is faithfully represented on the C^*-module $(\Gamma^2(F), \pi_r) \circledast L^2(G, \lambda)$, and the formula

$$\big(\delta_F(\lambda_f)(\xi \circledast \eta)\big)(x, y) := \int_{G^{r(x)}} f(z)\xi(z^{-1}x)\eta(z^{-1}y)d\lambda^{r(x)}(z),$$
$$(x, y) \in G\,{}_r{\times}_r\, G,\, \xi \in \Gamma_c(F),\, \eta \in C_c(G),\, f \in \Gamma_c(U, F),$$

defines an injective coaction δ_F of $(\boldsymbol{S}, \boldsymbol{\Delta})$ on $\big(C^*_r(F), \pi\big)$.

Proof. It is easy to see that π is a non-degenerate $*$-homomorphism $C_0(G^0) \to M(C^*_r(F))$.

Denote by π_s and π_r the representations of $C_0(G^0)$ on $\Gamma^2(F)$ induced by the source and range map of G, and by $p\colon G\,{}_s{\times}_r\, G \to G$ and $q\colon G\,{}_r{\times}_r\, G \to G$ the projections onto the first components, respectively. Then the C^*-module $(\Gamma^2(F), \pi_r) \circledast L^2(G, \lambda)$ identifies with the completion of the space $\Gamma_c(p^*F)$ with respect to the inner product

$$\langle f|g\rangle(u) := \int_{G_u}\int_{G^{r(x)}} f(x, y)^*g(x, y)d\lambda^{r(x)}(y)d\lambda_u^{-1}(x), \quad u \in G^0,\, f, g \in \Gamma_c(p^*F),$$

and the C^*-module $(\Gamma^2(F), \pi_r) \otimes L^2(G, \lambda^{-1})$ identifies with the completion of the space $\Gamma_c(q^*F)$ with respect to the inner product

$$\langle f|g\rangle(u) := \int_{G_u} \int_{G_{r(x)}} f(x,y)^* g(x,y) d\lambda^{-1}_{r(x)}(y) d\lambda^{-1}_u(x), \quad u \in G^0,\ f,g \in \Gamma_c(q^*F).$$

Consider the map $\hat{V}_{F,0} \colon \Gamma_c(p^*F) \to \Gamma_c(q^*F)$ given by $(\hat{V}_{F,0}f)(x,y) := f(yx,x)$. We show that it extends to a unitary

$$\hat{V}_F \colon (\Gamma^2(F), \pi_r) \otimes L^2(G, \lambda) \to (\Gamma^2(F), \pi_r) \otimes L^2(G, \lambda^{-1}).$$

For $f \in \Gamma_c(q^*F)$, $g \in \Gamma_c(p^*F)$ and $u \in G^0$, one has

$$\langle f | \hat{V}_{F,0} g\rangle(u) = \int_{G_u} \int_{G_{r(x)}} f(x,y)^* g(yx,y) d\lambda^{-1}_{r(x)}(y) d\lambda^{-1}_u(x).$$

Substituting $y' := yx$ and $x' := x^{-1}$, this expression becomes

$$\int_{G_u} \int_{G^u} f(x'^{-1}, y'x')^* g(y', y'x') d\lambda^{-1}_u(y') d\lambda^u(x').$$

Substituting $z := y'x'$ and retaining y', we obtain

$$\langle f | \hat{V}_{F,0} g\rangle(u) = \int_{G_u} \int_{G^{r(y')}} f(z^{-1}y', z)^* g(y', z) d\lambda^{r(y')}(z) d\lambda^{-1}_u(y') = \langle \hat{V}^{-1}_{F,0} f | g\rangle(u).$$

A calculation similar to the one carried out for the computation of the coproduct Δ on $\mathscr{S}$ in part ii) of proposition 3.19 shows that the map

$$C^*_r(F) \to L_{C_0}\big((\Gamma^2(F), \pi_r) \otimes L^2(G, \lambda)\big), \qquad T \mapsto \hat{V}^*_F (1 \otimes T) \hat{V}_F,$$

implements the formula for δ_F given above. Thus, δ_F is a well-defined injective morphism $\big(C^*_r(F), \pi\big) \to \boldsymbol{S}\big(C^*_r(F), \pi\big)$ of $C_0(G^0)$-algebras. The inclusion $\delta(C^*_r(F))(1 \otimes \mathscr{S}) \subset \boldsymbol{S}(C^*_r(F), \pi)$ and coassociativity of the map δ_F follow easily from the formula. □

Remark 3.26. The proof suggest to generalise the notion of a coaction unitary so that it accommodates the unitary $\hat{V}_F$ – a modification of the approach to coaction unitaries can be used to show that the coaction constructed above is in fact a regular $(\boldsymbol{S}, \boldsymbol{\Delta})$-algebra. If the underlying groupoid is r-discrete, this will follow from corollary 3.38.

3.3.2 The Haar mean for the right leg

In this section, we construct a Haar mean for the right leg $(\boldsymbol{S}, \boldsymbol{\Delta})$, following ideas of [2]. To do so, we first need to collect several preliminary results about the right leg which are particular to the r-discrete case. It may be helpful to bear in mind the special case where G is just a discrete group. A fundamental rôle is played by the following subsets of G.

Notation 3.27. *We denote by $\mathscr{G}$ the family of open G-sets of G for which the restrictions $r|_U$ and $s|_U$ are homeomorphisms onto open subsets of G^0. We denote by $\mathscr{G}^{cc}$ the subfamily of all open subsets $U \subset G$ whose closure is compact and contained in an element of $\mathscr{G}$. For each $U \in \mathscr{G}^{cc}$, we choose a function $\chi_U \in C_c(G)$ such that $\chi_U|_U = 1$ and the support of χ_U is contained in some G-set. For $U \in \mathscr{G}$, we denote by $r_U \colon U \to r(U)$ and $s_U \colon U \to s(U)$ the restrictions of the range and source map, respectively. Then the partial homeomorphism q_U defined in proposition 3.2 is given by $q_U := r_U \circ s_U^{-1}$.*

Since G is r-discrete, both families $\mathscr{G}$ and $\mathscr{G}^{cc}$ form open covers of G. In the following, we prefer to work with the family $\mathscr{G}^{cc}$. However, all statements apply and all arguments carry through with $\mathscr{G}$ instead of $\mathscr{G}^{cc}$ after straight-forward modifications. If G is just a discrete group, the families $\mathscr{G}$ and $\mathscr{G}^{cc}$ coincide with the family $\big(\{x\}\big)_{x\in G}$ of the singleton sets of all group elements.

The family $\mathscr{G}^{cc}$ facilitates a useful description of the Hopf C^*-family $(\mathscr{S}, \Delta)$ and of the functor $\boldsymbol{S}$.

Lemma 3.28. *The C^*-family $\mathscr{S}$ is generated by the subfamily*

$$\big\{\lambda(C_c(U)) \mid U \subset \mathrm{Cov}_{\phi\wedge\psi}(G),\, U \in \mathscr{G}^{cc}\big\} \subset \mathscr{S}_{\phi_*}^{\psi_*}, \quad \phi, \psi \in \mathrm{PHom}(G^0).$$

For each $U \in \mathscr{G}^{cc}$ and $f', f'' \in C_0(U)$, one has $\Delta\big(\lambda(f'f'')\big) = \lambda(f') \otimes \lambda(f'')$.

Proof. Let $\phi, \psi \in \mathrm{PHom}(G^0)$. By proposition 3.19, the space $\mathscr{S}_{\phi_*}^{\psi_*}$ is densely spanned by operators of the form $\lambda(f)$ where $f \in C_c(\mathrm{Cov}_{\phi\wedge\psi}(G))$. By assumption on G and a partition of unity argument, each function $f \in C_c(\mathrm{Cov}_{\phi\wedge\psi}(G))$ can be written as a linear combination of functions $f_i \in C_c(U_i)$ where $U_i \subset \mathrm{Cov}_{\phi\wedge\psi}(G)$, $U_i \in \mathscr{G}^{cc}$.

Let us prove the second statement. Recall the formula for the coproduct Δ given in proposition 3.19. Put $f = f'f''$ and let $\xi, \eta \in L^2(G, \lambda)$ and $(x, y) \in G\,{}_r\!\times_r G$. If $U \cap r^{-1}(x)$ is empty, then

$$\big(\Delta(\lambda(f))(\xi \otimes \eta)\big)(x, y) = 0 = (\lambda(f')\xi)(x) \cdot (\lambda(f'')\eta)(y).$$

Otherwise, the intersection $U \cap r^{-1}(x)$ contains exactly one point, z, say. Then

$$\begin{aligned}\big(\Delta(\lambda(f))(\xi \otimes \eta)\big)(x, y) &= f(z)\xi(z^{-1}x)\eta(z^{-1}y) \\ &= f'(z)\xi(z^{-1}x) \cdot f''(z)\eta(z^{-1}y) \\ &= (\lambda(f')\xi)(x) \cdot (\lambda(f'')\eta)(y).\end{aligned}$$

□

Let $(C, \pi) \in \boldsymbol{C^*_A}$. The following lemma describes the C^*-algebra $(C, \pi) \otimes \mathscr{S}$.

Lemma 3.29. *i) For all $c \otimes \lambda(f) \in \mathscr{C}ov(C, \pi) \otimes \mathscr{S}$ and all $g \in C_0(G^0)$, one has $(c \otimes \lambda(f))\pi_{r2}(g) = c \otimes \lambda(fs^*(g)) = c\pi(g) \otimes \lambda(f)$ and $\pi_{r2}(g)(c \otimes \lambda(f)) = \pi(g)c \otimes \lambda(f) = c \otimes \lambda(r^*(g)f)$.*

ii) The C^*-algebra $(C,\pi) \otimes \mathscr{S}$ is generated by elements of the form $c \otimes \lambda(f)$ where $f \in C_c(U)$ for some $U \in \mathscr{G}^{cc}$ and $c \in \mathscr{C}ov_{q_{U*}}(C,\pi)$.

Proof. i) This follows from the equations

$$\pi_{r2}(g)(c' \otimes \xi) = c' \otimes \pi_r(g)\xi = c' \otimes \xi g = \pi(g)c' \otimes \xi, \quad c' \in C,\, \xi \in L^2(G,\lambda),$$
$$\lambda(f)\pi_r(g)\xi = f \star (g \star \xi) = (f \star g) \star \xi = \lambda(fs^*(g))\xi, \quad \xi \in L^2(G,\lambda).$$

ii) By lemma 3.28, the internal tensor product $(C,\pi) \otimes \mathscr{S}$ is generated by elements of the form $c \otimes \lambda(f)$ where $c \in \mathscr{C}ov_{\phi_*}(C,\pi)$, $f \in C_c(U)$, $U \subset \mathrm{Cov}_{\phi \wedge \psi}(G)$, $U \in \mathscr{G}^{cc}$ and $\phi, \psi \in \mathrm{PHom}(G^0)$. Choose a function $g \in C_0(s(U))$ such that $f = fs^*(g)$. Then $c \otimes \lambda(f) = c \otimes \lambda(fs^*(g)) = c\pi(g) \otimes \lambda(f)$ by part i), and $c\pi(g)$ is contained in $\mathscr{C}ov_{q_{U*}}(C,\pi)$. The claim follows. ☐

Next, we describe the individual subspaces $\mathscr{C}ov_{q_{U*}}(C,\pi) \otimes \mathscr{S}$ for $U \in \mathscr{G}^{cc}$.

Lemma 3.30. *Let $U \in \mathscr{G}^{cc}$.*

i) Given a finite number of elements $c_i \otimes \lambda(f_i) \in \mathscr{C}ov_{q_{U*}}(C,\pi) \otimes \lambda(C_0(U))$, one has $\sum_i c_i \otimes \lambda(f_i) = c \otimes \lambda(\chi_U)$ where $c = \sum_i c_i \pi(s_{U*}(f_i))$.

ii) Given a sequence of elements $c_i \otimes g_i \in \mathscr{C}ov_{q_{U*}}(C,\pi) \otimes \lambda(C_0(U))$ which converges in norm to some operator d, there exists $c \in \mathscr{C}ov_{q_{U*}}(C,\pi)$ such that $d = c \otimes \lambda(\chi_U)$.

iii) The spaces $\{c \otimes \lambda(\chi_U) \mid c \in \mathscr{C}ov_{q_{U*}}(C,\pi)\}$, $\{c \otimes \lambda(f) \mid c \in \mathscr{C}ov_{q_{U*}}(C,\pi),\ f \in C_0(U)\}$ and $\mathscr{C}ov_{q_{U*}}(C,\pi) \otimes \lambda(C_0(U))$ are equal.

iv) Let $c \otimes \lambda(f) \in \mathscr{C}ov_{q_{U*}}(C,\pi) \otimes \lambda(C_0(U))$ and $g \in C_0(U')$, $U' \in \mathscr{G}^{cc}$. Then $c \otimes \lambda(fg) = c' \otimes \lambda(\chi_{U \cap U'})$ for some $c' \in \mathscr{C}ov_{q_{U \cap U'}}(C,\pi)$.

Proof. i) This follows from part i) of lemma 3.29.

ii) One has $c_i \otimes \lambda(g) = \pi(r_{U*}(g))c_i \otimes \lambda(\chi_U)$. By assumption, the sequence of elements

$$\pi(r_{U*}(g))c_i = \big(1 \otimes \langle \chi_U|\big) \cdot (c_i \otimes \lambda(g)) \cdot \big(1 \otimes |s_{U*}(\chi)\rangle\big)$$

converges to $c := \big(1 \otimes \langle \chi_U|\big) \cdot d \cdot \big(1 \otimes |s_{U*}(\chi_U)\rangle\big)$. Hence, $d = c \otimes \lambda(\chi_U)$.

iii) This follows from parts i) and ii).

iv) Part i) of lemma 3.29 and the inclusion $\operatorname{supp} fg \subset U \cap U'$ imply $c \otimes \lambda(fg) = c' \otimes \lambda(\chi_{U \cap U'})$ where $c' = c\pi\big(s_{(U \cap U')*}(fg)\big)$. Clearly, c' is contained in $C \cdot C_0(s(U \cap U')) = C\,\mathrm{Dom}(q_{(U \cap U')*})$. Since the restriction of q_U to $s(U \cap U')$ coincides with $q_{U \cap U'}$, the element c' belongs to $\mathscr{C}ov_{q_{(U \cap U')*}}(C,\pi)$. ☐

The Haar mean

The notion of a Haar mean for a Hopf C^*-algebra has been introduced in [2, definition 7.6, exemples 7.8(3)]. Adapted to our setting, it will be the main tool for the construction of a Fell bundle out of a coaction.

Proposition 3.31. *i) For each $f \in C_c(G)$, the following assignment defines a function $Wf \in C_c(G\,{}_r{\times}_r\,G)$:*

$$(Wf)(x,y) := \begin{cases} 0, & x \neq y, \\ f(x), & x = y, \end{cases} \qquad (x,y) \in G\,{}_r{\times}_r\,G,$$

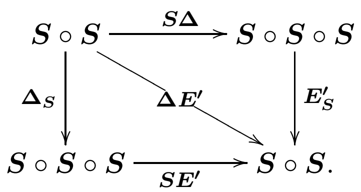
ii) The map $f \mapsto Wf$ extends to an isometry $W\colon L^2(G,\lambda) \to (L^2(G,\lambda),\pi_r) \otimes L^2(G,\lambda)$. Its adjoint is given by $(W^*\xi)(x) = \xi(x,x)$, $\xi \in C_c(G), x \in G$.

iii) Let $f,g \in C_c(G)$, and let $\xi',\xi'' \in C_c(U) \subset L^2(G,\lambda)$ for some $U \in \mathscr{G}$. Put $\xi = \xi'\xi''$. Then $W\xi = \xi' \otimes \xi''$ and $W^*(\lambda(f)\xi' \otimes \lambda(g)\xi'') = \lambda(fg)(\xi)$.

iv) Let $d \otimes \lambda(g) \in \boldsymbol{S}^2(C,\pi)$ where $d \in \mathscr{C}ov_{q_{U*}}\big((C,\pi) \otimes \mathscr{S},\pi_{r2}\big)$ and $g \in C_0(U)$, $U \in \mathscr{G}^{cc}$. Then $\mathrm{Ad}_{(1\otimes W^*)}(d \otimes \lambda(g)) = c \otimes \lambda(\chi_U)$ for some $c \in \mathscr{C}ov_{q_{U*}}(C,\pi)$.

v) The map $\mathrm{Ad}_{(1\otimes W^*)}$ defines a morphism $\boldsymbol{E'}_{(C,\pi)}\colon \boldsymbol{S}^2(C,\pi) \to \boldsymbol{S}(C,\pi)$, and the family $\Big(\boldsymbol{E'}_{(C,\pi)}\Big)_{(C,\pi)}$ defines a natural transformation $\boldsymbol{E'}\colon \boldsymbol{S}^2 \to \boldsymbol{S}$.

vi) One has $\boldsymbol{E'} \circ \boldsymbol{\Delta} = \mathrm{id}_{\boldsymbol{S}}$, and the following diagram commutes.



Proof. i,ii) We only need to show that for each $f \in C_c(G)$, the function Wf defined above is continuous. This follows from the fact that the diagonal $\{(x,x)|x \in G\}$ is closed in $G\,{}_r{\times}_r\,G$ because G is Hausdorff, and open in $G\,{}_r{\times}_r\,G$ because G is r-discrete.

iii) Let $x \in G$. Then $\big(W^*(\lambda(f)\xi' \otimes \lambda(g)\xi''\big)(x)$ is equal to

$$\big(\lambda(f)\xi'\big)(x) \cdot \big(\lambda(g)\xi''\big)(x) = \sum_{y \in G^{r(x)}} f(y)\xi'(y^{-1}x) \cdot \sum_{z \in G^{r(x)}} g(z)\xi''(z^{-1}x)$$

If $U \cap G^{r(X)}$ is empty, this expression is 0, and $(\lambda(fg)\xi)(x) = 0$ as well. Otherwise, the intersection consists of exactly one point, x_0, say. In that case,

$$\big(W^*(\lambda(f)\xi' \otimes \lambda(g)\xi''\big)(x) = f(x_0)\xi'(x_0^{-1}x)g(x_0)\xi''(x_0^{-1}x) = \big(\lambda(fg)\xi\big)(x).$$

iv) Write $d = \lim_n d_n$ where

$$d_n = \sum_i c_n^i \otimes \lambda(f_n^i), \quad c_n^i \otimes \lambda(f_n^i) \in \mathscr{C}ov_{q_{U_n^i *}}(C,\pi) \otimes \lambda(C_0(U_n^i)), \quad U_n^i \in \mathscr{G}^{cc}.$$

Let $c' \in C$ and $\xi = \xi' \cdot \xi'' \in C_c(V) \subset L^2(G,\lambda)$ where $V \in \mathscr{G}$. Then

$$\begin{aligned} \mathrm{Ad}_{(1 \otimes W^*)} \big(d \otimes \lambda(g)\big)(c' \otimes \xi) &= \lim_n \sum_i c_n^i c' \otimes W^* \big(\lambda(f_n^i)\xi' \otimes \lambda(g)\xi''\big) \\ &= \lim_n \sum_i c_n^i c' \otimes \lambda(f_n^i g)\xi. \end{aligned}$$

By part ii) of proposition 3.30, $c_n^i \otimes \lambda(f_n^i g) = \hat{c}_n^i \otimes \lambda(\chi_U)$ for some $\hat{c}_n^i \in \mathscr{C}ov_{q_{U*}}(C,\pi)$. Thus, $\mathrm{Ad}_{(1\otimes W^*)}(d \otimes \lambda(g)) = \lim_n \sum_i \hat{c}_n^i \otimes \lambda(\chi_U)$. By part v) of the same proposition, this element can be written in the form $c \otimes \lambda(\chi_U)$ with some $c \in \mathscr{C}ov_{q_{U*}}(C,\pi)$.

v) Consider the previous calculation. The equations

$$\pi_r(h)\lambda(g) = \lambda(r^*(h)g), \qquad \pi_r(h)\lambda(f_n^i g) = \lambda(f_n^i r^*(h) g), \quad h \in C_0(G^0)$$

show that $\mathrm{Ad}_{(1\otimes W^*)}$ intertwines the representation π_{r3} on $C \otimes L^2(G,\lambda)^{\pi_r} \otimes L^2(G,\lambda)$ with the representation π_{r2} on $C \otimes L^2(G,\lambda)$. Thus, $\mathrm{Ad}_{(1\otimes W^*)}$ defines a morphism $\boldsymbol{E}'_{(C,\pi)} \colon \boldsymbol{S}^2(C,\pi) \to \boldsymbol{S}(C,\pi)$. The fact that the family $\big(\boldsymbol{E}'_{(C,\pi)}\big)_{(C,\pi)}$ defines a natural transformation is clear from the construction.

vi) The equation $\boldsymbol{E}' \circ \boldsymbol{\Delta} = \mathrm{id}_{\boldsymbol{S}}$ follows from part iii) and lemma 3.28. Let us show that the diagram commutes. Consider an element $d \otimes \lambda(g)$ as in part iii). By lemma 3.28,

$$\begin{aligned} \boldsymbol{\Delta}_{(C,\pi)}\big(\boldsymbol{E}'_{(C,\pi)}(d \otimes \lambda(g))\big) &= \lim_n \sum_n c_n^i \otimes \Delta(\lambda(f_n^i g)), \\ \boldsymbol{S}(\boldsymbol{E}'_{(C,\pi)})\big(d \otimes \Delta(\lambda(g))\big) &= \boldsymbol{E}'_{(C,\pi)}\big(d \otimes \lambda(g)\big) \otimes \lambda(\chi_U), \\ \boldsymbol{E}'_{\boldsymbol{S}(C,\pi)}\big(\boldsymbol{\Delta}_{(C,\pi)}(d) \otimes \lambda(g)\big) &= \boldsymbol{E}'_{\boldsymbol{S}(C,\pi)}\Big(\lim_n \sum_i \big(c_n^i \otimes \lambda(\chi_{U_n^i})\big) \otimes \lambda(f_n^i) \otimes \lambda(g)\Big) \\ &= \lim_n \sum_i c_n^i \otimes \lambda(\chi_{U_n^i}) \otimes \lambda(f_n^i g). \end{aligned}$$

By the same lemma, these expressions are equal. □

Remark 3.32. In part iii), we can not simply write $\mathrm{Ad}_{W^*}(\lambda)(f) \otimes \lambda(g)) = \lambda(fg)$ because the internal tensor product $\lambda(f) \otimes \lambda(g)$ need not be well-defined. Parts iv) and v) of this proof could be substantially simplified if we had $\mathscr{C}ov((C,\pi) \otimes \mathscr{S}, \pi_{r2}) = \mathscr{C}ov(C,\pi) \otimes \mathscr{S}$.

Next, we carry over the construction of retractions for coactions from the group case [3, proposition 7.12, 7.13, lemme 7.14]. To do so, we need the following proposition and lemma.

Proposition 3.33. *Let (C,π,δ) be a coaction of $(\boldsymbol{S},\boldsymbol{\Delta})$. Then the image $\delta(C)$ is contained in the C^*-algebra $(C,\pi)\otimes\mathscr{S}$.*

Proof. Since π is non-degenerate, one has

$$\begin{aligned}\delta(C)=\delta\big(C\pi\big(C_0(G^0)\big)\big)&=\delta(C)\pi_{r2}(C_0(G^0))\\&=\delta(C)\big(1\otimes\lambda(C_0(G^0))\big)\subset(C,\pi)\otimes\mathscr{S}.\end{aligned}$$ ☐

Lemma 3.34. *Let (C,π,δ) be a coaction of $(\boldsymbol{S},\boldsymbol{\Delta})$ and let $d=c\otimes\lambda(f)\in\mathscr{C}ov_{q_U}(C,\pi)\otimes\lambda(C_0(U))$, $U\in\mathscr{G}^{cc}$. If $(\delta\otimes\mathrm{id})(d)=(\mathrm{id}\otimes\Delta)(d)$, then $d=\delta\big(r_{U*}(f)c\big)$.*

Proof. Put $\chi:=\chi_U$. By lemma 3.30 and the assumption, we have

$$\begin{aligned}d=c\otimes\pi_r(r_{U*}(\chi^3))\lambda(f)&=\langle\chi|_3\cdot(c\otimes\lambda(f)\otimes\lambda(\chi))\cdot|s_{U*}(\chi)\rangle_3\\&=\langle\chi|_3\cdot(c\otimes\Delta(\lambda(f)))\cdot|s_{U*}(\chi)\rangle_3\\&=\langle\chi|_3\cdot(\delta(c)\otimes\lambda(f))\cdot|s_{U*}(\chi)\rangle_3\\&=\pi_{r2}(r_{U*}(f))\delta(c)=\delta(r_{U*}(f)c).\end{aligned}$$ ☐

Proposition 3.35. *Let (C,π,δ) be an injective coaction of $(\boldsymbol{S},\boldsymbol{\Delta})$.*

i) There exists a unique morphism $\boldsymbol{E}_{(C,\pi,\delta)}\colon\boldsymbol{S}(C,\pi)\to(C,\pi)$ such that $\boldsymbol{E}_{(C,\pi,\delta)}\circ\delta=id_{(C,\pi)}$ and the following diagram commutes.

$$\begin{array}{ccc}\boldsymbol{S}(C,\pi)&\xrightarrow{\boldsymbol{S}(\delta)}&\boldsymbol{SS}(C,\pi)\\ {\scriptstyle\boldsymbol{\Delta}_{(C,\pi)}}\downarrow&\searrow{\scriptstyle\delta\boldsymbol{E}_{(C,\pi,\delta)}}&\downarrow{\scriptstyle\boldsymbol{E}'_{(C,\pi)}}\\ \boldsymbol{SS}(C,\pi)&\xrightarrow[\boldsymbol{SE}_{(C,\pi,\delta)}]{}&\boldsymbol{S}(C,\pi).\end{array}$$

ii) $\boldsymbol{E}_{(C,\pi,\delta)}\big(\mathscr{C}ov_{q_{U*}}(C,\pi)\otimes\lambda(C_0(U))\big)\subset\mathscr{C}ov_{q_{U*}}(C,\pi)$ for all $U\in\mathscr{G}^{cc}$.

iii) Let $\phi\colon(C,\pi,\delta)\to(C',\pi',\delta')$ be a morphism of injective coactions. Then $\phi\circ\boldsymbol{E}_{(C,\pi,\delta)}=\boldsymbol{E}_{(C',\pi',\delta')}\circ\boldsymbol{S}(\phi)$.

Proof. i,ii) Uniqueness follows from injectivity of δ and the commutativity condition on the upper right triangle in the diagram. By the previous proposition, the following maps on $\boldsymbol{S}(C,\pi)$ are equal.

$$\begin{aligned}\boldsymbol{S}(\delta)\boldsymbol{E}'_{(C,\pi)}\boldsymbol{S}(\delta)&=\boldsymbol{E}'_{\boldsymbol{S}(C,\pi)}\boldsymbol{S}^2(\delta)\boldsymbol{S}(\delta)\\&=\boldsymbol{E}'_{\boldsymbol{S}(C,\pi)}\boldsymbol{S}(\boldsymbol{\Delta}_{(C,\pi)})\boldsymbol{S}(\delta)=\boldsymbol{\Delta}_{\boldsymbol{S}(C,\pi)}\boldsymbol{E}'_{(C,\pi)}\boldsymbol{S}(\delta)\end{aligned}$$

Let $c\otimes\lambda(f)\in\mathscr{C}ov_{q_{U*}}(C,\pi)\otimes\lambda(C_0(U))$ where $U\in\mathscr{G}^{cc}$. By part iv) of the previous proposition, $\boldsymbol{E}'_{(C,\pi)}(\delta(c)\otimes\lambda(f))=c'\otimes\lambda(\chi_U)$ for some $c'\in\mathscr{C}ov_{q_{U*}}(C,\pi)$. By lemma 3.34 and the calculation above, $c'\otimes\lambda(\chi_U)=\delta(c')$. Hence, the image of

$\boldsymbol{E'}_{(C,\pi)}\boldsymbol{S}(\delta)$ is contained in the image of δ, and since δ is injective, the morphism $\boldsymbol{E}_{(C,\pi,\delta)} := \delta^{-1}\boldsymbol{E'}_{(C,\pi)}\boldsymbol{S}(\delta)$ is well-defined.

By part vi) of proposition 3.31, we have

$$\delta\boldsymbol{E}_{(C,\pi,\delta)}\delta = \boldsymbol{E'}_{\boldsymbol{S}(C,\pi)}\boldsymbol{S}(\delta)\delta = \boldsymbol{E'}_{\boldsymbol{S}(C,\pi)}\boldsymbol{\Delta}_{(C,\pi)}\delta = \delta.$$

Since δ is injective, this proves the equality $\boldsymbol{E}_{(C,\pi,\delta)}\delta = \mathrm{id}_{(C,\pi)}$. We show that $\delta\boldsymbol{E}_{(C,\pi,\delta)} = \boldsymbol{S}(\boldsymbol{E}_{(C,\pi,\delta)})\boldsymbol{\Delta}$. Since δ is injective, so is $\boldsymbol{S}(\delta)$. By the equation at the beginning of the proof and part v) of the previous proposition, we have

$$\begin{aligned}\boldsymbol{S}(\delta)\delta\boldsymbol{E}_{(C,\pi,\delta)} = \boldsymbol{\Delta}_{\boldsymbol{S}(C,\pi)}\boldsymbol{E'}_{(C,\pi)}\boldsymbol{S}(\delta) &= \boldsymbol{S}(\boldsymbol{E'}_{(C,\pi)})\boldsymbol{\Delta}_{\boldsymbol{S}(C,\pi)}\boldsymbol{S}(\delta) \\ &= \boldsymbol{S}(\boldsymbol{E'}_{(C,\pi)})\boldsymbol{S}^2(\delta)\boldsymbol{\Delta}_{(C,\pi)} = \boldsymbol{S}(\delta\boldsymbol{E}_{(C,\pi,\delta)})\boldsymbol{\Delta}_{(C,\pi)}.\end{aligned}$$

iii) This follows from a straight-forward calculation.

$$\begin{aligned}\phi\delta^{-1}\boldsymbol{E'}_{(C,\pi)}\boldsymbol{S}(\delta) &= \delta'^{-1}\boldsymbol{S}(\phi)\boldsymbol{E'}_{(C,\pi)}\boldsymbol{S}(\delta) \\ &= \delta'^{-1}\boldsymbol{E'}_{(C',\pi')}\boldsymbol{S}^2(\phi)\boldsymbol{S}(\delta) = \delta'^{-1}\boldsymbol{E'}_{(C',\pi')}\boldsymbol{S}(\delta')\boldsymbol{S}(\phi).\end{aligned}$$ □

Hooptedoodle 3.36. *The family* $\left(\boldsymbol{E}_{(C,\pi,\delta)}\right)$ *defines a natural transformation* $\boldsymbol{E}\colon \boldsymbol{SU}|_{\boldsymbol{Coact}^{*,inj}_{(\boldsymbol{S},\boldsymbol{\Delta})}} \to \boldsymbol{U}|_{\boldsymbol{Coact}^{*,inj}_{(\boldsymbol{S},\boldsymbol{\Delta})}}$, *where* $\boldsymbol{Coact}^{*,inj}_{(\boldsymbol{S},\boldsymbol{\Delta})}$ *denotes the full subcategory of* $\boldsymbol{Coact}_{(\boldsymbol{S},\boldsymbol{\Delta})}$ *consisting of the objects* (C,π,δ) *where* δ *is injective, and* $\boldsymbol{U}\colon \boldsymbol{Coact}_{(\boldsymbol{S},\boldsymbol{\Delta})} \to \boldsymbol{C}^*_{\boldsymbol{A}}$ *denotes the forgetful functor given by* $(C,\pi,\delta) \mapsto (C,\pi)$ *and* $\phi \mapsto \phi$. *The previous propositions also show that the morphisms comprising* $\boldsymbol{E'}$ *and* $\boldsymbol{E}$ *are equivariant with respect to the canonical coactions.*

3.3.3 From coactions to Fell bundles

In this subsection, we associate to each injective coaction an upper semi-continuous Fell bundle on G and show that the reduced C^*-algebra of this Fell bundle coincides with the C^*-algebra underlying the coaction. As a key tool, we will use the Haar mean constructed in the previous subsection.

Let (C,π,δ) be an injective coaction of $(\boldsymbol{S},\boldsymbol{\Delta})$. For each $U \in \mathscr{G}^{cc}$, put

$$C_U := \boldsymbol{E}_{(C,\pi,\delta)}\Big(\mathscr{C}ov_{q_{U*}}(C,\pi) \otimes \lambda(C_0(U))\Big).$$

The Fell bundle associated to (C,π,δ) will be constructed in such a way that the sections over a subset $U \in \mathscr{G}^{cc}$ identify with the space C_U. The next proposition collects some easy properties of the family $\left(C_U\right)_U$.

Proposition 3.37. *i)* $C_U = \{c' \in \mathscr{C}ov_{q_{U*}}(C,\pi) \mid \delta(c') = c' \otimes \lambda(\chi_U)\}$,

ii) $C_U C_{U'} \subset C_{UU'}$ *and* $C_U^* = C_{U^{-1}}$ *for all* $U,U' \in \mathscr{G}^{cc}$; *in particular,* $C_{U'}$ *is a* C^*-*algebra if* $U' \subset G^0$, *and* C_U *is a* C^*-*module over* $C_{s(U)}$, *iii)* C_U *is a non-degenerate convex* $C_0(s(U))$*-module with respect to the operation* $c \cdot f := c\pi(f)$.

iv) If $U' \in \mathscr{G}^{cc}$ *is contained in* U, *then* $C_{U'} = C_U C_0(s(U'))$.

Proof. i) Let $c \in \mathscr{C}ov_{q_{U*}}(C,\pi)$ and $c' = \boldsymbol{E}_{(C,\pi,\delta)}(c \otimes \lambda(\chi_U))$. By proposition 3.35, we have

$$\begin{aligned}\delta(c') &= \boldsymbol{S}\big(\boldsymbol{E}_{(C,\pi,\delta)}\big)\big(c \otimes \Delta(\lambda(\chi_U))\big) \\ &= \boldsymbol{S}\big(\boldsymbol{E}_{(C,\pi,\delta)}\big)(c \otimes \lambda(\chi_U) \otimes \lambda(\chi_U)) = c' \otimes \lambda(\chi_U).\end{aligned}$$

Conversely, if $\delta(c') = c' \otimes \lambda(\chi_U)$, then $c' = \boldsymbol{E}_{(C,\pi,\delta)}(\delta(c'))$ is contained in C_U.

ii) This follows immediately from part i).

iii) By part i) of lemma 3.29, $C_U\,\pi(C_0(s(U))) = C_U$. If $U \subset G^0$, the map π defines a $C_0(U)$-algebra structure on $\mathscr{C}ov_{q_{U*}}(C,\pi)$ and hence also on C_U. In the general case, convexity follows from lemma A.2 and part ii).

iv) One has $C_{U'} = C_{U'}C_0(s(U')) \subset C_U C_0(s(U'))$ because $C_{U'} \subset C_U$. The reverse inclusion follows from part iv) of lemma 3.30. □

Before constructing the Fell bundle, we use the results obtained so far to show that each injective coaction of $(\boldsymbol{S},\boldsymbol{\Delta})$ is a regular $(\boldsymbol{S},\boldsymbol{\Delta})$-algebra.

Corollary 3.38. *Let* (C,π,δ) *be an injective coaction of* $(\boldsymbol{S},\boldsymbol{\Delta})$.

i) $C = \overline{\operatorname{span}}_U C_U = \overline{\operatorname{span}}\,\mathscr{C}ov(C,\pi)$.

ii) Consider C *as a* C^*-$C_0(G^0)$-C*-bimodule and denote by* $\mathscr{C}$ *the* C^**-subfamily of* $\mathscr{L}_{\mathrm{id}}(C)$ *generated by the family* $(C_U)_U$. *Then* $\mathscr{C}$ *is non-degenerate, and* δ *restricts to a coaction of* $(\mathscr{S},\Delta)$ *on* $\mathscr{C}$.

iii) $(\mathscr{C},\delta)$ *is a regular* $(\mathscr{S},\Delta)$*-family and* (C,π,δ) *is a regular* $(\boldsymbol{S},\boldsymbol{\Delta})$*-algebra.*

Proof. i) By part i) of proposition 3.35, the morphism $\boldsymbol{E}_{(C,\pi,\delta)} \colon \boldsymbol{S}(C,\pi) \to (C,\pi)$ is surjective. By part ii) of lemma 3.29, $\boldsymbol{S}(C,\pi)$ is the closed linear span of the subspaces $\mathscr{C}ov_{q_{U*}}(C,\pi) \otimes \lambda(C_0(U))$ where $U \in \mathscr{G}^{cc}$. By definition, for each $U \in \mathscr{G}^{cc}$ the subspace $\mathscr{C}ov_{q_{U*}}(C,\pi) \otimes \lambda(C_0(U))$ gets mapped to $C_U \subset \mathscr{C}ov_{q_{U*}}(C,\pi)$.

ii) The fact that $\mathscr{C}$ is non-degenerate follows immediately from part i). The second part follows from part i) of proposition 3.37.

iii) First, we show that $\delta(\mathscr{C})(1 \otimes \mathscr{S}) = \mathscr{C} \otimes \mathscr{S}$. Let $\phi \in \mathrm{PHom}(G^0)$. By proposition 3.19 and by definition, one has

$$\begin{aligned}(\mathscr{C} \otimes \mathscr{S})^{\phi_*} &= \overline{\operatorname{span}}\{c \otimes \lambda(f) \mid c \in \mathscr{C}^{\phi_*},\, \lambda(f) \in \mathscr{S}^{\phi_*}_{\phi_*}\} \\ &= \overline{\operatorname{span}}\{c \otimes \lambda(f) \mid c \in C_U, U \in \mathscr{G}^{cc}, U \subset \mathrm{Cov}_\phi(G),\, \lambda(f) \in \mathscr{S}^{\phi_*}_{\phi_*}\}.\end{aligned}$$

Let $c \in C_U, U \in \mathscr{G}^{cc}, U \subset \mathrm{Cov}_\phi(G)$ and $\lambda(f) \in \mathscr{S}^{\phi_*}_{\phi_*}$. By part i) of proposition 3.37, one has $c \otimes \lambda(f) = \delta(c)\big(1 \otimes \lambda(\chi_{U^{-1}})\lambda(f)\big)$. This proves that $(\mathscr{C},\delta)$ is a $(\mathscr{S},\Delta)$-family. It is regular by proposition 2.36 and proposition 3.33. By part i), it follows that (C,π,δ) is a regular $(\boldsymbol{S},\boldsymbol{\Delta})$-algebra. □

As before, given a Fell bundle F on G, we denote by F^0 the restriction of F to G^0.

Proposition 3.39. *Let (C, π, δ) be an injective coaction of $(\boldsymbol{S}, \boldsymbol{\Delta})$.*

i) There exists an upper semi-continuous Fell bundle F on G with natural isomorphisms $j_U \colon \Gamma_0(U, F) \cong C_U$, $U \in \mathscr{G}^{cc}$. With respect to these identifications, the multiplication and involution on C coincide with the product and involution on sections of F induced by its Fell bundle structure.

ii) The family $(j_U)_U$ induces a dense embedding $j \colon \Gamma_c(F) \hookrightarrow C$. This embedding is equivariant with respect to the canonical coaction δ_F on $\Gamma_c(F) \subset C^*_r(F)$.

iii) $j(\Gamma_0(F^0)) = \mathscr{C}ov_{id}(C, \pi)$. Denote this C^*-algebra by C^0. The restriction map $\epsilon \colon \Gamma_c(F) \to \Gamma_c(F^0) \to C^0$ extends to a faithful conditional expectation $\epsilon \colon C \to C^0$.

iv) The embedding j extends to an isomorphism $\big(C^*_r(F), \pi, \delta_F\big) \cong (C, \pi, \delta)$.

Proof. i) The proof of this part follows the scheme of the proof of proposition 1.63.

Let $U \in \mathscr{G}^{cc}$. By part iii) of proposition 3.37, the space C_U corresponds to an upper semi-continuous Banach bundle on $s(U)$. Denote by F_U the pull-back of this bundle to U via the restriction of the source map $s_U \colon U \to s(U)$. By construction, we obtain an isomorphism $j_U \colon \Gamma_0(F_U) \cong C_U$.

Let $U' \subset U$ be an open subset. By part iv) of proposition 3.37, $C_{U'} = C_U C_0(s(U'))$. Therefore, the restriction of the bundle F_U to U' coincides with $F_{U'}$. Hence, there exists an upper semi-continuous Banach bundle F on G such that $F|_U = F_U$ for all $U \in \mathscr{G}^{cc}$.

By functoriality of the correspondence between modules and bundles, the multiplication and involution on the family $\big(C_U\big)_U$ induce families of multiplication and involution maps on the family $\big(F_U\big)_U$. These induce well-defined maps $F^2 \to F$ and $F \to F$ as in the definition of a Fell bundle because the multiplication and involution on $\big(C_U\big)_U$ is coherent with respect to inclusions. It is easy to check that these maps endow F with the structure of an upper semi-continuous Fell bundle.

ii) Let $(\phi_U)_U$ be a partition of unity subordinate to the open cover $\mathscr{G}^{cc}$ of G and put $j(f) := \sum_U j_U(\phi_U \cdot f)$, $f \in \Gamma_c(F)$. Note that this sum is finite since $(\phi_U)_U$ is locally finite and the support of f is compact. If $(\psi_V)_V$ is another partition of unity, we have $\phi_U = \sum_U \phi_U \psi_V$ for all $U \in \mathscr{G}^{cc}$ and hence

$$j(f) = \sum_U j_U(\phi_U \cdot f) = \sum_{U,V} j_{U \cap V}(\phi_U \psi_V f), \quad f \in \Gamma_c(F).$$

The last sum is symmetric with respect to the partitions $(\phi_U)_U$ and $(\psi_V)_V$, and hence the definition of j does not depend on the choice of the partition.

We show that j is equivariant. Let $U \in \mathscr{G}^{cc}$ and $f \in \Gamma_c(U, F)$. By construction, $j(f\pi(b)) = j(f)\pi(b)$ for all $b \in C_0(G^0)$. By part i) of proposition 3.37 and by the formula given in proposition 3.25, $\delta_F(f) = f \otimes \lambda(\chi_U)$ and $\delta(j_U(f)) = j_U(f) \otimes \lambda(\chi_U)$.

We show that the map j is injective. Let $f \in \Gamma_c(F)$ and $x \in G$ such that $f(x) \neq 0$. Choose a neighbourhood $U \in \mathscr{G}^{cc}$ of x such that $\phi_U(x) \neq 0$. Then

$$\begin{aligned}\big(1 \otimes \langle \chi_U|\big) \cdot \delta(j(f)) \cdot \big(1 \otimes |s_{U*}(\phi_U)\rangle\big) &= j\Big(\big(1 \otimes \langle \chi_U|\big) \cdot \delta_F(f) \cdot \big(1 \otimes |s_{U*}(\phi_U)\rangle\big)\Big) \\ &= j(\phi_U f) = j_U(\phi_U(f)).\end{aligned}$$

Since $\phi_U f \neq 0$ and j_U is an embedding, $j(f) \neq 0$. The image of j is dense in C by part i) of corollary 3.38.

iii) The first claim follows from part i). It is clear that ϵ is idempotent. We show that it extends to a continuous linear map $C \to C^0$ of norm 1. By Tomiyama's characterisation [54], this will imply that ϵ is a conditional expectation. So, let $f \in \Gamma_c(F)$. Choose a function $\chi \in C_0(G^0)$ with values between 0 and 1, which is equal to 1 on $\operatorname{supp} f \cap G^0$. By equivariance of the map j, we have

$$\epsilon(f) = \big(1 \otimes \langle \chi|\big) \cdot \delta(j(f)) \cdot \big(1 \otimes |\chi\rangle\big).$$

Since $\|\chi\| = 1$ and $\|\delta\| = 1$, we obtain $\|\epsilon(f)\| \leq \|j(f)\|$.

It remains to show that ϵ is faithful. Let $c \in C$ be a non-zero element. We want to show that $\epsilon(c^*c) \neq 0$. Choose $g \in C_c(G^0)$ such that $c' := c\pi(g) \neq 0$, and $\chi \in C_c(G^0)$ which is equal to 1 on $\operatorname{supp} g$. Assume that the map $\delta(c')(1 \otimes |\chi\rangle) \colon C \to C \otimes L^2(G, \lambda)$ given by $d \mapsto \delta(c')(d \otimes \chi)$ is 0. Then for each $d \in C$ and each $\xi \in C_c(G) \subset L^2(G, \lambda)$, one has

$$\begin{aligned}\delta(c')(d \otimes \xi) &= \delta(c)(d \otimes \pi_r(g)\xi) \\ &= \delta(c)(d \otimes \pi_r(g)\rho(\xi)\chi) = (1 \otimes \rho(\xi))\delta(c')(d \otimes \chi) = 0,\end{aligned}$$

where $\rho(\xi)$ is the operator on $L^2(G, \lambda)$ given by

$$\big(\rho(\xi)\zeta\big)(x) := \sum_{y \in G_{s(x)}} \zeta(xy^{-1})\xi(y), \quad x \in G,\ \zeta \in L^2(G, \lambda).$$

This implies $\delta(c') = 0$ and by injectivity of δ also $c' = 0$, a contradiction. Hence,

$$0 \neq \big(1 \otimes \langle \chi|\big) \cdot \delta(c')^*\delta(c') \cdot \big(1 \otimes |\chi\rangle\big) = \epsilon(c'^*c') = \rho(g)^*\epsilon(c^*c)\rho(g).$$

iv) By [25, fact 3.11], the norm on $\Gamma_c(F)$ inherited from $C^*_r(F)$ is the unique C^*-norm with respect to which the restriction map $\Gamma_c(F) \to \Gamma_c(G^0, F)$ extends to a faithful conditional expectation. Thus, by part iii), the map j extends to an isomorphism $C^*_r(F) \cong C$. By part ii), this isomorphism is equivariant. □

3.3.4 The correspondence between injective coactions and Fell bundles

In the previous subsections, we have shown that Fell bundles give rise to coactions and that coactions give rise to Fell bundles whose associated coaction coincides with the one we started with. It remains to prove the converse equality. After having done that, we discuss the functorial properties of the correspondence.

We denote the upper semi-continuous Fell bundle associated to a coaction (C, π, δ) in the previous proposition by $\mathrm{Fell}(C, \pi, \delta)$.

Proposition 3.40. *Let F be an upper semi-continuous Fell bundle on G. Then it is isomorphic to* $\mathrm{Fell}(C^*_r(F), \pi, \delta_F)$.

Proof. Let F be an upper semi-continuous Fell bundle on G. We show that $C^*_r(F)_U = \Gamma_0(U, F)$ for each $U \in \mathscr{G}^{cc}$, where

$$C^*_r(F)_U = \boldsymbol{E}_{(C^*_r(F),\pi,\delta_F)}\Big(\mathscr{C}ov_{q_{U*}}(C^*_r(F), \pi) \otimes \lambda(C_0(U))\Big).$$

Then the correspondence between bundles and modules implies that the Banach bundles F and $\mathrm{Fell}(C^*_r(F), \pi, \delta_F)$ are isomorphic. By part i) of proposition 3.37 and the formula for δ_F given in proposition 3.25, $\Gamma_0(U, F) \subset C^*_r(F)_U$. To prove the reverse inclusion, we determine the action of the map $E := \boldsymbol{E}_{(C^*_r(F),\pi,\delta_F)}$ on an element $c \otimes \lambda(f) \in \mathscr{C}ov_{q_{U*}}(C^*_r(F), \pi) \otimes \lambda(C_0(U))$ where $U \in \mathscr{G}^{cc}$. By proposition 3.35, $\delta_F E = \boldsymbol{E}'_{(C^*_r(F),\pi)}\boldsymbol{S}(\delta_F)$. Inserting the definitions, one obtains

$$1 \otimes E(c \otimes \lambda(f)) = V_F(1 \otimes W^*)(V_F^* \otimes 1)(1 \otimes c \otimes \lambda(f))(V_F^* \otimes 1)(1 \otimes W)V_F^*,$$

where W and V_F are the unitaries of propositions 3.31 and 3.25. A straightforward calculation shows that $E(c \otimes \lambda(f)) = cf$, where cf denotes the pointwise product over G. By definition, then, $C^*_r(F)_U \subset \Gamma_0(U, F)$.

The multiplication and involution maps on both Fell bundles correspond to the family of multiplication and involution maps on the family $\big(C^*_r(F)_U\big)_U$. Hence, they coincide. □

Theorem 3.41. *The maps $F \mapsto \big(C^*_r(F), \pi, \delta_F\big)$ and $(C, \pi, \delta) \mapsto \mathrm{Fell}(C, \pi, \delta)$ establish inverse bijections between the set of isomorphism classes of upper semi-continuous Fell bundles on G and the set of isomorphism classes of injective coactions of $(\boldsymbol{S}, \boldsymbol{\Delta})$.*

Proof. This follows from propositions 3.25, 3.39 and 3.40. □

Let F and F' be upper semi-continuous Fell bundles on G and let $\phi \colon F \to F'$ be a Fell bundle morphism. It is easy to see that composition with ϕ induces a $*$-homomorphism $\phi_* \colon \Gamma_c(F) \to \Gamma_c(F')$. In general, however, it is not clear whether ϕ_* extends to a $*$-homomorphism $C^*_r(F) \to C^*_r(F')$. In the reverse direction, one has the following result.

Proposition 3.42. *Let $\phi\colon (C,\pi,\delta) \to (C',\pi',\delta')$ be a morphism between injective coactions of $(\boldsymbol{S},\boldsymbol{\Delta})$. Then there exists a Fell bundle map* $\mathrm{Fell}(\phi)\colon \mathrm{Fell}(C,\pi,\delta) \to \mathrm{Fell}(C',\pi',\delta')$ *such that the following diagram commutes.*

$$\begin{array}{ccc} \Gamma_c(\mathrm{Fell}(C,\pi,\delta)) & \xrightarrow{\;j\;} & C \\ \downarrow{\scriptstyle \mathrm{Fell}(\phi)_*} & & \downarrow{\scriptstyle \phi} \\ \Gamma_c(\mathrm{Fell}(C',\pi',\delta')) & \xrightarrow{\;j'\;} & C'. \end{array}$$

Proof. Put $F := \mathrm{Fell}(C,\pi,\delta)$ and $F' := \mathrm{Fell}(C',\pi',\delta')$. Let $U \in \mathscr{G}^{cc}$. Since ϕ intertwines π with π', the image $\phi(\mathscr{C}ov_{q_{U*}}(C,\pi))$ is contained in $\mathscr{C}ov_{q_{U*}}(C',\pi')$. By part iii) of proposition 3.35, we obtain

$$\begin{aligned} \phi(C_U) &= \phi \circ \boldsymbol{E}_{(C,\pi,\delta)}\big(\mathscr{C}ov_{q_{U*}}(C,\pi) \otimes \lambda(\chi_U)\big) \\ &= \boldsymbol{E}_{(C',\pi',\delta')}\big(\phi(\mathscr{C}ov_{q_{U*}}(C,\pi)\big) \otimes \lambda(\chi_U)) \subset C'_U. \end{aligned}$$

Therefore, ϕ restricts to a morphism $\phi_U\colon C_U \to C'_U$ of $C_0(s(U))$-modules. By the correspondence between modules and bundles, the map $j'_U \phi_U j_U^{-1}\colon \Gamma_0(U,F) \to \Gamma_0(U,F')$ defines a bundle map $\mathrm{Fell}(\phi)_U\colon F|_U \to F'|_U$. By naturality of the construction, for each other set $U' \in \mathscr{G}^{cc}$, the restrictions of the map $\mathrm{Fell}(\phi)_U$ and $\mathrm{Fell}(\phi)_{U'}$ to $F|_{(U\cap U')}$ coincide. Thus, the family $\big(\mathrm{Fell}(\phi)_U\big)_U$ defines a bundle map $\mathrm{Fell}(\phi)\colon F \to F'$. This is a morphism of Fell bundles because ϕ is a $*$-homomorphism. Commutativity of the diagram follows immediately from the construction. □

3.4 Non-Hausdorff groupoids

In this section, we consider locally compact groupoids which are not Hausdorff. As indicated in the introduction to this chapter, they frequently arise in applications and require their own methods of study. Our initial aim was to extend the construction of the pseudo-Kac system of a Hausdorff groupoid to the non-Hausdorff case. To do so, one has to define the fundamental C^*-bimodule $L^2(G,\lambda)$. This has already been done by Jean-Louis Tu [55] and Mahmood Khoshkam and Georges Skandalis [23]. The first approach combines topological methods with algebro-analytic constructions, whereas the second approach is very abstract and does not provide a topological picture. The main result of this section is a lucid description of the C^*-module constructed via the second approach: it is the ordinary C^*-module associated to a certain Hausdorff groupoid which is constructed out of the initial non-Hausdorff groupoid in a functorial way, provided the latter is r-discrete. The methods used may be of individual interest.

First, we discuss a Hausdorff compactification of locally compact spaces introduced by James Fell [16] and provide a spectral picture which relates this construction to spaces of functions which are usually taken as substitutes for the space $C_c(X)$ when X is not Hausdorff [8, 23]. Applied to a locally compact groupoid, the Hausdorff compactification yields a (Hausdorff) groupoid. If the initial groupoid was r-discrete, so is the new one. These results are proved in the second subsection. Finally, we show that the classical C^*-module associated to the new groupoid coincides with the C^*-module associated to the initial groupoid in [23], provided the latter is r-discrete. For r-discrete groupoids, this answers our motivating question for a pseudo-Kac system mentioned in the beginning.

3.4.1 Preliminaries

Let us first recall some terminology. Let X be a topological space. A subset $Q \subset X$ is *quasi-compact* if every open cover of Q has a finite subcover, i.e. whenever Q is contained in a union of open sets, finitely many of these open sets already cover Q. A subset $K \subset X$ is *compact* if it is quasi-compact and Hausdorff with respect to the subspace topology. A topological space X is *locally compact* if every point in X has a compact neighbourhood [55, 23]. This definition differs from [7], where X is assumed to be Hausdorff. A continuous map $f\colon X \to Y$ of locally compact spaces is *proper* if the inverse image $f^{-1}(K)$ of every compact subset $K \subset Y$ is quasi-compact [55].

Lemma 3.43. *Let X be a locally compact space, $x \in X$ and V an open set containing x. Then there exists a compact neighbourhood K of x which is contained in V.*

Proof. Choose a compact neighbourhood K' of x and put $K := \overline{(K' \cap V)} \cap K'$. Since K' is compact, K is closed in K' and therefore compact as well. □

3.4.2 The Hausdorff compactification of a locally compact space

In [16], James Fell introduced a Hausdorff compactification for locally compact spaces. We summarise his definitions and results and present a new spectral approach to his construction. The main result is the identification of the algebra of quasi-continuous functions introduced in [16] with the algebra of functions generated by a certain replacement for $C_c(X)$ for non-Hausdorff spaces X [8]. This identification will be fundamental to the last subsection, where we describe a geometric approach to the C^*-module constructed in [23]. Next, we show that the Hausdorff compactification is functorial with respect to proper continuous maps and relate it to a construction introduced by Jean-Louis Tu [55].

The space $\mathfrak{H}X$

The Hausdorff compactification introduced in [16] is based on the notion of primitive nets which have several equivalent characterisations. Let X be a locally compact space. For a net $(x_\nu)_\nu$ in X, denote the set of its limit points by $\lim_\nu x_\nu$. Note that this set may contain more than one point because X need not be Hausdorff.

Lemma 3.44. *The following conditions on a net $(x_\nu)_\nu$ in X are equivalent.*

i) Each cluster point of the net is a limit point.

ii) For each point $x \in X$ which is not a limit point of the net, there exist a neighbourhood U of x and an index ν_0 such that $x_\nu \notin U$ for all $\nu \geq \nu_0$.

iii) Each compact set which does not contain a limit point of the net is eventually left by the net.

Proof. Obvious. □

Definition 3.45 ([16])**.** *A net $(x_\nu)_\nu$ in X is* primitive *if it satisfies the conditions of the previous lemma. The* Hausdorff compactification $\mathfrak{K}X$ of X *is set of all limit sets of primitive nets in X, equipped with the topology induced by the sub-basis of open sets*

$$\mathfrak{U}_V := \{A \in \mathfrak{K}X \mid A \cap V \neq \emptyset\}, \quad V \subset X \text{ open and Hausdorff,}$$
$$\mathfrak{U}^K := \{A \in \mathfrak{K}X \mid A \cap K = \emptyset\}, \quad K \subset X \text{ compact.}$$

Put $\mathfrak{H}X := \mathfrak{K}X \setminus \{\emptyset\}$.

Observe that by part ii) of lemma 3.44, the set of limit points of each primitive net in X is closed.

The space X embeds in $\mathfrak{K}X$, since each constant net is primitive. This embedding is dense, but not necessarily continuous. The subspace topology of $X \subset \mathfrak{K}X$ is the topology generated by the original topology of X and the family of all complements of compact sets. Since X is not necessarily Hausdorff, compact subsets need not be closed. However, if X is Hausdorff, $\mathfrak{H}X = X$ and $\mathfrak{K}X$ is its one-point compactification.

Sometimes it is more convenient to work with quasi-compact sets than with compact sets. The following lemmas show that in the definition of primitive nets and in the definition of the topology on $\mathfrak{K}X$, compact sets may be replaced by quasi-compact ones.

Lemma 3.46. *Let $Q \subset X$ be quasi-compact and let $A \in \mathfrak{H}X$ such that $Q \cap A = \emptyset$. Then there exist compact subsets $K_1, \dots, K_n$ of X such that $Q \subset \bigcup_i K_i$ and $A \cap \bigcup_i K_i = \emptyset$.*

Proof. By lemma 3.43 and part ii) of lemma 3.44, each point x in Q has a compact neighbourhood K_x which is disjoint to A. Since Q is quasi-compact, we can find finitely many points $x_i \in Q$ such that $Q \subset \bigcup_i K_i$ where $K_i := K_{x_i}$. □

Lemma 3.47. *A net $(x_\nu)_\nu$ in X is primitive if and only if each quasi-compact set which does not contain a limit point of $(x_\nu)_\nu$ is eventually left by the net.*

Proof. Combine part iii) of lemma 3.44 with lemma 3.46. □

Lemma 3.48. *The family of sets*

$$\mathfrak{U}^Q_{\mathcal{V}} := \{A \in \mathfrak{K}X \mid A \cap Q = \emptyset,\ A \cap V \neq \emptyset \text{ for all } V \in \mathcal{V}\},$$

where $Q \subset X$ is quasi-compact and $\mathcal{V}$ is a finite collection of open subsets of X, forms a basis for the topology on $\mathfrak{K}X$.

Proof. It is enough to show that the subset $\mathfrak{U}^Q := \{A \in \mathfrak{K}X \mid A \cap Q = \emptyset\}$ is open in $\mathfrak{K}X$ for each quasi-compact set $Q \subset X$. Let $A \in \mathfrak{U}^Q$ and $K_1, \dots K_n$ as in the lemma above. Then the intersection $\bigcap_i \mathfrak{U}^{K_i}$ is a neighbourhood of A and contained in $\mathfrak{U}^Q$. □

The following theorem justifies the terminology of definition 3.45.

Theorem 3.49 ([16])**.** *The space $\mathfrak{H}X$ is a locally compact Hausdorff space, and $\mathfrak{K}X$ is its one-point compactification.*

Convergence of nets in $\mathfrak{K}X$ has a nice characterisation which will be used later on frequently.

Lemma 3.50 ([16, lemma 2])**.** *Let $(x_\nu)_\nu$ be a net in X and let $A \in \mathfrak{K}X$. Then $\lim_\nu \{x_\nu\} = A$ in $\mathfrak{K}X$ if and only if the net $(x_\nu)_\nu$ is primitive and $\lim_\nu x_\nu = A$ in X.*

The spectral approach to the Hausdorff compactification

In [16], the Hausdorff compactification $\mathfrak{K}X$ is identified with the spectrum of the C^*-algebra of quasi-continuous functions on X. We identify the space with another C^*-algebra which arises naturally from a standard construction [8]. Before doing so, we recall some related definitions and results from [16].

Definition 3.51 ([16])**.** *Let Y be a locally compact Hausdorff space. A map $f\colon X \to Y$ is* w-quasi-continuous *if, for each primitive net $(x_\nu)_\nu$ of elements of X with non-empty limit set, the limit $\lim_\nu f(x_\nu)$ exists in Y. The map is* quasi-continuous *if, for each primitive net $(x_\nu)_\nu$ of elements of X, the limit $\lim_\nu f(x_\nu)$ exists in Y.*

The proof of the following corollary given in [16] is very brief. It can be completed by the following lemma which is of independent interest to us.

Lemma 3.52. *Let $f\colon Z \to Y$ be a map of topological spaces and assume that Y is locally compact and Hausdorff. Let $Z_0 \subset Z$ be a dense subset. If $\lim_\nu f(z_\nu) = f(z)$ for each net $(z_\nu)_\nu$ in Z_0 which converges to some point $z \in Z$, then f is continuous.*

Proof. Let $z \in Z$ be a point and $V \subset Y$ be a compact neighbourhood of $y := f(z)$. Let us assume that for each open neighbourhood U of z, there exists a point $z_U \in U$ such that $f(z_U)$ is not contained in V. Since the complement of V is open in Y and U is a neighbourhood of z_U, by assumption we find a point $x_U \in U \cap Z_0$

such that $f(x_U)$ belongs to the complement of V. Ordering the family of open neighbourhoods of z with respect to inclusion, we obtain a net $(x_U)_U$ in Z_0 which converges to z and whose image $\big(f(x_U)\big)_U$ does not converge to $f(z)$, contradicting the assumption of the statement. □

The following corollary is the first step towards a spectral description of the Hausdorff compactification.

Corollary 3.53 ([16, theorem 2]). *Let Y be a locally compact Hausdorff space.*

i) Each w-quasi-continuous map $X \to Y$ extends uniquely to a continuous map $\mathfrak{H}X \to Y$.

ii) The restriction of each continuous map $\mathfrak{H}X \to Y$ to $X \subset \mathfrak{H}X$ is w-quasi-continuous.

iii) The extension and restriction define a bijection between the set of w-quasi-continuous maps $X \to Y$ and the set of all continuous maps $\mathfrak{H}X \to Y$. Under this bijection, quasi-continuous maps $X \to Y$ correspond exactly to all those maps which extend to $\mathfrak{K}X \to Y$.

Proof. This follows from lemma 3.52 and 3.50. □

Our spectral description involves the following C^*-algebra. Recall that $C_c^{qc}(X)$ denotes the space of all functions on X which can be written as a finite sum $\sum f_i$, where $f_i \in C_c(U_i)$ for some Hausdorff open subsets $U_i \subset X$. Denote by $\mathfrak{C}_c(X)$ and $\mathfrak{C}_0(X)$ the $*$-algebra and the C^*-algebra, respectively, generated by $C_c^{qc}(X)$ with respect to pointwise multiplication, conjugation and the supremum norm.

Theorem 3.54. *Let X be a locally compact space.*

i) Each function $f \in \mathfrak{C}_c(X)$ is quasi-continuous.

ii) The algebra $\mathfrak{C}_c(X)$, considered as a subalgebra of $C_0(\mathfrak{H}X)$ via corollary 3.53 and part i), separates the points of $\mathfrak{H}X$.

iii) Extension of functions defines an isomorphism $\mathfrak{C}_0(X) \xrightarrow{\cong} C_0(\mathfrak{H}X)$. One has $\mathfrak{H}X \cong \operatorname{spec} \mathfrak{C}_0(X)$.

iv) Under the isomorphism $\mathfrak{C}_0(X) \cong C_0(\mathfrak{H}X)$, the subspace $\mathfrak{C}_c(X)$ gets identified with $C_c(\mathfrak{H}X)$.

Proof. i) Let $(x_\nu)_\nu$ be a primitive net in X with limit set A. We may assume that the support of f is compact and contained in a Hausdorff open set $U \subset X$. If $\operatorname{supp} f \cap A$ is empty, by quasi-compactness of $\operatorname{supp} f$ and primitivity of the net, there exists a ν_0 such that $x_\nu \notin \operatorname{supp} f$ for all $\nu \geq \nu_0$. Then the net $\big(f(x_\nu)\big)_\nu$ is eventually constant 0. Otherwise, the intersection consists of at most one point, x, say. Since f is continuous on U and the net $(x_\nu)_\nu$ converges to x, the net $\big(f(x_\nu)\big)_\nu$ converges to $f(x)$.

ii) Let $A, B \in \mathfrak{H}X$ be distinct points. Without loss of generality, there exists a point $x \in A \setminus B$. By part ii) of lemma 3.44, there exists a compact neighbourhood K of x which is disjoint to B. Choose a function $f \in C_c(K)$ such that $f(x) = 1$. Then the extension $\tilde{f}\colon \mathfrak{H}X \to \mathbb{C}$ of f satisfies $\tilde{f}(A) = 1$ and $\tilde{f}(B) = 0$.

iii) By corollary 3.53 and part i), one has an embedding $j\colon \mathfrak{C}_c(X) \to C_0(\mathfrak{H}X)$. Since $X \subset \mathfrak{H}X$ is dense, j is isometric with respect to the supremum norms and extends to an embedding $\mathfrak{C}_0(X) \hookrightarrow C_0(\mathfrak{H}X)$. By part ii) and the Stone-Weierstrass theorem, this embedding is an isomorphism. By the Gelfand-Naimark theorem, $\operatorname{spec} \mathfrak{C}_0(X) \cong \mathfrak{H}X$.

iv) Let $f \in C_c(\mathfrak{H}X)$. Then f vanishes on a neighbourhood of the point $\emptyset$ in $\mathfrak{K}X$, and by lemma 3.48, there exists a quasi-compact subset $Q \subset X$ such that $f(A) = 0$ whenever $A \cap Q = \emptyset$. Hence, the restriction $f|_X$ vanishes outside Q. Since X is locally compact, $f|_X$ belongs to $\mathfrak{C}_c(X)$. Conversely, assume that the support of a function $g \in \mathfrak{C}_c(X)$ is contained in a quasi-compact subset Q. Then $\lim_\nu g(x_\nu) = 0$ by part iii) of lemma 3.47. Hence, the extension of g vanishes on the neighbourhood $\mathfrak{U}^Q$ of the point $\emptyset$ in $\mathfrak{K}X$. □

Next, we summarise the functorial properties of the association $X \mapsto \mathfrak{H}X$.

Theorem 3.55. *Let $\phi\colon X \to Y$ be a proper continuous map of locally compact spaces.*

- *i) If $(x_\nu)_\nu$ is a primitive net in X with limit set A, then $(\phi(x_\nu))_\nu$ is a primitive net in Y with limit set $\phi(A)$.*
- *ii) The map $\mathfrak{K}\phi\colon A \mapsto \phi(A)$ defines a continuous map $\mathfrak{K}\phi\colon \mathfrak{K}X \to \mathfrak{K}Y$.*
- *iii) The map $\mathfrak{K}\phi$ restricts to a continuous proper map $\mathfrak{H}\phi\colon \mathfrak{H}X \to \mathfrak{H}Y$.*
- *iv) The map $X \mapsto \mathfrak{H}X$ extends to a functor from the category of locally compact spaces with proper continuous maps to the category of locally compact Hausdorff spaces with proper continuous maps.*

Proof. i) Since ϕ is continuous, each point in $\phi(A)$ is a limit point of the net $(\phi(x_\nu))_\nu$. Let K be a compact subset of $Y \setminus \phi(A)$. Then the inverse image $\phi^{-1}(K)$ is a quasi-compact subset of $X \setminus A$. By lemma 3.46, $\phi^{-1}(K)$ is eventually left by $(x_\nu)_\nu$, and therefore K is eventually left by $(\phi(x_\nu))_\nu$. By lemma 3.44, $\phi(x_\nu)_\nu$ is primitive.

ii) For each pair of subsets $A \subset X$ and $B \subset Y$, one has $\phi(A) \cap B = \emptyset$ if and only if $A \cap \phi^{-1}(B) = \emptyset$. For each open subset $V \subset Y$, the inverse image $\phi^{-1}(V)$ is open since ϕ is continuous, and $(\mathfrak{K}\phi)^{-1}\big(\mathfrak{U}_V\big) = \mathfrak{U}_{\phi^{-1}(V)}$. Likewise, for each quasi-compact subset $Q \subset Y$, the inverse image $\phi^{-1}(Q)$ is quasi-compact since ϕ is proper, and $(\mathfrak{K}\phi)^{-1}\big(\mathfrak{U}^Q\big) = \mathfrak{U}^{\phi^{-1}(Q)}$. Therefore, the map $\mathfrak{K}\phi$ is continuous.

iii) By construction, the image of $\mathfrak{H}X$ under the map $\mathfrak{K}\phi$ is contained in $\mathfrak{H}Y$. The restriction of $\mathfrak{K}\phi$ to $\mathfrak{H}X$ is proper because $\mathfrak{K}X$ and $\mathfrak{K}Y$ are the one-point compactifications of $\mathfrak{H}X$ and $\mathfrak{H}Y$, respectively. ☐

Corollary 3.56. *Let $\phi\colon X \to Y$ be a proper continuous map of locally compact spaces.*

i) For each quasi-continuous function f on Y, the pull-back ϕ^*f is quasi-continuous on X. Thus, one has a homomorphism $\phi^*\colon \mathfrak{C}_0(Y) \to \mathfrak{C}_0(X)$.

ii) Under the isomorphisms $\mathfrak{C}_0(X) \cong C_0(\mathfrak{H}X)$ and $\mathfrak{C}_0(Y) \cong C_0(\mathfrak{H}Y)$, the map ϕ^* coincides with $(\mathfrak{H}\phi)^*$.

Proof. i) Let $f \in \mathfrak{C}_0(Y)$ and denote by $\tilde{f}$ the extension of f to a continuous function on $\mathfrak{H}Y$. For each $x \in X$, one has

$$\big((\mathfrak{H}\phi)^*\tilde{f}\big)(\{x\}) = \tilde{f}\big(\mathfrak{H}\phi(\{x\})\big) = \tilde{f}\big(\{\phi(x)\}\big) = f(\phi(x)) = (\phi^*f)(x).$$

By the previous theorem, $(\mathfrak{H}\phi)^*\tilde{f}$ is contained in $C_0(\mathfrak{H}X)$. By part iii) of theorem 3.54, the pull-back ϕ^*f belongs to $\mathfrak{C}_0(X)$.

ii) Let $A \in \mathfrak{H}X$ be the limit set of a primitive net $(x_\nu)_\nu$ in X. Then $\mathfrak{H}\phi(A)$ is the limit set of the primitive net $\big(\phi(x_\nu)\big)_\nu$, and

$$\big((\mathfrak{H}\phi)^*\tilde{f}\big)(A) = \tilde{f}\big(\mathfrak{H}\phi(A)\big) = \lim_\nu f(\phi(x_\nu)) = \lim_\nu \phi^*f(x_\nu) = \widetilde{\phi^*f}(A).$$ ☐

Relation to Tu's construction of $\mathcal{H}X$

In [55], Jean-Louis Tu associates to each locally compact space a locally compact Hausdorff space $\mathcal{H}X$. We briefly describe the relation between his construction and the functor $\mathfrak{H}$. As a set, $\mathcal{H}X$ consists of all subsets $A \subset X$ satisfying the following condition: for every family $(V_x)_{x\in A}$ of open sets such that $x \in V_x$ and $V_x = X$ except perhaps for finitely many $x \in A$, one has $\bigcap_{x\in A} V_x \neq \emptyset$. This set is endowed with the topology generated by the families of subsets

$$\Omega_V := \{A \in \mathcal{H}X \mid A \cap V \neq \emptyset\}, \quad V \subset X \text{ open},$$
$$\Omega^Q := \{A \in \mathcal{H}X \mid A \cap Q = \emptyset\}, \quad Q \subset X \text{ quasi-compact}.$$

Proposition 3.57. *$\mathfrak{H}X$ is a subspace of $\mathcal{H}X$.*

Proof. We first show that $\mathfrak{H}X$ is a subset of $\mathcal{H}X$. Let $(x_\nu)_\nu$ be a primitive net in X with limit set A and let $\big(V_i\big)_i$ be a finite family of open subsets of X such that $V_i \cap A \neq \emptyset$ for all i. By lemma 3.44, the net $(x_\nu)_\nu$ is eventually contained in the intersection $\bigcap_i V_i$, which, therefore, can not be empty.

The fact that the topology on $\mathfrak{H}X$ coincides with the subspace topology inherited from $\mathcal{H}X$ follows from lemma 3.48 and the relations

$$\Omega_V \cap \mathfrak{H}X = \mathfrak{U}_V,\ V \subset X \text{ open}, \qquad \Omega^Q \cap \mathfrak{H}X = \mathfrak{U}^Q,\ Q \subset X \text{ quasi-compact}.$$ ☐

The following example shows that the space $\mathcal{H}X$ may be very large – from our point of view, too large – compared to the space $\mathfrak{H}X$.

Example 3.58. Let X be the quotient space of the product $[0,1] \times \mathbb{N}$ obtained by identifying (t,m) with (t,n) for each $t \in (0,1]$ and all $m,n \in \mathbb{N}$. Then $\mathfrak{H}X$ identifies with the disjoint union of the interval $[0,1]$ and the discrete space $\{0\}\times\mathbb{N}$. The point $0 \in [0,1]$ corresponds to the subset $\{0\} \times \mathbb{N} \subset X$ which is the limit of the primitive sequence $(1/n,0)_{n\in\mathbb{N}}$. The space $\mathcal{H}X$ identifies with the union of the interval $[0,1]$ with the discrete space of all non-empty subsets of $\mathbb{N}$, which is strictly larger than $\mathfrak{H}X$.

3.4.3 The Hausdorff functor on locally compact groupoids

The functor $\mathfrak{H}$ is compatible with groupoid structures: we show that applied to a locally compact groupoid, it yields a groupoid again, which, of course, is Hausdorff. This process preserves r-discreteness. Furthermore, every action of a locally compact groupoid on a locally compact space X extends to an action of the same groupoid on the space $\mathfrak{H}X$, provided some mild hypotheses are satisfied. We use the latter result to prove the first one. Then we present an alternative description of the Hausdorff groupoid obtained from a locally compact groupoid. The general plan of this subsection is inspired by [55], but the techniques and results are original.

Recall that an action of a topological groupoid G on a topological space X [18, 46] consists of a continuous map $\sigma\colon X \to G^0$ and a continuous map $\mu\colon X_\sigma\times_r G \to X$, written $\mu(x,y) =: xy$, where $X_\sigma\times_r G := \{(x,y) \in X \times G \mid \sigma(x) = r(y)\}$, such that

i) $\sigma(xy) = s(y)$ for all $(x,y) \in X_\sigma\times_r G$,

ii) $xv = x$ for all $(x,v) \in X_\sigma\times_r G^0$ and

iii) $(xy)y' = x(yy')$ for all $(x,y,y') \in X_\sigma\times_r G_s\times_r G$.

The Hausdorff functor is compatible with groupoid actions:

Proposition 3.59. *Let G be a locally compact groupoid with open range and source maps. Then each continuous action (σ,μ) of G on a locally compact space X extends to a continuous action $(\tilde\sigma,\tilde\mu)$ on $\mathfrak{H}X$.*

Proof. By corollary 3.53, σ extends to a continuous map $\tilde\sigma := \mathfrak{H}\sigma\colon \mathfrak{H}X \to G^0$.

Let $(A,y) \in \mathfrak{H}X_{\tilde\sigma}\times_r G$ and let $(x_\nu)_\nu$ be a primitive net in X with limit set A. We construct a primitive net in X with limit set Ay.
Denote by I the set of pairs (ν,U) where U is a neighbourhood of y such that $\sigma(x_{\nu'})$ belongs to $r(U)$ for all $\nu' \geq \nu$. Since the range map is open and $\big(\sigma(x_\nu)\big)_\nu$ converges to $\tilde\sigma(A) = r(y)$, for each U there exists an index ν_0 such that $(\nu,U) \in I$ for all $\nu \geq \nu_0$. Put $(\nu,U) \geq (\nu',U')$ if $\nu \geq \nu'$ and $U \subset U'$. Then I is a directed set.

By construction, for each index $(\nu, U) \in I$ one can choose an element $y_{\nu,U} \in U$ such that $r(y_{\nu,U}) = \sigma(x_\nu)$.
We show that the net $(x_\nu y_{\nu,U})_I$ in X is primitive with limit set Ay. By continuity of the action of G on X, the set Ay is contained in the limit set of this net. Let x' be a cluster point of this net. Then $\sigma(x') = \lim_I s(y_{\nu,U}) = s(y)$. By construction, the point $x'y^{-1}$ is a cluster point of the net $\big(x_\nu y_{\nu,U} y_{\nu,U}^{-1}\big)_I = (x_\nu)_I$. Therefore, it belongs to A, and hence x' is contained in Ay.

Thus, we may put $\tilde{\mu}(A, y) := Ay$. The construction shows that the subset $X_\sigma \times_r G$ is dense in $\mathfrak{H}X_\sigma \times_r G$. By lemma 3.52, the map $\tilde{\mu}$ thus defined is continuous.

The fact that $(\tilde{\sigma}, \tilde{\mu})$ defines an action is immediate. ☐

Let G be a locally compact groupoid with open range and source maps. We will show that the space $\mathfrak{H}G$ is a locally compact groupoid again. Denote by $\mathfrak{H}s, \mathfrak{H}r\colon \mathfrak{H}G \to G^0$ the extension of the maps $s, r\colon G \to G^0$. Let $(\mathfrak{H}G)^0$ denote the closure of G^0 in $\mathfrak{H}G$. First, we prove that the maps $r, s\colon G \to G^0$ extend to continuous maps $\mathfrak{r}, \mathfrak{s}\colon \mathfrak{H}G \to (\mathfrak{H}G)^0$.

Lemma 3.60. *Let $A \in \mathfrak{H}G$.*

i) One has $A \cdot A^{-1} \cdot A = A$.

ii) The sets $A \cdot A^{-1}$ and $A^{-1} \cdot A$ are subgroups of $G_{r(A)}^{r(A)}$ and $G_{s(A)}^{s(A)}$, respectively.

iii) The pointwise products $A \cdot A^{-1}$ and $A^{-1} \cdot A$ are points in $(\mathfrak{H}G)^0$.

iv) Let $B \in \mathfrak{H}G$ such that $A^{-1} \cdot A = B \cdot B^{-1}$. Then $A \cdot B$ is a point in $\mathfrak{H}G$.

Proof. Let $(x_\nu)_\nu$ be a primitive net in G with limit set A. Since the inversion on G is a homeomorphism, the net $(x_\nu^{-1})_\nu$ is primitive with limit set A^{-1}.

i,ii) Clearly, $A \subset A \cdot A^{-1} \cdot A$. Conversely, $A \cdot A^{-1} \cdot A$ is contained in the limit set of the net $(x_\nu x_\nu^{-1} x_\nu)_\nu = (x_\nu)_\nu$ which is A. Part ii) follows immediately from the equations $(AA^{-1})(AA^{-1}) = AA^{-1}$ and $(A^{-1}A)(A^{-1}A) = A^{-1}A$.

ii) Put $y_\nu = x_\nu x_\nu^{-1}$ for all ν. Then $A \cdot A^{-1} \subset \lim_\nu y_\nu$. On the other hand, if y is a cluster point of the net $(y_\nu)_\nu$, then $s(y) = r(A)$, and yA is a cluster point of the net $(y_\nu x_\nu)_\nu = (x_\nu)_\nu$ and therefore contained in A. Thus, y is contained in AA^{-1}. Therefore, the net $(y_\nu)_\nu$ is primitive, and $\lim_\nu y_\nu = A \cdot A^{-1}$. Furthermore, each y_ν is contained in G^0. This proves that the product $A \cdot A^{-1}$ is a point in $(\mathfrak{H}G)^0$. The corresponding result concerning $A^{-1} \cdot A$ follows similarly.

iv) Choose $x \in B$. By parts i) and ii), one has $A \cdot B = A \cdot BB^{-1}x = AA^{-1}Ax = Ax$. Consider the multiplication on G as an action of G on itself. Then Ax is a point in $\mathfrak{H}G$ by proposition 3.59. ☐

The following two propositions and the ensuing theorem are the key result of this subsection.

Proposition 3.61. *Let G be a locally compact groupoid with open range map. Then the maps*

$$\begin{aligned} &\mathfrak{r}\colon \mathfrak{H}G \to (\mathfrak{H}G)^0, && \mathfrak{s}\colon \mathfrak{H}G \to (\mathfrak{H}G)^0, && \mathfrak{m}\colon \mathfrak{H}G_{\mathfrak{s}}\times_{\mathfrak{r}} \mathfrak{H}G \to \mathfrak{H}G, && \mathfrak{i}\colon \mathfrak{H}G \to \mathfrak{H}G, \\ &\mathfrak{r}(A) := AA^{-1}, && \mathfrak{s}(A) := A^{-1}A, && \mathfrak{m}(A,B) := AB, && \mathfrak{i}(A) := A^{-1}, \end{aligned}$$

are well-defined. With these operations, $\mathfrak{H}G$ is a locally compact Hausdorff groupoid.

Proof. The maps $\mathfrak{r}$ and $\mathfrak{s}$ and the operations $\mathfrak{m}, \mathfrak{i}$ are well-defined by the previous lemma. It is clear that the multiplication is associative. The fact that with these operations, $\mathfrak{H}G$ becomes a groupoid, follows from the previous lemma, again.

We show that the map $\mathfrak{r}$ is continuous, using lemma 3.52. By definition of the map $\mathfrak{r}$ and continuity of the map r, one has for each net $(x_\nu)_\nu$ in G

$$\mathfrak{r}(\lim_\nu \{x_\nu\}) = \mathfrak{r}(\lim_\nu x_\nu) = \lim_\nu x_\nu x_\nu^{-1} = \lim_\nu \{x_\nu x_\nu^{-1}\} = \lim_\nu \mathfrak{r}(\{x_\nu\}).$$

Continuity of the other maps is proved using the same technique. □

Proposition 3.62. *If G is r-discrete, the groupoid $\mathfrak{H}G$ is r-discrete, too.*

Proof. Let $V \subset G$ be an open Hausdorff G-set. We claim that the restriction of the map $\mathfrak{r}$ to the open subset $\mathfrak{V} := \mathfrak{U}_V \subset \mathfrak{H}G$ is a homeomorphism with open image. Put $\mathfrak{U} := \big((\mathfrak{H}r)^{-1} \circ r(V)\big) \cap (\mathfrak{H}G)^0$ and $t := r_V^{-1} \circ \mathfrak{H}r\colon \mathfrak{U} \to V$. We show that the map $\mathfrak{t}\colon \mathfrak{U} \to \mathfrak{H}G$ given by $A \mapsto A \cdot t(A)$ is inverse to $\mathfrak{r}_{\mathfrak{V}} := \mathfrak{r}|_{\mathfrak{V}}$ and continuous.

Let $A \in \mathfrak{U}$. Then $\mathfrak{H}s(A) \in A$, and hence $t(A)$ is contained in $\mathfrak{t}(A) \cap V$. In particular, this set is not empty. Therefore, $\mathfrak{t}(A)$ belongs to $\mathfrak{V}$. Furthermore, $\mathfrak{r}(\mathfrak{t}(A)) = (A \cdot t(A))(A \cdot t(A))^{-1} = A$.

Let $B \in \mathfrak{V}$. By definition of $\mathfrak{V}$, the intersection $B \cap V$ is not empty. Since $\mathfrak{H}r(B)$ consists of just one point and V is a G-set, this intersection consists of the unique element $x \in V$ satisfying $r(x) = \mathfrak{H}r(B) = \mathfrak{H}r(B \cdot B^{-1})$. Hence, $x = t(B \cdot B^{-1})$. By parts i,ii) of lemma 3.60, $\mathfrak{t}(\mathfrak{r}(B)) = B \cdot B^{-1} \cdot t(B \cdot B^{-1}) = B$.

It remains to show that the map $\mathfrak{t}$ is continuous. Let $(x_\nu)_\nu$ be a primitive net in $r(V) = G \cap \mathfrak{U}$. Then

$$\lim_\nu \mathfrak{t}\big(\{x_\nu\}\big) = \lim_\nu \big(x_\nu t\big(\{x_\nu\}\big)\big) \overset{(1)}{=} \lim_\nu x_\nu \cdot \lim_\nu t\big(\{x_\nu\}\big) = \mathfrak{t}\big(\lim_\nu \{x_\nu\}\big),$$

where we used primitivity of the net for equality (1). By lemma 3.52, $\mathfrak{t}$ is continuous.

Since sets of the form $\mathfrak{V}$ form an open cover of $\mathfrak{H}G$, this groupoid is r-discrete. □

Theorem 3.63. *The map $G \mapsto \mathfrak{H}G$ extends to a functor from the category of locally compact groupoids with open range maps together with proper and continuous groupoid homomorphisms to the category of locally compact Hausdorff groupoids with proper continuous homomorphisms.*

Proof. Let $\phi\colon G \to G'$ be a proper continuous homomorphism of locally compact groupoids. By proposition 3.55, we only need to show that the map $\mathfrak{H}\phi\colon \mathfrak{H}G \to \mathfrak{H}G'$ is a groupoid homomorphism. This follows easily from the fact that the inclusions of G and G' in $\mathfrak{H}G$ and $\mathfrak{H}G'$ are groupoid homomorphisms with dense images and that the restriction of $\mathfrak{H}\phi$ to G coincides with ϕ. □

Another perspective on $\mathfrak{H}G$ The groupoid $\mathfrak{H}G$ can also be described as a quotient of the transformation groupoid $(\mathfrak{H}G)^0 \rtimes G$ of the adjoint action of G on $(\mathfrak{H}G)^0$.

Proposition 3.64. *There exists an action of G on $(\mathfrak{H}G)^0$ given by the maps*

$$\mathfrak{H}s\colon (\mathfrak{H}G)^0 \to G^0, \qquad \mathrm{Ad}\colon (\mathfrak{H}G)^0{}_{\mathfrak{H}s}\times_r G \to (\mathfrak{H}G)^0, \quad (A,x) \mapsto x^{-1}Ax.$$

The map $\mu\colon (\mathfrak{H}G)^0 \rtimes G \to \mathfrak{H}G$ given by $(A,x) \mapsto Ax$ is a continuous surjective groupoid homomorphism with kernel $N = \{(A,x) \in (\mathfrak{H}G)^0 \rtimes G \mid x \in A\}$. The induced map $\tilde{\mu}\colon \big((\mathfrak{H}G)^0 \rtimes G\big)/N \to \mathfrak{H}G$ is an isomorphism.

Proof. The subspace $G^{iso} := \{x \in G \mid r(x) = s(x)\}$ of G is closed because r, s are continuous and G^0 is Hausdorff. Hence, G^{iso} is locally compact. The restriction of the source map of G to G^{iso} and the map $G^{iso}{}_s\times_r G$, $(x,y) \mapsto y^{-1}xy$, define a continuous action of G on G^{iso}. By proposition 3.59, this action extends to an action of G on $\mathfrak{H}(G^{iso})$. Clearly, this action restricts to an action on G^0. Since $(\mathfrak{H}G)^0$ is the closure of G^0 in $\mathfrak{H}(G^{iso})$, the extended action restricts to $(\mathfrak{H}G)^0$.

We show that the map μ is a groupoid homomorphism. Let $(A,x) \in (\mathfrak{H}G)^0 \rtimes G$. Since each element $A \in (\mathfrak{H}G)^0$ is a group, one has

$$r((A,x)) = A = A \cdot A^{-1} = \mathfrak{r}(Ax), \quad s((A,x)) = \mathrm{Ad}(A,x) = x^{-1}Ax = \mathfrak{s}(Ax).$$

Furthermore, for each composable pair of elements $(A,x), (A',x') \in (\mathfrak{H}G)^0 \rtimes G$, one has $x^{-1}Ax = s(A,x) = r(A',x') = A'$ and therefore

$$\mu((A,x))\mu((A',x')) = AxA'x' = AAxx' = \mu((A,xx')) = \mu((A,x)(A',x')).$$

Next, let us show that the kernel of μ is N. If $(A,x) \in (\mathfrak{H}G)^0 \rtimes G$ and Ax is contained in $(\mathfrak{H}G)^0$, then x^{-1} belongs to A. Since A is a group, x must be an element of A.

To see that μ is surjective, let $A \in \mathfrak{H}G$. Then $A = AA^{-1}x$ for each element $x \in A$, and hence $A = \mathfrak{r}(A)x = \mu((\mathfrak{r}(A),x))$.

The proof of proposition 3.59 shows that $G^0{}_s\times_r G$ is dense in $(\mathfrak{H}G)^0{}_{\mathfrak{H}s}\times_r G$. An application of lemma 3.52 shows that μ is continuous. The induced map $\tilde{\mu}$ is a bijection by construction, and a homeomorphism by lemma 3.52, again. □

3.4.4 The Khoshkam-Skandalis construction

In [23], Mahmood Khoshkam and Georges Skandalis construct left regular representations of locally compact groupoids which are not necessarily Hausdorff. An important step is the definition of a C^*-module $L^2(G, \lambda^{-1})$ which carries the representation. Our interest in this module arose from the fact that it is the main ingredient for the construction of pseudo-multiplicative unitaries for non-Hausdorff groupoids – the unitary itself again will be given by the same formula as in the Hausdorff case. In the following, we show that if G is r-discrete and λ and λ' are the Haar systems on G and $\mathfrak{H}G$, respectively, given by the families of counting measures, the C^*-module $L^2(G, \lambda)$ coincides with $L^2(\mathfrak{H}G, \lambda'^{-1})$.

Let us first indicate the problem encountered when constructing $L^2(G, \lambda^{-1})$. A natural approach in the non-Hausdorff situation would be to equip the space $C_c^{qc}(G)$ with the operations

$$\langle \eta | \xi \rangle(v) := \int_{G_v} \overline{\eta(x)} \xi(x) d\lambda_v^{-1}(x), \qquad (\xi f)(x) := \xi(x) f(s(x)),$$
$$x \in G,\ \eta, \xi \in C_c^{qc}(G),$$

to obtain a pre-C^*-module over $C_0(G^0)$. However, $C_c^{qc}(G)$ need not be closed under multiplication, and hence the integrand need not be quasi-continuous. Therefore, the inner product, as a function on G^0, need not be continuous. A natural solution of this problem is to enlarge $C_0(G^0)$ suitably. The inner product can be rewritten using the convolution $*$-algebra structure on $C_c^{qc}(G)$ which is given by

$$(\eta \star \xi)(x) := \int_{G^{r(x)}} \eta(y) \xi(y^{-1}x) d\lambda^{r(x)}(y), \quad (\eta^*)(x) := \overline{\eta(x^{-1})},$$
$$x \in G,\ \eta, \xi \in C_c^{qc}(G).$$

Then for all $\eta, \xi \in C_c^{qc}(G)$, one has

$$\langle \eta | \xi \rangle(v) = \int_{G^v} \overline{\eta(x^{-1})} \xi(x^{-1}) d\lambda^v(x) = (\eta^* \star \xi)(v), \quad v \in G^0.$$

We briefly summarise the approach of Khoshkam and Skandalis [23]. Put $D := \{f|_{G^0} : f \in C_c^{qc}(G)\}$. Denote by $\mathfrak{B}_0(G^0)$ the C^*-algebra of all bounded Borel functions on G^0 vanishing at infinity, equipped with the supremum norm. Then $D \subset \mathfrak{B}_0(G^0)$, and the C^*-subalgebra generated by D is isomorphic to $C_0(Y)$ for some locally compact Hausdorff space Y. The algebraic tensor product $C_c^{qc}(G) \odot C_0(Y)$, equipped with the operations

$$\langle \eta \odot a | \xi \odot b \rangle := a^* \langle \eta | \xi \rangle b, \quad (\eta \odot a)b := \eta \odot ab, \quad \eta, \xi \in C_c^{qc}(G),\ a, b \in C_0(Y),$$

where the inner product $\langle \eta | \xi \rangle$ is the one given above, is a pre-C^*-module over $C_0(Y)$. If G is Hausdorff, the completion of this pre-C^*-module coincides with $L^2(G, \lambda^{-1})$. Following [23], we keep this notation also in the general case.

Now, the problem is that the space Y arises from an abstract definition with no illuminating connection to the topology on G. The groupoid $\mathfrak{H}G$ facilitates a geometric description of this space and the C^*-module $L^2(G,\lambda^{-1})$. Identify $C_0\big((\mathfrak{H}G)^0\big)$ and $C_0(Y)$ with subalgebras of $\mathfrak{B}_0(G^0)$ via the transpose of the dense inclusion $G^0 \to (\mathfrak{H}G)^0$ and the canonical isomorphism, respectively.

Proposition 3.65. *One has* $C_0\big((\mathfrak{H}G)^0\big) = C_0(Y)$ *as* C^**-subalgebras of* $\mathfrak{B}_0(G^0)$.

Proof. By the Tietze extension theorem, the restriction map $\mathfrak{C}_0(G) \cong C_0(\mathfrak{H}G) \to C_0\big((\mathfrak{H}G)^0\big) \subset \mathfrak{B}_0(G^0)$ is surjective. Since $\mathfrak{C}_0(G)$ is generated as a C^*-algebra by $C_c^{qc}(G)$, the C^*-algebra $C_0((\mathfrak{H}G)^0)$ is generated by the image of $C_c^{qc}(G)$ which is exactly D. ∎

We denote by $\pi_s\colon C_0(Y) \to \mathfrak{B}(G)$ the $*$-homomorphism given by $(\pi_s f)(x) = f(s(x))$.

Proposition 3.66. *Consider the map* $\Phi\colon C_c^{qc}(G) \odot C_0(Y) \to \mathfrak{B}_0(G)$ *given by* $\Phi(\xi \odot a)(x) := \pi_s(a)\xi(x)$, $x \in G$. *Let* G *be* r*-discrete and let* λ, λ' *denote the Haar systems on* G *and* $\mathfrak{H}G$*, respectively, given by the families of counting measures.*

i) $\Phi(C_c^{qc}(G) \odot C_0(Y)) = \mathfrak{C}_c(G)$.

ii) Φ *extends to an isomorphism* $L^2(G,\lambda^{-1}) \to L^2(\mathfrak{H}G,\lambda'^{-1})$.

Proof. i) We only need to show that the image of Φ is closed under multiplication. In fact, it is enough to show that it is equal to the pointwise product $C_c^{qc}(G) \cdot C_c^{qc}(G)$, since then

$$\begin{aligned}\operatorname{Im}\Phi \cdot \operatorname{Im}\Phi &= \pi_s(C_0(Y))\, C_c^{qc}(G) \cdot \pi_s(C_0(Y))\, C_c^{qc}(G) \\ &= \pi_s(C_0(Y)) \cdot C_c^{qc}(G)\, C_c^{qc}(G) = \pi_s(C_0(Y)) \operatorname{Im}(\Phi) = \operatorname{Im}(\Phi).\end{aligned}$$

The inclusion $\operatorname{Im}\Phi \subset C_c^{qc}(G) \cdot C_c^{qc}(G)$ is easy to see. Let us prove the reverse inclusion. Let U and V be Hausdorff open subsets of G and let $f \in C_c(U), g \in C_c(V)$. Choose a function $h \in C_c(U^{-1})$ which is equal to 1 on $(\operatorname{supp} f)^{-1}$. Then for each $x \in \operatorname{supp} f$, one has $f(x) = (h \star f)(s(x))$ and hence $(g \cdot f)(x) = g(x) \cdot (h \star f)(s(x))$. Since $(f \cdot g)(x) = 0$ whenever $x \notin \operatorname{supp} f$, we conclude that $f \cdot g = \Phi(g \odot \epsilon(h \star f))$, where ϵ denotes the restriction map $\mathfrak{B}_c(G) \to \mathfrak{B}_c(G^0)$.

ii) By part iv) of proposition 3.54 and part i), the image of Φ identifies with $C_c(\mathfrak{H}G)$. To see that Φ extends to an isomorphism, it is enough to show that

$$\big\langle \eta \odot a \big| \xi \odot b \big\rangle_{L^2(G,\lambda^{-1})} = \big\langle \Phi(\eta \odot a) \big| \Phi(\xi \odot b) \big\rangle_{L^2(\mathfrak{H}G,\lambda'^{-1})}$$

for all $\eta \odot a, \xi \odot b \in C_c^{qc}(G) \odot C_0(\mathfrak{H}G)$. Since the inclusion $G^0 \subset (\mathfrak{H}G)^0$ is dense, it suffices to check that the restrictions of both inner products, considered as functions on $(\mathfrak{H}G)^0$, to G^0 coincide. Let $v \in G^0$. Note that the fibre G_v is contained in $(\mathfrak{H}G)_{\{v\}}$, but also vice versa because for each $A \in \mathfrak{H}G$, the set

$\mathfrak{s}(A) = A^{-1} \cdot A$ consists of one element if and only if A consists of only one element. Hence,

$$\begin{aligned}\big\langle \eta \otimes a \big| \xi \otimes b \big\rangle_{L^2(G,\lambda^{-1})}(v) &= \overline{a(v)} b(v) \sum_{x \in G_v} \overline{\eta(x)} \xi(x) \\ &= \sum_{\{x\} \in (\mathfrak{H}G)_{\{v\}}} \overline{\Phi(\eta \odot a)(\{x\})} \cdot \Phi(\xi \odot b)(\{x\}) \\ &= \big\langle \Phi(\eta \odot a) \big| \Phi(\xi \odot b) \big\rangle_{L^2(\mathfrak{H}G, \lambda'^{-1})}(\{v\}). \qquad \square\end{aligned}$$

This proposition answers the question for the construction of a pseudo-multiplicative unitary or a pseudo-Kac system for an r-discrete non-Hausdorff groupoid G: starting from a C^*-module $L^2(G, \lambda)$ which is defined analogously to the C^*-module $L^2(G, \lambda^{-1})$ studied above, and proceeding in the canonical way, one obtains nothing else but the pseudo-multiplicative unitary and the pseudo-Kac system associated to the Hausdorff groupoid $\mathfrak{H}G$.

For a locally compact groupoid G which is not r-discrete, the question is still open. Again, given the C^*-bimodule $L^2(G, \lambda^{-1})$, it is clear how to proceed, but we do not see yet what the objects obtained would tell us about the groupoid.

Concluding remarks

Let us briefly mention some questions and points for further study.

Theory of pseudo-multiplicative unitaries Certainly, much of the theory developed for multiplicative unitaries on Hilbert spaces should generalise to pseudo-multiplicative unitaries on C^*-modules. It might be interesting to study the compact and the commutative case [3, 4] in detail.

Beyond homogeneity In this thesis, we approach to the problems arising from the internal tensor product of C^*-bimodules algebraically, introducing a kind of grading by semigroups. This approach is quite restricted – e.g. it does not cover the left regular representation of a groupoid on the C^*-module $L^2(G, \lambda)$. It would be very desirable to develop similar techniques analytically, without homogeneity restriction.

Locally compact groupoids Consider pseudo-multiplicative unitaries associated to locally compact groupoids. The main problem arising if the groupoid is no longer decomposable is the definition of the functor $\boldsymbol{S}$ which, in the decomposable case, is given by $(C, \rho) \mapsto ((C, \rho) \otimes \mathscr{S}, \pi_{r2})$. All other components of the right leg of the associated pseudo-Kac system still make sense: the formula for the map Δ, the algebra S_0 and the coaction δ_0. A rough approximation might be to take for $\boldsymbol{S}(C, \rho)$ the commutant of $1 \otimes \mathrm{Ad}_U(C_r^*(G))$ in $L_C(C \otimes L^2(G, \lambda))$, where $\mathrm{Ad}_U(C_r^*(G)) \subset L_{C_0(G^0)}(L^2(G, \lambda))$ denotes the C^*-algebra of the right regular representation.

Equivariant $\boldsymbol{KK}$-theory Parallel to [2], one could consider coactions of C^*-families on C^*-(bi)modules and introduce equivariant KK-theory with respect to coactions of C^*-families. Given a pseudo-Kac system, we expect the reduced crossed product constructions to generalise to coactions on C^*-(bi)modules and to yield descent homomorphisms on the equivariant KK-theories associated to the two legs of this pseudo-Kac system. As in the classical situation, the duality theorem should imply that these homomorphisms are, in certain cases, isomorphisms.

Appendix A

Background

A.1 Miscellaneous

An *inverse semigroup* is a semigroup Σ with an involution $\Sigma \to \Sigma, \sigma \mapsto \sigma^*$, such that $\sigma\sigma^*\sigma = \sigma$ for all $\sigma \in \Sigma$. The *natural partial order* on Σ is given by $\sigma \leq \tau :\Leftrightarrow \sigma = \tau\epsilon$ for some idempotent $\epsilon \in \Sigma$.

Let X be topological space. A *partial homeomorphism* of X is a homeomorphism $\phi\colon \mathrm{Dom}(\phi) \to \mathrm{Im}(\phi)$ where $\mathrm{Dom}(\phi)$ and $\mathrm{Im}(\phi)$ are open subsets of X. Together with composition and inversion, the set of partial homeomorphisms of a fixed topological space form an inverse semigroup. We denote this semigroup by $\mathrm{PHom}(X)$.

A partial homeomorphism $\phi \in \mathrm{PHom}(X)$ induces mutually inverse isomorphisms

$$\begin{aligned}
&\phi_*\colon C_0(\mathrm{Dom}(\phi)) \to C_0(\mathrm{Im}(\phi)), && (\phi_* f)(x) := f(\phi^{-1}(x)),\\
&\phi^*\colon C_0(\mathrm{Im}(\phi)) \to C_0(\mathrm{Dom}(\phi)), && (\phi^* f)(x) := f(\phi(x)).
\end{aligned}$$

For two partially defined maps ϕ, ψ defined on open subsets $\mathrm{Dom}(\phi), \mathrm{Dom}(\psi)$ on X with values in another topological space Y, we put $\phi \wedge \psi := \phi|_V = \psi|_V$ where $V = \mathrm{int}\{x \in \mathrm{Dom}(\phi) \cap \mathrm{Dom}(\psi) \mid \phi(x) = \psi(x)\}$. For $\phi, \psi \in \mathrm{PHom}(X)$, the map $\phi \wedge \psi$ is a partial homeomorphism onto its image.

Lemma A.1. *One has* $(\phi\wedge\psi)^{-1} = \phi^{-1}\wedge\psi^{-1}$ *and* $(\phi'\wedge\psi')\circ(\phi\wedge\psi) \leq (\phi'\circ\phi)\wedge(\psi'\circ\psi)$ *for all* $\phi, \phi', \psi, \psi' \in \mathrm{PHom}(X)$. □

A.2 C^*-algebras [9, 34]

A $*$-*algebra* is a complex algebra A with an anti-linear anti-automorphism $*\colon A \to A$ called the *involution* of A. A $*$-homomorphism between two $*$-algebras is a homomorphism of complex algebras intertwining the respective involutions. A

C^**-norm* on a $*$-algebra A is a norm which satisfies the C^**-equation* $\|a^*a\| = \|a\|^2$ for all $a \in A$. If A is complete, it is called a C^**-algebra*.

Let A be a C^*-algebra. An element $a \in A$ is a *partial isometry* if $aa^*a = a$, an *isometry* if $a^*a = 1$, a *unitary* if $a^*a = aa^* = 1$, *self-adjoint* if $a = a^*$, a *projection* if $a = a^*$ and $a^2 = a$, *positive* if it can be written in the form $a = b^*b$ for some $b \in A$. The *natural order* on A is given by $a \geq b :\Leftrightarrow a - b$ is positive. An element $a \in A$ is *central* if it commutes with every element of A, i.e. if $ab = ba$ for all $b \in A$. The set of all central elements of A is a commutative C^*-subalgebra, denoted by $Z(A)$.

An *approximate unit* for A is a net $(u_\nu)_\nu$ of elements of A satisfying $\lim_\nu u_\nu a = a = \lim_\nu a u_\nu$ for all $a \in A$. Every C^*-algebra contains an approximate unit consisting of positive elements which have norm less than 1.

An *ideal* of A is a two-sided ideal in the algebraic sense which, in addition, is closed with respect to the involution and with respect to the norm.

The *unitisation* of A, denoted by A^+, is the C^*-algebra with underlying vector space $A \oplus \mathbb{C}$, multiplication $(a, \lambda) \cdot (a', \lambda') = (aa' + \lambda a' + a\lambda', \lambda\lambda')$, involution $(a, \lambda)^* = (a^*, \overline{\lambda})$, and norm $\|(a, \lambda)\| = \sup\{\|aa' + \lambda a'\| : a' \in A, \|a'\| \leq 1\}$. The algebra A is contained in A^+ as a two-sided ideal.

A *multiplier* of A is a pair (S, T) of maps $A \to A$ satisfying

$$S(ab) = S(a)b, \qquad T(ab) = T(a)b, \qquad b^*S(a) = T(b)^*a, \qquad a, b \in A.$$

The set of all multipliers of A forms a C^*-algebra with respect to the operations $(S,T) + (S',T') := (S + S', T + T')$, $(S,T)(S',T') := (SS', T'T)$, $(S,T)^* := (T,S)$ and with respect to the operator norm $\|(S,T)\| = \sup\{\|Sa\| : a \in A, \|a\| \leq 1\}$. It is called the *multiplier algebra* of A and denoted by $M(A)$. The algebra A is contained in $M(A)$ as a two-sided ideal via $a \mapsto (L_a, L_{a^*})$ where $L_a b := ab, a, b \in A$.

Let A and B be C^*-algebras. A $*$*-homomorphism* from A to B is a homomorphism $\phi\colon A \to B$ of algebras which intertwines the involutions on A and B. A $*$-homomorphism is automatically continuous. A $*$-homomorphism $\phi\colon A \to M(B)$ is *non-degenerate* if $\phi(A)B = B$. In that case, ϕ extends uniquely to a $*$-homomorphism $M(A) \to M(B)$.

A linear map $\phi\colon A \to B$ of C^*-algebras is *positive* if it preserves the order. A positive map is always continuous.

An *automorphism* of A is a $*$-isomorphism. For each unitary element $u \in A$, the map $\mathrm{Ad}_u\colon A \to A$ given by $a \mapsto uau^*$ is an automorphism. A *partial automorphism* of A is a $*$-isomorphism $\alpha\colon \mathrm{Dom}(\alpha) \to \mathrm{Im}(\alpha)$ where $\mathrm{Dom}(\alpha)$ and $\mathrm{Im}(\alpha)$ are closed ideals of A. For notational convenience, we denote the inverse of a partial automorphism α by α^*. We denote the inverse semigroup of all partial automorphisms of A by $\mathrm{PAut}(A)$.

The category of all commutative C^*-algebras and non-degenerate $*$-homomorphisms is equivalent to the category of all locally compact Hausdorff spaces with proper continuous maps. This correspondence is established as follows:

Let X be a locally compact Hausdorff space. Then the algebra $C_0(X)$ of complex-valued functions vanishing at infinity, equipped with the supremum norm, is a commutative C^*-algebra. The multiplier algebra $M(C_0(X))$ is canonically isomorphic to the algebra $C_b(X)$ of all bounded continuous functions. A map $\phi\colon X \to Y$ between locally compact Hausdorff spaces induces a non-degenerate $*$-homomorphism $\phi^*\colon\ C_0(Y) \to M(C_0(X))$ via $(\phi^* f)(x) = f(\phi(x))$. The image of ϕ^* is contained in $C_0(X)$ if and only if ϕ is proper.

Let A be a commutative C^*-algebra. A *character* on A is a non-zero $*$-homomorphism $A \to \mathbb{C}$. Denote by $\widehat{A}$ the set of all characters of A endowed with the weakest topology making all functions of the form $\chi \mapsto \chi(a)$ where $a \in A$ and $\chi \in \widehat{A}$ continuous. Then $\widehat{A}$ is a locally compact Hausdorff space, and the Gelfand-Naimark theorem says that the natural map $A \to C_0\big(\widehat{A}\big)$ is a natural isomorphism. Each $*$-homomorphism $\phi\colon A \to B$ of commutative C^*-algebras induces a continuous map $\phi^*\colon \widehat{B} \to \widehat{A}$ via $\chi \mapsto \chi \circ \phi$.

A *representation* of a C^*-algebra on a Hilbert space H is a $*$-homomorphism into the C^*-algebra $B(H)$ of bounded operators on H. It is called *faithful* if it is injective.

A *state* on a C^*-algebra A is a positive linear functional of norm one. The *GNS-construction* associates to a state a representation as follows. Denote by H_τ the completion of A with respect to the semi-norm $a \mapsto \tau(a^*a)$, and by $a \mapsto \overline{a}$ the canonical map $A \to H_\tau$. Then H_τ is a Hilbert space with respect to the inner product $\langle \overline{a}|\overline{b}\rangle := \tau(a^*b)$, $a, b \in A$. For each $a \in A$, the map $\overline{b} \mapsto \overline{ab}$, $b \in A$, defines a bounded operator $\pi_\tau(a)$ on H_τ, and the map $\pi_\tau\colon A \to B(H)$ is a $*$-homomorphism.

The *enveloping C^*-algebra* of a $*$-algebra A is the completion of A with respect to the semi-norm

$$|a| := \sup\{\|\phi(a)\| : \phi\colon A \to B \text{ a } *\text{-homomorphism, } B \text{ a } C^*\text{-algebra}\}, \quad a \in A,$$

provided that the supremum is finite for each $a \in A$.

Let A and B be two C^*-algebras. Then the algebraic tensor product $A \odot B$ is a $*$-algebra. A *C^*-tensor product* of A and B is the completion of $A \odot B$ with respect to some C^*-norm. One example is the *maximal tensor product* $A \overset{max}{\otimes} B$ which is the enveloping C^*-algebra of $A \odot B$.

A C^*-algebra A is *nuclear* if for each C^*-algebra B there exists only one C^*-norm on $A \odot B$. For each Hilbert space H, the C^*-algebra $\mathcal{K}(H)$ of compact operators is nuclear. If X is a locally compact space, the C^*-algebra $A = C_0(X)$ is nuclear. For every C^*-algebra B, the C^*-tensor product $C_0(X)\otimes B$ is isomorphic to the C^*-algebra of all continuous functions on X with values in B vanishing at infinity, equipped with the supremum norm.

A.3 C^*-modules [47, 29]

Let A be a C^*-algebra. A *pre-C^*-module* over a dense $*$-subalgebra $A_0 \subset A$ is a complex vector space E equipped with a right module structure over A_0 and an inner product which is a sesqui-linear map $\langle -|-\rangle$ from E to A_0 satisfying

$$\langle \eta|\xi\rangle^* = \langle \xi|\eta\rangle, \quad \langle \eta|\xi a\rangle = \langle \eta|\xi\rangle a, \quad \langle \xi|\xi\rangle \geq 0 \quad \text{and} \quad \langle \xi|\xi\rangle = 0 \Leftrightarrow \xi = 0$$

for all $\eta, \xi \in E$ and all $a \in A_0$. The inner product induces a norm on E via $\|\xi\| := \|\langle \xi|\xi\rangle\|^{1/2}$, $\xi \in E$. If E is complete with respect to this norm and $A_0 = A$, it is called a *right C^*-module* over A. If not, the structure maps extend to the completion which becomes a C^*-module over A.

A basic example of a C^*-module is the C^*-algebra A itself, with inner product $\langle a|b\rangle := a^*b$ and the obvious right module structure.

Let E and F be C^*-modules over A. An *adjoint* of a map $T\colon E \to F$ is a map $S\colon F \to E$ satisfying $\langle \eta|T\xi\rangle = \langle S\eta|\xi\rangle$ for all $\eta \in F$ and $\xi \in E$. If it exists, the adjoint of T is unique; it is denoted by T^*. In this case, T and T^* are bounded, A-linear and satisfy the C^*-equation $\|T\|^2 = \|T^*T\| = \|T^*\|^2$. The space of all adjointable operators from E to F is denoted by $L_A(E,F)$. If $E = F$, this space is also denoted by $L_A(E)$. Equipped with the natural operations and the operator norm, it is a C^*-algebra. The space $L_A(E,F)$, equipped with the inner product $\langle S|T\rangle := S^*T$ and the right module structure given by composition $L_A(E,F) \circ L_A(E) \to L_A(E,F)$, becomes a C^*-module over $L_A(E)$.

For each pair of elements $\xi \in E$ and $\eta \in F$, the map $|\eta\rangle\langle\xi|\colon E \to F$ given by $\zeta \mapsto \eta\langle\xi|\zeta\rangle$ defines an operator $E \to F$. Its adjoint is given by the map $|\xi\rangle\langle\eta|\colon F \to E$. An operator $T \in L_A(E,F)$ is called *compact* if it can be approximated in norm by linear combinations of such elementary operators. The set of all compact operators is denoted by $K_A(E,F)$. If $E = F$, this space is also denoted by $K_A(E)$. If $E = A$, the space $K_A(A,F)$ identifies with F via $|\eta\rangle\langle a| \equiv \eta a^*$, $\eta \in F, a \in A$.

The composition of a compact operator with an arbitrary operator is compact again, whence $K_A(E)$ is an ideal in $L_A(E)$. One has $L_A(E) = M(K_A(E))$.

The *strict topology* on $L_A(E,F)$ is the topology generated by the family of sets

$$U_{\eta,\xi,\epsilon} := \{T \in L_A(E,F) : \|\eta - T\xi\| < \epsilon \text{ and } \|T^*\eta - \xi\| < \epsilon\}$$

where $\eta \in F$, $\xi \in E$ and $\epsilon > 0$. Thus, a net of operators $(T_\nu)_\nu$ converges to an operator T if and only if for each $\xi \in E$ and $\eta \in E'$, the nets $(T_\nu\xi)_\nu$ and $(T_\nu^*\eta)_\nu$ converge to $T\xi$ in F and $T\eta \in E$, respectively. The subset $K_A(E,F)$ is dense in $L_A(E,F)$ with respect to the strict topology.

Let E and F be C^*-modules over C^*-algebras A and B, respectively, and let $A \otimes B$ be a C^*-tensor product of A and B. Then the algebraic tensor product $E \odot F$, equipped with the structure maps

$$\langle \xi \odot \eta|\xi' \odot \eta'\rangle := \langle \xi|\xi'\rangle \odot \langle \eta|\eta'\rangle, \quad (\xi \odot \eta)(a \odot b) := \xi a \odot \eta b,$$
$$\xi, \xi' \in E,\ \eta, \eta' \in F,\ a \in A, b \in B,$$

is a pre-C^*-module over $A \odot B \subset A \otimes B$. Its completion is called the *external tensor product* of E and F and denoted by $E \otimes F$.

A *representation* of A on F is a $*$-homomorphism $\pi\colon A \to L_B(F)$. It is called *non-degenerate* if $\pi(A)F = F$. A C^*-module over B with a representation of A is called a *C^*-A-B-bimodule*. If the representation is not clear from the context, we denote it explicitly and write (F, π) for the C^*-B-A-bimodule. By a *C^*-bimodule over A* without further specification we mean a C^*-A-A-bimodule.

The *internal tensor product* $E \otimes_\pi F$ is defined as follows. The algebraic tensor product $E \odot F$ carries an inner product and a B-module structure given by

$$\langle \xi \odot \eta | \xi' \odot \eta' \rangle := \big\langle \eta \big| \pi\big(\langle \xi | \xi' \rangle_E\big) \eta' \big\rangle_F, \qquad (\xi \odot \eta) b := \xi \odot \eta b,$$
$$\xi, \xi' \in E,\ \eta, \eta' \in F,\ b \in B.$$

These operations descend to the quotient $(E \odot F)/N$ over $N = \{\zeta \in E \odot F \mid \langle \zeta | \zeta \rangle = 0\}$, which becomes a pre-$C^*$-module over B. Its completion is called the *internal tensor product of E and F*. It is denoted by $E \otimes_\pi F$ or $E \otimes_* F$ if the representation π is implicit.

If the representation π is non-degenerate, the map $a \otimes_* \eta \mapsto \pi(a)\eta$ defines an isomorphism $A \otimes_* F \xrightarrow{\cong} F$.

Let B be a C^*-subalgebra of A. A *conditional expectation* $\rho\colon A \to B$ is a positive linear projection satisfying $\rho(ba) = b\rho(a)$ and $\rho(ab) = \rho(a)b$ for all $a \in A$ and $b \in B$. Denote by A_ρ the completion of A with respect to the seminorm $|a| := \rho(a^*a)$, and by $a \mapsto \overline{a}$ the canonical map $A \to A_\rho$. Then A_ρ is a C^*-module over B with respect to the inner product $\langle \overline{a} | \overline{a}' \rangle := \rho(a^* a')$, $a, a' \in A$ and the right module structure $\overline{a} b := \overline{ab}$, $a \in A, b \in B$. For each $a \in A$, the map $\overline{a}' \mapsto \overline{aa'}$, $a' \in A$, defines an operator $\pi_\rho(a)$ on A_ρ, and the map $\pi_\rho\colon A \to L_B(A_\rho)$ is a $*$-homomorphism. The construction of A_ρ and π_ρ is called the *Rieffel construction.*

A.4 Bundles and modules over locally compact spaces

$C_0(X)$-Banach modules and $C_0(X)$-algebras [22, 10, 4] Let X be a locally compact Hausdorff space. A right $C_0(X)$-*Banach module* is a Banach space M with a right $C_0(X)$-module structure such that $\|mf\| \le \|m\|\|f\|$ for all $f \in C_0(X)$ and $m \in M$. It is called *non-degenerate* if $MC_0(X) = M$. Each C^*-module over $C_0(X)$ is a $C_0(X)$-Banach module. A $C_0(X)$-*algebra* is a C^*-algebra A with a fixed non-degenerate $*$-homomorphism $C_0(X) \to ZM(A)$. A $C_0(X)$-homomorphism of $C_0(X)$-algebras is a $*$-homomorphism which intertwines the respective module structures.

Let F be a closed subset of X and put $I_F := \{f \in C_0(X) : f|_F = 0\}$. Then the quotient $M_F := M/(MI_F)$ becomes a $C_0(F)$-module, called the *restriction of M to F*. The *fibre* of M at a point $x \in X$ is the Banach space $M_x := M_{\{x\}}$. The quotient map $M \to M_x$ is denoted by $m \mapsto m_x$.

Let Y be a locally compact Hausdorff space. Let A be a $C_0(X)$-algebra and B be a $C_0(Y)$-algebra. Then the maximal tensor product $A \overset{max}{\otimes} B$ becomes a $C_0(X \times Y)$-algebra. If $X = Y$, the restriction of $A \overset{max}{\otimes} B$ to the diagonal $\Delta(X) = \{(x, x) \mid x \in X\} \subset X \times X$ is called the *$C_0(X)$-tensor product of A and B* and denoted by $A \otimes_{C_0(X)} B$.

If $\phi\colon A \to A'$ and $\psi\colon B \to B'$ are $C_0(X)$-homomorphisms of $C_0(X)$-algebras, the functoriality of the maximal tensor product yields a $C_0(X)$-homomorphism $\phi \otimes_{C_0(X)} \psi\colon A \otimes_{C_0(X)} B \to A' \otimes_{C_0(X)} B'$. Likewise, if $\phi\colon A \to C$ and $\psi\colon B \to C$ are $C_0(X)$-homomorphisms of $C_0(X)$-algebras with commuting images, the map $c \otimes d \mapsto \phi(c)\psi(d)$ defines a $C_0(X)$-homomorphism $\phi \times_{C_0(X)} \psi\colon A \otimes_{C_0(X)} B \to C$.

If $p\colon Y \to X$ is a continuous map, the *pull-back* $p^*A = A \otimes_{C_0(X)} C_0(Y)$ is a $C_0(Y)$-algebra, where the $C_0(Y)$-action is given by multiplication on the second factor. Here, $C_0(Y)$ is considered as a $C_0(X)$-algebra via the homomorphism p^*.

Banach bundles, Hilbert bundles and C^*-bundles [9, 10] Let X be a locally compact space, not necessarily Hausdorff. An *upper semi-continuous/continuous Banach bundle* on X is a topological space E with an open continuous surjection $p\colon E \to X$ such that

i) for each point $x \in X$, the fibre $E_x := p^{-1}(x)$ is endowed with a Banach space structure,

ii) the maps $\mathbb{C} \times E \to E$ and $E \times_X E \to E$ given in each fibre by scalar multiplication and addition, respectively, are continuous,

iii) the map $E \to \mathbb{R}$ given by the norm in each fibre is upper semi-continuous/continuous,

iv) for each point $x \in X$ and each neighbourhood W of the zero 0_x over x in E, there exist a number $\epsilon > 0$ and a neighbourhood U of x such that the set $\{e \in p^{-1}(U) : \|e\| < \epsilon\}$ is contained in W.

A *bundle morphism* between two upper-semicontinuous/continuous Banach bundles E and E' on X is a continuous map $\phi\colon E \to E'$ which intertwines the projections onto X and whose restriction to each fibre $E_x, x \in X$, is a linear operator $\phi_x\colon E_x \to E'_x$ bounded by a constant C not depending on x.

A *continuous Hilbert bundle* is a continuous Banach bundle each of whose fibres is a Hilbert space, with the additional condition that the map $E \times_X E \to \mathbb{C}$ given in each fibre by the interior product should be continuous.

An *upper semi-continuous/continuous C^*-bundle* is an upper semi-continuous/continuous Banach bundle each of whose fibres is a C^*-algebra, with the additional condition that the maps $E \times_X E \to E$ and $E \to E$ given in each fibre by multiplication and the involution, respectively, are continuous.

For an open subset $U \subset X$, the space of all continuous sections of p over U is denoted by $\Gamma(U, E)$. The subspaces of compactly supported sections and of all sections vanishing at infinity are denoted by $\Gamma_c(U,E)$ and $\Gamma_0(U,E)$, respectively. We abbreviate $\Gamma(E) := \Gamma(X,E)$.

The correspondence [10] Let X be a locally compact Hausdorff space.

A $C_0(X)$-Banach module M is called *convex* if for all elements $m, m' \in M$ and all non-negative functions f and f' in $C_0(X)$ satisfying $f + f' \leq 1$, one has $\|mf+m'f'\| \leq \max\{\|m\|, \|m'\|\}$. Every $C_0(X)$-algebra as well as every C^*-module over $C_0(X)$, considered as a right Banach module over $C_0(X)$, is convex.

Lemma A.2. *Let E be a C^*-module over a $C_0(X)$-algebra A. Denote the map $C_0(X) \to ZM(A)$ by ρ. Then E is a $C_0(X)$-module via $\xi \cdot f := \xi\rho(f), \xi \in E, f \in C_0(X)$ and as such non-degenerate and convex.*

Proof. The $C_0(X)$-module E is non-degenerate since $EC_0(X) = EA \cdot C_0(X) = E \cdot A = E$. The linking algebra $K_A(A \oplus E)$, considered as a $C_0(X)$-algebra via the map $C_0(X) \to L_A(A \oplus E)$ given by $(a,\xi)f := \big(\rho(f)a, \xi\rho(f)\big)$ where $(a,\xi) \in A \oplus E$ and $f \in C_0(X)$, is convex, and hence so is the $C_0(X)$-submodule $E \subset K_A(A \oplus E)$. Here, we consider E embedded in $K_A(A \oplus E)$ via $\xi(a,\eta) := (0, \xi a)$. □

The category of non-degenerate convex Banach modules over $C_0(X)$ is equivalent to the category of upper semi-continuous Banach bundles over X. Under this equivalence, C^*-modules over $C_0(X)$ correspond exactly to continuous Hilbert bundles over X, and $C_0(X)$-algebras correspond exactly to upper semi-continuous C^*-bundles over X. The equivalence is established as follows.

Let E be a Banach bundle on X. Then the space of sections $\Gamma_0(E)$, equipped with the supremum norm, is a Banach space. Pointwise multiplication of sections with functions endows $\Gamma_0(E)$ with the structure of a convex $C_0(X)$-module. For each point $x \in X$, the evaluation at x defines an isomorphism of fibres $\Gamma_0(E)_x \cong E_x$.

Let M be a non-degenerate convex $C_0(X)$-module. Denote by $Bd(M)$ the disjoint union $\coprod_{x \in X} M_x$ of the fibres of M, by $p\colon Bd(M) \to X$ the obvious projection map, and for each element $m \in M$, by $\sigma(m)$ the map $X \to Bd(M), x \mapsto m_x$. Then the set $Bd(M)$ can be equipped with a topology such that $p\colon Bd(M) \to X$ becomes an upper semi-continuous Banach bundle and the map $m \mapsto \sigma_m$ is an isometric isomorphism $M \cong \Gamma_0(Bd(M))$.

Let E and F be C^*-modules over $C_0(X)$. Then the space $K_{C_0(X)}(E,F)$, considered as a Banach module over $C_0(X)$, corresponds to a continuous Banach bundle whose fibre over $x \in X$ is $K(E_x, F_x)$. If $E = F$, this bundle is a continuous C^*-bundle.

A.5 Locally compact groupoids [45, 41, 25]

Basic definitions A *groupoid* is a small category in which every morphism is invertible. Equivalently, a groupoid consists of a set of morphisms G, a set of *units* $G^0 \subset G$, two maps $r, s\colon G \to G^0$ called the *range* and *source* maps, and a composition map $\circ\colon G\,{}_s\!\times_r G \to G$, where $G\,{}_s\!\times_r G = \{(x,y) \subset G \times G \mid s(x) = r(y)\}$, subject to the following conditions:

i) $r(x \circ y) = r(x)$ and $s(x \circ y) = s(y)$ for all $(x,y) \in G\,{}_s\!\times_r G$,

ii) $r(x) \circ x = x = x \circ s(x)$ for all $x \in G$,

iii) $r(u) = u = s(u)$ for all $u \in G^0$,

iv) $(x \circ y) \circ z = x \circ (y \circ z)$ for all $(x,y,z) \in G\,{}_s\!\times_r G\,{}_s\!\times_r G$,

v) each $x \in G$ has an inverse $x^{-1} \in G$ such that $x \circ x^{-1} = r(x)$ and $x^{-1} \circ x = s(x)$.

Denote the fibres of the range and source maps by $G^u = r^{-1}(u)$ and $G_u = s^{-1}(u)$, $u \in G^0$, respectively.

A *topological groupoid* is a groupoid equipped with a topology on its set of morphisms for which the inversion and the composition are continuous. Then both the range and source map are continuous, too. The topological groupoid is called *locally compact* if its topology is second countable, locally compact and if additionally the space of units and the fibres of the range and source map are locally compact and Hausdorff. Here, a topological space is *locally compact* if each of its points has a compact neighbourhood [23, 55].

Let G be a locally compact groupoid. An subset $U \subset G$ is a *G-set* if the restrictions $r|_U\colon U \to r(U)$ and $s|_U\colon U \to s(U)$ are bijections. A locally compact groupoid G is called *r-discrete* if G can be covered by open G-sets $U \subset G$ for which the restrictions $r|_U$ and $s|_U$ are homeomorphisms onto open subsets of G^0.

Let G and G' be topological groupoids. A map $\phi\colon G \to G'$ is a *homomorphism* if it is continuous and a functor of small categories.

Let G be a locally compact groupoid. A *left Haar system* for G is a family of measures λ^u on G^u, $u \in G^0$, such that

i) for each $u \in G^0$, λ^u is a regular Borel measure with support G^u,

ii) for each open Hausdorff subset $U \subset G$ and each $f \in C_0(U)$, extended by 0 to a (not necessarily continuous) function on G, the function $u \mapsto \int_{G^u} f(x)\, d\lambda^u(x)$ on G^0 is continuous,

iii) for each $x \in G$, each open Hausdorff subset $U \subset G$ and each $f \in C_0(U)$, one has

$$\int_{G^{r(x)}} f(y)\, d\lambda^{r(x)}(y) = \int_{G^{s(x)}} f(xy')\, d\lambda^{s(x)}(y').$$

The *associated right Haar system* is the family of measures $\lambda_u^{-1} := i_*(\lambda^u)$ on $G_u, u \in G^0$, given by the push-forward of the family λ with respect to the inversion map $i\colon G \to G,\ x \mapsto x^{-1}$.

If G is r-discrete, the fibres of the range map are discrete, and the family of counting measures on these fibres forms a left Haar system on G.

Assume that G is Hausdorff. The Hilbert bundle $L^2(G, \lambda)$ over G^0 is constructed as follows. The space $C_c(G)$, equipped with the operations

$$\langle \eta | \xi \rangle (u) := \int_{G^u} \overline{\eta(x)} \xi(x) d\lambda^u(x), \qquad (\xi f)(x) := \xi(x) f(r(x)),$$

where $\xi, \eta \in C_c(G), f \in C_0(G^0)$ and $x \in G, u \in G^0$, is a pre-C^*-module over $C_0(G^0)$. We denote its completion and the Hilbert bundle corresponding to this C^*-module both by $L^2(G, \lambda)$. Its fibres are the Hilbert spaces $L^2(G^u, \lambda^u), u \in G^0$.

If G is not Hausdorff, this definition has to be adapted, see [23, 55] and section 3.4.

Fell bundles [25, 65] Let G be a locally compact groupoid, not necessarily Hausdorff. Given a Banach bundle E on G, we denote by E^0 the restriction of E to G^0, and by E^2 the restriction of $E \times E$ to $G\,{}_s\!\times_r G$. An *upper semi-continuous/continuous Fell bundle* on G is an upper semi-continuous/continuous Banach bundle $p\colon E \to G$ together with a continuous multiplication map $E^2 \to E$ and an involution $*\colon E \to E$ satisfying

i) $p(e_1 e_2) = p(e_1) p(e_2)$ and $p(e^*) = p(e)^{-1}$,

ii) the induced map $E_x \times E_y \to E_{xy}$ is bilinear for each pair $(x, y) \in G\,{}_s\!\times_r G$, and the induced map $E_x \to E_{x^{-1}}, e \mapsto e^*$, is conjugate linear for all $x \in G$,

iii) $(e_1 e_2) e_3 = e_1 (e_2 e_3)$, $(e_1 e_2)^* = e_2^* e_1^*$ and $(e^*)^* = e$,

iv) $\|e_1 e_2\| \le \|e_1\| \|e_2\|$ and $\|e^* e\| = \|e\|^2$ and

v) $e^* e \ge 0$

for all $e \in E$ and $(e_1, e_2) \in E^2$. For each $u \in G^0$, the fibre E_u is a C^*-algebra, and for each $x \in G$, the fibre E_x is a C^*-bimodule over $E_{r(x)}$ and $E_{s(x)}$. In particular, the restriction of E to G^0 is an upper semi-continuous/continuous C^*-bundle, and $\Gamma_0(E^0)$ is a C^*-algebra.

Let (E, p) and (E', p') be Fell bundles on G. A map $\phi\colon E \to E'$ is a *Fell bundle morphism* if it is a bundle morphism and intertwines the structure maps, i.e. $\phi(e)^* = \phi(e^*)$ for all $e \in E$ and $\phi(e_1)\phi(e_2) = \phi(e_1 e_2)$ for all $(e_1, e_2) \in E^2$. If these conditions are satisfied, then for each $v \in G^0$, the map $\phi_v\colon E_v \to E'_v$ is a $*$-homomorphism of C^*-algebras and bounded of norm less than or equal to 1. Since $\|e\| = \|e^* e\|$ for each $e \in E$, one has $\|\phi(e)\| \le \|e\|$ for all $e \in E$.

The *trivial Fell bundle on G* is the bundle $\mathbb{C} \times G$ on G with obvious structure maps.

Examples

i) The *effective groupoid* of a locally compact Hausdorff space X is constructed as follows. For each $x \in X$, let $\mathfrak{PHom}(X)_x$ denote the set of equivalence classes of all partial homeomorphisms defined on an open neighbourhood of x, where two maps are identified if they agree on some neighbourhood of x. The equivalence class of a partial homeomorphism ϕ is called the *germ of ϕ at x* and denoted by $[\phi, x]$. The disjoint union $\mathfrak{PHom}(X) := \bigcup_{x\in X} \mathfrak{PHom}(X)_x$ is a groupoid with unit space $\{[\mathrm{id}, x] \mid x \in X\} \cong X$. Its range and source map, the inversion and the composition are given by

$$r([\phi, x]) = \phi(x), \qquad s([\phi, x]) = x,$$
$$[\phi, x]^{-1} = [\phi^{-1}, \phi(x)], \qquad [\phi, \psi(y)] \circ [\psi, y] = [\phi \circ \psi, y].$$

Sets of the form $[\phi] := \{[\phi, x] \mid x \in \mathrm{Dom}(\phi)\}, \phi \in \mathrm{PHom}(X)$, form a basis for a topology on $\mathfrak{PHom}(X)$. With respect to this topology, $\mathfrak{PHom}(X)$ is r-discrete. We call $\mathfrak{PHom}(X)$ the *effective groupoid* of X. For $[\phi, x] \in \mathfrak{PHom}(X)$, put $[x, \phi] := [\phi, \phi^{-1}(x)]$.

ii) Let $\sim$ be an equivalence relation on a locally compact Hausdorff space such that $\sim$ is closed as a subset of $X \times X$. To such an equivalence relation, one can associate a locally compact Hausdorff groupoid $G_\sim$ as follows. As a locally compact space, $G_\sim$ is just the subspace $\sim$ of $X \times X$ with the induced topology. The space of units of $G_\sim$ is X. The range and source maps, inversion and composition are given by

$$r(x, y) := x, \quad s(x, y) := y, \quad (x, y)^{-1} := (y, x), \quad (x, y) \cdot (y, z) := (x, z),$$

for all $x, y, z \in X$ satisfying $x \sim y$ and $y \sim z$.

iii) Let X be a locally compact Hausdorff space with a right action α of a discrete group G by homeomorphisms. The *transformation groupoid* $X \rtimes_\alpha G$ is the locally compact Hausdorff space $X \times G$ endowed with the following structure. The space of units of $X \rtimes_\alpha G$ is $X \times \{e\} \cong X$, where e denotes the unit of G. The range and source maps, the inversion and the composition are given by

$$r(x, g) = x, \qquad s(x, g) = \alpha_g(x),$$
$$(x, g)^{-1} = (\alpha_g(x), g^{-1}), \qquad (x, g) \cdot (\alpha_g(x), g') = (x, gg'),$$

respectively.

Let λ_G denote the left Haar system on G. For each $x \in X$ one has $(X \rtimes_\alpha G)^x = \{x\} \times G$. The family $\lambda^{(x,e)} = \delta_x \times \lambda_G, x \in X$, is a left Haar system on $X \rtimes_\alpha G$.

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