Contributions to the Theory of Quantum Groupoids in the Setting of C^*-algebras

gemäß §4 Nr. 3
der Habilitationsordnung für den
Fachbereich Mathematik und Informatik
der Westfälischen Wilhelms-Universität Münster
als kumulative Habilitationsleistung
eingereichte Schriften

vorgelegt von
Thomas Timmermann
aus Leipzig
– 2013 –

I would like to thank
the research group of Joachim Cuntz and Siegfried Echterhoff in Münster,
Michel Enock, Erik Koelink, Stefaan Vaes and many others
for discussions, hospitality and support,
and my family.

Abstract

This thesis is concerned with quantum groupoids in the setting of operator algebras. It consists of a brief introduction to quantum groupoids, a synopsis of the articles [I.1]–[II.2] listed below, and the articles themselves in the form of appendices.

[I.1] T. Timmermann. The relative tensor product and a minimal fiber product in the setting of C^*-algebras. *J. Operator Theory* 68(2):101-140, 2012.

[I.2] T. Timmermann. C^*-pseudo-multiplicative unitaries, Hopf C^*-bimodules and their Fourier algebras. *J. Inst. Math. Jussieu* 11(1):189-220, 2012.

[I.3] T. Timmermann. A definition of compact C^*-quantum groupoids. *Contemp. Math.* 503:267-290, 2009.

[I.4] T. Timmermann. Coactions of Hopf C^*-bimodules. *J. Operator Theory* 68(1):19-66, 2012.

[II.1] T. Timmermann. Free dynamical quantum groups and the dynamical quantum group $\mathrm{SU}_q(2)$. *Banach Center Publications* 98:311-341, 2012.

[II.2] T. Timmermann. Measured quantum groupoids associated to proper dynamical quantum groups. arXiv:1206.6744, prelim. accepted by *J. Noncommut. Geom.*

The articles [I.1]–[I.4] lay foundations for the theory of quantum groupoids in the setting of C^*-algebras, while the articles [II.1] and [II.2] are concerned with dynamical quantum groups in the algebraic setting and connections to the setting of operator algebras. The logical dependence of the articles is depicted in the diagram below, where dotted lines indicate the provision of examples:

$$\begin{tikzcd}
 & {[I.1]} \arrow[d] & \\
 & {[I.2]} \arrow[dl] \arrow[d] \arrow[dr] & {[II.1]} \arrow[d, dotted] \\
{[I.3]} \arrow[r, dotted] & {[I.4]} & {[II.2]}
\end{tikzcd}$$

Contents

A brief introduction to quantum groupoids

This section gives a brief introduction to the concept of a quantum groupoid as it is used in this thesis and lists the related approaches found in the literature.

Quantum groupoids can be thought of as the "push-out" in the following diagram:

$$(*)\qquad \begin{tikzcd} \text{groups} \arrow[r] \arrow[d] & \text{groupoids} \arrow[d] \\ \text{quantum groups} \arrow[r] & \text{quantum groupoids} \end{tikzcd}$$

They generalize groupoids and quantum groups in a similar way like the latter generalize groups. To explain these statements, let us traverse the arrows in this diagram.

If one thinks of a group as describing the symmetries of one object, then groupoids capture the symmetries of a parameterized family of objects. For example, the former act on vector spaces and the latter on parameterized families or bundles of vector spaces. Formally, a groupoid is a small category where every morphism is invertible and thus consists of a set G^0 of objects or units, a set G of morphisms, two maps from G to G^0 assigning to each morphism its source and range, and a multiplication that assigns to each composable pair of morphisms the composition.

The passage from groups to quantum groups follows two principles that quantum physics, algebraic geometry and non-commutative geometry have in common, namely,

(1) "classical" (phase) spaces are replaced by "quantum" algebras (of observables) which usually do not commute and
(2) "classical" objects yield "quantum" counterparts via contravariant functors from certain classes of spaces to suitable algebras of functions.

Thus, a quantum group is not a set G with a multiplication map $G \times G \to G$, but an algebra A with a homomorphism $A \to A \otimes A$ called comultiplication that satisfies several conditions. The precise form of the latter depends on the context. In the algebraic setting, one usually assumes existence of a counit and antipode which correspond to the unit element and the inversion of a group and lead to the notion of a Hopf algebra. In the setting of operator algebras, one usually demands existence of a left- and a right-invariant weight which correspond to a left and a right Haar measure of a group, and then constructs a counit and antipode which may be unbounded maps.

Applying the same paradigm to groupoids instead of groups, one finds that a quantum groupoid should consist of algebras B and A corresponding to the sets of objects and morphisms, respectively, two maps $s, r \colon B \to A$ corresponding to the source and range map, and a comultiplication $\Delta \colon A \to A{*}A$ satisfying several conditions. The precise form of these conditions and the definition of the fiber product $A * A$ depend on the context and are not easy to guess from the analogy with groupoids unless B is commutative and $r(B)$ and $s(B)$ are central in A. The main variants of such quantum groupoids are the

- weak Hopf algebras [9], [45], [51], where the algebra B is separable and hence semi-simple and the target of the comultiplication is a subalgebra of $A \otimes A$;
- Hopf algebroids [8], [11], [40], [74], which provide a general algebraic framework beyond the case where B is semi-simple;

- weak multiplier Hopf algebras [67], [68] and multiplier Hopf algebroids [61], where in contrast to the variants above the algebras A and B need no longer be unital;
- measured quantum groupoids [20], [19], [39] in the setting of von Neumann algebras.

For example, every groupoid G yields a quantum groupoid (B, A, s, r, Δ), where B and A are suitable algebras of functions on G and s, r, Δ are the transposes of the source map, range map and the multiplication map of G, respectively. If the groupoid is (i) finite, (ii) infinite, (iii) algebraic, (iv) locally compact and Hausdorff, or (v) measured, one should take those functions that are (i) arbitrary, (ii) finitely supported, (iii) regular, (iv) continuous and vanishing at infinity, or (v) measurable and essentially bounded, respectively. Furthermore, such a groupoid yields a second quantum groupoid $(B, \hat{A}, \hat{s}, \hat{r}, \hat{\Delta})$, where B is as before and $\hat{A}$ is an associated groupoid algebra. If G is finite, then

$$B = \mathbb{C}(G^0), \qquad A = \mathbb{C}(G), \qquad \hat{A} = \mathbb{C}G$$

and these spaces have bases $(\delta_x)_{x\in G^0}$, $(\delta_\gamma)_{\gamma\in G}$ and $(\gamma)_{\gamma\in G}$, respectively, such that

$$\delta_x \cdot \delta_y = \delta_{x,y}\delta_x, \quad \delta_\gamma \cdot \delta_{\gamma'} = \delta_{\gamma,\gamma'}\delta_\gamma, \quad \gamma \cdot \gamma' = \begin{cases} \gamma\gamma', & \text{if the product is defined in } G, \\ 0, & \text{otherwise,} \end{cases}$$

$$\begin{aligned} s(\delta_x) &= \sum_{\gamma^{-1}\gamma = x} \delta_\gamma, & r(\delta_x) &= \sum_{\gamma\gamma^{-1} = x} \delta_\gamma, & \Delta(\delta_\gamma) &= \sum_{\gamma'\gamma''=\gamma} \delta_{\gamma'} \otimes \delta_{\gamma''}, \\ \hat{s}(\delta_x) &= x, & \hat{r}(\delta_x) &= x, & \hat{\Delta}(\gamma) &= \gamma \otimes \gamma. \end{aligned}$$

Here, the targets of Δ and $\hat{\Delta}$ are certain subalgebras of $A \otimes A$ and $\hat{A} \otimes \hat{A}$, respectively.

Examples of genuine quantum groupoids appeared in a variety of mathematical contexts and guises, for example, in the form of generalized Galois symmetries for depth 2 inclusions of factors or algebras [10], [19], [20], [30], [31], [44], dynamical quantum groups in connection with solutions of the quantum dynamical Yang-Baxter equation [18], [24], [33], Tannaka-Krein duals of certain monoidal categories of bimodules, and in connection with invariants of knots and 3-manifolds [42] and transverse geometry [14].

The heuristic explanation of the concept of a quantum groupoid given above suffices for the purpose of this thesis but does not touch the following important aspect of quantum groupoids — their close relation to certain monoidal categories of bimodules. Indeed, each (i) group, (ii) quantum group, (iii) groupoid and (iv) quantum groupoid has a naturally associated monoidal category of (i) representations on vector spaces, (ii) corepresentations on vector spaces, (iii) representations on vector bundles over the base space G^0 or (iv) bimodules over the base algebra B, respectively. Many properties of the initial object are reflected in the associated category, for example, in the “classical” cases (i) and (iii), the monoidal categories are symmetric, whereas in the “quantum” cases (ii) and (iv), one can only hope for a braiding; see §II. In certain cases, the initial object can be reconstructed from the associated category; in the case of compact groups, this is known as Tannaka-Krein duality [56]. Extensions of this duality to quantum groupoids were studied in [28], [41], [47], and the idea to define and study quantum groupoids in terms of the associated categories of representations is pursued in [16], [17], [22].

I Foundations for quantum groupoids in the setting of C^*-algebras

C^*-algebras and von Neumann algebras provide the right setting to study "quantum" counterparts of locally compact Hausdorff or measured groupoids. Indeed, for locally compact Hausdorff spaces and measure spaces, the principles (1) and (2) above amount to a passage to C^*-algebras or von Neumann algebras, respectively, where spaces correspond to commutative algebras via Gelfand duality.

For groups, locally compact topologies and invariant measures determine each other by a classical result of Weil [71]. Kustermans and Vaes showed that this result extends to quantum groups, where the same objects can equivalently be described on the level of universal C^*-algebras [34], reduced C^*-algebras [35] or von Neumann algebras [36]. This equivalence breaks down if one passes to groupoids because the latter include ordinary spaces, where topologies determine Borel structures but not vice versa. Therefore, the study of quantum groupoids in the setting of C^*-algebras provides a refinement to the setting of von Neumann algebras and one can only expect to pass from the former to the latter but not backwards.

The articles [I.1]–[I.4] lay foundations for a theory of quantum groupoids in the setting of C^*-algebras. The initial motivation for this work was to answer the following questions:

(1) Every locally compact, Hausdorff groupoid G with Haar systems yields C^*-algebras $B = C_0(G^0)$, $A = C_0(G)$ and $\hat{A} = C^*_r(G)$ with natural maps $s, r\colon B \to M(A)$ and $\hat{s} = \hat{r}\colon B \to M(\hat{A})$. In which sense do these form examples of quantum groupoids?

(2) Given an action α of a locally compact Abelian group G on a C^*-algebra C, one can form a crossed product C^*-algebra $C \rtimes G$ with a dual action $\hat{\alpha}$ of the dual group $\hat{G} = \mathrm{Hom}(G, \mathbb{T})$, and the iterated crossed product $C \rtimes G \rtimes \hat{G}$ is equivariantly Morita equivalent to C. This duality was extended to coactions of quantum groups on C^*-algebras by Baaj and Skandalis [1]. For actions of groupoids as in (1) on C^*-algebras, le Gall constructed reduced crossed products [38], and the question is whether the latter carry a dual action and whether there exists a duality for coactions of quantum groupoids on C^*-algebras.

In the setting of von Neumann algebras, partial answers to the corresponding questions were obtained by Vallin [63] and Yamanouchi [75], respectively. Later, Lesieur and Enock developed a comprehensive theory of measured quantum groupoids and actions of such objects on von Neumann algebras, see [39] and [20], [21]. This theory uses powerful tools like Connes' fusion of correspondences [13] and Haagerups theory of operator-valued weights [26], [27], which are or were not available for C^*-algebras.

The articles [I.1]–[I.4] provide answers to questions (1) and (2) and tools and concepts for a general theory of quantum groupoids in the setting of C^*-algebras. To complete this theory, a better understanding of operator-valued weights on C^*-algebras is needed.

I.1 Relative tensor products and fiber products in the setting of C^*-algebras.

Most approaches to quantum groupoids involve the construction of

(1) the fiber product $A \underset{B}{*} C$ of two algebras A and C relative to an algebra B which embeds anti-homomorphically into A and homomorphically into C,

(2) the relative tensor product $H \underset{B}{\otimes} K$ of an A-module H and a C-module K relative to B; this product will be an $(A \underset{B}{*} C)$-module.

These constructions are needed to define the notion of a comultiplication, coaction and corepresentation of a quantum groupoid. Variants were known in the setting of

- algebra, where (2) is obvious and (1) is due to Takeuchi [55];
- von Neumann algebras, where (2) is Connes fusion of correspondences and (1) is given by Sauvageot's bicommutant formula $A \underset{B}{*} C = (A' \underset{B}{\otimes} C')' \subseteq \mathcal{L}(H \underset{B}{*} K)$, see §5.B in [13] and [25], [49], [50];
- C^*-algebras, when $B = C_0(X)$ is commutative and central in A and C, where the algebras A, C and the modules H, K correspond to bundles over X and (1) and (2) correspond to the fiber-wise tensor products; see [5], [6].

None of these constructions suggests how to proceed in the general C^*-algebraic setting.

The article [I.1] proposes an approach which is based on an algebraic reformulation of Connes' fusion of correspondences. The main ideas are as follows.

(1) We fix commuting representations of the C^*-algebra B and its opposite B^{op} on a Hilbert space $\mathfrak{H}$ as a substitute for the standard form of a von Neumann algebra, using, for example, the GNS-construction for a KMS-weight on B.
(2) We define a left or right C^*-module relative to $(\mathfrak{H}, B, B^{\mathrm{op}})$ to be a Hilbert space H with a closed subspace $\alpha \subseteq \mathcal{L}(\mathfrak{H}, H)$ satisfying $[\alpha\mathfrak{H}] = H$ and $[\alpha^*\alpha] = B^{\mathrm{op}}$, $[\alpha B^{\mathrm{op}}] = \alpha$ or $[\alpha^*\alpha] = B$, $[\alpha B] = \alpha$ respectively, where B and B^{op} are identified with their images in $\mathcal{L}(\mathfrak{H})$ and $[-]$ denotes the closed linear span. Given (H, α), there exists a representation ρ_α of B or B^{op}, respectively, on H such that $\rho_\alpha(x)\xi = \xi x$ for all $\xi \in \alpha$. We define a C^*-bimodule to be a triple (H, α, β), where (H, α) is a left and (H, β) a right C^*-module such that $[\rho_\alpha(B)\beta] = \beta$ and $[\rho_\beta(B^{\mathrm{op}})\alpha] = \alpha$.
(3) We define the relative tensor product of two C^*-bimodules (H, α, β) and (K, γ, δ) as follows. Denote by $H_\beta \otimes_\gamma K$ the separated completion of the algebraic tensor product $\beta \otimes \mathfrak{H} \otimes \gamma$ with respect to the sesquilinear form given by
$$\langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta' \rangle = \langle \zeta | (\xi^*\xi')(\eta^*\eta')\zeta' \rangle = \langle \zeta | (\eta^*\eta')(\xi^*\xi')\zeta' \rangle.$$
Then γ and β yield subspaces $|\gamma\rangle_2 \subseteq \mathcal{L}(H, H_\beta \otimes_\gamma K)$ and $|\beta\rangle_1 \subseteq \mathcal{L}(K, H_\beta \otimes_\gamma K)$ such that $H_\beta \otimes_\gamma K$ and the spaces $[|\gamma\rangle_2\alpha]$ and $[|\beta\rangle_1\delta]$ form a C^*-bimodule.
(4) We define the fiber product $A_\beta *_\gamma C$ of two C^*-algebras $A \subseteq \mathcal{L}(H)$ and $C \subseteq \mathcal{L}(K)$ to be the C^*-algebra formed by all $T \in \mathcal{L}(H_\beta \otimes_\gamma K)$ satisfying
$$T|\beta\rangle_1 + T^*|\beta\rangle_1 \subseteq [|\beta\rangle_1 C] \text{ and } T|\gamma\rangle_2 + T^*|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A]$$
as subsets of $\mathcal{L}(K, H_\beta \otimes_\gamma K)$ or $\mathcal{L}(H, H_\beta \otimes_\gamma K)$, respectively.

The articles [I.2]–[I.4], [II.2] and the following results of [I.1] show that these definitions and constructions fulfill their purpose:

(5) The fiber product describes the target of the comultiplication for the C^*-algebras $C_0(G)$ and $C_r^*(G)$ associated to a locally compact Hausdorff groupoid G.

(6) The definitions above carry over to the setting of von Neumann algebras, where norm closures get replaced by σ-weak closures. Then the spaces α and β above are determined by the representations ρ_α and ρ_β, the relative tensor product can be identified with Connes' fusion of correspondences and the fiber product reduces to Sauvageot's construction.

(7) If $B = B^{\mathrm{op}} = C_0(X)$ for some locally compact, Hausdorff space X and $\mathfrak{H} = L^2(X, \mu)$ for some measure μ, then the subcategory of all C^*-bimodules (H, α, β) with $\alpha = \beta$ is monoidally equivalent to the category of all continuous bundles of Hilbert spaces on X with the fibrewise tensor product, and the fiber product contains Blanchard's represented $C_0(X)$-tensor product introduced in [6].

I.2 C^*-pseudo-multiplicative unitaries. A main motivation for the development of the theory of quantum groups in the setting of operator algebras was to extend the classical Pontryagin duality of locally compact Abelian groups to the non-Abelian case. Fundamental to this generalization and the whole theory of locally compact quantum groups is the notion of a multiplicative unitary introduced by Baaj and Skandalis in [1]. In the theory of measured quantum groupoids of Enock, Lesieur and Vallin, the corresponding concept of a pseudo-multiplicative unitary was introduced by Vallin [64]. Briefly, (pseudo-)multiplicative unitaries are used to pass from a locally compact quantum group or measured quantum groupoid to its generalized Pontryagin dual and, in the quantum group case, to switch between the level of reduced or universal C^*-algebras and von Neumann algebras; see [35], [36], [37], [39], [52], [73] and [59].

In more detail, a multiplicative unitary is a unitary operator V on the tensor product $H \otimes H$ of some Hilbert space H with itself satisfying the pentagon equation $V_{12}V_{13}V_{23} = V_{23}V_{12}$, where each V_{ij} acts on $H \otimes H \otimes H$ like V at the positions i and j.

To every reasonable quantum group (A, Δ), one can associate a multiplicative unitary V on the L^2-space for the right Haar weight which, roughly, is given by $a \otimes b \mapsto \Delta(a)(1 \otimes b)$; see [35], [36], [37]. This unitary generalizes the canonical pairing of a locally compact Abelian group G with its dual group $\hat{G} = \mathrm{Hom}(G, \mathbb{T})$, regarded as a function on $\hat{G} \times G$ that is represented on $L^2(G) \otimes L^2(G)$ using partial Fourier transformation. In the case where $A = C_0(G)$ for a locally compact group G, the formula above reduces to $(Vf)(x, y) = f(xy, y)$ for all $f \in L^2(G \times G)$. The same formulas can be used to associate pseudo-multiplicative unitaries to measured (quantum) groupoids; see [39] and [64].

Conversely, if V is a multiplicative unitary for a Hilbert space H that is regular [1] or modular [73], the spaces $A_V = [\{(\omega \otimes \iota)(V) : \omega \in \mathcal{L}(H)_*\}]$ and $\hat{A}_V = [\{(\iota \otimes \omega)(V) : \omega \in \mathcal{L}(H)_*\}]$ are C^*-subalgebras of $\mathcal{L}(H)$ and carry comultiplications Δ_V and $\hat{\Delta}_V$ given by $\Delta_V \colon a \mapsto V(a \otimes 1)V^*$ and $\hat{\Delta}_V \colon \hat{a} \mapsto V^*(1 \otimes a)V$, respectively. Here, $[-]$ denotes the norm closure, $\mathcal{L}(H)_*$ the space of normal functionals on $\mathcal{L}(H)$, and $\omega \otimes \iota$ and $\iota \otimes \omega$ certain slice maps. For the multiplicative unitary associated to a locally compact group G or quantum group, one recovers $C_0(G)$ and $C_r^*(G)$ or the initial quantum group and its generalized Pontryagin dual, respectively. Corresponding results hold for pseudo-multiplicative unitaries; see [39].

The article [I.2] introduces C^*-pseudo-multiplicative unitaries for the study of quantum groupoids in the setting of C^*-algebras, refining the definition and some of the constructions outlined above on the basis of the concepts developed in [I.1]:

(1) We define a C^*-pseudo-multiplicative unitary to be a unitary $V\colon H_{\hat{\beta}} \otimes {}_{\alpha}H \to H_{\alpha} \otimes {}_{\beta}H$, where H is a Hilbert space with compatible left or right C^*-module structures $\alpha, \beta, \hat{\beta}$ relative to some triple $(\mathfrak{H}, B, B^{\mathrm{op}})$ as in §I.1, satisfying

$$(\dagger)\qquad V[|\alpha\rangle_1\alpha] = [|\alpha\rangle_2\alpha],\ V[|\hat{\beta}\rangle_2\beta] = [|\hat{\beta}\rangle_1\beta],\ V[|\hat{\beta}\rangle_2\hat{\beta}] = [|\alpha\rangle_2\hat{\beta}],\ V[|\beta\rangle_1\alpha] = [\beta\rangle_1\beta]$$

in $\mathcal{L}(\mathfrak{H}, H_{\alpha} \otimes {}_{\beta}H)$ and $V_{12}V_{13}V_{23} = V_{23}V_{12}$. Here, the relations (†) are necessary for the V_{ij} to be well-defined. This definition refines or generalizes the variants and generalizations of multiplicative unitaries considered in [6], [46], [58], [64].

(2) In [I.2], [I.3] and [II.2], we construct C^*-pseudo-multiplicative unitaries for locally compact Hausdorff groupoids, compact C^*-quantum groupoids, and proper dynamical quantum groups with integrals, respectively. In each of these cases, the main difficulty is to prove that the relations (†) hold.

(3) We adapt the construction of the spaces $A_V, \hat{A}_V$ and maps $\Delta_V, \hat{\Delta}_V$ to a general C^*-pseudo-multiplicative unitary V. In the regular case which includes the examples mentioned in (2), the former are C^*-algebras and the latter take values in the fiber products $(A_V)_{\alpha} * {}_{\beta}(A_V)$ and $(\hat{A}_V)_{\hat{\beta}} * {}_{\alpha}(\hat{A}_V)$ defined in §I.1.

(4) Extending corresponding definitions and results from [1], we associate in [I.2] and in the extended preprint [60] to every C^*-pseudo-multiplicative unitary Fourier algebras and monoidal categories of (co-)representations.

I.3 Compact C^*-quantum groupoids. The theory of locally compact quantum groups [36] and measured quantum groupoids [39] suggest that a quantum groupoid in the setting of C^*-algebras should be given by

- C^*-algebras B, A with commuting non-degenerate embeddings $r\colon B \to M(A)$ and $s\colon B^{\mathrm{op}} \to M(A)$,
- a comultiplication Δ on A which takes values in a fiber product $A * A$ formed with respect to s and r and which satisfies certain density conditions, and
- weights ϕ and ψ from A to B that are left- or right-invariant with respect to Δ,
- a KMS-weight μ on B such that the compositions $\nu := \mu \circ \phi$ and $\nu^{-1} := \mu \circ \psi$ are related by a modular element δ and satisfy a KMS-condition.

To develop a theory in this generality, a better understanding of unbounded operator-valued weights on C^*-algebras would be needed.

Building on the theory developed in [I.1] and [I.2], the article [I.3] introduces quantum groupoids in the setting of C^*-algebras that are compact in the sense that A and B are unital and ϕ, ψ and μ are bounded. It assumes in addition the existence of a unitary antipode, that is,

- an anti-automorphism R on A which satisfies $R \circ s = r$, $\psi \circ R = \phi$ and is determined by a certain strong invariance condition.

In the theory of locally compact quantum groups and measured quantum groupoids, the existence of such a unitary antipode is a result and not an axiom.

Examples of compact quantum groupoids in the sense of [I.3] include the algebras $A = C(G)$ and $\hat{A} = C^*_r(G)$ associated to a locally compact Hausdorff groupoid G, where G has to be compact or étale with compact space of units G^0, respectively. There, the algebra B is $C(G^0)$, the weights ϕ and ψ are given by fibrewise integration along Haar systems or by the restriction of functions in $C_c(G) \subseteq C_r^*(G)$ to G^0, respectively, and R is induced by the groupoid inversion.

The main results of [I.3] are as follows. For every compact quantum groupoid, we construct (1) a regular C^*-pseudo-fundamental unitary and, using this unitary, (2) a generalized Pontryagin dual and (3) a completion in the form of a measured quantum groupoid. Furthermore, we prove (4) essential uniqueness of the weights ϕ and ψ and (5) triviality of the modular element δ after modification of μ.

I.4 Coactions of Hopf C^*-bimodules. Coactions of quantum groupoids generalize coactions of quantum groups and actions of groupoids and were studied in various settings, including that of weak Hopf algebras or finite quantum groupoids [53], [54], Hopf algebroids or algebraic quantum groupoids [7], [29], and Hopf-von Neumann bimodules or measured quantum groupoids [20], [21]. In these settings, each coaction of a quantum groupoid on some algebra gives rise to a crossed product algebra that carries a coaction of the generalized Pontryagin dual. Iterating this construction, one obtains a bidual coaction that, under suitable assumptions, is Morita equivalent to the initial one.

The article [I.4] establishes a similar duality for coactions of quantum groupoids on C^*-algebras within the framework developed in [I.1] and [I.2], and answers question (2) in §I. It is based on the article [1] of Baaj and Skandalis, which develops the corresponding duality for coactions of quantum groups on C^*-algebras, and results in the thesis [57]. The main definitions and results are as follows.

(1) We start with a regular C^*-pseudo-multiplicative unitary $V\colon H_{\hat{\beta}}\otimes_{\alpha} H \to H_{\alpha}\otimes_{\beta} H$ with associated Hopf C^*-bimodules (A_V, Δ_V) and $(\hat{A}_V, \hat{\Delta}_V)$ (see §I.2), and a symmetry U on H satisfying a few relations. Examples arise from locally compact Hausdorff groupoids and compact C^*-quantum groupoids, where U is related to the groupoid inversion or the product of the modular conjugation and the unitary antipode, respectively.
(2) Given a coaction of (A_V, Δ_V), that is, a C^*-algebra C on a left C^*-module (K, γ) with a $*$-homomorphism $\delta\colon C \to C_{\gamma} * {}_{\beta}(A_V)$ satisfying $(\delta * \iota) \circ \delta = (\iota * \Delta_V) \circ \delta$, we obtain a crossed product $C \rtimes_{\delta} \hat{A}_V = [\delta(C)(1 \otimes \hat{A}_V] \subseteq \mathcal{L}(K_{\gamma} \otimes {}_{\beta}H)$ with a dual coaction $\hat{\delta}$ of $(\hat{A}_V, \hat{\Delta}_V)$, given by $\delta(c)(1 \otimes \hat{a}) \mapsto (\delta(c) \otimes 1)(1 \otimes \hat{\Delta}_V(\hat{a}))$. The corresponding construction for coactions of $(\hat{A}_V, \hat{\Delta}_V)$ involves the symmetry U.
(3) Under a few natural assumptions on δ, we obtain an isomorphism between the bidual coaction $\hat{\hat{\delta}}$ on $C \rtimes_{\delta} \hat{A}_V \rtimes_{\hat{\delta}} A_V$ and a stabilization of the coaction δ on C.
(4) Let G be a locally compact Hausdorff groupoid G. We show that then coactions of $C_0(G)$ essentially correspond to actions of G on bundles of C^*-algebras over G^0, and that each Fell bundle on G, which is a bundle of Banach spaces over G with a multiplication and involution that cover the multiplication and inversion on G, yields a coaction of $C_r^*(G)$ on the reduced convolution C^*-algebra. If G is étale, coactions of $C_r^*(G)$ essentially correspond to Fell bundles on G this way.

II Examples of dynamical quantum groups and the passage to operator algebras

For every group and groupoid, the associated category of representations is symmetric in the sense that for any two representations u and v, the flip Σ on the underlying vector spaces or bundles yields isomorphisms $u \otimes v \leftrightarrows v \otimes u$. For non-commutative quantum group(oid)s, the categories of corepresentations are no longer symmetric but may carry a braiding which is a coherent family of isomorphisms $\hat{R}_{u,v} \colon u \otimes v \to v \otimes u$, where each automorphism $\hat{R} = \hat{R}_{u,u}$ satisfies the braid relation $\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$; see, for example [12] or [32].

Conversely, the FRT-construction of Faddeev, Reshetikhin and Takhtajan associates to each R-matrix, that is, each endomorphism R of $\mathbb{C}^n \otimes \mathbb{C}^n$ satisfying the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, a bialgebra and possibly a quantum group with a fundamental corepresentation u and a braiding such that $R = \hat{R}_{u,u}\Sigma$ [48].

For example, this construction yields the Hopf $*$-algebra $\mathcal{O}(\mathrm{SU}_q(2))$ of Soibelman, Vaksman [62] and Woronowicz [72] which is probably the most fundamental and best-studied quantum group. This is the universal algebra generated by the entries of a matrix

$$(\ddagger) \qquad u = (u_{ij})_{i,j} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

with the single condition that u is unitary. The comultiplication is given by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ so that u is a corepresentation.

The FRT-construction was extended by Etingof and Varchenko to dynamical R-matrices arising in mathematical physics; see [23], [24]. Instead of a bialgebra or quantum group, this generalization yields a dynamical quantum group with a corepresentation on a dynamical vector space. More precisely,

- "dynamical" refers to a fixed action of a group Γ on a commutative algebra B,
- a dynamical vector space is a Γ-graded B-bimodule $V = \bigoplus_\gamma V_\gamma$, where $vb = \gamma(b)v$ if $v \in V_\gamma$,
- a dynamical quantum group over (B, Γ) is a quantum groupoid (B, A, r, s, Δ), where
  - B is commutative and equipped with an action of a group Γ as above,
  - A is graded by $\Gamma \times \Gamma$ such that $ar(b)s(b') = r(\gamma(b))s(\gamma'(b'))a$ if $a \in A_{\gamma,\gamma'}$,
  - Δ maps A to $A \tilde{\otimes} A = \big(\sum_{\gamma,\gamma',\gamma''} A_{\gamma,\gamma'} \otimes A_{\gamma',\gamma''}\big)/(s(b) \otimes 1 - 1 \otimes r(b) : b \in B)$.

Applying this construction to a trigonometric dynamical R-matrix, Koelink and Rosengren obtained in [33] a dynamical analogue of the quantum group $\mathcal{O}(\mathrm{SU}_q(2))$, where B is the field of meromorphic functions on the complex plane, the group $\Gamma = \mathbb{Z}$ acts by shifts, and A is generated by $r(B), s(B)$ and the entries of a corepresentation u as in $(\ddagger)$. Here, the relations imposed on the generators involve special meromorphic functions and the matrix u is longer unitary.

II.1 Free dynamical quantum groups and the dynamical $\mathrm{SU}_q(2)$. The main results of the article [II.1] are as follows.

(1) The non-dynamical quantum group $\mathcal{O}(\mathrm{SU}_q(2))$ belongs to the family of free orthogonal quantum groups. The latter were introduced along with free unitary quantum groups by Wang [70] and Wang and Van Daele [66] and have been studied intensely; see, for example, [2], [3], [4], [15], [69]. Briefly, the free orthogonal quantum group with parameter $F \in \mathrm{GL}_n(\mathbb{C})$ is the universal quantum group (A, Δ) with a matrix $u \in M_n(A)$ such that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ and $u = F\bar{u}F^{-1}$, that is, u is a corepresentation of (A, Δ) and F intertwines u and its conjugate $\bar{u}$.

The article [II.1] introduces dynamical analogues of the free orthogonal and free unitary quantum groups and shows that the dynamical analogue of $\mathcal{O}(\mathrm{SU}_q(2))$ of Koelink and Rosengren is an example of a free orthogonal dynamical quantum group.

(2) Koelink and Rosengren remarked that in a sense, the dynamical analogue of $\mathcal{O}(\mathrm{SU}_q(2))$ contains the non-dynamical $\mathcal{O}(\mathrm{SU}_q(2))$, $\mathcal{O}(\mathrm{SU}_{q^{-1}}(2))$ and further quantum groups as limit cases. In the article [II.1], we make this precise in three steps. First, we extend the assignment that associates to a fixed dynamics (B, Γ) the category of dynamical quantum groups over (B, Γ) to a functor that associates to each equivariant homomorphism π of commutative algebras a base change functor π_* on dynamical quantum groups. Second, we refine the definition of the dynamical analogue of $\mathcal{O}(\mathrm{SU}_q(2))$, replacing the field B of meromorphic functions by a smaller algebra B'. Finally, we show that the limit cases above correspond to base changes, applied to the refined analogue of $\mathcal{O}(\mathrm{SU}_q(2))$ and certain homomorphisms from B' to $\mathbb{C}$.

(3) The free orthogonal and free unitary quantum groups of Wang and Van Daele can easily be defined on the level of universal C^*-algebras. In [II.1], we show that the same is possible for their dynamical analogues. The main step is to construct a C^*-algebraic analogue of the product $A\tilde{\otimes}A$ which appears as the target of the comultiplication.

II.2 Measured quantum groupoids and proper dynamical quantum groups. To every quantum group in the algebraic setting that has a positive left- or right-invariant functional, that is, an analogue of a left or right Haar measure, one can associate a locally compact quantum group in the setting of operator algebras. The basic idea is to use the GNS-construction for the functional. To show that this construction yields bounded operators and to extend the comultiplication, however, one needs a multiplicative unitary, see also §I.2.

For quantum groupoids, the connection between the algebraic and the operator-algebraic setting had only been studied in the finite-dimensional case [43], [65]. In the article [II.2], we carry the construction above over to dynamical quantum groups that are compact or, borrowing terminology from groupoids, proper in a certain sense.

The motivating example for the construction in [II.2] is the dynamical analogue (B, A, r, s, Δ) of $\mathcal{O}(\mathrm{SU}_q(2))$ mentioned above. Studying its representation theory, Koelink and Rosengren obtained in [33] a Peter-Weyl decomposition of A as a direct sum of matrix coefficients of irreducible corepresentations. The subspace of A corresponding to the trivial corepresentation is $r(B)s(B) \cong B \otimes B$, and the projection onto this subspace is bi-invariant with respect to the comultiplication in a suitable sense.

The main ideas and results of the article [II.2] are as follows.

(1) We fix a dynamical quantum group that is equipped with a left- and a right-invariant map $\phi, \psi \colon A \to B$ and a functional $\mu \colon B \to \mathbb{C}$ that is quasi-invariant with respect to the action of Γ and satisfies $\mu \circ \phi = \mu \circ \psi$. In the case of the

dynamical analogue of $\mathcal{O}(\mathrm{SU}_q(2))$, suitable maps ϕ and ψ can be obtained from the bi-invariant projection $A \to r(B)s(B)$. We then establish the existence of a modular automorphism θ on A satisfying $\nu(aa') = \nu(a'\theta(a))$ for all $a, a' \in A$, where $\nu := \mu \circ \phi = \mu \circ \psi$.

(2) We assume that μ and ν are positive and faithful. Then they yield natural inner products on B and A and corresponding Hilbert space completions $\mathfrak{H}$ and H. We furthermore assume that left multiplication on $B \subseteq \mathfrak{H}$ extends to a representation $B \to \mathcal{L}(\mathfrak{H})$. Then the second main result is the existence of C^*-pseudo-multiplicative unitaries $V\colon H_\alpha \otimes_{\hat{\beta}} H \to H_\beta \otimes_\alpha H$ and $W\colon H_\beta \otimes_\alpha H \to H_\alpha \otimes_{\hat{\beta}} H$, where, roughly, the subspaces $\alpha, \beta, \hat{\beta} \subseteq \mathcal{L}(\mathfrak{H}, H)$ are spanned by operators of the form $b \mapsto r(b)a$, $b \mapsto s(b)a$ or $b \mapsto ar(b)$ associated to elements $a \in A$, and V and W^* are suitable closures of the maps $a \otimes a' \mapsto \Delta(a)(1 \otimes a')$ and $a \otimes a' \mapsto \Delta(a')(a \otimes 1)$. For some background, see §I.1 and §I.2.

(3) Using the C^*-pseudo-multiplicative unitaries V and W, we obtain completions of the dynamical quantum group and a generalized Pontryagin dual on the level of C^*-algebras and von Neumann algebras. In particular, we show that the GNS-representation for ν is via bounded operators and that the comultiplication extends to the completion.

(4) Under mild assumptions on the algebra B and the functional μ, we lift the maps ϕ, ψ to the level of von Neumann algebras and obtains a measured quantum groupoid in the sense of Enock and Lesieur [20], [39].

References

[1] S. Baaj and G. Skandalis. Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres. *Ann. Sci. École Norm. Sup. (4)*, 26(4):425–488, 1993.

[2] T. Banica. Théorie des représentations du groupe quantique compact libre $\mathrm{O}(n)$. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(3):241–244, 1996.

[3] T. Banica. Le groupe quantique compact libre $\mathrm{U}(n)$. *Comm. Math. Phys.*, 190(1):143–172, 1997.

[4] T. Banica and R. Speicher. Liberation of orthogonal Lie groups. *Adv. Math.*, 222(4):1461–1501, 2009.

[5] E. Blanchard. Tensor products of $C(X)$-algebras over $C(X)$. *Astérisque*, (232):81–92, 1995. Recent advances in operator algebras (Orléans, 1992).

[6] É. Blanchard. Déformations de C^*-algèbres de Hopf. *Bull. Soc. Math. France*, 124(1):141–215, 1996.

[7] G. Böhm. Galois theory for Hopf algebroids. *Ann. Univ. Ferrara Sez. VII (N.S.)*, 51:233–262, 2005.

[8] G. Böhm. Hopf algebroids. In *Handbook of algebra. Vol. 6*, volume 6 of *Handb. Algebr.*, pages 173–235. Elsevier/North-Holland, Amsterdam, 2009.

[9] G. Böhm, F. Nill, and K. Szlachányi. Weak Hopf algebras. I. Integral theory and C^*-structure. *J. Algebra*, 221(2):385–438, 1999.

[10] G. Böhm and K. Szlachányi. Hopf algebroid symmetry of abstract Frobenius extensions of depth 2. *Comm. Algebra*, 32(11):4433–4464, 2004.

[11] G. Böhm and K. Szlachányi. Hopf algebroids with bijective antipodes: axioms, integrals, and duals. *J. Algebra*, 274(2):708–750, 2004.

[12] V. Chari and A. Pressley. *A guide to quantum groups.* Cambridge University Press, Cambridge, 1994.

[13] A. Connes. *Noncommutative geometry.* Academic Press Inc., San Diego, CA, 1994.

[14] A. Connes and H. Moscovici. Rankin-Cohen brackets and the Hopf algebra of transverse geometry. *Mosc. Math. J.*, 4(1):111–130, 311, 2004.

[15] S. Curran and R. Speicher. Quantum invariant families of matrices in free probability. *J. Funct. Anal.*, 261(4):897–933, 2011.

[16] B. Day and R. Street. Monoidal bicategories and Hopf algebroids. *Adv. Math.*, 129(1):99–157, 1997.

[17] B. Day and R. Street. Quantum categories, star autonomy, and quantum groupoids. In *Galois theory, Hopf algebras, and semiabelian categories*, volume 43 of *Fields Inst. Commun.*, pages 187–225. Amer. Math. Soc., Providence, RI, 2004.

[18] J. Donin and A. Mudrov. Quantum groupoids and dynamical categories. *J. Algebra*, 296(2):348–384, 2006.

[19] M. Enock. Inclusions of von Neumann algebras and quantum groupoïds. III. *J. Funct. Anal.*, 223(2):311–364, 2005.

[20] M. Enock. Measured quantum groupoids in action. *Mém. Soc. Math. Fr. (N.S.)*, (114):ii+150 pp. (2009), 2008.

[21] M. Enock. The unitary implementation of a measured quantum groupoid action. *Ann. Math. Blaise Pascal*, 17(2):233–302, 2010.

[22] P. Etingof, D. Nikshych, and V. Ostrik. On fusion categories. *Ann. of Math. (2)*, 162(2):581–642, 2005.

[23] P. Etingof and O. Schiffmann. *Lectures on quantum groups*. Lectures in Mathematical Physics. International Press, Somerville, MA, second edition, 2002.

[24] P. Etingof and A. Varchenko. Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups. *Comm. Math. Phys.*, 196(3):591–640, 1998.

[25] T. Falcone. L^2-von Neumann modules, their relative tensor products and the spatial derivative. *Illinois J. Math.*, 44(2):407–437, 2000.

[26] U. Haagerup. Operator-valued weights in von Neumann algebras. I. *J. Funct. Anal.*, 32(2):175–206, 1979.

[27] U. Haagerup. Operator-valued weights in von Neumann algebras. II. *J. Funct. Anal.*, 33(3):339–361, 1979.

[28] P. H. Hai. Tannaka-Krein duality for Hopf algebroids. *Israel J. Math.*, 167:193–225, 2008.

[29] L. Kadison. Hopf algebroids and H-separable extensions. *Proc. Amer. Math. Soc.*, 131(10):2993–3002 (electronic), 2003.

[30] L. Kadison. Skew Hopf algebras, irreducible extensions and the Π-method. *Münster J. Math.*, 2:183–199, 2009.

[31] L. Kadison and K. Szlachányi. Bialgebroid actions on depth two extensions and duality. *Adv. Math.*, 179(1):75–121, 2003.

[32] A. Klimyk and K. Schmüdgen. *Quantum groups and their representations*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.

[33] E. Koelink and H. Rosengren. Harmonic analysis on the SU(2) dynamical quantum group. *Acta Appl. Math.*, 69(2):163–220, 2001.

[34] J. Kustermans. Locally compact quantum groups in the universal setting. *Internat. J. Math.*, 12(3):289–338, 2001.

[35] J. Kustermans and S. Vaes. Locally compact quantum groups. *Ann. Sci. École Norm. Sup. (4)*, 33(6):837–934, 2000.

[36] J. Kustermans and S. Vaes. Locally compact quantum groups in the von Neumann algebraic setting. *Math. Scand.*, 92(1):68–92, 2003.

[37] J. Kustermans and A. van Daele. C^*-algebraic quantum groups arising from algebraic quantum groups. *Internat. J. Math.*, 8(8):1067–1139, 1997.

[38] P.-Y. Le Gall. Théorie de Kasparov équivariante et groupoïdes. I. *K-Theory*, 16(4):361–390, 1999.

[39] F. Lesieur. Measured quantum groupoids. *Mém. Soc. Math. Fr. (N.S.)*, (109):iv+158 pp. (2008), 2007.

[40] J.-H. Lu. Hopf algebroids and quantum groupoids. *Internat. J. Math.*, 7(1):47–70, 1996.

[41] M. B. McCurdy. Graphical methods for Tannaka duality of weak bialgebras and weak Hopf algebras. *Theory Appl. Categ.*, 26:No. 9, 233–280, 2012.

[42] D. Nikshych, V. Turaev, and L. Vainerman. Invariants of knots and 3-manifolds from quantum groupoids. In *Proceedings of the Pacific Institute for the Mathematical Sciences Workshop "Invariants of Three-Manifolds" (Calgary, AB, 1999)*, volume 127, pages 91–123, 2003.

[43] D. Nikshych and L. Vainerman. Algebraic versions of a finite-dimensional quantum groupoid. In *Hopf algebras and quantum groups (Brussels, 1998)*, volume 209 of *Lecture Notes in Pure and Appl. Math.*, pages 189–220. Dekker, New York, 2000.

[44] D. Nikshych and L. Vainerman. A characterization of depth 2 subfactors of II_1 factors. *J. Funct. Anal.*, 171(2):278–307, 2000.

[45] D. Nikshych and L. Vainerman. Finite quantum groupoids and their applications. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 211–262. Cambridge University Press, Cambridge, 2002.

[46] M. O'uchi. Pseudo-multiplicative unitaries on Hilbert C^*-modules. *Far East J. Math. Sci. (FJMS)*, Special Volume, Part II:229–249, 2001.

[47] H. Pfeiffer. Tannaka-Kreĭn reconstruction and a characterization of modular tensor categories. *J. Algebra*, 321(12):3714–3763, 2009.

[48] N. Y. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev. Quantization of Lie groups and Lie algebras. *Algebra i Analiz*, 1(1):178–206, 1989.

[49] J.-L. Sauvageot. Sur le produit tensoriel relatif d'espaces de Hilbert. *J. Operator Theory*, 9(2):237–252, 1983.

[50] J.-L. Sauvageot. Produits tensoriels de Z-modules et applications. In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, volume 1132 of *Lecture Notes in Mathematics*, pages 468–485. Springer, Berlin, 1985.

[51] P. Schauenburg. Weak Hopf algebras and quantum groupoids. In *Noncommutative geometry and quantum groups (Warsaw, 2001)*, volume 61 of *Banach Center Publ.*, pages 171–188. Polish Acad. Sci., Warsaw, 2003.

[52] P. M. Sołtan and S. L. Woronowicz. From multiplicative unitaries to quantum groups. II. *J. Funct. Anal.*, 252(1):42–67, 2007.

[53] K. Szlachányi. Weak Hopf algebras. In *Operator algebras and quantum field theory (Rome, 1996)*, pages 621–632. Int. Press, Cambridge, MA, 1997.

[54] K. Szlachányi. Galois actions by finite quantum groupoids. In *Locally compact quantum groups and groupoids (Strasbourg, 2002)*, volume 2 of *IRMA Lect. Math. Theor. Phys.*, pages 105–125. de Gruyter, Berlin, 2003.

[55] M. Takeuchi. Groups of algebras over $A \otimes \overline{A}$. *J. Math. Soc. Japan*, 29(3):459–492, 1977.

[56] T. Tannaka. *Sôtai genri.* Iwanami Shoten, Tokyo, 1951.

[57] T. Timmermann. *Pseudo-multiplicative unitaries and pseudo-Kac systems on C^*-modules.* PhD thesis, Westfälische Wilhelms-Universität Münster, 2005. Available at `www.math.uni-muenster.de/sfb/about/publ/heft394.ps`.

[58] T. Timmermann. Pseudo-multiplicative unitaries on C^*-modules and Hopf C^*-families. *J. Noncomm. Geom.*, 1:497–542, 2007.

[59] T. Timmermann. *An invitation to quantum groups and duality.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[60] T. Timmermann. C^*-pseudo-multiplicative unitaries and Hopf C^*-bimodules. Technical report, arXiv.0908.1850, 2009.

[61] T. Timmermann and A. Van Daele. Regular multiplier hopf algebroids. basic theory and examples. 35 pages, in preparation.

[62] L. L. Vaksman and Y. S. Soibelman. An algebra of functions on the quantum group SU(2). *Funktsional. Anal. i Prilozhen.*, 22(3):1–14, 96, 1988.

[63] J.-M. Vallin. Bimodules de Hopf et poids opératoriels de Haar. *J. Operator Theory*, 35(1):39–65, 1996.

[64] J.-M. Vallin. Unitaire pseudo-multiplicatif associé à un groupoïde. Applications à la moyennabilité. *J. Operator Theory*, 44(2):347–368, 2000.

[65] J.-M. Vallin. Groupoïdes quantiques finis. *J. Algebra*, 239(1):215–261, 2001.

[66] A. van Daele and S. Wang. Universal quantum groups. *Internat. J. Math.*, 7(2):255–263, 1996.

[67] A. Van Daele and S. Wang. Weak multiplier hopf algebras. preliminaries, motivation and basic examples. In P. M. Soltan and W. Pusz, editors, *Operator Algebras and Quantum Groups*, volume 98

of *Banach Center Publ.*, pages 367–415. Polish Academy of Sciences, Polish Academy of Sciences, Institute of Mathematics, Warsaw, 2012.

[68] A. Van Daele and S. Wang. Weak multiplier Hopf algebras. The main theory. Technical report, Preprint University of Leuven (Belgium) and Southeast University of Nanjing (China), 2012. arXiv:1210.4395.

[69] C. Voigt. The Baum-Connes conjecture for free orthogonal quantum groups. *Adv. Math.*, 227(5):1873–1913, 2011.

[70] S. Wang. *General constructions of compact quantum groups*. PhD thesis, University of California, Berkeley, 1993.

[71] A. Weil. *L'intègration dans les groupes topologiques et ses applications*. Actual. Sci. Ind., no. 869. Hermann et Cie., Paris, 1940. [This book has been republished by the author at Princeton, N. J., 1941.].

[72] S. L. Woronowicz. Twisted SU(2) group. An example of a noncommutative differential calculus. *Publ. Res. Inst. Math. Sci.*, 23(1):117–181, 1987.

[73] S. L. Woronowicz. From multiplicative unitaries to quantum groups. *Internat. J. Math.*, 7(1):127–149, 1996.

[74] P. Xu. Quantum groupoids. *Comm. Math. Phys.*, 216(3):539–581, 2001.

[75] T. Yamanouchi. Duality for actions and coactions of measured groupoids on von Neumann algebras. *Mem. Amer. Math. Soc.*, 101(484):vi+109, 1993.

APPENDIX I.1

THE RELATIVE TENSOR PRODUCT AND A MINIMAL FIBER PRODUCT IN THE SETTING OF C^*-ALGEBRAS

THOMAS TIMMERMANN

J. Operator Theory *68(2):101-140, 2012.*

Abstract. We introduce a relative tensor product of C^*-bimodules and a spatial fiber product of C^*-algebras that are analogues of Connes' fusion of correspondences and the fiber product of von Neumann algebras introduced by Sauvageot, respectively. These new constructions form the basis for our approach to quantum groupoids in the setting of C^*-algebras that is published separately.

Contents

2000 *Mathematics Subject Classification.* Primary 46L08; Secondary 46L06, 46L55.
Key words and phrases. Hilbert module, bimodule, relative tensor product, fiber product, quantum groupoid.

1. Introduction

The relative tensor product of Hilbert modules over von Neumann algebras was introduced by Connes in an unpublished manuscript [4], [10], [20] and later used by Sauvageot to define a fiber product of von Neumann algebras relative to a common (commutative) von Neumann subalgebra [21]. These constructions and Haagerups theory of operator-valued weights on von Neumann algebras [12], [13] form the basis for the theory of measured quantum groupoids developed by Enock, Lesieur and Vallin [8], [9], [18], [30], [31].

In this article, we introduce a new notion of a bimodule in the setting of C^*-algebras, construct relative tensor products of such bimodules, and define a fiber product of C^*-algebras represented on such bimodules. These constructions form the basis for a series of articles on quantum groupoids in the setting of C^*-algebras, individually addressing fundamental unitaries [29], axiomatics of the compact case [25], and coactions of quantum groupoids on C^*-algebras [28]. Moreover, our previous approach to quantum groupoids in the setting of C^*-algebras [27] embeds functorially into this new framework [26], and the latter overcomes the serious restrictions of the former one.

Already in the definition of a quantum groupoid, the relative tensor product and a fiber product appear as follows. Roughly, such an object consists of the following ingredients: an algebra B, thought of as the functions on the unit space, an algebra A, thought of as functions on the total space, a homomorphism $r\colon B \to A$ and an antihomomorphism $s\colon B \to A$ corresponding to the range and the source map, and a comultiplication $\Delta\colon B \to A \underset{B}{*} A$ corresponding to the multiplication of the quantum groupoid. Here, $A \underset{B}{*} A$ is a fiber product whose precise definition depends on the class of the algebras involved. In the setting of operator algebras, A acts naturally on some bimodule H and product $A \underset{B}{*} A$ is a certain subalgebra of operators acting on a relative tensor product $H \underset{B}{\otimes} H$. This relative tensor product is important also because it forms the domain or range of the fundamental unitary of the quantum groupoid.

Let us now sketch the problems and constructions studied in this article.

The first problem is the construction of a tensor product $H \underset{B}{\otimes} K$ of modules H, K over some algebra B. In the algebraic setting, $H \underset{B}{\otimes} K$ is simply a quotient of the full tensor product $H \otimes K$. In the setting of von Neumann algebras, H and K are Hilbert spaces, and Connes explained that the right tensor product is not a completion of the algebraic one but something more complicated. If B is commutative and of the form $B = L^\infty(X, \mu)$, then the modules H, K can be disintegrated into two measurable fields of Hilbert spaces in the form $H = \int_X^\oplus H_x d\mu(x)$ and $K = \int_X^\oplus K_x d\mu(x)$, and $H \underset{B}{\otimes} K$ is obtained by taking tensor products of the fibers and integrating again: $H \underset{B}{\otimes} K = \int_X^\oplus H_x \otimes K_x d\mu(x)$. For the situation where B is a C^*-algebra, we propose an approach that is based on the internal tensor product of Hilbert C^*-modules and essentially consists of an algebraic reformulation of Connes' fusion. Central to this approach is a new notion of a bimodule in the setting of C^*-algebras.

The second problem is the construction of a fiber product $A \underset{B}{*} C$ of two algebras A, C relative to a subalgebra B. If B is central in A and the opposite B^{op} is central in C, this fiber product is just a relative tensor product. In the algebraic setting, it coincides with the tensor product of modules; in the setting of operator algebras, it can be obtained via disintegration and a fiberwise tensor product again. This approach was studied by Sauvageot for Neumann algebras [21], and by Blanchard [1] for C^*-algebras.

The case where the subalgebra $B^{(op)}$ is no longer central in A or C is more difficult. In the algebraic setting, the fiber product was introduced by Takeuchi [24] and is, roughly, the largest subalgebra of the relative tensor product $A \underset{B}{\otimes} C$ where componentwise multiplication is still well defined. In the setting of von Neumann algebras, Sauvageot's definition of the fiber product carries over to the general case and takes the form $A \underset{B}{*} C = (A' \underset{B}{\otimes} C')'$, where A and C are represented on Hilbert spaces H and K, respectively, and $A' \underset{B}{\otimes} C'$ acts on Connes' relative tensor product $H \underset{B}{\otimes} K$. Here, it is important to note that $A' \underset{B}{\otimes} C'$ is a completion of an algebraic tensor product spanned by elementary tensors, but in general, $A \underset{B}{*} C$ is not. Similarly, in the setting of C^*-algebras, one can not start from some algebraic tensor product and define the fiber product to be some completion; rather, a new idea is needed. We propose such a new fiber product for C^*-algebras represented on the new class of modules mentioned above. Unfortunately, several important questions concerning this construction remain open, but the applications in [25], [28], [29] already prove its usefulness.

This article is organized as follows.

The introduction ends with a short summary on terminology and some background on Hilbert C^*-modules.

Section 2 is devoted to the relative tensor product in the setting of C^*-algebras. It starts with some motivation, then presents a new notion of modules and bimodules in the setting of C^*-algebras, and finally gives the construction and its formal properties like functoriality, associativity and unitality.

Section 3 introduces a minimal fiber product of C^*-algebras. It begins with an overview and then proceeds to C^*-algebras represented on the class of modules and bimodules introduced in Section 2. The fiber product is first defined and studied for such represented C^*-algebras, including a discussion of functoriality, slice maps, lack of associativity, and unitality. A natural extension to non-represented C^*-algebras is indicated at the end.

Section 4 relates our constructions for the setting of C^*-algebras to the corresponding constructions for the setting of von Neumann algebras. Adapting our constructions to von Neumann algebras, one recovers Connes fusion and Sauvageot's fiber product; moreover, the constructions are related by functors going from the C^*-level to the W^*-level. The section ends with a categorical interpretation of Sauvageot's fiber product.

Section 5 shows that for a commutative base $B = C_0(X)$, the relative tensor product of the new class of modules corresponds to the fiberwise tensor product of continuous Hilbert bundles over X, and the fiber product of represented C^*-algebras is related to the relative tensor product of continuous $C_0(X)$-algebras studied by Blanchard.

We use the following conventions and notation.

Given a category $\mathbf{C}$, we write $A, B \in \mathbf{C}$ to indicate that A, B are objects of $\mathbf{C}$, and denote by $\mathbf{C}(A, B)$ the associated set of morphisms.

Given a subset Y of a normed space X, we denote by $[Y] \subset X$ the closed linear span of Y.

All sesquilinear maps like inner products on Hilbert spaces are assumed to be conjugate-linear in the first component and linear in the second one.

Given a Hilbert space H and an element $\xi \in H$, we define ket-bra operators $|\xi\rangle\colon \mathbb{C} \to H$, $\lambda \mapsto \lambda\xi$, and $\langle\xi| = |\xi\rangle^*\colon H \to \mathbb{C}$, $\xi' \mapsto \langle\xi|\xi'\rangle$.

We shall make extensive use of (right) Hilbert C^*-modules; a standard reference is [16].

Let A and B be C^*-algebras. Given Hilbert C^*-modules E and F over B, we denote by $\mathcal{L}(E, F)$ the space of all adjointable operators from E to F. Let E and F be Hilbert C^*-modules over A and B, respectively, and let $\pi\colon A \to \mathcal{L}(F)$ be a $*$-homomorphism. Then the internal tensor product $E \otimes_\pi F$ is a Hilbert C^*-module over B [16, §4] and the closed linear span of elements $\eta \otimes_\pi \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle\eta \otimes_\pi \xi|\eta' \otimes_\pi \xi'\rangle = \langle\xi|\pi(\langle\eta|\eta'\rangle)\xi'\rangle$ and $(\eta \otimes_\pi \xi)b = \eta \otimes_\pi \xi b$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$. We denote the internal tensor product by "$\varotimes$" and drop the index π if the representation is understood; thus, $E \varotimes F = E \varotimes_\pi F = E \otimes_\pi F$.

We define a *flipped internal tensor product* $F_\pi \varolessthan E$ as follows. We equip the algebraic tensor product $F \odot E$ with an product $\langle\xi \odot \eta|\xi' \odot \eta'\rangle := \langle\xi|\pi(\langle\eta|\eta'\rangle)\xi'\rangle$ and a module structure via $(\xi \odot \eta)b := \xi b \odot \eta$, form the separated completion, and obtain a Hilbert C^*-B-module $F_\pi \varolessthan E$ which is the closed linear span of elements $\xi_\pi \varolessthan \eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle\xi_\pi \varolessthan \eta|\xi'_\pi \varolessthan \eta'\rangle = \langle\xi|\pi(\langle\eta|\eta'\rangle)\xi'\rangle$ and $(\xi_\pi \varolessthan \eta)b = \xi b_\pi \varolessthan \eta$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$. As above, we usually drop the index π and simply write "$\varolessthan$" instead of "${}_\pi\varolessthan$". Evidently, there exists a unitary $\Sigma\colon F \varolessthan E \xrightarrow{\cong} E \varotimes F$, $\eta \varolessthan \xi \mapsto \xi \varotimes \eta$.

Let E_1, E_2 be Hilbert C^*-modules over A, let F_1, F_2 be Hilbert C^*-modules over B with $*$-homomorphisms $\pi_i\colon A \to \mathcal{L}(F_i)$ for $i = 1, 2$, and let $S \in \mathcal{L}(E_1, E_2)$, $T \in \mathcal{L}(F_1, F_2)$ such that $T\pi_1(a) = \pi_2(a)T$ for all $a \in A$. Then there exists a unique operator $S \varotimes T \in \mathcal{L}(E_1 \varotimes F_1, E_2 \varotimes F_2)$ such that $(S \varotimes T)(\eta \varotimes \xi) = S\eta \varotimes T\xi$ for all $\eta \in E_1$, $\xi \in F_1$, and $(S \varotimes T)^* = S^* \varotimes T^*$ [7, Proposition 1.34].

2. The relative tensor product in the setting of C^*-algebras

2.1. **Motivation.** The aim of this section is to construct a relative tensor product of suitably defined left and right modules over a general C^*-algebra B such that i) the construction shares the main properties of the ordinary tensor product of bimodules over rings like functoriality and associativity and ii) the modules admit representations of C^*-algebras that do not commute with the module structures. The latter condition will be needed to construct fiber products of C^*-algebras; see Section 3.

The internal tensor product of Hilbert C^*-modules meets condition i) but not ii) because C^*-algebras represented on such modules necessarily commute with the right module structure. An approach to quantum groupoids based on the internal tensor product was developed in [27] but remained restricted to very special cases.

What we are looking for is an analogue of Connes' fusion of correspondences. Here, B is a von Neumann algebra, and left and right modules are Hilbert spaces equipped with suitable representation or antirepresentation of B, respectively. The relative tensor product of a right module H and a left module K is then constructed as follows. Choose a normal, semi-finite, faithful (n.s.f.) weight μ on B, construct a B-valued inner product $\langle \cdot | \cdot \rangle_\mu$ on the dense subspace $H_0 \subseteq H$ of all bounded vectors, and define $H \underset{\mu}{\otimes} K$ to be the separated completion of the algebraic tensor product $H_0 \odot K$ with respect to the sesquilinear form given by $\langle \xi \odot \eta | \xi' \odot \eta' \rangle = \langle \eta | \langle \xi | \xi' \rangle_\mu \eta' \rangle$. The definition of bounded vectors involves the GNS-space $\mathfrak{H} := H_\mu$ for μ which — by Tomita-Takesaki theory — is bimodule over B, and each bounded vector $\xi \in H_0$ gives rise to a map $L(\xi) \in \mathcal{L}(\mathfrak{H}_B, H_B)$ of right B-modules such that $\langle \xi | \xi' \rangle_\mu = L(\xi)^* L(\xi') \in B \subseteq \mathcal{L}(\mathfrak{H})$.

Example 2.1. *Assume that $B = L^\infty(X, \mu)$ for some nice measure space (X, μ), and denote the weight on B given by integration by μ as well. Then $\mathfrak{H} = L^2(X, \mu)$, and we can disintegrate H and K into measurable fields $(H_x)_x$ and $(K_x)_x$ of Hilbert spaces over X such that $H \cong \int_X^\oplus H_x d\mu(x)$ and $K \cong \int_X^\oplus K_x d\mu(x)$. Each vector ξ of H or K corresponds to a measurable section $x \mapsto \xi(x)$ with square-integrable norm function $|\xi| \colon x \mapsto \|\xi_x\|$, and is bounded with respect to μ if and only if this norm function is essentially bounded. Then for all $\xi, \xi' \in H_0$, $x \in X$, $\eta, \eta' \in K$,*

$$\langle \xi | \xi' \rangle_\mu(x) = \langle \xi(x) | \xi'(x) \rangle_{H_x},$$

$$\langle \xi \odot \eta | \xi' \odot \eta' \rangle = \int_X \langle \xi(x) | \xi'(x) \rangle \langle \eta(x) | \eta'(x) \rangle d\mu(x),$$

and $H \underset{\mu}{\otimes} K \cong \int_X^\oplus H_x \otimes K_x d\mu(x)$. Note that the sesquilinear form above need not extend to $H \odot K$ because the integrand need not be in $L^1(X, \mu)$ for arbitrary $\xi, \xi' \in H$ and $\eta, \eta' \in K$.

For our purpose, the following algebraic description of $H \underset{\mu}{\otimes} K$ is useful. This relative tensor product can be identified with the separated completion of algebraic tensor product

$$\mathcal{L}(\mathfrak{H}_B, H_B) \odot \mathfrak{H} \odot \mathcal{L}({}_B\mathfrak{H}, {}_BK) \tag{2.1}$$

with respect to the sesquilinear form

$$\langle S \odot \zeta \odot T | S' \odot \zeta' \odot T' \rangle = \langle \zeta | S^* S' T^* T' \zeta' \rangle = \langle \zeta | T^* T' S^* S' \zeta' \rangle,$$

where $\mathcal{L}(\mathfrak{H}_B, H_B)$ and $\mathcal{L}({}_B\mathfrak{H}, {}_BK)$ are all bounded maps of right or left B-modules, respectively. We adapt this definition to the setting of C^*-algebras, making the following modifications:

(A) The construction above depends on the choice of some n.s.f. weight μ or, more precisely, the triple $(H_\mu, \pi_\mu(B), \pi_\mu(B)')$, but any other μ yields a triple which is unitarily equivalent. In the setting of C^*-algebras, such a canonical triple does not exist but has to be chosen.

(B) The module structure of H and K can equivalently be described in terms of (anti)representations of B or in terms of the spaces $\mathcal{L}(\mathfrak{H}_B, H_B)$ and $\mathcal{L}({}_B\mathfrak{H}, {}_BK)$. In the setting of C^*-algebras, this equivalence breaks down, and we shall make suitable closed subspaces of intertwiners the primary object. In the commutative case, a representation corresponds to a measurable field of Hilbert spaces, and the subspaces fix a continuous structure.

(C) If H and K are bimodules, then so is $H \underset{\mu}{\otimes} K$. Here, a bimodule structure on H is given by the additional choice of a representation of some von Neumann algebra A that commutes with the antirepresentation of B or, equivalently, satisfies $A\mathcal{L}(\mathfrak{H}_B, H_B) = \mathcal{L}(\mathfrak{H}_B, H_B)$. If we pass to C^*-algebras, then commutation is too weak, and we shall adopt the second condition, where $\mathcal{L}(\mathfrak{H}_B, H_B)$ is replaced by the subspace of intertwiners mentioned above.

2.2. **Modules and bimodules over C^*-bases.** Observation (A) leads us to adopt the following terminology.

Definition 2.2. *A C^*-*base $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ *consists of a Hilbert space $\mathfrak{H}$ and commuting nondegenerate C^*-algebras $\mathfrak{B}, \mathfrak{B}^\dagger \subseteq \mathcal{L}(\mathfrak{K})$, respectively. The* opposite *of $\mathfrak{b}$ is the C^*-base $\mathfrak{b}^\dagger := (\mathfrak{K}, \mathfrak{B}^\dagger, \mathfrak{B})$. A C^*-base $(\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ is* equivalent *to $\mathfrak{b}$ if $\mathrm{Ad}_V(\mathfrak{A}) = \mathfrak{B}$ and $\mathrm{Ad}_V(\mathfrak{A}^\dagger) = \mathfrak{B}^\dagger$ for some unitary $V \in \mathcal{L}(\mathfrak{H}, \mathfrak{K})$.*

Clearly, the Hilbert space $\mathbb{C}$ and twice the algebra $\mathbb{C} \equiv \mathcal{L}(\mathbb{C})$ form a trivial C^*-base $\mathfrak{t} = (\mathbb{C}, \mathbb{C}, \mathbb{C})$.

Example 2.3. *Let μ be a proper, faithful KMS-weight on a C^*-algebra A* [15] *with GNS-space H_μ, GNS-representation $\pi_\mu \colon A \to \mathcal{L}(H_\mu)$, modular conjugation $J_\mu \colon H_\mu \to H_\mu$, and opposite GNS-representation $\pi_{\mu^{op}} \colon A^{op} \to \mathcal{L}(H_\mu)$, $a \mapsto J_\mu \pi_\mu(a^*) J_\mu$. Then the triple $(H_\mu, \pi_\mu(A), \pi_{\mu^{op}}(A^{op}))$ is a C^*-base. Its opposite is equivalent to the C^*-base associated to the opposite weight μ^{op} on A^{op}. Indeed, H_μ can be considered as the GNS-space for μ^{op} via the opposite GNS-map $\Lambda_{\mu^{op}} \colon \mathfrak{N}_{\mu^{op}} \to H_\mu$, $a^{op} \mapsto J_\mu \Lambda_\mu(a^*)$, and then $J_{\mu^{op}} \pi_{\mu^{op}}(A^{op}) J_{\mu^{op}} = \pi_\mu(A)$.*

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. We define C^*-modules over $\mathfrak{b}$ as indicated in comment (B).

Definition 2.4. *A C^*-$\mathfrak{b}$-*module $H_\alpha = (H, \alpha)$ *is a Hilbert space H with a closed subspace $\alpha \subseteq \mathcal{L}(\mathfrak{K}, H)$ satisfying $[\alpha\mathfrak{K}] = H$, $[\alpha\mathfrak{B}] = \alpha$, $[\alpha^*\alpha] = \mathfrak{B}$. A* semi-morphism *between C^*-$\mathfrak{b}$-modules H_α and K_β is an operator $T \in \mathcal{L}(H, K)$ satisfying $T\alpha \subseteq \beta$. If additionally $T^*\beta \subseteq \alpha$, we call T a* morphism. *We denote the set of all (semi-)morphisms by $\mathcal{L}_{(s)}(H_\alpha, K_\beta)$.*

Evidently, the class of all C^*-$\mathfrak{a}$-modules forms a category with respect to all semi-morphisms, and a C^*-category in the sense of [11] with respect to all morphisms.

Lemma 2.5. (i) *Let H, K be Hilbert spaces and $I \subseteq \mathcal{L}(H, K)$ such that $[IH] = K$. Then there exists a unique normal, unital $*$-homomorphism $\rho_I \colon (I^*I)' \to (II^*)'$ such that $\rho_I(x)S = Sx$ for all $x \in (I^*I)'$, $S \in I$.*

(ii) *Let H, K, L be Hilbert spaces and $I \subseteq \mathcal{L}(H, K)$, $J \subseteq \mathcal{L}(K, L)$ such that $[IH] = K$, $[JK] = L$, and $J^*JI \subseteq I$. Then $\rho_I((I^*I)') \subseteq (J^*J)'$ and $\rho_J \circ \rho_I = \rho_{JI}$.*

Proof. (i) Uniqueness is evident. Let $x \in (I^*I)'$ and $S_1, \ldots, S_n \in I$, $\xi_1, \ldots, \xi_n \in H$. Since x^*x commutes with each $S_i^* S_j$, the matrix $(S_i^* S_j x^* x)_{i,j} \in M_n(\mathcal{L}(H))$ is dominated by $\|x^*x\|(S_i^* S_j)_{i,j}$, and

$$\|\sum_i S_i x \xi_i\|^2 = \sum_{i,j} \langle \xi_i | S_i^* S_j x^* x \xi_j \rangle \leq \|x\|^2 \sum_{i,j} \langle \xi_i | S_i^* S_j \xi_j \rangle = \|x\|^2 \|\sum_i S_i \xi_i\|^2.$$

Hence, there exists an operator $\rho_I(x) \in \mathcal{L}(K)$ as claimed. One easily verifies that the assignment $x \mapsto \rho_I(x)$ is a $*$-homomorphism. It is normal because $[IH] = K$ and for all $S, T \in I$, $\xi, \eta \in K$, the functional $x \mapsto \langle S\xi | \rho_I(x) T\eta \rangle = \langle \xi | x S^* T \eta \rangle$ is normal.

(ii) Let $x \in (I^*I)'$. Then $\rho_I(x) \in J^*J$ since $S^*T\rho_I(x)R = S^*TRx = \rho_I(x)S^*TR$ for all $S, T \in J$, $R \in I$, and $\rho_{JI}(x) = \rho_J(\rho_I(x))$ because $\rho_{JI}(x)TR = TRx = \rho_J(\rho_I(x))TR$ for all $T \in J$, $R \in I$. □

Lemma 2.6. *Let H_α be a C^*-$\mathfrak{b}$-module.*

(i) *α is a Hilbert C^*-$\mathfrak{B}$-module with inner product $(\xi, \xi') \mapsto \xi^* \xi'$.*
(ii) *There exist isomorphisms $\alpha \otimes \mathfrak{K} \to H$, $\xi \otimes \zeta \mapsto \xi\zeta$, and $\mathfrak{K} \otimes \alpha \to H$, $\zeta \otimes \xi \mapsto \xi\zeta$.*
(iii) *There exists a unique normal, unital and faithful representation $\rho_\alpha \colon \mathfrak{B}' \to \mathcal{L}(H)$ such that $\rho_\alpha(x)(\xi\zeta) = \xi x \zeta$ for all $x \in \mathfrak{B}'$, $\xi \in \alpha$, $\zeta \in \mathfrak{K}$.*
(iv) *Let K_β be a C^*-$\mathfrak{b}$-module and $T \in \mathcal{L}_s(H_\alpha, K_\beta)$. Then $T\rho_\alpha(x) = \rho_\beta(x)T$ for all $x \in \mathfrak{B}'$. If additionally $T \in \mathcal{L}(H_\alpha, K_\beta)$, then left multiplication by T defines an operator in $\mathcal{L}_{\mathfrak{B}}(\alpha, \beta)$, again denoted by T.*

Proof. Assertions (i) and (ii) are obvious, and (iii) follows from the preceding lemma. To prove (iv), let $x \in \mathfrak{B}', \xi \in \alpha, \zeta \in \mathfrak{K}$. Then $T\xi \in \beta$ and $T\rho_\alpha(x)\xi\zeta = T\xi x\zeta = \rho_\beta(x)T\xi\zeta$. □

Example 2.7. *Let Z be a locally compact Hausdorff space, μ a Radon measure on Z of full support, and $\mathcal{H} = (H_z)_z$ a continuous bundle of Hilbert spaces on Z with full support. Then the Hilbert space $\mathfrak{K} = L^2(Z, \mu)$ together with the C^*-algebras $\mathfrak{B} = \mathfrak{B}^\dagger = C_0(Z) \subseteq \mathcal{L}(\mathfrak{K})$ forms a C^*-base. Let $H = \int_Z^\oplus H_z d\mu(z)$ and $\alpha = m(\Gamma_0(\mathcal{H}))$, where for each section $\xi \in \Gamma_0(\mathcal{H})$, the operator $m(\xi) \in \mathcal{L}(\mathfrak{K}, H)$ is given by pointwise multiplication, $m(\xi)f = (\xi(z)f(z))_{z \in Z}$. Then H_α is a C^*-$\mathfrak{b}$-module and $\rho_\alpha \colon \mathfrak{B}' = L^\infty(Z, \mu) \to \mathcal{L}(H)$ is given by pointwise multiplication of sections by functions. Every C^*-$\mathfrak{b}$-module arises in this way from a continuous bundle; see Section 5.*

Let also $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ be a C^*-base. We define C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-bimodules as indicated in (C).

Definition 2.8. *A* C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-module *is a triple ${}_\alpha H_\beta = (H, \alpha, \beta)$, where H is a Hilbert space, (H, α) a C^*-$\mathfrak{a}^\dagger$-module, (H, β) a C^*-$\mathfrak{b}$-module, and $[\rho_\alpha(\mathfrak{A})\beta] = \beta$ and $[\rho_\beta(\mathfrak{B}^\dagger)\alpha] = \alpha$. The set of* (semi-)morphisms *between C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-modules ${}_\alpha H_\beta$ and ${}_\gamma K_\delta$ is the intersection $\mathcal{L}_{(s)}({}_\alpha H_\beta, {}_\gamma K_\delta) := \mathcal{L}_{(s)}(H_\alpha, K_\gamma) \cap \mathcal{L}_{(s)}(H_\beta, K_\delta)$.*

Remark 2.9. *By Lemma 2.6, we have $[\rho_\alpha(\mathfrak{A}), \rho_\beta(\mathfrak{B}^\dagger)] = 0$ for every C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-module ${}_\alpha H_\beta$.*

Again, the class of all C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-modules forms a category with respect to all semi-morphisms, and a C^*-category with respect to all morphisms.

Example 2.10. (i) *$\mathfrak{H}_{\mathfrak{A}}$ is a C^*-$\mathfrak{a}$-module, $\rho_{\mathfrak{A}}(x) = x$ for all $x \in \mathfrak{A}'$, and ${}_{\mathfrak{A}^\dagger}\mathfrak{H}_{\mathfrak{A}}$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{a})$-module because $[\rho_{\mathfrak{A}^\dagger}(\mathfrak{A})\mathfrak{A}] = [\mathfrak{A}\mathfrak{A}] = \mathfrak{A}$ and $[\rho_{\mathfrak{A}}(\mathfrak{A}^\dagger)\mathfrak{A}^\dagger] = \mathfrak{A}^\dagger$.*
(ii) *Let H_β be a C^*-$\mathfrak{b}$-module, let $\mathfrak{t} = (\mathbb{C}, \mathbb{C}, \mathbb{C})$ be the trivial C^*-base, and let $\alpha = \mathcal{L}(\mathbb{C}, H)$. Then ${}_\alpha H_\beta$ is a C^*-$(\mathfrak{t}, \mathfrak{b})$-module.*
(iii) *Let $(\mathcal{H}_i)_i$ be a family of C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-modules, where $\mathcal{H}_i = (H_i, \alpha_i, \beta_i)$ for each i. Denote by $\boxplus_i \alpha_i \subseteq \mathcal{L}(\mathfrak{H}, \oplus_i H_i)$ the norm-closed linear span of all operators of the form $\zeta \mapsto (\xi_i \zeta)_i$, where $(\xi_i)_i$ is in the algebraic direct sum $\bigoplus_i^{\mathrm{alg}} \alpha_i$, and*

similarly define $\boxplus_i\beta_i \subseteq \mathcal{L}(\mathfrak{K}, \oplus_i H_i)$. Then the triple $\boxplus_i\mathcal{H}_i := (\oplus_i H_i, \boxplus_i\alpha_i, \boxplus_i\beta_i)$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-module, for each j, the canonical inclusions $\iota_j\colon H_j \to \oplus_i H_i$ and projection $\pi_j\colon \oplus_i H_i \to H_j$ are morphisms $\mathcal{H}_j \to \boxplus_i\mathcal{H}_i$ and $\boxplus_i\mathcal{H}_i \to \mathcal{H}_j$, and with respect to these maps, $\boxplus_i\mathcal{H}_i$ is the direct sum of the family $(\mathcal{H}_i)_i$.

The following example shows how bimodules arise from conditional expectations.

Example 2.11. *Let B be a C^*-algebra with a KMS-state μ and associated C^*-base $\mathfrak{b}$ (Example 2.3), let A be a unital C^*-algebra containing B such that $1_A \in B$, and let $\phi\colon A \to B$ be a faithful conditional expectation such that $\nu := \mu \circ \phi$ is a KMS-state and $\phi \circ \sigma^\nu_t = \sigma^\mu_t \circ \phi$ for all $t \in \mathbb{R}$. Fix a GNS-construction $\pi_\nu\colon A \to \mathcal{L}(H_\nu)$ for ν with modular conjugation $J_\nu\colon H_\nu \to H_\nu$, and define $\pi^{op}_\nu\colon A^{op} \to \mathcal{L}(H_\nu)$ by $a \mapsto J_\nu\pi_\nu(a^*)J_\nu$. Then the inclusion $B \hookrightarrow A$ extends to an isometry $\zeta\colon \mathfrak{K} = H_\mu \hookrightarrow H_\nu = H$, and we obtain a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-module ${}_\alpha H_\beta$, where $H = H_\nu$, $\alpha = [J_\nu\pi_\nu(A)\zeta]$, $\beta = [\pi_\nu(A)\zeta]$, and $\rho_\alpha \circ \pi_{\mu^{op}} = \pi^{op}_\nu$, $\rho_\beta \circ \pi_\mu = \pi_\nu$. Moreover, $\pi_\nu(A) + \pi^{op}_\nu((A \cap B')^{op}) \subseteq \mathcal{L}(H_\alpha)$, $\pi_{\nu^{op}}(A^{op}) + \pi_\nu(A \cap B') \subseteq \mathcal{L}(H_\beta)$. For details, see* [25, §2–3].

2.3. **The relative tensor product.** The concepts introduced above allow us to adapt the algebraic formulation of Connes' fusion to the setting of C^*-algebras as follows. Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, H_β a C^*-$\mathfrak{b}$-module, and K_γ a C^*-$\mathfrak{b}^\dagger$-module. Then the *relative tensor product* of H_β and K_γ is the Hilbert space

$$H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K := \beta \otimes \mathfrak{K} \otimes \gamma,$$

which is spanned by elements $\xi \otimes \zeta \otimes \eta$, where $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, the inner product being given by $\langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta' \rangle = \langle \zeta | \xi^*\xi'\eta^*\eta'\zeta' \rangle = \langle \zeta | \eta^*\eta'\xi^*\xi'\zeta' \rangle$ for all $\xi, \xi' \in \beta$, $\zeta, \zeta' \in \mathfrak{K}$, $\eta, \eta' \in \gamma$.

Example 2.12. (i) *If $\mathfrak{b}$ is the trivial C^*-base $\mathfrak{t} = (\mathbb{C}, \mathbb{C}, \mathbb{C})$, then $\beta = \mathcal{L}(\mathbb{C}, H)$, $\gamma = \mathcal{L}(\mathbb{C}, K)$, and $H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K \cong H \otimes K$ via $\xi \otimes \zeta \otimes \eta \mapsto \xi\zeta \otimes \eta 1 = \xi 1 \otimes \eta\zeta$.*

(ii) *Let Z be a locally compact Hausdorff space, μ a Radon measure on Z of full support, $\mathcal{H} = (H_z)_z$ and $\mathcal{K} = (K_z)_z$ continuous bundles of Hilbert spaces on Z with full support, and H_α, K_β the associated C^*-$\mathfrak{b}$-modules as defined in Example 2.7. One easily checks that then we have an isomorphism*

$$H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K \to \int_Z^\oplus H_z \otimes K_z \, d\mu(z), \quad m(\xi) \otimes \zeta \otimes m(\eta) \mapsto (\xi(z)\zeta(z) \otimes \eta(z))_{z \in Z}.$$

Let us list some easy observations and a few definitions.

(i) The isomorphisms in Lemma 2.6 (ii), applied to H_β and K_γ, respectively, yield the following identifications which we shall use without further notice:

$$\beta \otimes_{\rho_\gamma} K \cong H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K \cong H_{\rho_\beta} \otimes \gamma, \quad \xi \otimes \eta\zeta \equiv \xi \otimes \zeta \otimes \eta \equiv \xi\zeta \otimes \eta.$$

(ii) For each $\xi \in \beta$ and $\eta \in \gamma$, there exist bounded linear operators

$$|\xi\rangle_1\colon K \to \beta \otimes_{\rho_\gamma} K = H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K, \ \omega \mapsto \xi \otimes \omega,$$
$$|\eta\rangle_2\colon H \to H_{\rho_\beta} \otimes \gamma = H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K, \ \omega \mapsto \omega \otimes \eta,$$

whose adjoints $\langle\xi|_1 := |\xi\rangle_1^*$ and $\langle\eta|_2 := |\eta\rangle_2^*$ are given by

$$\langle\xi|_1 \colon \xi' \ogreaterthan \omega \mapsto \rho_\gamma(\xi^*\xi')\omega, \qquad \langle\eta|_2 \colon \omega \olessthan \eta' \mapsto \rho_\beta(\eta^*\eta')\omega.$$

We put $|\beta\rangle_1 := \{|\xi\rangle_1 \,|\, \xi \in \beta\} \subseteq \mathcal{L}(K, H{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K)$ and similarly define $\langle\beta|_1$, $|\gamma\rangle_2$, $\langle\gamma|_2$.

(iii) For all $S \in \rho_\beta(\mathfrak{B}^\dagger)'$ and $T \in \rho_\gamma(\mathfrak{B})'$, we have operators

$$S \olessthan \mathrm{id} \in \mathcal{L}(H_{\rho_\beta} \olessthan \gamma) = \mathcal{L}(H{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K), \qquad \mathrm{id} \ogreaterthan T \in \mathcal{L}(\beta \ogreaterthan_{\rho_\gamma} K) = \mathcal{L}(H{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K).$$

If these operators commute, we let $S \underset{\mathfrak{b}}{\otimes} T := (S \olessthan \mathrm{id})(\mathrm{id} \ogreaterthan T) = (\mathrm{id} \ogreaterthan T)(S \olessthan \mathrm{id})$. The commutativity condition holds in each of the following cases:

(a) $S \in \mathcal{L}_s(H_\beta)$; then $(S \underset{\mathfrak{b}}{\otimes} T)(\xi \ogreaterthan \omega) = S\xi \ogreaterthan T\omega$ for each $\xi \in \beta, \omega \in K$;

(b) $T \in \mathcal{L}_s(K_\gamma)$; then $(S \underset{\mathfrak{b}}{\otimes} T)(\omega \olessthan \eta) = S\omega \olessthan T\eta$ for each $\omega \in H, \eta \in \gamma$;

(c) $(\mathfrak{B}^\dagger)' = \mathfrak{B}''$; then for all $\xi, \xi' \in \beta$ and $\eta, \eta' \in \gamma$, the elements $\eta^* T\eta' \in \mathfrak{B}'$ and $\xi^* S\xi' \in (\mathfrak{B}^\dagger)'$ commute, and if $\zeta, \zeta' \in \mathfrak{K}$ and $\omega = \xi \ogreaterthan \zeta \olessthan \eta$, $\omega' = \xi' \ogreaterthan \zeta' \olessthan \eta'$, then

$$\begin{aligned}\langle\omega|(\mathrm{id} \ogreaterthan T)(S \olessthan \mathrm{id})\omega'\rangle &= \langle\zeta|(\eta^* T\eta')(\xi^* S\xi')\zeta'\rangle \\ &= \langle\zeta|(\xi^* S\xi')(\eta^* T\eta')\zeta'\rangle = \langle\omega|(S \olessthan \mathrm{id})(\mathrm{id} \ogreaterthan T)\omega'\rangle.\end{aligned}$$

Let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ and $\mathfrak{c} = (\mathfrak{L}, \mathfrak{C}, \mathfrak{C}^\dagger)$ be further C^*-bases. Then the relative tensor product of bimodules over $(\mathfrak{a}^\dagger, \mathfrak{b})$ and $(\mathfrak{b}^\dagger, \mathfrak{c})$ is a bimodule over $(\mathfrak{a}^\dagger, \mathfrak{c})$:

Proposition 2.13. *Let $\mathcal{H} = {}_\alpha H_\beta$ be a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-module, $\mathcal{K} = {}_\gamma K_\delta$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-module, and*

$$\alpha \lhd \gamma := [|\gamma\rangle_2 \alpha] \subseteq \mathcal{L}(\mathfrak{H}, H{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K), \qquad \beta \rhd \delta := [|\beta\rangle_1 \delta] \subseteq \mathcal{L}(\mathfrak{L}, H{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K). \tag{2.2}$$

Then $\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K} := {}_{(\alpha\lhd\gamma)}(H{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K)_{(\beta\rhd\delta)}$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{c})$-module and

$$\rho_{(\alpha\lhd\gamma)}(x) = \rho_\alpha(x) \olessthan \mathrm{id} \text{ for all } x \in (\mathfrak{A}^\dagger)', \quad \rho_{(\beta\rhd\delta)}(y) = \mathrm{id} \ogreaterthan \rho_\delta(y) \text{ for all } y \in \mathfrak{C}'. \tag{2.3}$$

Proof. The pair $(H{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K, \alpha \lhd \gamma)$ is a C^*-$\mathfrak{a}^\dagger$-module since $[\alpha^*\langle\gamma|_2|\gamma\rangle_2\alpha] = [\alpha^*\rho_\beta(\mathfrak{B}^\dagger)\alpha] = \mathfrak{A}^\dagger$, $[|\gamma\rangle_2\alpha\mathfrak{A}^\dagger] = [|\gamma\rangle_2\alpha]$, and $[|\gamma\rangle_2\alpha\mathfrak{H}] = [|\gamma\rangle_2 H] = H{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K$. Likewise, $(H{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K, \beta \rhd \delta)$ is a C^*-$\mathfrak{c}$-module. For all $x \in (\mathfrak{A}^\dagger)'$, $\zeta \in \mathfrak{H}$, $\theta \in \alpha$, $\eta \in \gamma$, we have $|\eta\rangle_2\theta \in \alpha \lhd \gamma$ and hence

$$\begin{aligned}\rho_{(\alpha\lhd\gamma)}(x)(\theta\zeta \olessthan \eta) &= \rho_{(\alpha\lhd\gamma)}(x)|\eta\rangle_2\theta\zeta \\ &= |\eta\rangle_2\theta x\zeta = \rho_\alpha(x)\theta\zeta \olessthan \eta = (\rho_\alpha(x) \olessthan \mathrm{id})(\theta\zeta \olessthan \eta).\end{aligned}$$

The first equation in (2.3) follows, and a similar agument proves the second one. Finally, ${}_{(\alpha\lhd\gamma)}(H{}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K)_{(\beta\rhd\delta)}$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{c})$-module because $[\rho_{(\alpha\lhd\gamma)}(\mathfrak{A})|\beta\rangle_1\delta] = [|\rho_\alpha(\mathfrak{A})\beta\rangle_1\delta] = [|\beta\rangle_1\delta]$ and $[\rho_{(\beta\rhd\delta)}(\mathfrak{C}^\dagger)|\gamma\rangle_2\alpha] = [|\gamma\rangle_2\alpha]$. □

In the situation above, we call $\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}$ the *relative tensor product* of $\mathcal{H}$ and $\mathcal{K}$. Note the following commutative diagram of Hilbert spaces and closed spaces of operators between them:

$$\begin{tikzcd}
\mathfrak{H} \arrow[dr, "\alpha"] \arrow[ddrr, bend right, "\alpha \triangleleft \gamma"'] & & \mathfrak{K} \arrow[dl, "\beta"'] \arrow[dr, "\gamma"] & & \mathfrak{L} \arrow[dl, "\delta"'] \arrow[ddll, bend left, "\beta \triangleright \delta"] \\
& H \arrow[dr, "|\gamma\rangle_2"] & & K \arrow[dl, "|\beta\rangle_1"'] & \\
& & H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K & &
\end{tikzcd}$$

Given a C^*-$\mathfrak{b}$-module $\mathcal{H} = H_\beta$ and a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-module $\mathcal{K} = {}_\gamma K_\delta$, we abbreviate $H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K_\delta := (H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)_{\beta \triangleright \delta}$. Likewise, we write ${}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K$ for $(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)_{\alpha \triangleleft \gamma}$ and ${}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K_\delta$ for ${}_{\alpha \triangleleft \gamma}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)_{\beta \triangleright \delta}$.

The relative tensor product is functorial, associative, unital, and compatible with direct sums in the following sense:

Proposition 2.14. *Let $\mathcal{H} = {}_\alpha H_\beta$ and $\mathcal{H}^1 = {}_{\alpha_1} H^1_{\beta_1}, \mathcal{H}^2 = {}_{\alpha_2} H^2_{\beta_2}$ be C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-modules, $\mathcal{K} = {}_\gamma K_\delta$, $\mathcal{K}^1 = {}_{\gamma_1} K^1_{\delta_1}$, $\mathcal{K}^2 = {}_{\gamma_2} K^2_{\delta_2}$ C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-modules, and $\mathcal{L} = {}_\epsilon L_\phi$ a C^*-$(\mathfrak{c}^\dagger, \mathfrak{d})$-module.*

(i) *$S \underset{\mathfrak{b}}{\otimes} T \in \mathcal{L}\big(\mathcal{H}^1 \underset{\mathfrak{b}}{\otimes} \mathcal{K}^1, \mathcal{H}^2 \underset{\mathfrak{b}}{\otimes} \mathcal{K}^2\big)$ for all $S \in \mathcal{L}(\mathcal{H}^1, \mathcal{H}^2)$, $T \in \mathcal{L}(\mathcal{K}^1, \mathcal{K}^2)$.*

(ii) *The composition of the isomorphisms $(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta) \underset{\mathfrak{c}}{\otimes} {}_\epsilon L \cong (H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K)_{\rho_{(\beta \triangleright \delta)}} \olessthan \epsilon \cong \beta \ogreaterthan_{\rho_\gamma} K_{\rho_\delta} \olessthan \epsilon$ and $\beta \ogreaterthan_{\rho_\gamma} K_{\rho_\delta} \olessthan \epsilon \cong \beta \ogreaterthan_{\rho_{(\gamma \triangleleft \epsilon)}} (K_\delta \underset{\mathfrak{c}}{\otimes} {}_\epsilon L) \cong H_\beta \underset{\mathfrak{b}}{\otimes} ({}_\gamma K_\delta \underset{\mathfrak{c}}{\otimes} {}_\epsilon L)$ is an isomorphism of C^*-$(\mathfrak{a}^\dagger, \mathfrak{c})$-modules $a_{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}(\mathcal{L}, \mathcal{K}, \mathcal{H}) \colon (\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}) \underset{\mathfrak{c}}{\otimes} \mathcal{L} \to \mathcal{H} \underset{\mathfrak{b}}{\otimes} (\mathcal{K} \underset{\mathfrak{c}}{\otimes} \mathcal{L})$.*

(iii) *Put $\mathcal{U} := {}_{\mathfrak{B}^\dagger} \mathfrak{K}_{\mathfrak{B}}$. Then there exist isomorphisms*
$$r_{\mathfrak{a},\mathfrak{b}}(\mathcal{H}) \colon \mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{U} \to \mathcal{H}, \ \xi \ogreaterthan \zeta \olessthan b^\dagger \mapsto \xi b^\dagger \zeta = \rho_\beta(b^\dagger) \xi \zeta,$$
$$l_{\mathfrak{b},\mathfrak{c}}(\mathcal{K}) \colon \mathcal{U} \underset{\mathfrak{b}}{\otimes} \mathcal{K} \to \mathcal{K}, \ b \ogreaterthan \zeta \olessthan \eta \mapsto \eta b \zeta = \rho_\gamma(b) \eta \zeta.$$

(iv) *Let $(\mathcal{H}^i)_i$ be a family of C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-modules and $(\mathcal{K}^j)_j$ a family of C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-modules. For each i, j, denote by $\iota^i_{\mathcal{H}} \colon \mathcal{H}^i \to \boxplus_{i'} \mathcal{H}^{i'}$, $\iota^j_{\mathcal{K}} \colon \mathcal{K}^j \to \boxplus_{j'} \mathcal{K}^{j'}$ and $\pi^i_{\mathcal{H}} \colon \boxplus_{i'} \mathcal{H}^{i'} \to \mathcal{H}^i$, $\pi^j_{\mathcal{K}} \colon \boxplus_{j'} \mathcal{K}^{j'} \to \mathcal{K}^j$ the canonical inclusions and projections, respectively. Then there exist inverse isomorphisms $\boxplus_{i,j} (\mathcal{H}^i \underset{\mathfrak{b}}{\otimes} \mathcal{K}^j) \leftrightarrows (\boxplus_i \mathcal{H}^i) \underset{\mathfrak{b}}{\otimes} (\boxplus_j \mathcal{K}^j)$, given by $(\omega_{i,j})_{i,j} \mapsto \sum_{i,j} (\iota^i_{\mathcal{H}} \underset{\mathfrak{b}}{\otimes} \iota^j_{\mathcal{K}})(\omega_{i,j})$ and $\big((\pi^i_{\mathcal{H}} \underset{\mathfrak{b}}{\otimes} \pi^j_{\mathcal{K}})(\omega)\big)_{i,j} \mapsfrom \omega$, respectively.*

Proof. (i) If S, T are as above and $\mathcal{H}^i = {}_{\alpha_i} H^i_{\beta_i}$, $\mathcal{K}^j = {}_{\gamma_j} K^j_{\delta_j}$ for $i, j = 1, 2$, then $(S \underset{\mathfrak{b}}{\otimes} T) |\gamma_1\rangle_2 \alpha_1 = |T\gamma_1\rangle_2 S \alpha_1 \subseteq |\gamma_2\rangle_2 \alpha_2$ and similarly $(S \underset{\mathfrak{b}}{\otimes} T) |\beta_1\rangle_1 \delta_1 \subseteq |\beta_2\rangle_1 \delta_2$, $(S \underset{\mathfrak{b}}{\otimes} T)^* |\gamma_2\rangle_2 \alpha_2 \subseteq |\gamma_1\rangle_2 \alpha_1$, $(S \underset{\mathfrak{b}}{\otimes} T)^* |\beta_2\rangle_1 \delta_2 \subseteq |\beta_1\rangle_1 \delta_1$.

(ii) Straightforward.

(iii) $r_{\mathfrak{a},\mathfrak{b}}(\mathcal{H}) \cdot (\alpha \triangleleft \mathfrak{B}^\dagger) = [\rho_\beta(\mathfrak{B}^\dagger)\alpha] = \alpha$ and $r_{\mathfrak{a},\mathfrak{b}}(\mathcal{H}) \cdot (\beta \triangleright \mathfrak{B}) = [\beta \mathfrak{B}] = \beta$. For $l_{\mathfrak{b},\mathfrak{c}}(\mathcal{K})$, the arguments are similar.

(iv) Straightforward. $\square$

Remark 2.15. *The relative tensor product of modules and morphisms can be considered as composition in a bicategory as follows. Recall that a bicategory $\mathbf{B}$ consists of a class of objects* $\operatorname{ob} \mathbf{B}$, *a category $\mathbf{B}(A, B)$ for each $A, B \in \operatorname{ob} \mathbf{B}$ whose objects and morphisms are called* 1-cells *and* 2-cells, *respectively, a functor $c_{A,B,C} \colon \mathbf{B}(B, C) \times \mathbf{B}(A, B) \to$*

$\mathbf{B}(A,C)$ ("composition") for each $A,B,C \in \operatorname{ob}\mathbf{B}$, an object $1_A \in \mathbf{B}(A,A)$ ("identity") for each $A \in \operatorname{ob}\mathbf{B}$, an isomorphism $a_{A,B,C,D}(f,g,h)\colon c_{A,B,D}(c_{B,C,D}(h,g),f) \to c_{A,C,D}(h,c_{A,B,C}(g,f))$ in $\mathbf{B}(A,D)$ ("associativity") for each triple of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\mathbf{B}$, and isomorphisms $l_A(f)\colon c_{A,A,B}(f,1_A) \to f$ and $r_B(f)\colon c_{A,B,B}(1_B,f) \to f$ in $\mathbf{B}(A,B)$ for each 1-cell $A \xrightarrow{f} B$ in $\mathbf{B}$, subject to several axioms [17]. *Tedious but straightforward calculations show that there exists a bicategory $\mathbf{C^*}$-**bimod** such that*

(i) *the objects are all C^*-bases and $\mathbf{C^*}$-$\mathbf{bimod}(\mathfrak{a},\mathfrak{b})$ is the category of all C^*-$(\mathfrak{a}^\dagger,\mathfrak{b})$-modules with morphisms (not semi-morphisms) for all C^*-bases $\mathfrak{a},\mathfrak{b}$;*
(ii) *the functor $c_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}$ is given by $({}_\gamma K_\delta, {}_\alpha H_\beta) \mapsto {}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta$ and $(T,S) \mapsto S \underset{\mathfrak{b}}{\otimes} T$, respectively, and the identity $1_{\mathfrak{a}}$ is ${}_{\mathfrak{A}^\dagger}\mathfrak{H}_{\mathfrak{A}}$ for all C^*-bases $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$;*
(iii) *a,r,l are as in Proposition 2.14.*

3. The spatial fiber product of C^*-algebras

3.1. **Background.** We now use the relative tensor product to construct a fiber product of C^*-algebras that are represented on C^*-modules over C^*-bases. To motivate our approach, let us first review several related constructions. In each case, the task is to construct a relative tensor product or "fiber product" of two algebras A and C with respect to a common subalgebra B.

First, assume that we are working in the category of unital commutative rings. Then the fiber product is just the push-out of the diagram formed by A,B,C. Explicitly, it is the algebraic tensor product $A \underset{B}{\odot} C$, where A and C are considered as modules over B, and the multiplication is defined componentwise. In the category of commutative C^*-algebras, the push-out is the maximal completion of the algebraic tensor product $A \underset{B}{\odot} C$ and, as usual in the setting of C^*-algebras, also other interesting completions exist [1]. For example, if $B = C_0(X)$ for some locally compact Hausdorff space and if A and C are represented on Hilbert spaces H and K, respectively, then H and K can be disintegrated over X with respect to some measure μ (see Subsection 2.1), and the algebra $A \underset{B}{\odot} C$ has a natural representation π on the relative tensor product $H \underset{\mu}{\otimes} K = \int_X^\oplus H_x \otimes K_x d\mu(x)$, leading to a minimal completion $\overline{\pi(A \underset{B}{\odot} C)}$. In the setting of von Neumann algebras, H and K are intrinsic, and the desired fiber product is $\pi(A \underset{B}{\odot} C)'' \subseteq \mathcal{L}(H \underset{\mu}{\otimes} K)$. Note that all of these constructions do not depend on commutativity of A and C and make sense as long as B is central in A and in C.

Next, consider the case where A,B,C are non-commutative, B is a subalgebra of A, and the opposite B^{op} is a subalgebra of C. Then one can consider A and C as modules over B via right multiplication, and form the algebraic tensor product $A \underset{B}{\odot} C$, but componentwise multiplication is well defined only on the subspace $A \underset{B}{\times} C \subseteq A \underset{B}{\odot} C$ which consists of all elements $\sum_i a_i \odot c_i$ satisfying $\sum_i ba_i \odot c_i = \sum_i a_i \odot b^{op} c_i$ for all $b \in B$. This subspace was first considered by Takeuchi and provides the right notion of a fiber product for the algebraic theory of quantum groupoids [2], [32]. In the setting of

C^*-algebras, the Takeuchi product $A \underset{B}{\times} C$ may be 0 even when we expect a nontrivial fiber product on the level of C^*-algebras; therefore, the latter can not be obtained as the completion of the former. In the setting of von Neumann algebras, a fiber product can be constructed as follows [21]. If A and C act on Hilbert spaces H and K, respectively, one can form the Connes fusion $H \underset{\mu}{\otimes} K$ with respect to some weight μ on B and the actions of B on H and B^{op} on K which — by functoriality — carries a representation $\pi\colon A' \odot C' \to \mathcal{L}(H \underset{\mu}{\otimes} K)$, and the desired fiber product is $A \underset{\mu}{*} C = \pi(A' \odot C')'$. A categorical interpretation of this construction is given in 4.3.

We modify the last construction to define a fiber product for C^*-algebras A and C as follows.

(A) We assume that A and C are represented on a C^*-$\mathfrak{b}$-module H_β and a C^*-$\mathfrak{b}^\dagger$-module K_γ, respectively, where $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ is a C^*-base, such that $\rho_\beta(\mathfrak{B})$ and $\rho_\gamma(\mathfrak{B}^\dagger)$ take the places of B and B^{op}, respectively.

(B) On $H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K$, we define two C^*-algebras $\mathrm{Ind}_{|\gamma\rangle_2}(A)$ and $\mathrm{Ind}_{|\beta\rangle_1}(C)$ which, roughly, take the places of $\pi(A' \odot \mathrm{id}_K)'$ and $\pi(\mathrm{id}_H \odot C')'$.

(C) The fiber product is then $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B = \mathrm{Ind}_{|\gamma\rangle_2}(A) \cap \mathrm{Ind}_{|\beta\rangle_1}(C) \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$.

3.2. **C^*-algebras represented on C^*-modules.** Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. As indicated in step (A), we adopt the following terminology.

Definition 3.1. *A* C^*-$\mathfrak{B}^\dagger$-algebra (A, ρ), *briefly written* A_ρ, *is a* C^**-algebra* A *with a* $*$*-homomorphism* $\rho\colon \mathfrak{B}^\dagger \to M(A)$. *A* morphism *of* C^*-$\mathfrak{B}^\dagger$*-algebras* A_ρ *and* B_σ *is a* $*$*-homomorphism* $\pi\colon A \to B$ *satisfying* $\sigma(x)\pi(a) = \pi(\rho(x)a)$ *for all* $x \in \mathfrak{B}^\dagger, a \in A$. *We denote the category of all* C^*-$\mathfrak{B}^\dagger$*-algebras by* $\mathbf{C}^*_{\mathfrak{B}^\dagger}$.

A (nondegenerate) C^*-$\mathfrak{b}$-algebra *is a pair* $A^\alpha_H = (H_\alpha, A)$, *where* H_α *is a* C^*-$\mathfrak{b}$*-module,* $A \subseteq \mathcal{L}(H)$ *a (nondegenerate)* C^**-algebra, and* $\rho_\alpha(\mathfrak{B}^\dagger)A \subseteq A$. *A (semi-)morphism between* C^*-$\mathfrak{b}$*-algebras* A^α_H, B^β_K *is a* $*$*-homomorphism* $\pi\colon A \to B$ *satisfying the condition* $\beta = [\mathcal{L}^\pi_{(s)}(H_\alpha, K_\beta)\alpha]$, *where* $\mathcal{L}^\pi_{(s)}(H_\alpha, K_\beta) := \{T \in \mathcal{L}_{(s)}(H_\alpha, K_\beta) \mid \forall a \in A : Ta = \pi(a)T\}$. *We denote the category of all* C^*-$\mathfrak{b}$*-algebras together with all (semi-)morphisms by* $\mathbf{C}^{*(s)}_{\mathfrak{b}}$.

We first give some examples of C^*-$\mathfrak{b}$-algebras and then study the relation between $\mathbf{C}^*_{\mathfrak{B}^\dagger}$ and $\mathbf{C}^*_{\mathfrak{b}}$.

Example 3.2. (i) *If* H *is a Hilbert space and* $A \subseteq \mathcal{L}(H)$ *a* C^**-algebra, then* A^α_H *is a* C^*-$\mathfrak{t}$*-algebra, where* $\mathfrak{t} = (\mathbb{C}, \mathbb{C}, \mathbb{C})$ *denotes the trivial* C^**-base and* $\alpha = \mathcal{L}(\mathbb{C}, H)$.

(ii) *Let* A^α_H *be a nondegenerate* C^*-$\mathfrak{b}$*-algebra. If we identify* $M(A)$ *with a* C^**-subalgebra of* $\mathcal{L}(H)$ *in the canonical way,* $M(A)^\alpha_H$ *becomes a* C^*-$\mathfrak{b}$*-algebra.*

(iii) *Let* $(\mathcal{A}_i)_i$ *be a family of* C^*-$\mathfrak{b}$*-algebras, where* $\mathcal{A}_i = (\mathcal{H}_i, A_i)$ *for each* i. *Then the* c_0*-sum* $\bigoplus_i A_i$ *and the* l^∞*-product* $\prod_i A_i$ *are naturally represented on the underlying Hilbert space of* $\boxplus_i \mathcal{H}_i$*, and we obtain* C^*-$\mathfrak{b}$*-algebras* $\boxplus_i \mathcal{A}_i := \big(\boxplus_i \mathcal{H}_i, \bigoplus_i A_i\big)$ *and* $\prod_i \mathcal{A}_i := \big(\boxplus_i \mathcal{H}_i, \prod_i A_i\big)$. *For each* j*, the canonical maps* $A_j \to \bigoplus_i A_i \to \prod_i A_i \to A_j$ *are evidently morphisms of* C^*-$\mathfrak{b}$*-algebras* $\mathcal{A}_j \to \boxplus_i \mathcal{A}_i \to \prod_i \mathcal{A}_i \to \mathcal{A}_j$.

The following example is a continuation of Example 2.11.

Example 3.3. *Let B be a C^*-algebra with a KMS-state μ and associated C^*-base $\mathfrak{b}$, and let A be a C^*-algebra containing B with a conditional expectation $\phi\colon A \to B$ as in Example 2.11. With the notation introduced before, $\pi_\nu(A)_H^\beta$ is a nondegenerate C^*-$\mathfrak{b}$-algebra because $\rho_\beta(\mathfrak{B})\pi_\nu(A) = \pi_\nu(B)\pi_\nu(A) \subseteq \pi_\nu(A)$, and similarly, $(\pi_\nu^{op}(A^{op}))_H^\alpha$ is a nondegenerate C^*-$\mathfrak{b}^\dagger$-algebra* [25, §2–3].

The categories $\mathbf{C}_{\mathfrak{b}}^{*s}$ and $\mathbf{C}_{\mathfrak{B}^\dagger}^{*}$ are related by a pair of adjoint functors, as we shall see now.

Lemma 3.4. *Let π be a semi-morphism of C^*-$\mathfrak{b}$-algebras A_H^α and B_K^β. Then π is normal and $\pi(a\rho_\alpha(x)) = \pi(a)\rho_\beta(x)$ for all $x \in \mathfrak{B}^\dagger$, $a \in A$.*

Proof. Let $T, T' \in \mathcal{L}_s^\pi(H_\alpha, K_\beta)$, $\xi, \xi' \in \alpha$, $\zeta, \zeta' \in \mathfrak{K}$, $a \in A$, and $x \in \mathfrak{B}^\dagger$. Then $\langle T\xi\zeta|\pi(a)T'\xi'\zeta'\rangle = \langle \xi\zeta|aT^*T'\xi'\zeta'\rangle$ and

$$\pi(a\rho_\alpha(x))T\xi\zeta = Ta\rho_\alpha(x)\xi\zeta = \pi(a)T\xi x\zeta = \pi(a)\rho_\beta(x)T\xi\zeta$$

because $T\xi \in \beta$. Now, the assertions follow since $K = [\mathcal{L}_s^\pi(H_\alpha, K_\beta)\alpha\mathfrak{K}]$. □

The preceding lemma shows that there exists a forgetful functor

$$\mathbf{U}_{\mathfrak{b}}\colon \mathbf{C}_{\mathfrak{b}}^{*s} \to \mathbf{C}_{\mathfrak{B}^\dagger}^{*}, \qquad \begin{cases} A_H^\alpha \mapsto A_{\rho_\alpha} & \text{for each object } A_H^\alpha, \\ \pi \mapsto \pi & \text{for each morphism } \pi. \end{cases}$$

We shall see that this functor has a partial adjoint that associates to a C^*-$\mathfrak{B}^\dagger$-algebra a universal representation on a C^*-$\mathfrak{b}$-module. For the discussion, we fix a C^*-$\mathfrak{B}^\dagger$-algebra C_σ.

Definition 3.5. *A* representation of C_σ in $\mathbf{C}_{\mathfrak{b}}^{*s}$ *is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A} = A_H^\alpha \in \mathbf{C}_{\mathfrak{b}}^{*s}$ and $\phi \in \mathbf{C}_{\mathfrak{B}^\dagger}^{*}(C_\sigma, \mathbf{U}\mathcal{A})$. Denote by $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ the category of all such representations, where the* morphisms *between objects $(\mathcal{A}, \phi)$ and $(\mathcal{B}, \psi)$ are all $\pi \in \mathbf{C}_{\mathfrak{b}}^{*s}(\mathcal{A}, \mathcal{B})$ satisfying $\psi = \mathbf{U}\pi \circ \phi$.*

Note that $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ is just the comma category $(C_\sigma \downarrow \mathbf{U}_{\mathfrak{b}})$ [19]. Unfortunately, we have no general method like the GNS-construction to produce representations of C_σ in in $\mathbf{C}_{\mathfrak{b}}^{*s}$. In particular, we have no good criteria to decide whether there are any and, if so, whether there exists a faithful one. However, we now show that if there are any representations, then there also is a universal one. The proof involves the following direct product construction.

Example 3.6. *Let $(\mathcal{A}_i, \phi_i) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ for all i, where $\mathcal{A}_i = (\mathcal{H}_i, A_i)$, and define $\phi\colon C \to \prod_i A_i$ by $c \mapsto (\phi_i(c))_i$. Then $\prod_i(\mathcal{A}_i, \phi_i) := (\prod_i \mathcal{A}_i, \phi) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$, and the canonical maps $\mathcal{A}_j \to \prod_i \mathcal{A}_i \to \mathcal{A}_j$ are morphisms between $(\mathcal{A}_j, \phi_j)$ and $(\prod_i \mathcal{A}_i, \phi)$ for each j.*

Proposition 3.7. *If the category $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ is non-empty, then it has an initial object.*

Proof. Assume that $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ is non-empty. We first use a cardinality argument to show that $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ has an initial set of objects, and then apply the direct product construction to this set to obtain an initial object.

Given a topological vector space X and a cardinal number c, let us call X *c-separable* if X has a linearly dense subset of cardinality c. Choose a cardinal number d such that $\mathfrak{B}$ and $C \times \mathfrak{K}$ are d-separable, and let $e := |\mathbb{N}| \sum_n d^n$. Then the isomorphism classes of e-separable Hilbert C^*-$\mathfrak{B}$-modules form a set, and hence there exists a set $\mathcal{R}$ of objects in $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ such that each $(A^\alpha_H, \phi) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ with e-separable α is isomorphic to some element of $\mathcal{R}$. Let $(A^\alpha_H, \phi) = \boxplus_{R\in\mathcal{R}} R$. We show that $(\phi(C)^\alpha_H, \phi)$ is initial in $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$.

Let $(B^\beta_K, \psi) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$. We show that there exists a morphism $\pi \in \mathbf{C}^{*s}_{\mathfrak{b}}(\phi(C)^\alpha_H, B^\beta_K)$ such that $\psi = \pi \circ \phi$. Uniqueness of such a π is evident. Let $\xi \in \beta$ be given. Since $\mathfrak{B}$ and $C \times \mathfrak{K}$ are d-separable, we can inductively choose subspaces $\beta_0 \subseteq \beta_1 \subseteq \cdots \subseteq \beta$ and cardinal numbers $d_0, d_1, \ldots$ such that $\xi \in \beta_0$, $[\beta_0^*\beta_0] = \mathfrak{B}$, $d_0 \leq 2d+1$, β_0 is d_0-separable and for all $n \geq 0$,

$$\beta_n\mathfrak{B} \subseteq \beta_{n+1}, \quad \psi(C)\beta_n\mathfrak{K} \subseteq [\beta_{n+1}\mathfrak{K}], \quad d_{n+1} \leq |\mathbb{N}|dd_n, \quad \beta_{n+1} \text{ is } d_{n+1}\text{-separable}.$$

Let $\tilde{\beta} := [\bigcup_n \beta_n] \subseteq \beta$ and $\tilde{K} := [\tilde{\beta}\mathfrak{K}] \subseteq K$. By construction, $[\tilde{\beta}^*\tilde{\beta}] = \mathfrak{B}$, $\tilde{\beta}\mathfrak{B} \subseteq \tilde{\beta}$, $\psi(C)\tilde{K} \subseteq \tilde{K}$, so that $(\psi(C)|_{\tilde{K}})^{\tilde{\beta}}_{\tilde{K}}$ is in $\mathbf{C}^*_{\mathfrak{b}}$. Define $\tilde{\psi} \colon C \to \psi(C)|_{\tilde{K}}$ by $c \mapsto \psi(c)|_{\tilde{K}}$. Then $(\tilde{\psi}(C)^{\tilde{\beta}}_{\tilde{K}}, \tilde{\psi})$ is in $\mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$. Since $\tilde{\beta}$ is e-separable, $(\tilde{\psi}(C)^{\tilde{\beta}}_{\tilde{K}}, \tilde{\psi})$ is isomorphic to some element of $\mathcal{R}$. Hence, there exists a surjection $\tilde{T} \colon H \to \tilde{K}$ such that $\tilde{T}\alpha = \tilde{\beta}$, and the composition with the inclusion $\tilde{K} \to K$ gives an operator $T \in \mathcal{L}_s(H_\alpha, K_\beta)$ such that $\psi(c)T = T\phi(c)$ for all $c \in C$. Since $\xi \in \tilde{\beta} = T\alpha$ and $\xi \in \beta$ was arbitrary, we can conclude the existence of π as desired. □

Evidently, every morphism Φ between C^*-$\mathfrak{B}^\dagger$-algebras C_σ and D_τ yields a functor

$$\Phi^* \colon \mathbf{Rep}_{\mathfrak{b}}(D_\tau) \to \mathbf{Rep}_{\mathfrak{b}}(C_\sigma), \quad \begin{cases} (A^\alpha_H, \phi) \mapsto (A^\alpha_H, \phi \circ \Phi) & \text{for objects } (A^\alpha_H, \phi), \\ \pi \mapsto \pi & \text{for morphisms } \pi. \end{cases}$$

Denote by $\mathbf{C}^{*r}_{\mathfrak{B}^\dagger}$ the full subcategory of $\mathbf{C}^*_{\mathfrak{B}^\dagger}$ formed by all objects C_σ for which $\mathbf{Rep}(C_\sigma)$ is non-empty.

Theorem 3.8. *There exist a functor $\mathbf{R}_{\mathfrak{b}} \colon \mathbf{C}^{*r}_{\mathfrak{B}^\dagger} \to \mathbf{C}^{*s}_{\mathfrak{b}}$ and natural transformations $\eta \colon \mathrm{id}_{\mathbf{C}^{*r}_{\mathfrak{B}^\dagger}} \to \mathbf{U}_{\mathfrak{b}}\mathbf{R}_{\mathfrak{b}}$ and $\epsilon \colon \mathbf{R}_{\mathfrak{b}}\mathbf{U}_{\mathfrak{b}} \to \mathrm{id}_{\mathbf{C}^{*s}_{\mathfrak{b}}}$ such that for every $C_\sigma, D_\tau \in \mathbf{C}^{*r}_{\mathfrak{B}^\dagger}$, $\Phi \in \mathbf{C}^{*r}_{\mathfrak{B}^\dagger}(C_\sigma, D_\tau)$, $A^\alpha_H \in \mathbf{C}^{*s}_{\mathfrak{b}}$,*

(i) *$\mathbf{R}_{\mathfrak{b}}(C_\sigma) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ is an initial object and $\mathbf{R}_{\mathfrak{b}}(\Phi)$ is the unique morphism from $\mathbf{R}_{\mathfrak{b}}(C_\sigma)$ to $\Phi^*(\mathbf{R}_{\mathfrak{b}}(D_\tau))$;*

(ii) *$\eta_{C_\sigma} = \phi$ if $\mathbf{R}_{\mathfrak{b}}(C_\sigma) = (B^\beta_K, \phi)$, and $\epsilon_{A^\alpha_H}$ is the unique morphism from $\mathbf{R}_{\mathfrak{b}}\mathbf{U}_{\mathfrak{b}}(A^\alpha_H)$ to $(A^\alpha_H, \mathrm{id}_A)$.*

Moreover, $\mathbf{R}_{\mathfrak{b}}$ is left adjoint to $\mathbf{U}_{\mathfrak{b}}$ and η, ϵ are the unit and counit of the adjunction, respectively.

Proof. This follows from Proposition 3.7 and [19, §IV Theorem 2]. □

We next consider C^*-algebras acting on C^*-bimodules. Let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ be a C^*-base.

Definition 3.9. *A* C^*-$(\mathfrak{A}, \mathfrak{B}^\dagger)$-algebra *is a triple (A, ρ, σ), briefly written $A_{\rho,\sigma}$, where A_ρ is a C^*-$\mathfrak{A}$-algebra, A_σ is a C^*-$\mathfrak{B}^\dagger$-algebra, and $[\rho(\mathfrak{A}), \sigma(\mathfrak{B}^\dagger)] = 0$. A* morphism *of*

C^*-$(\mathfrak{A}, \mathfrak{B}^\dagger)$-algebras is a morphism of the underlying C^*-$\mathfrak{A}$-algebras and C^*-$\mathfrak{B}^\dagger$-algebras. We denote the category of all C^*-$(\mathfrak{A}, \mathfrak{B}^\dagger)$-algebras by $\mathbf{C}^*_{(\mathfrak{A},\mathfrak{B}^\dagger)}$.

A (nondegenerate) C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebra *is a pair $A^{\alpha,\beta}_H = ({}_\alpha H_\beta, A)$, where ${}_\alpha H_\beta$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-module, A^α_H is a (nondegenerate) C^*-$\mathfrak{a}^\dagger$-algebra, and A^β_H is a C^*-$\mathfrak{b}$-algebra. A (semi-)morphism of C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebras $A^{\alpha,\beta}_H$ and $B^{\gamma,\delta}_K$ is a $*$-homomorphism $\pi \colon A \to B$ satisfying $\gamma = [\mathcal{L}^\pi_{(s)}({}_\alpha H_\beta, {}_\gamma K_\delta)\alpha]$ and $\delta = [\mathcal{L}^\pi_{(s)}({}_\alpha H_\beta, {}_\gamma K_\delta)\beta]$, where $\mathcal{L}^\pi_{(s)}({}_\alpha H_\beta, {}_\gamma K_\delta) := \{T \in \mathcal{L}_{(s)}({}_\alpha H_\beta, {}_\gamma K_\delta) \mid \forall a \in A : Ta = \pi(a)T\}$. We denote the category of all C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebras together with all (semi-)morphisms by $\mathbf{C}^{*(s)}_{(\mathfrak{a}^\dagger,\mathfrak{b})}$.*

Remark 3.10. *Note that the condition on a (semi-)morphism between C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebras above is stronger than just being a (semi-)morphism of the underlying C^*-$\mathfrak{a}^\dagger$-algebras and C^*-$\mathfrak{b}$-algebras.*

Examples 3.2 (ii) and (iii) naturally extend to C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebras, and the categories $\mathbf{C}^*_{(\mathfrak{A},\mathfrak{B}^\dagger)}$ and $\mathbf{C}^{*s}_{(\mathfrak{a}^\dagger,\mathfrak{b})}$ are again related by a pair of adjoint functors.

Theorem 3.11. *There exists a functor $\mathbf{U}_{(\mathfrak{a}^\dagger,\mathfrak{b})} \colon \mathbf{C}^{*s}_{(\mathfrak{a}^\dagger,\mathfrak{b})} \to \mathbf{C}^*_{(\mathfrak{A},\mathfrak{B}^\dagger)}$, given by $A^{\alpha,\beta}_H \mapsto A_{\rho_\alpha,\rho_\beta}$ on objects and $\pi \mapsto \pi$ on morphisms. Denote by $\mathbf{C}^{*r}_{(\mathfrak{A},\mathfrak{B}^\dagger)}$ the full subcategory of $\mathbf{C}^*_{(\mathfrak{A},\mathfrak{B}^\dagger)}$ formed by all objects $C_{\sigma,\rho}$ for which the comma category $(C_{\sigma,\rho} \downarrow \mathbf{U}_{(\mathfrak{a}^\dagger,\mathfrak{b})})$ is non-empty. Then the corestriction of $\mathbf{U}_{(\mathfrak{a}^\dagger,\mathfrak{b})}$ to $\mathbf{C}^{*r}_{(\mathfrak{A},\mathfrak{B}^\dagger)}$ has a left adjoint $\mathbf{R}_{(\mathfrak{a}^\dagger,\mathfrak{b})} \colon \mathbf{C}^{*r}_{(\mathfrak{A},\mathfrak{B}^\dagger)} \to \mathbf{C}^{*s}_{(\mathfrak{a}^\dagger,\mathfrak{b})}$.*

Proof. The proof proceeds as in the case of C^*-$\mathfrak{b}$-algebras with straightforward modifications, so we only indicate the necessary changes for the second half of the proof of Proposition 3.7. Given a C^*-$(\mathfrak{A}, \mathfrak{B}^\dagger)$-algebra $C_{\sigma,\tau}$ and a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebra $B^{\gamma,\delta}_K$ with a morphism $\psi \colon C_{\sigma,\tau} \to B_{\rho_\gamma,\rho_\delta}$, one constructs $\tilde{\gamma} \subseteq \gamma$ and $\tilde{\delta} \subseteq \delta$ for given $\xi \in \gamma$, $\eta \in \delta$ as follows. One first fixes a cardinal number d such that $\mathfrak{A}, \mathfrak{A}^\dagger, \mathfrak{H}, \mathfrak{B}, \mathfrak{B}^\dagger, \mathfrak{H}$ are d-separable, and then inductively chooses cardinal numbers $d_0, d_1, \ldots$ and closed subspaces $\gamma_0 \subseteq \gamma_1 \subseteq \cdots \subseteq \gamma$ and $\delta_0 \subseteq \delta_1 \subseteq \cdots \subseteq \delta$ such that

$$\begin{gathered}
\xi \in \gamma_0, \quad \eta \in \delta_0, \quad [\gamma_0^*\gamma_0] = \mathfrak{A}^\dagger, \quad [\delta_0^*\delta_0] = \mathfrak{B}, \\
\qquad\qquad\qquad d_0 \leq 2d + 1, \quad \gamma_0, \delta_0 \text{ are } d_0\text{-separable}, \\
\rho_\delta(\mathfrak{B}^\dagger)\gamma_n + \gamma_n\mathfrak{A}^\dagger \subseteq \gamma_{n+1}, \quad \rho_\gamma(\mathfrak{A})\gamma_n + \delta_n\mathfrak{B} \subseteq \delta_{n+1}, \\
\psi(C)\gamma_n\mathfrak{H} + \psi(C)\delta_n\mathfrak{K} \subseteq [\gamma_{n+1}\mathfrak{H}] \cap [\delta_{n+1}\mathfrak{K}], \\
d_{n+1} \leq |\mathbb{N}|d^2 d_n, \quad \gamma_{n+1}, \delta_{n+1} \text{ are } d_{n+1}\text{-separable}
\end{gathered}$$

for all $n \geq 0$, and finally lets $\tilde{\gamma} := [\bigcup_n \gamma_n], \tilde{\delta} := [\bigcup_n \delta_n], \tilde{K} := [\tilde{\gamma}\mathfrak{H}] = [\tilde{\delta}\mathfrak{K}]$. □

Remark 3.12. *Let $C_{\rho,\sigma}$ be a C^*-$(\mathfrak{A}, \mathfrak{B}^\dagger)$-algebra, $A^{\alpha,\beta}_H = \mathbf{R}_{(\mathfrak{a}^\dagger,\mathfrak{b})}(C_{\rho,\sigma})$ and let $\phi = \eta_{C_{\rho,\sigma}} \colon C_{\rho,\sigma} \to A_{\rho_\alpha,\rho_\beta}$ be the morphism given by the unit of the adjunction above. Then $(A^\alpha, \phi) \in \mathbf{Rep}_{\mathfrak{a}^\dagger}(C_\rho)$ and $(A^\beta, \phi) \in \mathbf{Rep}_{\mathfrak{b}}(C_\sigma)$ and therefore, we have semi-morphisms $\mathbf{R}_{\mathfrak{a}^\dagger}(C_\sigma) \to A^\alpha_H$ and $\mathbf{R}_{\mathfrak{b}}(C_\rho) \to A^\beta_H$.*

3.3. **The spatial fiber product for C^*-algebras on C^*-modules.** Our definition of the fiber product of C^*-algebras represented on C^*-modules — more precisely, step (B) in the introduction — involves the following construction.

Let H and K be Hilbert spaces, $I \subseteq \mathcal{L}(H,K)$ a subspace and $A \subseteq \mathcal{L}(H)$ a C^*-algebra such that $[IH] = K$, $[I^*K] = H$, $[II^*I] = I$, $I^*IA \subseteq A$. We define a new C^*-algebra

$$\mathrm{Ind}_I(A) := \{T \in \mathcal{L}(K) \mid TI + T^*I \subseteq [IA]\} \subseteq \mathcal{L}(K).$$

Definition 3.13. *The* I-strong-$*$, I-strong, and I-weak topology *on* $\mathcal{L}(K)$ *are the topologies induced by the families of semi-norms* $T \mapsto \|T\xi\|+\|T^*\xi\|$ *(*$\xi \in I$*),* $T \mapsto \|T\xi\|$ $(\xi \in I)$, *and* $T \mapsto \|\xi^*T\xi'\|$ $(\xi, \xi' \in I)$, *respectively. Given a subset* $X \subseteq \mathcal{L}(K)$*, denote by* $[X]_I$ *the closure of* $\operatorname{span} X$ *with respect to the* I*-strong-*$*$ *topology.*

Evidently, the multiplication in $\mathcal{L}(K)$ is separately continuous with respect to the topologies introduced above, and the involution $T \mapsto T^*$ is continuous with respect to the I-strong-$*$ and the I-weak topology. Define $\rho_I \colon (I^*I)' \to \mathcal{L}(K)$ as in Lemma 2.5.

Lemma 3.14. (i) $[I^* \mathrm{Ind}_I(A) I] \subseteq A$ *and* $\mathrm{Ind}_I(A) = [IAI^*]_I$.
(ii) $\mathrm{Ind}_I(M(A)) \subseteq M(\mathrm{Ind}_I(A))$.
(iii) $\mathrm{Ind}_I(A) \subseteq \mathcal{L}(K)$ *is nondegenerate if and only if* $A \subseteq \mathcal{L}(H)$ *is nondegenerate.*
(iv) *If* $A \subseteq \mathcal{L}(H)$ *is nondegenerate, then* $A' \subseteq (I^*I)'$ *and* $\mathrm{Ind}_I(A) \subseteq \rho_I(A')'$.

Proof. (i) First, $[I^* \mathrm{Ind}_I(A) I] \subseteq [I^*IA] \subseteq A$ by definition and $[IAI^*]_I \subseteq \mathrm{Ind}_I(A)$ because $[IAI^*]_I I \subseteq [IAI^*I] \subseteq [IA]$. To see that $[IAI^*]_I \supseteq \mathrm{Ind}_I(A)$, choose a bounded approximate unit $(u_\nu)_\nu$ for the C^*-algebra $[II^*]$ and observe that for each $T \in \mathrm{Ind}_I(A)$, the net $(u_\nu T u_\nu)_\nu$ lies in the space $[II^* \mathrm{Ind}_I(A) II^*] \subseteq [IAI^*]$ and converges to T in the I-strong-$*$ topology because $\lim_\nu T^{(*)}u_\nu\xi = T^{(*)}\xi \in [IA]$ for all $\xi \in I$ and $\lim_\nu u_\nu\omega = \omega$ for all $\omega \in [IA]$.

(ii) If $S \in \mathrm{Ind}_I(M(A))$ and $T \in \mathrm{Ind}_I(A)$, then $ST \in \mathrm{Ind}_I(A)$ because $STI \subseteq [SIA] \subseteq [IM(A)A] = [IA]$ and $T^*S^*I \subseteq [TIM(A)] \subseteq [IAM(A)] = [IA]$.

(iii) If $\mathrm{Ind}_I(A) \subseteq \mathcal{L}(K)$ is nondegenerate, then we have $[AH] \supseteq [I^* \mathrm{Ind}_I(A) IH] = [I^* \mathrm{Ind}_I(A) K] = [I^*K] = H$. Conversely, if A is nondegenerate, then $[IAI^*]$ and hence also $\mathrm{Ind}_I(A)$ is nondegenerate.

(iv) Assume that A is nondegenerate. Then $I^*I \subseteq M(A) \subseteq \mathcal{L}(H)$ and hence $A' \subseteq (I^*I)'$. For all $x \in \mathrm{Ind}_I(A)$, $y \in A'$, $S, T \in I$, we have $S^*x\rho_I(y)T = S^*xTy = yS^*xT = S^*\rho_I(y)xT$ because $S^*xT \in A$, and since $[IH] = K$, we can conclude that $x\rho_I(y) = \rho_I(y)x$. $\square$

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, A_H^β a C^*-$\mathfrak{b}$-algebra, and B_K^γ a C^*-$\mathfrak{b}^\dagger$-algebra. We apply the construction above to A, B and $|\gamma\rangle_2 \subseteq \mathcal{L}(H, H {}_\beta\!\underset{\mathfrak{b}}{\otimes}{}_\gamma K)$, $|\beta\rangle_1 \subseteq \mathcal{L}(K, H {}_\beta\!\underset{\mathfrak{b}}{\otimes}{}_\gamma K)$, respectively, and define the *fiber product* of A_H^β and B_K^γ to be the C^*-algebra

$$\begin{aligned} A {}_\beta\!\underset{\mathfrak{b}}{*}{}_\gamma B &:= \mathrm{Ind}_{|\gamma\rangle_2}(A) \cap \mathrm{Ind}_{|\beta\rangle_1}(B) \\ &= \{T \in \mathcal{L}(H {}_\beta\!\underset{\mathfrak{b}}{\otimes}{}_\gamma K) \mid T^{(*)}|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A] \text{ and } T^{(*)}|\beta\rangle_1 \subseteq [|\beta\rangle_1 B]\}. \end{aligned}$$

The spaces of operators involved are visualized as arrows in the following diagram:

$$\begin{tikzcd} H \arrow[r, "|\gamma\rangle_2"] \arrow[d, "A"'] & H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K \arrow[d, "A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B"] & K \arrow[l, "|\beta\rangle_1"'] \arrow[d, "B"] \\ H \arrow[r, "|\gamma\rangle_2"] & H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K & K \arrow[l, "|\beta\rangle_1"'] \end{tikzcd}$$

Even in very special situations, it seems to be difficult to give a more explicit description of the fiber product. The main drawback of the definition above is that apart from special situations, we do not know how to produce elements of the fiber product.

Let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ and $\mathfrak{c} = (\mathfrak{L}, \mathfrak{C}, \mathfrak{C}^\dagger)$ be further C^*-bases.

Proposition 3.15. *Let $\mathcal{A} = A_H^{\alpha,\beta}$ be a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebra and $\mathcal{B} = B_K^{\gamma,\delta}$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebra. Then $\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B} := ({}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta, A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B)$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{c})$-algebra.*

Proof. The product $X := \rho_{(\alpha\lhd\gamma)}(\mathfrak{A}^\dagger)(A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B)$ is contained in $A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B$ because

$$X|\beta\rangle_1 \subseteq [|\rho_\alpha(\mathfrak{A})\beta\rangle_1 B] = [|\beta\rangle_1 B], \qquad X^*|\beta\rangle_1 = (A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B)|\rho_\alpha(\mathfrak{A})\beta\rangle_1 \subseteq [|\beta\rangle_1 B],$$

$$X|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 \rho_\alpha(\mathfrak{A}) A] \subseteq [|\gamma\rangle_2 A], \qquad X^*|\gamma\rangle_2 = (A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B)|\gamma\rangle_2 \rho_\alpha(\mathfrak{A}) \subseteq [|\gamma\rangle_2 A]$$

by equation (2.3). A similar argument proves $\rho_{(\beta\rhd\delta)}(\mathfrak{C}^\dagger)(A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B) \subseteq A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B$. □

In the situation above, we call $\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B}$ the *fiber product* of $\mathcal{A}$ and $\mathcal{B}$. Forgetting α or δ, we obtain a C^*-$\mathfrak{c}$-algebra $A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B_\delta := A_H^\beta \underset{\mathfrak{b}}{*} B_H^{\gamma,\delta} := (H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K_\delta, A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B)$ and a C^*-$\mathfrak{a}^\dagger$-algebra ${}_\alpha A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B = A_H^{\alpha,\beta} \underset{\mathfrak{b}}{*} B_K^\gamma$.

Denote by $A' \subseteq \mathcal{L}(H)$ and $B' \subseteq \mathcal{L}(K)$ the commutants of A and B, respectively, and let

$$A^{(\beta)} := A \cap \mathcal{L}(H_\beta), \quad B^{(\gamma)} := B \cap \mathcal{L}(K_\gamma), \quad X := (A^{(\beta)} \underset{\mathfrak{b}}{\otimes} \mathrm{id}) + (\mathrm{id} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)}),$$

$$M_s(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)}) := \{T \in \mathcal{L}(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K) \mid TX, XT \subseteq A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)}\}.$$

Lemma 3.16. *The following relations hold:*

(i) *$\langle\beta|_1(A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B)|\beta\rangle_1 \subseteq B$, $\langle\gamma|_2(A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B)|\gamma\rangle_2 \subseteq A$ and $M(A) {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma M(B) \subseteq M(A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B)$.*

(ii) *$A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)} \subseteq A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B$.*

(iii) *If $[A^{(\beta)}\beta] = \beta$ and $[B^{(\gamma)}\gamma] = \gamma$, then $A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B$ is nondegenerate and $M_s(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)}) \subseteq A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B$.*

(iv) *If $\rho_\beta(\mathfrak{B}^\dagger) \subseteq A$, then $\mathrm{id}_H \underset{\mathfrak{b}}{\otimes} B^{(\gamma)} \subseteq A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B$. If $\rho_\gamma(\mathfrak{B}) \subseteq B$, then $A^{(\beta)} \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K \subseteq A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B$.*

(v) *$\mathrm{id}_{(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K)} \in A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B$ if and only if $\rho_\beta(\mathfrak{B}^\dagger) \subseteq A$ and $\rho_\gamma(\mathfrak{B}) \subseteq B$.*

(vi) *If $A_H^{\alpha,\beta}$ is a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebra and $B_K^{\gamma,\delta}$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebra such that $\rho_\alpha(\mathfrak{A}) + \rho_\beta(\mathfrak{B}^\dagger) \subseteq A$ and $\rho_\gamma(\mathfrak{B}) + \rho_\delta(\mathfrak{C}^\dagger) \subseteq B$, then $\rho_{(\alpha\lhd\gamma)}(\mathfrak{A}) + \rho_{(\beta\rhd\delta)}(\mathfrak{C}^\dagger) \subseteq A {}_\beta\underset{\mathfrak{b}}{*}{}_\gamma B$.*

(vii) *If $A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B$ is nondegenerate, then the C^*-algebra $[\beta^*A\beta] \cap [\gamma^*B\gamma] \subseteq \mathcal{L}(\mathfrak{K})$ is nondegenerate.*

(viii) *If A and B are nondegenerate, then $A' \subseteq \rho_\beta(\mathfrak{B}^\dagger)'$, $B' \subseteq \rho_\gamma(\mathfrak{B})'$, and $A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B \subseteq \rho_{|\gamma\rangle_2}(A') \cap \rho_{|\beta\rangle_1}(B') = (A' \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K)' \cap (\mathrm{id}_H \underset{\mathfrak{b}}{\otimes} B')'$.*

Proof. (i) Immediate from Lemma 3.14.

(ii) Follows from $(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)})|\beta\rangle_1 \subseteq [|A^{(\beta)}\beta\rangle_1 B^{(\gamma)}] \subseteq [|\beta\rangle_1 B]$ and $(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)})|\gamma\rangle_2 \subseteq [|B^{(\gamma)}\gamma\rangle_1 A^{(\beta)}] \subseteq [|\gamma\rangle_2 A]$.

(iii) Assume $[A^{(\beta)}\beta] = \beta$ and $[B^{(\gamma)}\gamma] = \gamma$. Then $A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)} \subseteq A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B$ is nondegenerate and for each $T \in M_s(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)})$, we have $T|\beta\rangle_1 \subseteq [T(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} \mathrm{id})|\beta\rangle_1] \subseteq [(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} B^{(\gamma)})|\beta\rangle_1] \subseteq [|\beta\rangle_1 B]$ and similarly $T^*|\beta\rangle_1 \subseteq [|\beta\rangle_1 B]$, $T|\gamma\rangle_2 + T^*|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A]$.

(iv) If $\rho_\gamma(\mathfrak{B}) \subseteq B$, then $(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K)|\gamma\rangle_2 = |\gamma\rangle_2 A^{(\beta)}$ and $[(A^{(\beta)} \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K)|\beta\rangle_1] \subseteq |\beta\rangle_1 = [|\beta\mathfrak{B}\rangle_1] = [|\beta\rangle_1 \rho_\gamma(\mathfrak{B})] \subseteq [|\beta\rangle_1 B]$. The second assertion follows similarly.

(v) If $\mathrm{id}_{(H_{\beta}\underset{\mathfrak{b}}{\otimes}{}_{\gamma}K)} \in A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B$, then $\rho_\beta(\mathfrak{B}^\dagger) = [\langle\gamma|_2|\gamma\rangle_2] \subseteq A$, $\rho_\gamma(\mathfrak{B}) = [\langle\beta|_1|\beta\rangle_1] \subseteq B$ by i). Conversely, if the last two inclusions hold, then $|\gamma\rangle_2 = [|\gamma\mathfrak{B}^\dagger\rangle_2] = [|\gamma\rangle_2\rho_\beta(\mathfrak{B}^\dagger)] \subseteq [|\gamma\rangle_2 A]$ and similarly $|\beta\rangle_1 \subseteq [|\beta\rangle_1 B]$, whence $\mathrm{id}_{(H_{\beta}\underset{\mathfrak{b}}{\otimes}{}_{\gamma}K)} \in A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B$.

(vi) Immediate from (iv).

(vii) The C^*-algebra $C := [\beta^*A\beta] \cap [\gamma^*B\gamma]$ contains the space $\beta^*\langle\gamma|_2(A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B)|\gamma\rangle_2\beta = \gamma^*\langle\beta|_1(A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B)|\beta\rangle_1\gamma$. Hence, $[C\mathfrak{K}] \supseteq [\beta^*\langle\gamma|_2(A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B)(H_{\beta}\underset{\mathfrak{b}}{\otimes}{}_{\gamma}K)] = \mathfrak{K}$ if $A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B$ is nondegenerate.

(viii) Immediate from Lemma 3.14. □

Even in the case of a trivial C^*-base, we have no explicit description of the fiber product.

Example 3.17. *Let H and K be Hilbert spaces, $\beta = \mathcal{L}(\mathbb{C}, H)$, $\gamma = \mathcal{L}(\mathbb{C}, K)$, $\mathfrak{b} = \mathfrak{t}$ the trivial C^*-base $(\mathbb{C}, \mathbb{C}, \mathbb{C})$, and identify $H_{\beta}\underset{\mathfrak{b}}{\otimes}{}_{\gamma}K$ with $H \otimes K$ as in Example 2.12.*

(i) *Let $A \subseteq \mathcal{L}(H)$ and $B \subseteq \mathcal{L}(K)$ be nondegenerate C^*-algebras. Then $A^{(\beta)} = A$, $B^{(\gamma)} = B$, and by Lemma 3.16, $A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B$ contains the minimal tensor product $A \otimes B \subseteq \mathcal{L}(H \otimes K)$ and $M_s(A \otimes B) = \{T \in \mathcal{L}(H \otimes K) \mid T^{(*)}(1 \otimes B), T^{(*)}(A \otimes 1) \subseteq A \otimes B\}$. If A or B is non-unital, then $\mathrm{id}_{H\otimes K} \notin A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B$ by Lemma 3.16 and so $M(A \otimes B) \not\subseteq A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B$. In Example 5.3 (iii), we shall see that also $A_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}B \not\subseteq M(A \otimes B)$ is possible.*

(ii) *Assume that $H = K = l^2(\mathbb{N})$ and identify $\beta = \gamma = \mathcal{L}(\mathbb{C}, H)$ with H. Then the flip $\Sigma\colon H \otimes H \to H \otimes H$, $\xi \otimes \eta \mapsto \eta \otimes \xi$, is not contained in $\mathcal{L}(H)_{\beta}\underset{\mathfrak{b}}{*}{}_{\gamma}\mathcal{L}(H)$. Indeed, let $(\xi_\nu)_\nu$ be an orthonormal basis for H and let $\eta \in H$ be non-zero. Then $\langle\xi_\nu|_1\Sigma|\eta\rangle_1 = |\eta\rangle\langle\xi_\nu|$ for each ν and hence $\sum_\nu \langle\xi_\nu|_1\Sigma|\eta\rangle_1$ does not converge in norm. On the other hand, one easily verifies that $\sum_\nu \langle\xi_\nu|_1 S$ converges in norm for each $S \in [|H\rangle_1\mathcal{L}(H)]$. Hence, $\Sigma|\eta\rangle_1 \notin [|H\rangle_1\mathcal{L}(H)]$.*

3.4. **Functoriality and slice maps.** We show that the fiber product is functorial, and consider various slice maps. The results concerning functoriality were stated in slightly different form in [25], [28], [29] with proofs referring to unpublished material. We use the opportunity to rectify this situation. As before, let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$, $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$, $\mathfrak{c} = (\mathfrak{L}, \mathfrak{C}, \mathfrak{C}^\dagger)$ be C^*-bases.

Lemma 3.18. *Let π be a (semi-)morphism of C^*-$\mathfrak{b}$-algebras A_H^β and C_L^λ, let ${}_\gamma K_\delta$ be a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-module, and let $I := \mathcal{L}^\pi_{(s)}(H_\beta, L_\lambda) \underset{\mathfrak{b}}{\otimes} \mathrm{id} \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K, L_\lambda \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$.*

(i) *The pairs $\mathcal{X} := (H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K_\delta, (I^*I)')$ and $\mathcal{Y} := (L_\lambda \underset{\mathfrak{b}}{\otimes}{}_\gamma K_\delta, (II^*)')$ are nondegenerate C^*-$\mathfrak{c}$-algebras.*
(ii) *There is a unique $\rho_I \in \mathrm{Mor}_{(s)}(\mathcal{X}, \mathcal{Y})$ satisfying $\rho_I(x)S = Sx$ for all $x \in (I^*I)', S \in I$.*
(iii) *There is a unique linear contraction $j_\pi \colon [|\gamma\rangle_2 A] \to [|\gamma\rangle_2 C]$ given by $|\eta\rangle_2 a \mapsto |\eta\rangle_2 \pi(a)$.*
(iv) *$\mathrm{Ind}_{|\gamma\rangle_2}(A) \subseteq (I^*I)'$ and $\rho_I(x)|\eta\rangle_2 = j_\pi(x|\eta\rangle_2)$ for all $x \in \mathrm{Ind}_{|\gamma\rangle_2}(A)$, $\eta \in \gamma$.*
(v) *Let B_K^γ be a C^*-$\mathfrak{b}^\dagger$-algebra. Then $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \subseteq (I^*I)'$ and $\rho_I(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B) \subseteq C_\lambda \underset{\mathfrak{b}}{*}{}_\gamma B$.*

Proof. (i) First, $(I^*I)'$ and $(II^*)'$ are nondegenerate C^*-algebras, and second, $\rho_{(\beta \triangleright \delta)}(\mathfrak{C}^\dagger) = \mathrm{id}_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma \rho_\delta(\mathfrak{C}^\dagger) \subseteq (I^*I)'$ and $\rho_{(\lambda \triangleright \delta)}(\mathfrak{C}^\dagger) = \mathrm{id}_\lambda \underset{\mathfrak{b}}{\otimes}{}_\gamma \rho_\delta(\mathfrak{C}^\dagger) \subseteq (II^*)'$.

(ii) There exists a unique $*$-homomorphism $\rho_I \colon (I^*I)' \to (II^*)'$ satisfying the formula above by Lemma 2.5, and this is a (semi-)morphism because $[I(\beta \triangleright \delta)] = [\lambda \triangleright \delta]$ by assumption on π.

(iii) For all $\eta_1, \ldots, \eta_n \in \gamma$ and $a_1, \ldots, a_n \in A$, we have

$$\begin{aligned}\|\sum_j |\eta_j\rangle_2 \pi(a_j)\|^2 &= \|\sum_{i,j} \pi(a_i^*)\rho_\lambda(\eta_i^*\eta_j)\pi(a_j)\| \\ &\leq \|\sum_{i,j} a_i^* \rho_\beta(\eta_i^*\eta_j) a_j\| = \|\sum_j |\eta_j\rangle_2 a_j\|^2\end{aligned}$$

by Lemma 3.4. The claim follows.

(iv) The first assertion follows from Lemma 3.14 and the relation $I^*I \subseteq A' \underset{\mathfrak{b}}{\otimes} \mathrm{id} = \rho_{|\gamma\rangle_2}(A')$, and the second one from the fact that for all $x \in \mathrm{Ind}_{|\gamma\rangle_2}(A), \eta \in \gamma, S \in \mathcal{L}^\pi_{(s)}(H_\beta, L_\lambda)$, we have $\rho_I(x)|\eta\rangle_2 S = \rho_I(x)(S \underset{\mathfrak{b}}{\otimes} \mathrm{id})|\eta\rangle_2 = (S \underset{\mathfrak{b}}{\otimes} \mathrm{id})x|\eta\rangle_2 = j_\pi(x|\eta\rangle_2)S$.

(v) First, $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \subseteq (I^*I)'$ by Lemma 3.16. The second assertion follows from the relations

$$\begin{aligned}&\rho_I(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)|\gamma\rangle_2 \subseteq \rho_I(\mathrm{Ind}_{|\gamma\rangle_2}(A))|\gamma\rangle_2 \subseteq j_\pi([|\gamma\rangle_2 A]) = [|\gamma\rangle_2 C], \\ &\rho_I(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)|\lambda\rangle_1 = \rho_I(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)[I|\beta\rangle_1] \subseteq [I(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)|\beta\rangle_1] \subseteq [I|\beta\rangle_1 B] = [|\lambda\rangle_1 B]. \qquad \square\end{aligned}$$

Theorem 3.19. *Let ϕ be a (semi-)morphism of C^*-$(\mathfrak{a}, \mathfrak{b})$-algebras $\mathcal{A} = A_H^{\alpha,\beta}$ and $\mathcal{C} = C_L^{\kappa,\lambda}$, and ψ a (semi-)morphism of C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebras $\mathcal{B} = B_K^{\gamma,\delta}$ and $\mathcal{D} = D_M^{\mu,\nu}$. Then there exists a unique (semi-)morphism of C^*-$(\mathfrak{a}, \mathfrak{c})$-algebras $\phi * \psi$ from $\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B}$ to $\mathcal{C} \underset{\mathfrak{b}}{*} \mathcal{D}$*

such that

$$(\phi * \psi)(x)R = Rx \quad \textit{for all } x \in A_{\beta} \underset{\mathfrak{b}}{*}{}_{\gamma} B \textit{ and } R \in I_M J_H + J_L I_K,$$

where $I_X = \mathcal{L}^{\phi}_{(s)}(H_\beta, L_\lambda) \underset{\mathfrak{b}}{\otimes} \mathrm{id}_X$ *and* $J_Y = \mathrm{id}_Y \underset{\mathfrak{b}}{\otimes} \mathcal{L}^{\psi}_{(s)}(K_\gamma, M_\mu)$ *for* $X \in \{K, M\}, Y \in \{H, L\}$.

Proof. By Lemma 3.18, we can define $\phi * \psi$ to be the restriction of $\rho_{I_M} \circ \rho_{J_H}$ or of $\rho_{J_L} \circ \rho_{I_K}$ to $A_{\beta} \underset{\mathfrak{b}}{*}{}_{\gamma} B$. Uniqueness follows from the fact that $[I_M J_H (H_{\beta} \underset{\mathfrak{b}}{\otimes}{}_{\gamma} K)] = [J_L I_K (H_{\beta} \underset{\mathfrak{b}}{\otimes}{}_{\gamma} K)] = L_{\lambda} \underset{\mathfrak{b}}{\otimes}{}_{\mu} M$. □

Remark 3.20. *Let* A^{β}_H, C^{λ}_L *be* C^*-$\mathfrak{b}$*-algebras,* B^{γ}_K, D^{μ}_M C^*-$\mathfrak{b}^\dagger$*-algebras and let* $\phi \in \mathrm{Mor}(A^{\beta}_H, M(C)^{\lambda}_L)$, $\psi \in \mathrm{Mor}(B^{\gamma}_K, M(D)^{\mu}_M)$ *such that* $[\phi(A)C] = C$, $[\psi(B)D] = D$. *Then there exists a* $*$*-homomorphism* $\phi \underset{\mathfrak{b}}{*} \psi \colon A_{\beta} \underset{\mathfrak{b}}{*}{}_{\gamma} B \to M(C)_{\lambda} \underset{\mathfrak{b}}{*}{}_{\mu} M(D) \hookrightarrow M(C_{\lambda} \underset{\mathfrak{b}}{*}{}_{\mu} D)$, *but in general, we do not know whether this is nondegenerate.*

Next, we briefly discuss two kinds of slice maps on fiber products. For applications and further details, see [29]. The first class of slice maps arises from a completely positive map on one factor and takes values in operators on a certain KSGNS-construction, that is, an internal tensor product with respect to a completely positive linear map [16, §4–§5].

Proposition 3.21. *Let* A^{β}_H *be a* C^*-$\mathfrak{b}$*-algebra,* K_γ *a* C^*-$\mathfrak{b}^\dagger$*-module,* L *a Hilbert space,* $\phi \colon [A + \rho_\beta(\mathfrak{B}^\dagger)] \to \mathcal{L}(L)$ *a c.p. map, and* $\theta = \phi \circ \rho_\beta \colon \mathfrak{B}^\dagger \to \mathcal{L}(L)$. *Then there exists a unique c.p. map* $\phi * \mathrm{id} \colon \mathrm{Ind}_{|\gamma\rangle_2}(A) \to \mathcal{L}(L_\theta \otimes \gamma)$ *such that for all* $\zeta, \zeta' \in L, \eta, \eta' \in \gamma, x \in \mathrm{Ind}_{|\gamma\rangle_2}(A)$,

$$\langle \zeta \otimes \eta | (\phi * \mathrm{id})(x)(\zeta' \otimes \eta') \rangle = \langle \zeta | \phi(\langle \eta |_2 x | \eta' \rangle_2) \zeta' \rangle. \tag{3.1}$$

If B^{γ}_K *is a* C^*-$\mathfrak{b}^\dagger$*-algebra, then*

$$(\phi * \mathrm{id})(A_{\beta} \underset{\mathfrak{b}}{*}{}_{\gamma} B) \subseteq (\phi(A)'{}_{\theta} \otimes (B' \cap \mathcal{L}(K_\gamma))' \subseteq \mathcal{L}(L_\theta \otimes \gamma).$$

Proof. Let $x = (x_{ij})_{i,j} \in M_n(\mathrm{Ind}_{|\gamma\rangle_2}(A))$ be positive, let $\zeta_1, \ldots, \zeta_n \in L$, $\eta_1, \ldots, \eta_n \in \gamma$, where $n \in \mathbb{N}$, and let $d = \mathrm{diag}(|\eta_1\rangle_2, \ldots, |\eta_n\rangle_2)$. Then $0 \leq (\langle \eta_i |_2 x_{ij} | \eta_j \rangle_2)_{i,j} = d^* x d \leq \|x\| d^* d$ and hence $0 \leq (\phi(\langle \eta_i |_2 x_{ij} | \eta_j \rangle_2))_{i,j} \leq \|x\| \phi(d^* d)$ and

$$0 \leq \sum_{i,j} \langle \zeta_i | \phi(\langle \eta_i |_2 x_{ij} | \eta_j \rangle_2) \zeta_j \rangle \leq \|x\| \sum_{i,j} \langle \zeta_i \otimes \eta_i | \zeta_j \otimes \eta_j \rangle.$$

Hence, there exists a map $\phi * \mathrm{id}$ as claimed. The verification of the assertion concerning B^{γ}_K is straightforward. □

Remark 3.22. *If* C^{λ}_L *is a* C^*-$\mathfrak{b}^\dagger$*-algebra and* $\phi|_A$ *is a semi-morphism of* C^*-$\mathfrak{b}^\dagger$*-algebras, then the map* $\phi * \mathrm{id}$ *extends the fiber product* $\phi * \mathrm{id}$ *defined in Theorem 3.19.*

Second, we show that the fiber product is functorial with respect to the following class of maps. A *spatially implemented* map of C^*-$\mathfrak{b}$-algebras A^{β}_H and C^{λ}_L is a map $\phi \colon A \to C$

admitting sequences $(S_n)_n$ and $(T_n)_n$ in $\mathcal{L}(L_\lambda, H_\beta)$ such that

(3.2)
$$\text{(i)} \quad \sum_n S_n^* S_n \text{ and } \sum_n T_n^* T_n \text{ converge in norm,} \quad \text{(ii)} \quad \phi(a) = \sum_n S_n^* a T_n \text{ for all } a \in A.$$

Note that condition (i) implies norm-convergence of the sum in (ii). Evidently, such a map is linear, it extends to a normal map $\bar{\phi}\colon A'' \to C''$, its norm is bounded by $\| \sum_n S_n^* S_n \|^{1/2} \cdot \| \sum_n T_n^* T_n \|^{1/2}$, and the composition of spatially implemented maps is spatially implemented again.

Proposition 3.23. *Let ϕ be a spatially implemented map of C^*-$\mathfrak{b}$-algebras A_H^β and C_L^λ, and let $B_K^{\gamma,\delta}$ be a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebra. Then there exists a spatially implemented map from $A_H^\beta \underset{\mathfrak{b}}{*} B_K^{\gamma,\delta}$ to $C_H^\lambda \underset{\mathfrak{b}}{*} B_K^{\gamma,\delta}$ such that $\langle \eta|_2 (\phi * \mathrm{id})(x) |\eta'\rangle_2 = \phi(\langle \eta|_2 x|\eta'\rangle_2)$ for all $x \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$, $\eta, \eta' \in \gamma$.*

Proof. Uniqueness is clear. Fix sequences $(S_n)_n$, $(T_n)_n$ as in (3.2) and let $\tilde{S}_n := S_n \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K$, $\tilde{T}_n := T_n \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K$ for all n. Then $\tilde{S}_n, \tilde{T}_n \in \mathcal{L}(L_\lambda \underset{\mathfrak{b}}{\otimes}{}_\gamma K_\delta, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K_\delta)$ for all n, we have $\| \sum_n \tilde{S}_n^* \tilde{S}_n \| = \| \sum_n S_n^* S_n \|$, $\| \sum_n \tilde{T}_n^* \tilde{T}_n \| = \| \sum_n T_n^* T_n \|$, and the map $\phi * \mathrm{id}\colon A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \to \mathcal{L}(L_\lambda \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$ given by $x \mapsto \sum_n \tilde{T}_n^* x \tilde{S}_n$ has the desired properties. Indeed, let $x \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$, $\eta, \eta' \in \gamma$. Then $\tilde{S}_n |\eta\rangle_2 = |\eta\rangle_2 S_n$ and $\tilde{T}_n |\eta'\rangle_2 = |\eta'\rangle_2 T_n$ for all n, and hence $\langle \eta|_2 (\phi * \mathrm{id})(x)|\eta'\rangle_2 = \phi(\langle \eta|_2 x |\eta'\rangle_2)$. It remains to show that $(\phi * \mathrm{id})(x) \in C_\lambda \underset{\mathfrak{b}}{*}{}_\gamma B$. Consider the expression $(\phi * \mathrm{id})(x)|\eta'\rangle_2 = \sum_n \tilde{S}_n^* x |\eta'\rangle_2 T_n$. This sum converges in norm and each summand lies in $[|\gamma\rangle_2 \mathcal{L}(H)]$ because $x|\eta'\rangle_2 \in [|\gamma\rangle_2 A]$ and $[\tilde{S}_n^* |\gamma\rangle_2] = [|\gamma\rangle_2 S_n^*]$. Since $\langle \eta''|_2 (\phi * \mathrm{id})(x) |\eta'\rangle_2 \in C$ for each $\eta'' \in \gamma$, we can conclude that the sum lies in $[|\gamma\rangle_2 C]$. Finally, consider the expression $(\phi * \mathrm{id})(x)|\xi\rangle_1 = \sum_n \tilde{S}_n x \tilde{T}_n |\xi\rangle_1$, where $\xi \in \lambda$. Again, the sum converges in norm and each summand lies in $[|\lambda\rangle_1 B]$ because $\tilde{S}_n^* x \tilde{T}_n |\xi\rangle_1 = \tilde{S}_n^* x |T_n \xi\rangle_1 \in \tilde{S}_n^* (A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)|\beta\rangle_1 \subseteq [\tilde{S}_n^* |\beta\rangle_1 B] \subseteq [|\lambda\rangle_1 B]$. ☐

Remark 3.24. (i) *The map $\phi * \mathrm{id}$ constructed above is a "slice map" in the case where $C_L^\lambda = \mathcal{L}(\mathfrak{K})_{\mathfrak{K}}^{\mathfrak{B}}$ and $S_n, T_n \in \beta \subseteq \mathcal{L}(\mathfrak{K}_{\mathfrak{B}}, H_\beta)$ for all n. Then, we can identify $C_\lambda \underset{\mathfrak{b}}{*}{}_\gamma B$ with a C^*-subalgebra of $\mathcal{L}(K)$, and $\phi * \mathrm{id}$ is just the map $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \to B$ given by $x \mapsto \sum_n \langle S_n|_1 X |T_n\rangle_1$.*

(ii) *Assume that the extension $\tilde{\phi}\colon [A + \rho_\beta(\mathfrak{B}^\dagger)] \to C$ given by $x \mapsto \sum_n S_n^* x T_n$ is completely positive. Here, we use the notation of the proof above. Then the map $\tilde{\phi} * \mathrm{id}$ constructed in Proposition 3.21 extends the map $\phi * \mathrm{id}$ of Proposition 3.23 because then $\theta = \rho_\lambda$ and hence $\langle \eta|_2 (\tilde{\phi} * \mathrm{id})(x)|\eta'\rangle_2 = \tilde{\phi}(\langle \eta|_2 x|\eta'\rangle_2)$ for all $x \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$ and $\eta, \eta' \in \gamma$.*

Of course, slice maps of the form $\mathrm{id} * \phi$ can be constructed in a similar way.

3.5. **Further categorical properties.** The fiber product of C^*-algebras is neither associative, unital, nor compatible with infinite sums.

We first discuss non-associativity. Let $\mathcal{A} = A_H^{\alpha,\beta}$ be a C^*-$(\mathfrak{a}^\dagger, \mathfrak{b})$-algebra, $\mathcal{B} = B_K^{\gamma,\delta}$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebra, and $\mathcal{C} = C_L^{\epsilon,\phi}$ a C^*-$(\mathfrak{c}^\dagger, \mathfrak{d})$-algebra. Then we can form the fiber products $(\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B}) \underset{\mathfrak{c}}{*} \mathcal{C}$ and $\mathcal{A} \underset{\mathfrak{b}}{*} (\mathcal{B} \underset{\mathfrak{c}}{*} \mathcal{C})$. The following example shows that these C^*-algebras need *not* be identified by the canonical isomorphism $a_{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}({}_\epsilon L_\phi, {}_\gamma K_\delta, {}_\alpha H_\beta)$ of Proposition 2.14. A similar phenomenon occurs in the purely algebraic setting with the Takeuchi $\times_R$-product [24].

Example 3.25. *Let $\mathfrak{a} = \mathfrak{b} = \mathfrak{c} = \mathfrak{d}$ be the trivial C^*-base, $H = l^2(\mathbb{N})$, $\alpha = \mathcal{L}(\mathbb{C}, H)$, $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{L}(H)_H^{\alpha,\alpha}$. Identify $H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\alpha K_\alpha \underset{\mathfrak{c}}{\otimes}{}_\alpha L \cong \alpha \otimes H \otimes \alpha$ with $H \otimes H \otimes H$ via $|\xi\rangle \otimes \zeta \otimes |\eta\rangle \equiv \xi \otimes \zeta \otimes \eta$, fix an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H, and define $T \in \mathcal{L}(H^{\otimes 3})$ by*

$$T(e_k \otimes e_l \otimes e_m) = \begin{cases} e_k \otimes e_l \otimes e_m & \text{for all } k,l,m \in \mathbb{N} \text{ s.t. } m \leq k+l, \\ e_l \otimes e_k \otimes e_m & \text{for all } k,l,m \in \mathbb{N} \text{ s.t. } m > k+l. \end{cases}$$

We show that T belongs to the underlying C^*-algebra of $(\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B}) \underset{\mathfrak{c}}{*} \mathcal{C}$, but not of $\mathcal{A} \underset{\mathfrak{b}}{*} (\mathcal{B} \underset{\mathfrak{c}}{*} \mathcal{C})$.

For each $\xi \in H$ and $\omega \in H^{\otimes 2}$, we define $|\xi\rangle_1, |\xi\rangle_3 \in \mathcal{L}(H^{\otimes 2}, H^{\otimes 3})$ and $|\omega\rangle_{12} \in \mathcal{L}(H, H^{\otimes 3})$ by $\upsilon \mapsto \xi \otimes \upsilon$, $\upsilon \mapsto \upsilon \otimes \xi$, and $\zeta \mapsto \omega \otimes \zeta$, respectively. Then for all $k,l,m \in \mathbb{N}$,

$$\begin{aligned} T|e_k \otimes e_l\rangle_{12} &= |e_k \otimes e_l\rangle_{12} P_{l+k} + |e_l \otimes e_k\rangle_{12}(\mathrm{id} - P_{l+k}), \\ &\qquad \text{where } P_{l+k} := \sum_{m \leq k+l} |e_m\rangle\langle e_m|, \\ T|e_m\rangle_3 &= |e_m\rangle_3(\mathrm{id} + \Sigma_m), \quad \text{where } \Sigma_m := \sum_{\substack{k,l \\ k+l<m}} |e_l \otimes e_k - e_k \otimes e_l\rangle\langle e_k \otimes e_l|, \end{aligned}$$

and therefore,

$$\begin{aligned} T|H^{\otimes 2}\rangle_{12} &\in [|H^{\otimes 2}\rangle_{12} \mathcal{L}(H)], \\ T|\alpha\rangle_3 &\in [|\alpha_3\rangle(\mathrm{id} + \mathcal{K}(H) \otimes \mathcal{K}(H))] \subseteq [|\alpha\rangle_3(\mathcal{L}(H)_\alpha \underset{\mathfrak{b}}{*}{}_\alpha \mathcal{L}(H))]. \end{aligned}$$

Since $T = T^*$, we can conclude that T belongs to $(\mathcal{L}(H)_\alpha \underset{\mathfrak{b}}{*}{}_\alpha \mathcal{L}(H)_\alpha) \underset{\mathfrak{b}}{*}{}_\alpha \mathcal{L}(H)$. However,

$$\begin{aligned} T|e_0\rangle_1 = |e_0\rangle_1 Q + \sum_l |e_l\rangle_1 Q_l, \quad &\text{where } Q = \sum_{m \leq l} |e_l \otimes e_m\rangle\langle e_l \otimes e_m| \\ &\text{and } Q_l = \sum_{m>l} |e_0 \otimes e_m\rangle\langle e_l \otimes e_m|, \end{aligned}$$

and $|e_0\rangle_1 Q \in [|\alpha\rangle_1 \mathcal{L}(H \otimes H)]$, but $\sum_l |e_l\rangle_1 Q_l \notin [|\alpha\rangle_1 \mathcal{L}(H \otimes H)]$ because the sum

$$\sum_l Q_l^* Q_l = \sum_l \sum_{m>l} |e_l \otimes e_m\rangle\langle e_l \otimes e_m|$$

does not converge in norm. Therefore, we have $T|e_0\rangle_1 \notin [|\alpha\rangle_1 \mathcal{L}(H \otimes H)]$ and $T \notin \mathcal{L}(H)_\alpha \underset{\mathfrak{b}}{*}({}_\alpha \mathcal{L}(H)_\alpha \underset{\mathfrak{b}}{*}{}_\alpha \mathcal{L}(H))$.

We next discuss unitality. A unit for the fiber product relative to $\mathfrak{b}$ would be a C^*-$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebra $\mathcal{U} = \mathfrak{U}_{\mathfrak{K}}^{\mathfrak{B}^\dagger,\mathfrak{B}}$ such that for all C^*-$(\mathfrak{a}^\dagger,\mathfrak{b})$-algebras $\mathcal{A} = A_H^{\alpha,\beta}$ and all C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-algebras $\mathcal{B} = B_K^{\gamma,\delta}$, we have $\mathcal{A} = \mathrm{Ad}_r(\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{U})$ and $\mathcal{B} = \mathrm{Ad}_l(\mathcal{U} \underset{\mathfrak{b}}{*} \mathcal{B})$, where $r = r_{\mathfrak{a},\mathfrak{b}}({}_\alpha H_\beta)$ and $l = l_{\mathfrak{b},\mathfrak{c}}({}_\gamma K_\delta)$ (see Proposition 2.14). The relations $r|\beta\rangle_1 = \beta$, $r|\mathfrak{B}^\dagger\rangle_2 = \rho_\beta(\mathfrak{B}^\dagger)$, $l|\gamma\rangle_2 = \gamma$, $l|\mathfrak{B}\rangle_1 = \rho_\gamma(\mathfrak{B})$ imply

$$\begin{aligned} \mathrm{Ad}_r(A_\beta \underset{\mathfrak{b}}{*} {}_{\mathfrak{B}^\dagger}\mathfrak{U}) &= \mathrm{Ind}_\beta(\mathfrak{U}) \cap \mathrm{Ind}_{\rho_\beta(\mathfrak{B}^\dagger)}(A), \\ \mathrm{Ad}_l(\mathfrak{U}_{\mathfrak{B}} \underset{\mathfrak{b}}{*} {}_\gamma B) &= \mathrm{Ind}_{\rho_\gamma(\mathfrak{B})}(B) \cap \mathrm{Ind}_\gamma(\mathfrak{U}). \end{aligned} \tag{3.3}$$

If $\mathfrak{B}^\dagger$ and $\mathfrak{B}$ are unital, then $\mathrm{Ind}_{\rho_\beta(\mathfrak{B}^\dagger)}(A) = A$ and $\mathrm{Ind}_{\rho_\gamma(\mathfrak{B})}(B) = B$, and then the C^*-$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebra $\mathcal{L}(\mathfrak{K})_{\mathfrak{K}}^{\mathfrak{B}^\dagger,\mathfrak{B}}$ is a unit for the fiber product on the full subcategories of all $A_H^{\alpha,\beta}$ and $B_K^{\gamma,\delta}$ satisfying $A \subseteq \mathrm{Ind}_\beta(\mathcal{L}(\mathfrak{K}))$ and $B \subseteq \mathrm{Ind}_\gamma(\mathcal{L}(\mathfrak{K}))$.

Remark 3.26. (i) *If $A \subseteq \mathrm{Ind}_\alpha(\mathcal{L}(\mathfrak{H}))$ and $B \subseteq \mathrm{Ind}_\gamma(\mathcal{L}(\mathfrak{L}))$, then we have $A_\beta \underset{\mathfrak{b}}{*} {}_\gamma B \subseteq \mathrm{Ind}_{(\alpha \lhd \gamma)}(\mathcal{L}(\mathfrak{H})) \cap \mathrm{Ind}_{(\beta \rhd \delta)}(\mathcal{L}(\mathfrak{K}))$.*

(ii) $\mathrm{Ind}_\beta(\mathfrak{B}^\dagger) = \mathcal{L}(H_\beta)$, *and if $\mathfrak{B}^\dagger$ is unital, then* $\mathrm{Ad}_r(A_\beta \underset{\mathfrak{b}}{*} {}_{\mathfrak{B}^\dagger}\mathfrak{B}^\dagger) = A \cap \mathcal{L}(H_\beta) = A^{(\beta)}$.

(iii) $\mathrm{Ad}_r(\mathfrak{B}_{\mathfrak{B}} \underset{\mathfrak{b}}{*} {}_{\mathfrak{B}^\dagger}\mathfrak{B}^\dagger) = \mathcal{L}(\mathfrak{K}_{\mathfrak{B}}) \cap \mathcal{L}(\mathfrak{K}_{\mathfrak{B}^\dagger}) = M(\mathfrak{B}) \cap M(\mathfrak{B}^\dagger)$.

We finally discuss compatibility with sums and products. First, the fiber product is compatible with finite sums in the following sense. Let $(\mathcal{A}^i)_i$ be a finite family of C^*-$(\mathfrak{a}^\dagger,\mathfrak{b})$-algebras and $(\mathcal{B}^j)_j$ a finite family of C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-algebras. For each i,j, denote by $\iota^i_{\mathcal{A}} \colon \mathcal{A}^i \to \boxplus_{i'}\mathcal{A}^{i'}$, $\iota^j_{\mathcal{B}} \colon \mathcal{B}^j \to \boxplus_{j'}\mathcal{B}^{j'}$ and $\pi^i_{\mathcal{A}} \colon \boxplus_{i'} \mathcal{A}^{i'} \to \mathcal{A}^i$, $\pi^j_{\mathcal{B}} \colon \boxplus_{j'} \mathcal{B}^{j'} \to \mathcal{B}^j$ the canonical inclusions and projections, respectively. One easily verifies that there exist inverse isomorphisms $\boxplus_{i,j}\mathcal{A}^i \underset{\mathfrak{b}}{*} \mathcal{B}^j \leftrightarrows (\boxplus_i\mathcal{A}^i) \underset{\mathfrak{b}}{*} (\boxplus_j\mathcal{B}^j)$, given by $(x_{i,j})_{i,j} \mapsto \sum_{i,j}(\iota^i_{\mathcal{A}} \underset{\mathfrak{b}}{*} \iota^j_{\mathcal{B}})(x_{i,j})$ and $\big((\pi^i_{\mathcal{A}} \underset{\mathfrak{b}}{*} \pi^j_{\mathcal{B}})(y)\big)_{i,j} \mapsfrom y$, respectively. However, the fiber product is neither compatible with infinite sums nor infinite products:

Example 3.27. *Let $\mathfrak{t} = (\mathbb{C},\mathbb{C},\mathbb{C})$ be the trivial C^*-base.*

(i) *For each $i,j \in \mathbb{N}$, let $\mathcal{A}^i$ and $\mathcal{B}^j$ be the C^*-$\mathfrak{t}$-algebra $\mathbb{C}^{\mathbb{C}}_{\mathbb{C}}$. Identify $\bigoplus_{i,j} \mathbb{C}_{\mathbb{C}} \underset{\mathfrak{t}}{\otimes} {}_{\mathbb{C}}\mathbb{C}$ with $l^2(\mathbb{N}\times\mathbb{N})$ in the canonical way. Then $\bigoplus_{i,j}(\mathcal{A}^i \underset{\mathfrak{t}}{*} \mathcal{B}^j)$ corresponds to $C_0(\mathbb{N}\times\mathbb{N})$, represented on $l^2(\mathbb{N}\times\mathbb{N})$ by multiplication operators, but $(\bigoplus_i \mathcal{A}^i) \underset{\mathfrak{t}}{*} (\bigoplus_j \mathcal{B}^j) \cong C_0(\mathbb{N}) \underset{\mathfrak{t}}{*} C_0(\mathbb{N})$ is strictly larger and contains, for example, the characteristic function of the diagonal $\{(x,x) \mid x \in \mathbb{N}\}$ (see Example 5.3).*

(ii) *Let $H = l^2(\mathbb{N})$, $\alpha = \mathcal{L}(\mathbb{C},H)$, and let $\mathcal{A}$ and $\mathcal{B}^j$ be the C^*-$\mathfrak{t}$-algebra $\mathcal{K}(H)^\alpha_H$ for all j. Identify $H_\alpha \underset{\mathfrak{t}}{\otimes} {}_\alpha H$ with $H \otimes H$ as in Example 2.12 (i), choose an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H, and put $y_j := |e_j \otimes e_0\rangle\langle e_0 \otimes e_0| \in \mathcal{K}(H \otimes H)$ for each $j \in \mathbb{N}$. Then $y := (y_j)_j \in \prod_j \mathcal{A} \underset{\mathfrak{t}}{*} \mathcal{B}^j$ because $y_j \in \mathcal{K}(H) \otimes \mathcal{K}(H) \subset \mathcal{A} \underset{\mathfrak{t}}{*} \mathcal{B}^j$ for all $j \in \mathbb{N}$, but with respect to the canonical identification $\bigoplus_j H \otimes H \cong H \otimes \left(\bigoplus_j \otimes H\right)$, we have $y \notin \mathcal{A} \underset{\mathfrak{t}}{*} (\prod_j \mathcal{B}^j)$ because $y|e_0\rangle_1$ corresponds to the family*

$(|e_j\rangle_1|e_0\rangle\langle e_0|)_j \in \prod_j \mathcal{L}(H, H\otimes H) \subseteq \mathcal{L}(\bigoplus_j H, \bigoplus_j H\otimes H)$ *which is not contained in the space* $[|\alpha\rangle_1\mathcal{L}(\bigoplus_j H)]$.

3.6. **A fiber product of non-represented C^*-algebras.** The spatial fiber product of C^*-algebras represented on C^*-modules yields a fiber product of non-represented C^*-algebras as follows.

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. In Subsection 3.2, we constructed a functor $\mathbf{R}_{\mathfrak{b}}\colon \mathbf{C}^{*r}_{\mathfrak{B}^\dagger} \to \mathbf{C}^{*s}_{\mathfrak{b}}$ that associates to each C^*-$\mathfrak{B}^\dagger$-algebra a universal representation in form of a C^*-$\mathfrak{b}$-algebra. Replacing $\mathfrak{b}$ by $\mathfrak{b}^\dagger$, we obtain a functor $\mathbf{R}_{\mathfrak{b}^\dagger}\colon \mathbf{C}^{*r}_{\mathfrak{B}} \to \mathbf{C}^{*s}_{\mathfrak{b}}$, and composition of these with the spatial fiber product gives a fiber product of non-represented C^*-algebras in form of a functor

$$\mathbf{C}^{*r}_{\mathfrak{B}^\dagger} \times \mathbf{C}^{*r}_{\mathfrak{B}} \xrightarrow{\mathbf{R}_{\mathfrak{b}}\times\mathbf{R}_{\mathfrak{b}^\dagger}} \mathbf{C}^{*s}_{\mathfrak{b}} \times \mathbf{C}^{*s}_{\mathfrak{b}^\dagger} \to \mathbf{C}^*_{,} \quad (C_\sigma, D_\tau) \mapsto \mathbf{R}_{\mathfrak{b}}(C_\sigma) \underset{\mathfrak{b}}{*} \mathbf{R}_{\mathfrak{b}^\dagger}(D_\tau),$$

where $\mathbf{C}^*$ denotes the category of C^*-algebras and $*$-homomorphisms. In categorical terms, this is the right Kan extension of the spatial fiber product on $\mathbf{C}^{*s}_{\mathfrak{b}} \times \mathbf{C}^{*s}_{\mathfrak{b}^\dagger}$ along the product of the forgetful functors $\mathbf{U}_{\mathfrak{b}} \times \mathbf{U}_{\mathfrak{b}^\dagger}\colon \mathbf{C}^{*s}_{\mathfrak{b}} \times \mathbf{C}^{*s}_{\mathfrak{b}^\dagger} \to \mathbf{C}^{*r}_{\mathfrak{B}^\dagger} \times \mathbf{C}^{*r}_{\mathfrak{B}}$ [19, §X].

Given further C^*-bases $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ and $\mathfrak{c} = (\mathfrak{L}, \mathfrak{C}, \mathfrak{C}^\dagger)$, we similarly obtain a functor

$$\mathbf{C}^{*r}_{(\mathfrak{A},\mathfrak{B}^\dagger)} \times \mathbf{C}^{*r}_{(\mathfrak{B},\mathfrak{C}^\dagger)} \xrightarrow{\mathbf{R}_{(\mathfrak{a}^\dagger,\mathfrak{b})}\times\mathbf{R}_{(\mathfrak{b}^\dagger,\mathfrak{c})}} \mathbf{C}^{*s}_{(\mathfrak{a}^\dagger,\mathfrak{b})} \times \mathbf{C}^{*s}_{(\mathfrak{b}^\dagger,\mathfrak{c})} \to \mathbf{C}^{*s}_{(\mathfrak{a}^\dagger,\mathfrak{c})} \xrightarrow{\mathbf{U}_{(\mathfrak{a}^\dagger,\mathfrak{c})}} \mathbf{C}^{*r}_{(\mathfrak{A},\mathfrak{C}^\dagger)},$$

and, using Remark 3.12, a natural transformation between the compositions in the square

$$\begin{tikzcd}
\mathbf{C}^{*r}_{(\mathfrak{A},\mathfrak{B}^\dagger)} \times \mathbf{C}^{*r}_{(\mathfrak{B},\mathfrak{C}^\dagger)} \arrow[r] \arrow[d] & \mathbf{C}^{*r}_{(\mathfrak{A},\mathfrak{C}^\dagger)} \arrow[d] \arrow[dl, Rightarrow] \\
\mathbf{C}^{*r}_{\mathfrak{B}^\dagger} \times \mathbf{C}^{*r}_{\mathfrak{B}} \arrow[r] & \mathbf{C}^*,
\end{tikzcd}$$

where the vertical maps are the forgetful functors.

4. Relation to the setting of von Neumann algebras

In this section, let N be a von Neumann algebra with a n.s.f. weight μ, denote by $\mathfrak{N}_\mu, H_\mu, \pi_\mu, J_\mu$ the usual objects of Tomita-Takesaki theory [23], and define the antirepresentation $\pi_\mu^{op}\colon N \to \mathcal{L}(H_\mu)$ by $x \mapsto J_\mu\pi_\mu(x^*)J_\mu$.

4.1. **Adaptation to von Neumann algebras.** The definitions and constructions presented in Sections 2 and 3 can be adapted to a variety of other settings. We now briefly explain what happens when we pass to the setting of von Neumann algebras. Instead of a C^*-base, we start with the triple $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$, where $\mathfrak{K} = H_\mu$, $\mathfrak{B} = \pi_\mu(N)$, and $\mathfrak{B}^\dagger = J_\mu\pi_\mu(N)J_\mu$. Next, we define W^*-$\mathfrak{b}$-modules, W^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-modules, their relative tensor product, W^*-$\mathfrak{b}$-algebras, and the fiber product by just replacing the norm closure $[\,\cdot\,]$ by the closure with respect to the weak operator topology $[\,\cdot\,]_w$ everywhere in Sections 2 and 3. We then recover Connes' fusion of Hilbert bimodules over N and Sauvageot's fiber product as follows.

MODULES. Let H be some Hilbert space. If (H, ρ) is a right N-module, then

$$\alpha = \mathcal{L}((\mathfrak{K}, \pi_\mu^{op}), (H, \rho)) := \{T \in \mathcal{L}(\mathfrak{K}, H) \mid \forall x \in N : T\pi_\mu^{op}(x) = \rho(x)T\}$$

satisfies $[\alpha\mathfrak{K}] = H, [\alpha^*\alpha]_w = \mathfrak{B}, \alpha\mathfrak{B} \subseteq \alpha$, and $\rho_\alpha \circ \pi_\mu^{op}$ (see Lemma 2.5) coincides with ρ. Conversely, if $\alpha \subseteq \mathcal{L}(\mathfrak{K}, H)$ is a weakly closed subspace satisfying the three preceding equations, then $(H, \rho_\alpha \circ \pi_\mu^{op})$ is a right N-module and $\alpha = \mathcal{L}((\mathfrak{K}, \pi_\mu^{op}), (H, \rho_\alpha \circ \pi_\mu^{op}))$ [22]. We thus obtain a bijective correspondence between right N-modules and W^*-$\mathfrak{b}$-modules. This correspondence is an isomorphism of categories since for every other right N-module (K, σ), an operator $T \in \mathcal{L}(H, K)$ intertwines ρ and σ if and only if $T\alpha$ is contained in $\beta := \mathcal{L}((\mathfrak{K}, \pi_\mu^{op}), (K, \sigma))$. For W^*-$\mathfrak{b}$-modules, the notions of morphisms and semi-morphisms coincide.

ALGEBRAS. Let H, ρ, α be as above and let $A \subseteq \mathcal{L}(H)$ be a von Neumann algebra. Then $\rho(N) \subseteq A$ if and only if $\rho_\alpha(\mathfrak{B})A \subseteq A$. Thus, W^*-$\mathfrak{b}$-algebras correspond with von Neumann algebras equipped with a normal unital embedding of N. Moreover, let K, σ, β be as above, let $B \subseteq \mathcal{L}(K)$ be a von Neumann algebra, assume $\rho(N) \subseteq A$ and $\sigma(N) \subseteq B$, and let $\pi \colon A \to B$ be a $*$-homomorphism satisfying $\pi \circ \rho = \sigma$. Then π is normal if and only if $[\mathcal{L}^\pi(H_\alpha, K_\beta)\alpha]_w = \beta$. Indeed, the "if" part is straightforward (see Lemma 3.4), and the "only if" part follows easily from the fact that every normal $*$-homomorphism is the composition of an amplification, reduction, and unitary transformation [5, §4.4].

BIMODULES. Let (H, ρ) be a left N-module, let (H, σ) be a right N-module, and let $\alpha = \mathcal{L}((\mathfrak{K}, \pi_\mu), (H, \rho))$ and $\beta = \mathcal{L}((\mathfrak{K}, \pi_\mu^{op}), (H, \sigma))$. Then (H, ρ, σ) is an N-bimodule if and only if $\rho(N)\beta = \beta$ and $\sigma(N)\alpha = \alpha$, and thus we obtain an isomorphism between the category of N-bimodules and the category of W^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-modules.

FUSION. The preceding considerations and (2.1) show that the relative tensor product of W^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-modules corresponds to Connes' fusion of N-bimodules.

FIBER PRODUCT. Let (H, ρ) be a right N-module, (K, σ) a left N-module, $\alpha = \mathcal{L}((\mathfrak{K}, \pi_\mu^{op}), (H, \rho))$ and $\beta = \mathcal{L}((\mathfrak{K}, \pi_\mu), (K, \sigma))$, and let $A \subseteq \mathcal{L}(H)$ and $B \subseteq \mathcal{L}(K)$ be von Neumann algebras satisfying $\rho(N) \subseteq H$ and $\sigma(N) \subseteq K$. One easily verifies the equivalence of the following conditions for each $x \in \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$:

(i) $x|\alpha\rangle_1 \subseteq [|\alpha\rangle_1 B]_w$, (ii) $\langle\alpha|_1 x|\alpha\rangle_1 \subseteq B$, (iii) $x \in (\mathrm{id}_H \underset{\mathfrak{b}}{\otimes} B')'$.

Consequently, the fiber product of A and B, considered as a W^*-$\mathfrak{b}$-algebra and a W^*-$\mathfrak{b}^\dagger$-algebra, coincides with the fiber product $(\mathrm{id}_H \underset{\mathfrak{b}}{\otimes} B')' \cap (A' \underset{\mathfrak{b}}{\otimes} \mathrm{id}_K)' = (A' \underset{\mathfrak{b}}{\otimes} B')'$ of Sauvageot.

4.2. **Relation to Connes' fusion and Sauvageot's fiber product.** Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base such that $\mathfrak{K} = H_\mu$, $\mathfrak{B}'' = \pi_\mu(N)$, $(\mathfrak{B}^\dagger)'' = \pi_\mu^{op}(N) = \mathfrak{B}'$. Denote by $\mathbf{C^*\text{-}mod}_{(\mathfrak{b}^\dagger, \mathfrak{b})}$ the category of all C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-modules with all semi-morphisms, and by $\mathbf{W^*\text{-}bimod}_{(N, N^{op})}$ the category of all N-bimodules, respectively. Lemmas 2.5 and 2.6 imply:

Lemma 4.1. *There is a faithful functor* $\mathbf{F} \colon \mathbf{C^*\text{-}mod}_{(\mathfrak{b}^\dagger, \mathfrak{b})} \to \mathbf{W^*\text{-}bimod}_{(N, N^{op})}$, *given by* ${}_\alpha H_\beta \mapsto (H, \rho_\alpha \circ \pi_\mu, \rho_\beta \circ \pi_\mu^{op})$ *on objects and* $T \mapsto T$ *on morphisms.* □

The categories $\mathbf{C^*\text{-}mod}_{(\mathfrak{b}^\dagger, \mathfrak{b})}$ and $\mathbf{W^*\text{-}bimod}_{(N, N^{op})}$ carry the structure of monoidal categories [19], and we now show that the functor $\mathbf{F}$ above is monoidal. Let H_β be a C^*-$\mathfrak{b}$-module, K_γ a C^*-$\mathfrak{b}^\dagger$-module, and let

$$\rho = \rho_\beta \circ \pi_\mu^{op}, \quad X = \mathcal{L}((\mathfrak{K}, \pi_\mu^{op}), (H, \rho)), \quad \sigma = \rho_\gamma \circ \pi_\mu, \quad Y = \mathcal{L}((\mathfrak{K}, \pi_\mu), (K, \sigma)).$$

Given subspaces $X_0 \subseteq X$ and $Y_0 \subseteq Y$, we define a sesquilinear form $\langle \cdot | \cdot \rangle$ on the algebraic tensor product $X_0 \odot \mathfrak{K} \odot Y_0$ such that for all $\xi, \xi' \in X_0, \zeta, \zeta' \in \mathfrak{K}, \eta, \eta' \in Y_0$,

$$\langle \xi \odot \zeta \odot \eta | \xi' \odot \zeta' \odot \eta' \rangle = \langle \zeta | (\xi^* \xi')(\eta^* \eta') \eta' \rangle = \langle \zeta | (\eta^* \eta')(\xi^* \xi') \eta' \rangle$$

Denote by $X_0 \ogreaterthan \mathfrak{K} \olessthan Y_0$ the Hilbert space obtained by forming the separated completion.

Lemma 4.2. *Let $X_0 \subseteq X$, $Y_0 \subseteq Y$ be subspaces satisfying $[X_0 \mathfrak{K}] = H$ and $[Y_0 \mathfrak{K}] = K$. Then the natural map $X_0 \ogreaterthan \mathfrak{K} \olessthan Y_0 \to X \ogreaterthan \mathfrak{K} \olessthan Y$ is an isomorphism.*

Proof. Injectivity is clear. The natural map $X_0 \ogreaterthan \mathfrak{K} \olessthan Y_0 \to X \ogreaterthan \mathfrak{K} \olessthan Y_0$ is surjective because both spaces coincide with the separated completion of the algebraic tensor product $H \odot Y_0$ with respect to the sesquilinear inner form given by $\langle \omega \odot \eta | \omega' \odot \eta' \rangle = \langle \omega | \rho_\beta(\eta^* \eta') \omega' \rangle$, and a similar argument shows that the natural map $X \ogreaterthan \mathfrak{K} \olessthan Y_0 \to X \ogreaterthan \mathfrak{K} \olessthan Y$ is surjective. $\square$

We conclude that Connes' original definition of the relative tensor product $H_\rho \underset{\mu}{\otimes}_\sigma K$ via bounded vectors coincides with the algebraic one given in (2.1) and with the relative tensor product $H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K$.

Theorem 4.3. *There exists a natural isomorphism between the compositions in the square*

$$\begin{tikzcd}
\mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \times \mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \arrow[r, "-\underset{\mathfrak{b}}{\otimes}-"] \arrow[d, "\mathbf{F}\times\mathbf{F}"'] & \mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \arrow[d, "\mathbf{F}"] \arrow[dl, phantom, "\Leftarrow"] \\
\mathbf{W}^*\text{-}\mathbf{bimod}_{(N,N^{op})} \times \mathbf{W}^*\text{-}\mathbf{bimod}_{(N,N^{op})} \arrow[r, "-\underset{\mu}{\otimes}-"'] & \mathbf{W}^*\text{-}\mathbf{bimod}_{(N,N^{op})},
\end{tikzcd}$$

given for each object $({}_\alpha H_\beta, {}_\gamma K_\delta) \in \mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \times \mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})}$ by the natural map

$$H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K = \beta \ogreaterthan \mathfrak{K} \olessthan \gamma \to X \ogreaterthan \mathfrak{K} \olessthan Y = H_\rho \underset{\mu}{\otimes}_\sigma K. \tag{4.1}$$

With respect to this isomorphism, the functor $\mathbf{F} \colon \mathbf{C}^*\text{-}\mathbf{mod}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \to \mathbf{W}^*\text{-}\mathbf{bimod}_{(N,N^{op})}$ is monoidal.

Proof. Lemma 4.2 implies that the map (4.1) is an isomorphism. Evidently, this map is natural with respect to ${}_\alpha H_\beta$ and ${}_\gamma K_\delta$. The verification of the assertion concerning $\mathbf{F}$ is now tedious but straightforward. $\square$

Denote by $\mathbf{C}^{*s,nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})}$ the category formed by all C^*-$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebras $A_H^{\alpha,\beta}$ satisfying $\rho_\alpha(\mathfrak{B}) + \rho_\beta(\mathfrak{B}^\dagger) \subseteq A$ and all semi-morphisms, and by $\mathbf{W}^*{}_{(N,N^{op})}$ the category of all von Neumann algebras A equipped with a normal, unital embedding and anti-embedding $\iota_A^{(op)} \colon N \to A$ such that $[\iota_A(N), \iota_A^{op}(N)] = 0$, together with all morphisms preserving these (anti-)embeddings. Lemma 3.4 implies:

Proposition 4.4. *There exists a faithful functor $\mathbf{G} \colon \mathbf{C}^{*s,nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \to \mathbf{W}^*{}_{(N,N^{op})}$, given by $({}_\alpha H_\beta, A) \mapsto (A'', \rho_\alpha \circ \pi_\mu, \rho_\beta \circ \pi_\mu^{op})$ on objects and $\phi \mapsto \phi''$ on morphisms, where ϕ'' denotes the normal extension of ϕ.* $\square$

By Lemma 3.16, $\mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B} \in \mathbf{C}^{*s,nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})}$ for all $\mathcal{A}, \mathcal{B} \in \mathbf{C}^{*s,nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})}$, but $\mathbf{C}^{*s,nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})}$ is not a monoidal category with respect to the fiber product because the latter is not associative (see Subsection 3.5).

Proposition 4.5. *There exists a natural transformation*

$$\begin{tikzcd} \mathbf{C}^{*s,nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \times \mathbf{C}^{*s,nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \arrow[r, "-\underset{\mathfrak{b}}{*}-"] \arrow[d, "\mathbf{G}\times\mathbf{G}"'] & \mathbf{C}^{*s,nd}_{(\mathfrak{b}^\dagger,\mathfrak{b})} \arrow[d, "\mathbf{G}"] \arrow[dl, Rightarrow] \\ \mathbf{W}^*{}_{(N,N^{op})} \times \mathbf{W}^*{}_{(N,N^{op})} \arrow[r, "-\underset{\mu}{*}-"'] & \mathbf{W}^*{}_{(N,N^{op})}, \end{tikzcd}$$

given for each object $A^{\alpha,\beta}_H$ and $B^{\gamma,\delta}_K$ by conjugation with the isomorphism (4.1).

Proof. Immediate from Theorem 4.3 and Lemma 3.16. $\square$

4.3. **A categorical interpretation of the fiber product of von Neumann algebras.** We keep the notation introduced above, denote by **Hilb** the category of Hilbert spaces and bounded linear operators, and call a subcategory of $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{op})}$ a $*$-*subcategory* if it is closed with respect to the involution $T \mapsto T^*$ of morphisms.

Definition 4.6. *A* category over $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{op})}$ *is a category* $\mathbf{C}$ *equipped with a functor* $\mathbf{U_C}\colon \mathbf{C} \to \mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{op})}$ *such that* $\mathbf{U_C}\mathbf{C}$ *is a $*$-subcategory of* $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{op})}$. *Let* $(\mathbf{C}, \mathbf{U_C})$ *be such a category. We loosely refer to* $\mathbf{C}$ *as a category over* $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{op})}$ *without mentioning* $\mathbf{U_C}$ *explicitly, and denote by* $\mathbf{H_C}$ *the composition of* $\mathbf{U_C}$ *with the forgetful functor* $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{op})} \to \mathbf{Hilb}$. *We call an object* $G \in \mathbf{C}$ separating *if it satisfies* $[\mathbf{H_C}\mathbf{C}(G,X)(\mathbf{H_C}G)] = \mathbf{H_C}X$ *for each* $X \in \mathbf{C}$.

We denote by $\mathbf{Cat}_{(N,N^{op})}$ *the category of all categories over* $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{op})}$ *having a separating object, where the morphisms between objects* $(\mathbf{C}, \mathbf{U_C})$ *and* $(\mathbf{D}, \mathbf{U_D})$ *are all functors* $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ *satisfying* $\mathbf{U_D}\mathbf{F} = \mathbf{U_C}$.

Example 4.7. *For each* $A \in \mathbf{W}^*{}_{(N,N^{op})}$, *denote by* $\mathbf{W}^*\text{-}\mathbf{mod}_A$ *the category of all normal, unital representations* $\pi\colon A \to \mathcal{L}(H)$ *for which* $\pi \circ \iota_A$ *and* $\pi \circ \iota_A^{op}$ *are faithful, and all intertwiners. This is a category over* $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{op})}$, *where* $\mathbf{U}_A\colon \mathbf{W}^*\text{-}\mathbf{mod}_A \to \mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{op})}$ *is given by* $(L,\pi) \mapsto (L, \pi \circ \iota_A, \pi \circ \iota_A^{op})$ *on objects and* $T \mapsto T$ *on morphisms. The only non-trivial thing to check is that* $\mathbf{W}^*\text{-}\mathbf{mod}_A$ *has a separating object; by* [3, Lemma 2.10] *or* [23, IX Theorem 1.2 (iv)], *one can take the GNS-representation for a n.s.f. weight on A.*

Each morphism $\phi\colon A \to B$ *in* $\mathbf{W}^*{}_{(N,N^{op})}$ *yields a functor* $\phi^*\colon \mathbf{W}^*\text{-}\mathbf{mod}_B \to \mathbf{W}^*\text{-}\mathbf{mod}_A$, *given by* $(L,\pi) \mapsto (L, \pi \circ \phi)$ *on objects and* $T \mapsto T$ *on morphisms.*

Remark 4.8. *In the definition above,* $\mathbf{Cat}_{(N,N^{op})}(\mathbf{C},\mathbf{D})$ *need not be a set, and this may cause problems. There are several possible solutions: we can fix a "universe" to work in, or replace the category* $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{op})}$ *by a small subcategory and require categories over* $\mathbf{W}^*\text{-}\mathbf{mod}_{(N,N^{op})}$ *to be small, too. It is clear how to modify the preceding example in that case.*

Proposition 4.9. *There exists a contravariant functor* $\mathbf{Mod}\colon \mathbf{W}^*{}_{(N,N^{op})} \to \mathbf{Cat}_{(N,N^{op})}$ *given by* $A \mapsto \mathbf{Mod}(A) := (\mathbf{W}^*\text{-}\mathbf{mod}_A, \mathbf{U}_A)$ *on objects and by* $\phi \mapsto \mathbf{Mod}(\phi) := \phi^*$ *on morphisms.* $\square$

For each category $\mathbf{C} \in \mathbf{Cat}_{(N,N^{op})}$, choose a separating object $G_{\mathbf{C}}$. Fix some $\mathbf{C} \in \mathbf{Cat}_{(N,N^{op})}$, let $\mathbf{U} = \mathbf{U}_{\mathbf{C}}$, $\mathbf{H} = \mathbf{H}_{\mathbf{C}}$, $G = G_{\mathbf{C}}$, $(H, \rho, \sigma) = \mathbf{U}G$, and define $\mathbf{End}(\mathbf{C}) := \mathbf{H}(\mathbf{C}(G,G))' \subseteq \mathcal{L}(H)$. Then $\rho(N) + \sigma(N) \subseteq \mathbf{End}(\mathbf{C})$ because $\mathbf{H}(\mathbf{C}(G,G)) \subseteq (\rho(N) + \sigma(N))'$, and we can consider $\mathbf{End}(\mathbf{C})$ as an element of $\mathbf{W^*}_{(N,N^{op})}$ with respect to ρ and σ.

Lemma 4.10. *There exists a morphism $\eta_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{Mod}(\mathbf{End}(\mathbf{C}))$ in $\mathbf{Cat}_{(N,N^{op})}$, given by $X \mapsto (\mathbf{U}X, \rho^X)$ on objects and $T \mapsto \mathbf{H}T$ on morphisms, where $\rho^X = \rho_{\mathbf{HC}(G,X)}$ for each $X \in \mathbf{C}$. In particular, $\rho^X(\mathbf{End}(\mathbf{C})) \subseteq \mathbf{H}(\mathbf{C}(X,X))'$ for each $X \in \mathbf{C}$.*

Proof. Let $X \in \mathbf{C}$ and $(K, \phi, \psi) = \mathbf{U}X$. Then Lemma 2.5, applied to $I := \mathbf{HC}(G,X) \subseteq \mathcal{L}(\mathbf{H}G, \mathbf{H}X)$, gives a normal representation $\rho_I \colon (I^*I)' \to \mathcal{L}(K)$. Since $I^*I \subseteq \mathbf{HC}(G,X)$ by assumption on $\mathbf{C}$, we have $\mathbf{End}(\mathbf{C}) \subseteq (I^*I)'$ and can define $\rho^X = \rho_I|_{\mathbf{End}(\mathbf{C})}$. Each element of I intertwines ρ with ϕ and σ with ψ, whence $\mathbf{U}X = (K, \rho_I \circ \rho, \rho_I \circ \sigma) = \mathbf{U}_{\mathbf{End}(\mathbf{C})}(\eta_{\mathbf{C}}X)$.

Let $Y \in \mathbf{C}$, $T \in \mathbf{C}(X,Y)$, $J := \mathbf{HC}(G,Y)$. Then $\mathbf{H}(T)\rho_I(S) = \rho_J(S)\mathbf{H}(T)$ for all $S \in \mathbf{End}(G)$ because $\mathbf{H}(T)I \in J$, and therefore $\mathbf{H}(T)$ is a morphism from $(\mathbf{H}X, \rho^X)$ to $(\mathbf{H}Y, \rho^Y)$. By definition, $\mathbf{H}_{\mathbf{End}(\mathbf{C})}(\eta_{\mathbf{C}}(T)) = \mathbf{H}T$. □

Remark 4.11. *If $G' \in \mathbf{C}$ is another separating object, $\rho^{G'} \colon \mathbf{H}(\mathbf{C}(G,G))' \to \mathbf{H}(\mathbf{C}(G',G'))'$ is an isomorphism with inverse $\rho_{\mathbf{HC}(G',G)}$.*

We shall eventually show that the assignment $\mathbf{C} \mapsto \mathbf{End}(\mathbf{C})$ extends to a functor $\mathbf{End} \colon \mathbf{Cat}_{(N,N^{op})} \to \mathbf{W^*}_{(N,N^{op})}$ that is adjoint to $\mathbf{Mod}$. The key is a more careful analysis of functors from a category $\mathbf{C} \in \mathbf{Cat}_{(N,N^{op})}$ to categories of the form $\mathbf{Mod}(A)$, where $A \in \mathbf{W^*}_{(N,N^{op})}$. Such functors themselves can be considered as objects of a category as follows.

For all $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}_{(N,N^{op})}$, the elements of $\mathbf{Cat}_{(N,N^{op})}(\mathbf{C}, \mathbf{D})$ are the objects of a category, where the morphisms are all natural transformations with the usual composition.

Similarly, for all $A, B \in \mathbf{Cat}_{(N,N^{op})}$, the morphisms in $\mathbf{W^*}_{(N,N^{op})}(A,B)$ can be considered as objects of a category, where the morphisms between ϕ, ψ are all $b \in B$ satisfying $b\phi(a) = \psi(a)b$ for all $a \in A$, and where composition is given by multiplication.

Proposition 4.12. *Let $A \in \mathbf{W^*}_{(N,N^{op})}$ and $\mathbf{C} \in \mathbf{Cat}_{(N,N^{op})}$. Then there exists an isomorphism $\Phi_{\mathbf{C},A} \colon \mathbf{Cat}_{(N,N^{op})}(\mathbf{C}, \mathbf{Mod}(A)) \to \mathbf{W^*}_{(N,N^{op})}(A, \mathbf{End}(\mathbf{C}))$ with inverse $\Psi_{\mathbf{C},A} := \Phi^{-1}_{\mathbf{C},A}$ such that*

(i) *$\Phi_{\mathbf{C},A}(\mathbf{F})$ is defined by $\mathbf{F}G_{\mathbf{C}} = (\mathbf{H}_{\mathbf{C}}G_{\mathbf{C}}, \Phi_{\mathbf{C},A}(\mathbf{F}))$ for each functor $\mathbf{F} \colon \mathbf{C} \to \mathbf{Mod}(A)$ and $\Phi_{\mathbf{C},A}(\alpha) = \alpha_{G_{\mathbf{C}}}$ for each $\alpha \in \mathbf{Cat}_{(N,N^{op})}(\mathbf{C}, \mathbf{Mod}(A))$;*

(ii) *$\Psi_{\mathbf{C},A}(\pi) = \mathbf{Mod}(\pi) \circ \eta_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{Mod}(\mathbf{End}(\mathbf{C})) \to \mathbf{Mod}(A)$ for each object π and $\Psi_{\mathbf{C},A}(S) = (\rho^X(S))_{X \in \mathbf{C}}$ for each morphism S in $\mathbf{W^*}_{(N,N^{op})}(A, \mathbf{End}(\mathbf{C}))$.*

Explicitly, $\Psi_{\mathbf{C},A}(\pi)$ is given by $X \mapsto (\mathbf{H}_{\mathbf{C}}X, \rho^X \circ \pi)$ on objects and $T \mapsto \mathbf{H}_{\mathbf{C}}T$ on morphisms.

The proof of Proposition 4.12 involves the following result.

Lemma 4.13. *Write $\mathbf{U}_{\mathbf{C}}G_{\mathbf{C}} = (\mathbf{H}_{\mathbf{C}}G_{\mathbf{C}}, \rho, \sigma)$. Then the assignments $\alpha \mapsto \alpha_{G_{\mathbf{C}}}$ and $(\rho^X(S))_{X \in \mathbf{C}} \leftarrow S$ are inverse bijections between all natural transformations α of $\mathbf{H}_{\mathbf{C}}$ (or $\eta_{\mathbf{C}}$) and all elements $S \in \mathbf{End}(G_{\mathbf{C}})$ (or $S \in \mathbf{End}(G_{\mathbf{C}}) \cap (\rho(N) + \sigma(N))'$, respectively).*

Proof. A family of morphisms $(\alpha_X \colon \mathbf{H_C}X \to \mathbf{H_C}X)_{X\in\mathbf{C}}$ is a natural transformation of $\mathbf{H_C}$ if and only if $\alpha_X T = T\alpha_X$ for all $X \in \mathbf{C}$ and $T \in \mathbf{H_C}(G_\mathbf{C}, X)$, that is, if $\alpha_X = \rho^X(\alpha_{G_\mathbf{C}})$ and $\alpha_{G_\mathbf{C}} \in \mathbf{End}(\mathbf{C})$. Such a family is a natural transformation of $\eta_\mathbf{C}$ if and only if additionally, $\alpha_X = \rho^X(\alpha_{G_\mathbf{C}})$ is a morphism of $\mathbf{U_C}X$ for each $X \in \mathbf{C}$ or, equivalently, if $\alpha_{G_\mathbf{C}} \in (\rho(N) + \sigma(N))'$. □

Proof of Proposition 4.12. Lemma 4.13 implies that $\Psi := \Psi_{\mathbf{C},A}$ is well defined by (ii). Let us show that $\Phi := \Phi_{\mathbf{C},A}$ is well defined by (i). For each $\mathbf{F}$ as above, the image $\mathbf{H}_{\mathbf{Mod}(A)}(\mathbf{F}(\mathbf{C}(G_\mathbf{C}, G_\mathbf{C}))) = \mathbf{H_C}(\mathbf{C}(G_\mathbf{C}, G_\mathbf{C}))$ consists of intertwiners for $\Phi(\mathbf{F})$ and hence $(\Phi(\mathbf{F}))(A) \subseteq \mathbf{H_C}(\mathbf{C}(G_\mathbf{C}, G_\mathbf{C}))' = \mathbf{End}(\mathbf{C})$. Likewise, for each α as above, $\alpha_{G_\mathbf{C}}$ intertwines $\mathbf{H_C}(\mathbf{C}(G_\mathbf{C}, G_\mathbf{C}))$ and hence $\alpha_{G_\mathbf{C}} \in \mathbf{End}(\mathbf{C})$. Finally, $\Phi(\alpha\circ\beta) = \alpha_{G_\mathbf{C}}\circ\beta_{G_\mathbf{C}} = \Phi(\alpha)\Phi(\beta)$ for all composable α, β.

Next, $\Phi \circ \Psi = \mathrm{id}$ because for each π as above, $\Psi(\pi)(G_\mathbf{C}) = (\mathbf{H_C}G_\mathbf{C}, \rho^{G_\mathbf{C}} \circ \pi)$ so that $\Phi(\Psi(\pi)) = \rho^{G_\mathbf{C}} \circ \pi = \pi$, and for each S as above, the component of $(\rho^X(S))_{X\in\mathbf{C}}$ at $X = G_\mathbf{C}$ is $\rho^{G_\mathbf{C}}(S) = S$.

Finally, we prove $\Psi \circ \Phi = \mathrm{id}$. Let $\mathbf{F}$ be as above and define ϕ^X by $\mathbf{F}X = (\mathbf{H}_C X, \phi^X)$ for each $X \in \mathbf{C}$. Then $\Phi(\mathbf{F}) = \phi^{G_\mathbf{C}}$, and for each $a \in A$, the family $(\phi^X(a))_{X\in\mathbf{C}}$ is a natural transformation of $\mathbf{H}_{\mathbf{Mod}(A)} \circ \mathbf{F} = \mathbf{H_C}$ which coincides by Lemma 4.13 with $(\rho^X(\phi^{G_\mathbf{C}}(a)))_{X\in\mathbf{C}}$. Therefore, $\mathbf{F}X = (\mathbf{H_C}X, \phi^X) = (\mathbf{H_C}X, \rho^X \circ \Phi(\mathbf{F})) = \Psi(\Phi(\mathbf{F}))(X)$ for each $X \in \mathbf{C}$. On morphisms, $\Psi(\Phi(\mathbf{F}))$ and $\mathbf{F}$ coincide anyway. For each α as above, $\Psi(\Phi(\alpha)) = (\rho^X(\alpha_{G_\mathbf{C}}))_{X\in\mathbf{C}} = \alpha$ by Lemma 4.13. □

Corollary 4.14. (i) *Let $A \in \mathbf{W^*}_{(N,N^{op})}$ and consider id_A as an object of $\mathbf{C} := \mathbf{Mod}(A)$. Then $\Phi_{\mathbf{C},A}(\mathrm{id}_\mathbf{C}) \colon A \to \mathbf{End}(\mathbf{Mod}(A))$ is an isomorphism in $\mathbf{W^*}_{(N,N^{op})}$ with inverse $\epsilon_A := \rho^{\mathrm{id}_A}$.*

(ii) *Let $A, B \in \mathbf{W^*}_{(N,N^{op})}$. Then the isomorphism $\mathbf{Mod}_{(A,B)}$ obtained by composing*

$$(\epsilon_B^{-1})_* \colon \mathbf{W^*}_{(N,N^{op})}(A, B) \to \mathbf{W^*}_{(N,N^{op})}(A, \mathbf{End}(\mathbf{Mod}(B)))$$

with

$$\Psi_{\mathbf{Mod}(B),A} \colon \mathbf{W^*}_{(N,N^{op})}(A, \mathbf{End}(\mathbf{Mod}(B))) \to \mathbf{Cat}_{(N,N^{op})}(\mathbf{Mod}(B), \mathbf{Mod}(A)),$$

is given by $\phi \mapsto \mathbf{Mod}(\phi)$ on objects and $b \mapsto (\pi(b))_{(L,\pi)}$ on morphisms.

(iii) *Let $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}_{(N,N^{op})}$. Then the functor $\mathbf{End}_{(\mathbf{C},\mathbf{D})}$ obtained by composing*

$$(\eta_\mathbf{D})_* \colon \mathbf{Cat}_{(N,N^{op})}(\mathbf{C}, \mathbf{D}) \to \mathbf{Cat}_{(N,N^{op})}(\mathbf{C}, \mathbf{Mod}(\mathbf{End}(\mathbf{D})))$$

with

$$\Phi_{\mathbf{C},\mathbf{End}(\mathbf{D})} \colon \mathbf{Cat}_{(N,N^{op})}(\mathbf{C}, \mathbf{Mod}(\mathbf{End}(\mathbf{D}))) \to \mathbf{W^*}_{(N,N^{op})}(\mathbf{End}(\mathbf{D}), \mathbf{End}(\mathbf{C}))$$

is given by $\mathbf{F} \mapsto \rho^{\mathbf{F}G_\mathbf{C}}$ on objects and $\alpha \mapsto \mathbf{H_D}(\alpha_{G_\mathbf{C}})$ on morphisms.

Proof. Assertions (i) and (iii) follow immediately from the definitions and Proposition 4.12.

Let us prove (ii). For each object ϕ, we have $G_{\mathbf{Mod}(B)} = (\mathbf{H}_{\mathbf{Mod}(B)}, \epsilon_B^{-1})$ and $\Phi_{\mathbf{Mod}(B),A}(\mathbf{Mod}(\phi)) = \epsilon_B^{-1} \circ \phi$, whence $\Psi_{\mathbf{Mod}(B),A}(\epsilon_B^{-1} \circ \phi) = \mathbf{Mod}(\phi)$, and for each morphism b, the family $\alpha := (\pi(b))_{(L,\pi)}$ is a natural transformation and $\Phi_{\mathbf{Mod}(B),A}(\alpha) = \alpha_{G_{\mathbf{Mod}(B)}} = \epsilon_B^{-1}(b)$. □

The relative tensor product on $\mathbf{W^*\text{-}mod}_{(N,N^{op})}$ yields a product on the category $\mathbf{Cat}_{(N,N^{op})}$ as follows. Let $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}_{(N,N^{op})}$. Then $\mathbf{C} \times \mathbf{D}$ and the functor

$$\mathbf{U}_{\mathbf{C}\times\mathbf{D}} = (- \underset{\mu}{\otimes} -) \circ (\mathbf{U}_{\mathbf{C}} \times \mathbf{U}_{\mathbf{D}}) \colon \mathbf{C} \times \mathbf{D} \to \mathbf{W^*\text{-}mod}_{(N,N^{op})},$$

form a category over $\mathbf{W^*\text{-}mod}_{(N,N^{op})}$ with separating object $(G_{\mathbf{C}}, G_{\mathbf{D}})$. Thus, we obtain a monoidal structure on $\mathbf{Cat}_{(N,N^{op})}$, given by $(\mathbf{C}, \mathbf{D}) \mapsto \mathbf{C} \times \mathbf{D}$ on objects and $(\mathbf{F}, \mathbf{G}) \mapsto \mathbf{F} \times \mathbf{G}$ on morphisms.

Corollary 4.15. *For all $A, B, C \in \mathbf{W^*}_{(N,N^{op})}$, there exists an isomorphism*

$$\Xi \colon \mathbf{W^*}_{(N,N^{op})}(A, B \underset{\mu}{*} C) \to \mathbf{Cat}_{(N,N^{op})}(\mathbf{Mod}(B) \times \mathbf{Mod}(C), \mathbf{Mod}(A))$$

such that

(i) *for each object π, the functor $\Xi(\pi) \colon \mathbf{Mod}(B) \times \mathbf{Mod}(C) \to \mathbf{Mod}(A)$ is given by $((L,\tau),(M,\upsilon)) \mapsto (L \underset{\mu}{\otimes} M, (\tau \underset{\mu}{*} \upsilon) \circ \pi)$ and $(S,T) \mapsto S \underset{\mu}{\otimes} T$;*

(ii) *for each morphism $x \colon \pi_1 \to \pi_2$, the transformation $\Xi(b) \colon \Xi(\pi_1) \to \Xi(\pi_2)$ is given by $\Xi(b)_{((L,\tau),(M,\upsilon))} = (\tau \underset{\mu}{*} \upsilon)(x)$.*

Proof. Let $\mathbf{B} := \mathbf{Mod}(B)$, $\mathbf{C} := \mathbf{Mod}(C)$, $\mathbf{D} := \mathbf{B} \times \mathbf{C}$. Then $G := (G_{\mathbf{B}}, G_{\mathbf{C}})$ is separating and

$$\begin{aligned}\rho^G \colon \mathbf{End}(\mathbf{D}) \to \mathbf{H}_{\mathbf{D}}(\mathbf{D}(G,G))' &= (\mathbf{End}(\mathbf{B})' \underset{\mu}{\otimes} \mathbf{End}(\mathbf{C})')' \\ &= \mathbf{End}(\mathbf{B}) \underset{\mu}{*} \mathbf{End}(\mathbf{C}) \cong B \underset{\mu}{*} C\end{aligned}$$

is an isomorphism by Remark 4.11.

Let $X = (L,\tau) \in \mathbf{B}$ and $Y = (M,\upsilon) \in \mathbf{C}$. Then $\rho^{(X,Y)} = (\tau \underset{\mu}{*} \upsilon) \circ \rho^G$ by Lemma 2.5 because $\tau \underset{\mu}{*} \upsilon = \rho_J$, where $J = \mathbf{H}_{\mathbf{B}}(\mathbf{B}(G_{\mathbf{B}}, X)) \underset{\mu}{\otimes} \mathbf{H}_{\mathbf{C}}(\mathbf{C}(G_{\mathbf{C}}, Y))$, and $J \cdot \mathbf{H}_{\mathbf{D}}(\mathbf{D}(G_{\mathbf{D}}, G)) \subseteq \mathbf{H}_{\mathbf{D}}(\mathbf{D}(G_{\mathbf{D}}, (X,Y)))$. Now, the assertion follows from Proposition 4.12. $\square$

The categories $\mathbf{W^*}_{(N,N^{op})}$ and $\mathbf{Cat}_{(N,N^{op})}$ are enriched over the monoidal category $\mathbf{Cat}$ of small categories [14], or, equivalently, are 2-categories, meaning that the morphisms between fixed objects are themselves objects of a small category, as explained before Proposition 4.12, and that the composition of morphisms between fixed objects extends to a functor, where

$$\begin{tikzcd} B \arrow[r, bend left, "\psi_1"] \arrow[r, bend right, "\psi_2"'] \arrow[r, phantom, "\Downarrow c"] & C \end{tikzcd} \circ \begin{tikzcd} A \arrow[r, bend left, "\phi_1"] \arrow[r, bend right, "\phi_2"'] \arrow[r, phantom, "\Downarrow b"] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\psi_1\circ\phi_1"] \arrow[r, bend right, "\psi_2\circ\phi_2"'] \arrow[r, phantom, "\Downarrow \psi_2(b)c"] & C \end{tikzcd} \text{ in } \mathbf{W^*}_{(N,N^{op})}, \tag{4.2}$$

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "\mathbf{G}_1"] \arrow[r, bend right, "\mathbf{G}_2"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathbf{D} \end{tikzcd} \circ \begin{tikzcd} \mathbf{B} \arrow[r, bend left, "\mathbf{F}_1"] \arrow[r, bend right, "\mathbf{F}_2"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathbf{C} \end{tikzcd} = \begin{tikzcd} \mathbf{B} \arrow[r, bend left, "\mathbf{G}_1\circ\mathbf{F}_1"] \arrow[r, bend right, "\mathbf{G}_2\circ\mathbf{F}_2"'] \arrow[r, phantom, "\Downarrow \beta_{\mathbf{F}_2}\circ\mathbf{G}_1\alpha"] & \mathbf{D} \end{tikzcd} \text{ in } \mathbf{Cat}_{(N,N^{op})}. \tag{4.3}$$

Recall that a contravariant functor between two enriched categories $\mathbf{C}, \mathbf{D}$ consists of an assignment $\mathbf{F} \colon \operatorname{ob}\mathbf{C} \to \operatorname{ob}\mathbf{D}$ and, for each pair of objects $X, Y \in \mathbf{C}$, a functor $\mathbf{F}_{(X,Y)} \colon \mathbf{C}(X,Y) \to \mathbf{D}(\mathbf{F}Y, \mathbf{F}X)$ that is compatible with composition in a natural sense. We now show that the assignments $\mathbf{Mod}, \mathbf{End}$ defined above are functors in this sense and that the isomorphisms in Proposition 4.12 form part of an adjunction between **Mod** and **End**. For background on enriched categories, see [14].

Theorem 4.16. *The assignments* **Mod** *and* **End** *define two contravariant functors* $\mathbf{Mod}\colon \mathbf{W}^*{}_{(N,N^{op})} \to \mathbf{Cat}_{(N,N^{op})}$ *and* $\mathbf{End}\colon \mathbf{Cat}_{(N,N^{op})} \to \mathbf{W}^*{}_{(N,N^{op})}$ *of enriched categories. The isomorphisms* $(\Phi_{\mathbf{C},A})_{\mathbf{C},A}$ *define an adjunction whose unit is* $(\eta_{\mathbf{C}})_{\mathbf{C}\in\mathbf{Cat}_{(N,N^{op})}}$ *and counit is* $(\epsilon_A)_{A\in\mathbf{W}^*{}_{(N,N^{op})}}$.

Proof. We first show that **Mod** and **End** are functors of enriched categories. By Corollary 4.14, it suffices to prove this for **End**. Consider a diagram as in (4.3) and let $a = \mathbf{End}_{(\mathbf{B},\mathbf{C})}(\alpha)$, $b = \mathbf{End}_{(\mathbf{C},\mathbf{D})}(\beta)$, $c = \mathbf{End}_{(\mathbf{B},\mathbf{D})}(\beta_{\mathbf{F}_2} \circ \mathbf{G}_1\alpha)$. We have to show that then the cells

$$\begin{tikzcd}
\mathbf{End}(\mathbf{C}) \arrow[r, bend left, "\mathbf{End}_{(\mathbf{B},\mathbf{C})}(\mathbf{F}_1)"] \arrow[r, bend right, "\mathbf{End}_{(\mathbf{B},\mathbf{C})}(\mathbf{F}_2)"'] \arrow[r, phantom, "\Downarrow a"] & \mathbf{End}(\mathbf{B})
\end{tikzcd} \circ \begin{tikzcd}
\mathbf{End}(\mathbf{D}) \arrow[r, bend left, "\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{G}_1)"] \arrow[r, bend right, "\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{G}_2)"'] \arrow[r, phantom, "\Downarrow b"] & \mathbf{End}(\mathbf{C})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{End}(D) \arrow[r, bend left, "\mathbf{End}_{(\mathbf{B},\mathbf{D})}(\mathbf{G}_1\mathbf{F}_1)"] \arrow[r, bend right, "\mathbf{End}_{(\mathbf{B},\mathbf{D})}(\mathbf{G}_2\mathbf{F}_2)"'] \arrow[r, phantom, "\Downarrow c"] & \mathbf{End}(B)
\end{tikzcd}$$

are equal. By definition, $a = \mathbf{H}_{\mathbf{C}}(\alpha_{G_{\mathbf{B}}})$, $b = \mathbf{H}_{\mathbf{D}}(\beta_{G_{\mathbf{C}}})$, and by Lemma 4.13,

$$c = \mathbf{H}_{\mathbf{D}}(\beta_{\mathbf{F}_2 G_{\mathbf{B}}} \cdot \mathbf{G}_1(\alpha_{G_{\mathbf{B}}})) = \rho^{\mathbf{F}_2 G_{\mathbf{B}}}(\mathbf{H}_{\mathbf{D}}(\beta_{G_{\mathbf{C}}})) \cdot \mathbf{H}_{\mathbf{C}}(\alpha_{G_{\mathbf{B}}}) = \mathbf{End}(\mathbf{F}_2)(b) \cdot a.$$

It remains to show that for all morphisms $\phi\colon A \to B$ in $\mathbf{W}^*{}_{(N,N^{op})}$ and $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ in $\mathbf{Cat}_{(N,N^{op})}$, the diagram

$$\begin{tikzcd}
\mathbf{Cat}_{(N,N^{op})}(\mathbf{D}, \mathbf{Mod}(B)) \arrow[r, "\Phi_{\mathbf{D},B}"] \arrow[d] & \mathbf{W}^*{}_{(N,N^{op})}(B, \mathbf{End}(\mathbf{D})) \arrow[d] \\
\mathbf{Cat}_{(N,N^{op})}(\mathbf{C}, \mathbf{Mod}(A)) \arrow[r, "\Phi_{\mathbf{C},A}"] & \mathbf{W}^*{}_{(N,N^{op})}(A, \mathbf{End}(\mathbf{C}))
\end{tikzcd}$$

commutes, where the vertical maps are induced by $\mathbf{F}$ and $\mathbf{Mod}_{(A,B)}(\phi)$ on the left and ϕ and $\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{F})$ on the right, respectively, or, more precisely, that for each object $\mathbf{G}$ and each morphism α in $\mathbf{Cat}_{(N,N^{op})}(\mathbf{D}, \mathbf{Mod}(B))$,

$$\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{F}) \circ \Phi_{\mathbf{D},B}(\mathbf{G}) \circ \phi = \Phi_{\mathbf{C},A}(\mathbf{Mod}_{(A,B)}(\phi) \circ \mathbf{G} \circ \mathbf{F}),$$
$$\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{F})(\alpha) = \mathbf{Mod}_{(A,B)}(\phi)(\alpha_{\mathbf{F}}).$$

The second equation holds because of Lemma 4.13 and the relation

$$\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{F})(\alpha_{G_{\mathbf{C}}}) = \rho^{\mathbf{F}G_{\mathbf{C}}}(\alpha_{G_{\mathbf{D}}}) = \alpha_{\mathbf{F}G_{\mathbf{C}}} = \mathbf{Mod}_{(A,B)}(\phi)(\alpha_{\mathbf{F}G_{\mathbf{C}}})$$

first one holds because by Corollary 4.14,

$$\mathbf{End}_{(\mathbf{C},\mathbf{D})}(\mathbf{F}) \circ \Phi_{\mathbf{D},B}(\mathbf{G}) \circ \phi = \rho^{\mathbf{F}G_{\mathbf{C}}} \circ \Phi_{\mathbf{D},B}(\mathbf{G}) \circ \phi,$$
$$(\mathbf{Mod}_{(A,B)}(\phi) \circ \mathbf{G} \circ \mathbf{F})(G_{\mathbf{C}}) = (\mathbf{H}_{\mathbf{C}} G_{\mathbf{C}}, \rho^{\mathbf{F}G_{\mathbf{C}}} \circ \Phi_{\mathbf{D},B}(\mathbf{G}) \circ \phi). \qquad \square$$

5. The special case of a commutative base

Let Z be a locally compact Hausdorff space with a Radon measure μ of full support, and identify $C_0(Z)$ with multiplication operators on $\mathcal{L}(L^2(Z,\mu))$. Then the relative tensor product and the fiber product over the C^*-base $\mathfrak{b} = (L^2(Z,\mu), C_0(Z), C_0(Z))$ can be related to the fiberwise product of bundles as follows.

Denote by $\mathbf{Mod}_{\mathfrak{b}}$, $\mathbf{Mod}_{C_0(Z)}$, and $\mathbf{Bdl}_Z$ the categories of all C^*-$\mathfrak{b}$-modules with all morphisms, of all Hilbert C^*-modules over $C_0(Z)$, and of all continuous Hilbert bundles over Z; for the precise definition of the latter, see [6]. Each of these categories carries a monoidal structure, where the product

(i) of $E, F \in \mathbf{Mod}_{C_0(Z)}$ is the separated completion of $E \odot F$ with respect to the inner product $\langle \xi \odot \eta | \xi' \odot \eta' \rangle = \langle \xi | \xi' \rangle \langle \eta | \eta' \rangle$, denoted by $E \underset{C_0(Z)}{\otimes} F$;
(ii) of $\mathcal{E}, \mathcal{F} \in \mathbf{Bdl}_Z$ is the fibrewise tensor product of $\mathcal{E}$ and $\mathcal{F}$;
(iii) of $H_\beta, K_\gamma \in \mathbf{Mod}_{\mathfrak{b}}$ is $(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K, \beta \bowtie \gamma)$, where $\beta \bowtie \gamma := [|\gamma\rangle_2 \beta] = [|\beta\rangle_1 \gamma]$; here, note that ${}_\beta H_\beta, {}_\gamma K_\gamma$ are C^*-$(\mathfrak{b},\mathfrak{b})$-modules.

There exist equivalences of monoidal categories

$$\mathbf{Mod}_{\mathfrak{b}} \underset{\mathbf{F}}{\overset{\mathbf{U}}{\rightleftarrows}} \mathbf{Mod}_{C_0(Z)} \underset{\Gamma_0}{\overset{\mathbf{B}}{\rightleftarrows}} \mathbf{Bdl}_Z$$

such that for each $E \in \mathbf{Mod}_{C_0(Z)}$, $\mathcal{F} \in \mathbf{Bdl}_Z$, $H_\beta \in \mathbf{Mod}_{\mathfrak{b}}$,

(i) $\mathbf{U} H_\beta = \beta \in \mathbf{Mod}_{C_0(Z)}$;
(ii) $\mathbf{F}E = (E \otimes_{C_0(Z)} L^2(Z,\mu), l(E))$, where $l(\xi)\eta = \xi \otimes_{C_0(Z)} \eta$ for each $\xi \in E, \eta \in L^2(Z,\mu)$;
(iii) $\mathbf{B}E = \coprod_{z \in Z} E_z$ and $\Gamma_0(\mathbf{B}E) = \{(\xi_z)_z \mid \xi \in E\}$, where E_z is the completion of E with respect to the inner product $(\xi, \eta) \mapsto \langle \xi | \eta \rangle (z)$, and $\xi \mapsto \xi_z$ denotes the quotient map $E \to E_z$;
(iv) the operations on the space of sections $\Gamma_0(\mathcal{F}) \in \mathbf{Mod}_{C_0(Z)}$ are defined fiberwise.

The equivalence on the left is easily verified, and the equivalence on the right is explained in [6]. Compare also Examples 2.7 and 2.12 (ii).

Denote by $\mathbf{C}^*_{C_0(Z)}$ the category of all continuous $C_0(Z)$-algebras with full support [6], where the morphisms between $A, B \in \mathbf{C}^*_{C_0(Z)}$ are all $C_0(Z)$-linear nondegenerate $*$-homomorphisms $\pi\colon A \to M(B)$, and by $\tilde{\mathbf{C}}^*_{\mathfrak{b}}$ the category of all C^*-$\mathfrak{b}$-algebras A^β_H satisfying $[\rho_\beta(C_0(Z))A] = A$ and $[A\beta] = \beta$, where the morphisms between $A^\beta_H, B^\gamma_K \in \tilde{\mathbf{C}}^*_{\mathfrak{b}}$ are all $\pi \in \mathbf{C}^*_{\mathfrak{b}}(A^\beta_H, M(B)^\gamma_K)$ satisfying $[\pi(A)B] = B$. Then there exists a functor $\tilde{\mathbf{C}}^*_{\mathfrak{b}} \to \mathbf{C}^*_{C_0(Z)}$, given by $A^\beta_H \mapsto (A, \rho_\alpha)$ and $\pi \mapsto \pi$, and this functor has a full and faithful left adjoint which embeds $\mathbf{C}^*_{C_0(Z)}$ into $\tilde{\mathbf{C}}^*_{\mathfrak{b}}$ [28, Theorem 6.6].

We finally consider the fiber product of commutative C^*-$\mathfrak{b}$-algebras and start with preliminaries. Let Z be a locally compact space, E a Hilbert C^*-module over $C_0(Z)$, and $\mathbf{B}E = \coprod_{z \in Z} E_z$ the corresponding Hilbert bundle. The topology on $\mathbf{B}E$ is generated by all open sets of the form $U_{V,\eta,\epsilon} = \{\zeta | z \in V, \zeta \in E_z, \|\eta_z - \zeta\|_{E_z} < \epsilon\}$, where $V \subseteq Z$ is open, $\eta \in E$, $\epsilon > 0$. Denote by $q\colon \coprod_{z \in Z} \mathcal{L}(E_z) \to Z$ the natural projection and define for each $\eta, \eta' \in E$ maps

$$\omega_{\eta,\eta'} : \bigsqcup_{z\in Z} \mathcal{L}(E_z) \to \mathbb{C}, \qquad T \mapsto \langle \eta_{q(T)} | T\eta'_{q(T)} \rangle,$$
$$\upsilon_\eta^{(*)} : \bigsqcup_{z\in Z} \mathcal{L}(E_z) \to \bigsqcup_{z\in Z} E_z, \quad T \mapsto T^{(*)}\eta_{q(T)}.$$

The *weak topology (strong-*-topology)* on $\bigsqcup_{z\in Z} \mathcal{L}(E_z)$ is the weakest one that makes q and all maps of the form $\omega_{\eta,\eta'}$ (of the form $\upsilon_\eta^{(*)}$) continuous.

Let A be a commutative C^*-algebra, let $\pi\colon C_0(Z) \to M(A)$ be a $*$-homomorphism, and let $\chi \in \widehat{A}$. Then we identify $E \otimes_{\phi^*} A \otimes_\chi \mathbb{C}$ with E_z, where $z \in Z$ corresponds to $\chi \circ \pi \in \widehat{C_0(Z)}$, via $\eta \otimes_\pi a \otimes_\chi \lambda \mapsto \lambda\chi(a)\eta_z$. A map $T\colon \widehat{A} \to \bigsqcup_{z\in Z} \mathcal{L}(E_z)$ is *weakly vanishing (strong-$*$-vanishing) at infinity* if for all $\eta, \eta' \in E$, the map $\omega_{\eta,\eta'} \circ T$ (the maps $\chi \mapsto \|\upsilon_\eta^{(*)}(T(\chi))\|$) vanish at infinity.

Lemma 5.1. *Let A_H^β be a C^*-$\mathfrak{b}$-algebra, K_γ a C^*-$\mathfrak{b}^\dagger$-module, $x \in \mathcal{L}(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K)$. Assume that A is commutative, $[\rho_\beta(C_0(Z))A] = A$, and $\langle\gamma|_2 x|\gamma\rangle_2 \subseteq A$. Define $F_x\colon \widehat{A} \to \bigsqcup_{z\in Z} \mathcal{L}(\gamma_z)$ by $\chi \mapsto (\chi * \mathrm{id})(x)$. Then:*

(i) *F_x is weakly continuous, weakly vanishing at infinity.*
(ii) *$x \in \mathrm{Ind}_{|\gamma\rangle_2}(A)$ if and only if F_x is strong-$*$ continuous, strong-$*$-vanishing at infinity.*

Proof. First, note that for all $\eta, \eta' \in \gamma$ and $\chi \in \hat{A}$,

$$\chi(\langle\eta|_2 x|\eta'\rangle_2) = \langle 1_{(\chi\circ\rho_\beta)} \otimes \eta | (\chi * \mathrm{id})(x)(1_{(\chi\circ\rho_\beta)} \otimes \eta')\rangle = \langle \eta_{(\chi\circ\rho_\beta)} | F_x(\chi)\eta'_{(\chi\circ\rho_\beta)}\rangle.$$

(i) For each $\eta', \eta \in \gamma$, the map $\chi \mapsto \langle \eta_{(\chi\circ\rho_\beta)} | F_x(\chi)\eta'_{(\chi\circ\rho_\beta)}\rangle$ equals $\langle\eta|_2 x|\eta'\rangle_2 \in A$.

(ii) Assume that F_x is strong-$*$ continuous vanishing at infinity and let $\eta \in \gamma$. Then the map $\chi \mapsto F_x(\chi)\eta_{(\chi\circ\rho_\beta)}$ lies in $\Gamma_0(\gamma \otimes_{\rho_\beta} A)$. Hence, there exists an $\omega \in \gamma \otimes_{\rho_\beta} A$ such that $F_x(\chi)\eta_{(\chi\circ\rho_\beta)} = \omega_\chi$ for all $\chi \in \widehat{A}$. We identify $\gamma \otimes_{\rho_\beta} A$ with $[|\gamma\rangle_2 A] \subseteq \mathcal{L}(H, H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K)$ in the canonical manner and find that $x|\eta\rangle_2 = \omega$ because $\chi(\langle\eta'|_2 x|\eta\rangle_2) = \langle \eta'_{(\chi\circ\rho_\beta)} | \omega_{(\chi\circ\rho_\beta)}\rangle = \chi(\langle\eta'|_2\omega)$ for all $\chi \in \hat{A}$, $\eta' \in \gamma$. Since $\eta \in \gamma$ was arbitrary, we can conclude $x|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A]$. A similar argument, applied to x^* instead of x, shows that $x^*|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A]$, and therefore $x \in \mathrm{Ind}_{|\gamma\rangle_2}(A)$. Reversing the arguments, we obtain the reverse implication. □

Let X be a locally compact Hausdorff space with a continuous surjection $p\colon X \to Z$ and a family of Radon measures $\phi = (\phi_z)_{z\in Z}$ such that (i) $\operatorname{supp}\phi_z = X_z := p^{-1}(z)$ for each $z \in Z$ and (ii) the map $\phi_*(f)\colon z \mapsto \int_{X_z} f d\phi_z$ is continuous for each $f \in C_c(X)$. Define a Radon measure ν_X on X such that $\int_X f\, \mathrm{d}\nu_X = \int_Z \phi_*(f) d\mu$ for all $f \in C_c(X)$. Then there exists a unique map $j_X\colon C_c(X) \to \mathcal{L}(L^2(Z,\mu), L^2(X,\nu_X))$ such that $j_X(f)h = fp^*(h)$ and $j_X(f)^*g = \phi_*(\overline{f}g)$ for all $f, g \in C_c(X), h \in C_c(Z)$. Similarly, let Y be a locally compact Hausdorff space with a continuous map $q\colon Y \to Z$ and a family of measures $\psi = (\psi_z)_{z\in Z}$ satisfying the same conditions as X, p, ϕ, and define a Radon measure ν_Y on Y and an embedding $j_Y\colon C_c(Y) \to \mathcal{L}(L^2(Z,\mu), L^2(Y,\nu_Y))$ as above. Let

$$\begin{aligned} H &:= L^2(X,\nu_X), & \beta &:= [j_X(C_c(X))], & A &:= C_0(X) \subseteq \mathcal{L}(L^2(X,\nu_X) = \mathcal{L}(H), \\ K &:= L^2(Y,\nu_Y), & \gamma &:= [j_Y(C_c(Y))], & B &:= C_0(Y) \subseteq \mathcal{L}(L^2(Y,\nu_Y)) = \mathcal{L}(K). \end{aligned}$$

Then H_β, K_γ are C^*-$\mathfrak{b}$-modules and A_H^β, B_K^γ are C^*-$\mathfrak{b}$-algebras, as one can easily check. Considering β and γ as Hilbert C^*-modules over $C_0(Z)$, we can canonically identify $\beta_z \cong L^2(X_z,\phi_z)$ and $\gamma_z \cong L^2(Y_z,\psi_z)$. Finally, define a Radon measure ν on $X_p\underset{Z}{\times}{}_qY$ such that for all $h \in C_c(X_p\underset{Z}{\times}{}_qY)$,

$$\int_{X_p\underset{Z}{\times}{}_qY} h \,\mathrm{d}\nu = \int_Z \int_{X_z} \int_{Y_z} h(x,y)\,\mathrm{d}\psi_z(y)\,\mathrm{d}\phi_z(x)\,\mathrm{d}\mu(z).$$

Proposition 5.2. (i) *There exists a unitary* $U\colon H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K \to L^2(X_p\underset{Z}{\times}{}_qY,\nu)$ *such that* $(U(j_X(f) \otimes h \otimes j_Y(g)))(x,y) = f(x)h(p(x))g(y)$ *for all* $f \in C_c(X)$, $g \in C_c(Y)$, $h \in C_c(Z)$, $(x,y) \in X_p\underset{Z}{\times}{}_qY$.

(ii) $\mathrm{Ad}_U(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)$ *is the* C^**-algebra of all* $f \in L^\infty(X_p\underset{Z}{\times}{}_qY,\nu)$ *that have representatives* f_X, f_Y *such that the maps* $X \to \operatorname{Tot}\mathcal{L}(\gamma)$ *and* $Y \to \operatorname{Tot}\mathcal{L}(\beta)$ *given by* $x \mapsto f_X(x,\cdot) \in L^\infty(Y_{p(x)},\psi_{p(x)})$ *and* $y \mapsto f_Y(\cdot,y) \in L^\infty(X_{q(y)},\phi_{q(y)})$ *respectively, are strong-$*$ continuous vanishing at infinity.*

Proof. The proof of assertion (i) is straightforward, and assertion (ii) follows immediately from Proposition Lemma 3.16 (viii) and Lemma 5.1 (ii). □

Example 5.3. (i) *Let X,Y be discrete, $Z=\{0\}$, and let ϕ_0, ψ_0 be the counting measures on X,Y, respectively. Then*

$$\begin{aligned} C_0(X)_\beta \underset{\mathfrak{b}}{*}{}_\gamma C_0(Y) \cong \{f \in C_b(X\times Y)\,|\, &f(x,\cdot\,) \in C_0(Y) \text{ for all } x \in X, \\ &f(\,\cdot\,,y) \in C_0(X) \text{ for all } y \in Y\}. \end{aligned}$$

This follows from Proposition 5.2 and the fact that for each $f \in C_b(X\times Y)$, the maps $X \to \mathcal{L}(l^2(Y))$, $x \mapsto f(x,\cdot\,)$, and $Y \to \mathcal{L}(l^2(X))$, $y \mapsto f(\,\cdot\,,y)$, are strong-$*$ continuous vanishing at infinity if and only if $f(\,\cdot\,,y) \in C_0(X)$ and $f(x,\cdot\,) \in C_0(Y)$ for each $y \in Y$ and $x \in X$.

(ii) *Let $X = \mathbb{N}$, $Z=\{0\}$, and let ϕ_0 be the counting measure. Then*

$$\begin{aligned} C_0(\mathbb{N})_\beta \underset{\mathfrak{b}}{*}{}_\gamma C_0(Y) \cong \{f \in C_b(\mathbb{N}\times Y) \mid (f(x,\cdot))_x &\text{ is a sequence in } C_0(Y) \\ &\text{that converges strongly to } 0 \in \mathcal{L}(L^2(Y,\psi_0))\} \end{aligned}$$

because for each $f \in L^\infty(\mathbb{N}\times Y)$, the map $Y \to \mathcal{L}(l^2(\mathbb{N}))$, $y \mapsto f(\,\cdot\,,y)$, is strong-$*$ continuous vanishing at infinity if and only if $f(x,\cdot\,) \in C_0(Y)$ for all $x \in \mathbb{N}$.

(iii) *Let $X = Y = [0,1]$, $Z = \{0\}$, and let $\phi_0 = \psi_0$ be the Lebesgue measure. For each subset $I \subseteq [0,1]$, denote by χ_I its characteristic function. Then the function $f \in L^\infty([0,1]\times[0,1])$ given by $f(x,y) = 1$ if $y \le x$ and $f(x,y) = 0$ otherwise belongs* *to* $C([0,1])_\beta \underset{\mathfrak{b}}{*}{}_\gamma C([0,1])$ *because the functions* $[0,1] \to L^\infty([0,1]) \subseteq \mathcal{L}(L^2([0,1]))$ *given by* $x \mapsto f(x,\cdot\,) = \chi_{[0,x]}$ *and* $y \mapsto f(\,\cdot\,,y) = \chi_{[y,1]}$ *are strong-$*$ continuous. In particular, we see that* $C([0,1])_\beta \underset{\mathfrak{b}}{*}{}_\gamma C([0,1]) \not\subseteq C([0,1]\times[0,1]) = C([0,1]) \otimes C([0,1])$.

References

[1] É. Blanchard. Déformations de C^*-algèbres de Hopf. *Bull. Soc. Math. France*, 124(1):141–215, 1996.

[2] G. Böhm. Hopf algebroids. In *Handbook of algebra. Vol. 6*, volume 6 of *Handb. Algebr.*, pages 173–235. Elsevier/North-Holland, Amsterdam, 2009.

[3] A. Connes. On the spatial theory of von Neumann algebras. *J. Funct. Anal.*, 35(2):153–164, 1980.

[4] A. Connes. *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994.

[5] J. Dixmier. *Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann)*. Cahiers Scientifiques, Fasc. XXV. Gauthier-Villars Éditeur, Paris, 1969. Deuxième édition, revue et augmentée.

[6] M. J. Dupré and R. M. Gillette. *Banach bundles, Banach modules and automorphisms of C^*-algebras*, volume 92 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1983.

[7] S. Echterhoff, S. Kaliszewski, J. Quigg, and I. Raeburn. A categorical approach to imprimitivity theorems for C^*-dynamical systems. *Mem. Amer. Math. Soc.*, 180(850):viii+169, 2006.

[8] M. Enock. Measured quantum groupoids in action. *Mém. Soc. Math. Fr. (N.S.)*, (114):ii+150 pp. (2009), 2008.

[9] M. Enock and J.-M. Vallin. Inclusions of von Neumann algebras, and quantum groupoids. *J. Funct. Anal.*, 172(2):249–300, 2000.

[10] T. Falcone. L^2-von Neumann modules, their relative tensor products and the spatial derivative. *Illinois J. Math.*, 44(2):407–437, 2000.

[11] P. Ghez, R. Lima, and J. E. Roberts. W^*-categories. *Pacific J. Math.*, 120(1):79–109, 1985.

[12] U. Haagerup. Operator-valued weights in von Neumann algebras. I. *J. Funct. Anal.*, 32(2):175–206, 1979.

[13] U. Haagerup. Operator-valued weights in von Neumann algebras. II. *J. Funct. Anal.*, 33(3):339–361, 1979.

[14] G. M. Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1982.

[15] J. Kustermans. KMS-weights on C^*-algebras. Technical report, Odense Universitet, 1997. Available at `arxiv.org/funct-an/9704008`.

[16] E. C. Lance. *Hilbert C^*-modules. A toolkit for operator algebraists*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995.

[17] T. Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.

[18] F. Lesieur. Measured quantum groupoids. *Mém. Soc. Math. Fr. (N.S.)*, (109):iv+158 pp. (2008), 2007.

[19] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[20] J.-L. Sauvageot. Sur le produit tensoriel relatif d'espaces de Hilbert. *J. Operator Theory*, 9(2):237–252, 1983.

[21] J.-L. Sauvageot. Produits tensoriels de Z-modules et applications. In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, volume 1132 of *Lecture Notes in Mathematics*, pages 468–485. Springer, Berlin, 1985.

[22] M. Skeide. von Neumann modules, intertwiners and self-duality. *J. Operator Theory*, 54(1):119–124, 2005.

[23] M. Takesaki. *Theory of operator algebras. II*, volume 125 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003.

[24] M. Takeuchi. Groups of algebras over $A \otimes \overline{A}$. *J. Math. Soc. Japan*, 29(3):459–492, 1977.

[25] T. Timmermann. A definition of compact C^*-quantum groupoids. In *Operator Structures and Dynamical Systems, Leiden 2008*, *Contemp. Math.* 503:267-290, 2009.

[26] T. Timmermann. From Hopf C^*-families to concrete Hopf C^*-bimodules. Technical report, SFB 478, WW-University Muenster, December 2007.

[27] T. Timmermann. Pseudo-multiplicative unitaries on C^*-modules and Hopf C^*-families. *J. Noncomm. Geom.*, 1:497–542, 2007.

[28] T. Timmermann. Coactions of Hopf C^*-bimodules. *J. Operator Theory* 68(1):19-66, 2012.

[29] T. Timmermann. C^*-pseudo-multiplicative unitaries, Hopf C^*-bimodules, and their Fourier algebras. *J. Inst. Math. Jussieu*, available on CJO 2011, doi:10.1017/S1474748010000290.

[30] J.-M. Vallin. Bimodules de Hopf et poids opératoriels de Haar. *J. Operator Theory*, 35(1):39–65, 1996.

[31] J.-M. Vallin. Unitaire pseudo-multiplicatif associé à un groupoïde. Applications à la moyennabilité. *J. Operator Theory*, 44(2):347–368, 2000.

[32] P. Xu. Quantum groupoids. *Comm. Math. Phys.*, 216(3):539–581, 2001.

T. TIMMERMANN, University of Muenster,, Einsteinstr. 62, 48149 Muenster, Germany
E-mail address: `timmermt@math.uni-muenster.de`

APPENDIX I.2

C^*-PSEUDO-MULTIPLICATIVE UNITARIES, HOPF C^*-BIMODULES, AND THEIR FOURIER ALGEBRAS

THOMAS TIMMERMANN

J. Inst. Math. Jussieu *11(1):189-220, 2012.*

Abstract. We introduce C^*-pseudo-multiplicative unitaries and concrete Hopf C^*-bimodules for the study of quantum groupoids in the setting of C^*-algebras. These unitaries and Hopf C^*-bimodules generalize multiplicative unitaries and Hopf C^*-algebras and are analogues of the pseudo-multiplicative unitaries and Hopf–von Neumann-bimodules studied by Enock, Lesieur and Vallin. To each C^*-pseudo-multiplicative unitary, we associate two Fourier algebras with a duality pairing and in the regular case two Hopf C^*-bimodules. The theory is illustrated by examples related to locally compact Hausdorff groupoids. In particular, we obtain a continuous Fourier algebra for a locally compact Hausdorff groupoid.

Contents

2000 *Mathematics Subject Classification.* 46L55.
Key words and phrases. quantum groupoid, groupoid, duality.

1. Introduction

Multiplicative unitaries, which were first systematically studied by Baaj and Skandalis [3], are fundamental to the theory of quantum groups in the setting of operator algebras and to generalizations of Pontrjagin duality [28]. First, one can associate to every locally compact quantum group a multiplicative unitary [13, 14, 17]. Out of this unitary, one can construct two Hopf C^*-algebras, where one coincides with the initial quantum group, while the other is the generalized Pontrjagin dual of the quantum group. The duality manifests itself by a pairing on dense Fourier subalgebras of the two Hopf C^*-algebras. These Hopf C^*-algebras can be completed to Hopf–von Neumann algebras and are reduced in the sense that they correspond to the regular representations of the quantum group and of its dual, respectively.

Much of the theory of quantum groups has been generalized for quantum groupoids in a variety of settings, for example, for finite quantum groupoids in the setting of finite-dimensional C^*-algebras by Böhm, Szlachányi, Nikshych and others [5, 6, 7, 18] and for measurable quantum groupoids in the setting of von Neumann algebras by Enock, Lesieur and Vallin [9, 10, 11, 16]. Fundamental for the second theory are the Hopf–von Neumann bimodules and pseudo-multiplicative unitaries introduced by Vallin [32, 33].

In this article, we introduce generalizations of multiplicative unitaries and Hopf C^*-algebras that are suited for the study of locally compact quantum groupoids in the setting of C^*-algebras, and extend some of the results on multiplicative unitaries that were obtained by Baaj and Skandalis in [3]. In particular, we associate to every regular C^*-pseudo-multiplicative unitary two Hopf C^*-bimodules and two Fourier algebras with a duality pairing.

Our concepts are related to their von Neumann-algebraic counterparts as follows. In the theory of quantum groups, one can use the multiplicative unitary to pass between the setting of von Neumann algebras and the setting of C^*-algebras and thus obtains a bijective correspondence between measurable and locally compact quantum groups. This correspondence breaks down for quantum groupoids — already for ordinary spaces, considered as groupoids consisting entirely of units, a measure does not determine a topology. In particular, one can not expect to pass from a measurable quantum groupoid in the setting of von Neumann algebras to a locally compact quantum groupoid in the setting of C^*-algebras in a canonical way. The reverse passage, however, is possible, at least on the level of the unitaries and the Hopf bimodules.

Fundamental to our approach is the framework of modules, relative tensor products and fiber products in the setting of C^*-algebras introduced in [25]. That article also explains in detail how the theory developed here can be reformulated in the setting of von Neumann algebras, where we recover Vallin's notions of a pseudo-multiplicative unitary and a Hopf–von Neumann bimodule, and how to pass from the level of C^*-algebras to the setting of von Neumann algebras by means of various functors.

The theory presented here overcomes several restrictions of our former generalizations of multiplicative unitaries and Hopf C^*-algebras [27]; see also [26]. It was applied already in [31] to the definition and study of compact C^*-quantum groupoids, and in [30] to the study of reduced crossed products for coactions of Hopf C^*-bimodules on C^*-algebras and to an extension of the Baaj-Skandalis duality theorem. In [29], we furthermore associate to every C^*-pseudo-multiplicative unitary a C^*-tensor category of (co)representations

and two universal Hopf C^*-bimodules that are related to the reduced Hopf C^*-bimodules studied here similarly like the universal to the reduced C^*-algebra of a group or groupoid.
This work was supported by the SFB 478 "Geometrische Strukturen in der Mathematik"[1] and initiated during a stay at the "Special Programme on Operator Algebras" at the Fields Institute in Toronto, Canada, in July 2007.

Organization. This article is organized as follows. We start with preliminaries, summarizing notation, terminology and some background on Hilbert C^*-modules.

In Section 2, we recall the notion of a multiplicative unitary and define C^*-pseudo-multiplicative unitaries. This definition involves C^*-modules over C^*-bases and their relative tensor product, which were introduced in [25] and which we briefly recall. As an example, we construct the C^*-pseudo-multiplicative unitary of a locally compact Hausdorff groupoid. We shall come back to this example frequently.

In Section 3, we associate to every well behaved C^*-pseudo-multiplicative unitary two Hopf C^*-bimodules. These Hopf C^*-bimodules are generalized Hopf C^*-algebras, where the target of the comultiplication is no longer a tensor product but a fiber product that is taken relative to an underlying C^*-base. Inside these Hopf C^*-bimodules, we identify dense convolution subalgebras which can be considered as generalized Fourier algebras, and construct a dual pairing on these subalgebras. To illustrate the theory, we apply all constructions to the unitary associated to a groupoid G, where one recovers the reduced groupoid C^*-algebra of G on one side and the function algebra of G on the other side.

In Section 4, we show that every C^*-pseudo-multiplicative unitary satisfying a certain regularity condition is well behaved. This condition is satisfied, for example, by the unitaries associated to groupoids and by the unitaries associated to compact quantum groupoids. Furthermore, we collect some results on proper and étale C^*-pseudo-multiplicative unitaries.

Terminology and notation. Given a subset Y of a normed space X, we denote by $[Y] \subset X$ the closed linear span of Y. We call a linear map ϕ between normed spaces *contractive* or a *linear contraction* if $\|\phi\| \leq 1$.

All sesquilinear maps like inner products of Hilbert spaces are assumed to be conjugate-linear in the first component and linear in the second one. Let H, K be Hilbert spaces. We denote by X' the commutant of a subset $X \subseteq \mathcal{L}(H)$. Given a C^*-subalgebra $A \subseteq \mathcal{L}(H)$ and a $*$-homomorphism $\pi\colon A \to \mathcal{L}(K)$, we put

$$\mathcal{L}^{\pi}(H,K) := \{T \in \mathcal{L}(H,K) \mid Ta = \pi(a)T \text{ for all } a \in A\}. \tag{1}$$

We shall use some theory of groupoids; for background, see [22] or [20]. Given a groupoid G, we denote its unit space by G^0, its range map by r, its source map by s, and let $G_r{\times_r}G = \{(x,y) \in G \times G \mid r(x) = r(y)\}$, $G_s{\times_r}G = \{(x,y) \in G \times G \mid s(x) = r(y)\}$ and $G^u = r^{-1}(u)$, $G_u = s^{-1}(u)$ for each $u \in G^0$.

We shall make extensive use of (right) Hilbert C^*-modules and the internal tensor product; a standard reference is [15]. Let A and B be C^*-algebras. Given Hilbert C^*-modules E and F over B, we denote by $\mathcal{L}_B(E,F)$ the space of all adjointable operators from E to F. Let E and F be C^*-modules over A and B, respectively, and let $\pi\colon A \to \mathcal{L}_B(F)$ be a $*$-homomorphism. Recall that the internal tensor product $E \otimes_\pi F$ is the Hilbert C^*-module over B which is the closed linear span of elements $\eta \otimes_\pi \xi$, where $\eta \in E$ and

---

[1] funded by the Deutsche Forschungsgemeinschaft (DFG)

$\xi \in F$ are arbitrary and $\langle \eta \otimes_\pi \xi | \eta' \otimes_\pi \xi' \rangle = \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\eta \otimes_\pi \xi)b = \eta \otimes_\pi \xi b$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$ [15, §4]. We denote the internal tensor product by "$\otimes$" and drop the index π if the representation is understood; thus, for example, $E \otimes F = E \otimes_\pi F = E \otimes_\pi F$.

We also define a *flipped internal tensor product* $F {}_\pi\!\otimes E$ as follows. We equip the algebraic tensor product $F \odot E$ with the structure maps $\langle \xi \odot \eta | \xi' \odot \eta' \rangle := \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$, $(\xi \odot \eta)b := \xi b \odot \eta$, form the separated completion, and obtain a Hilbert C^*-module $F {}_\pi\!\otimes E$ over B which is the closed linear span of elements $\xi {}_\pi\!\otimes \eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary and $\langle \xi {}_\pi\!\otimes \eta | \xi' {}_\pi\!\otimes \eta' \rangle = \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\xi {}_\pi\!\otimes \eta)b = \xi b {}_\pi\!\otimes \eta$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$. As above, we drop the index π and simply write "$\otimes$" instead of "${}_\pi\!\otimes$" if the representation π is understood. Evidently, the usual and the flipped internal tensor product are related by a unitary map $\Sigma \colon F \otimes E \xrightarrow{\cong} E \otimes F$, $\eta \otimes \xi \mapsto \xi \otimes \eta$.

For each $\xi \in E$, the maps $F \to E \otimes F$ and $F \to F \otimes E$ given by $\eta \mapsto \xi \otimes \eta$ and $\eta \mapsto \eta \otimes \xi$, respectively, are adjointable, and the adjoints are given by $\xi' \otimes \eta \mapsto \pi(\langle \xi | \xi' \rangle)\eta$ and $\eta \otimes \xi' \mapsto \pi(\langle \xi | \xi' \rangle)\eta$, respectively.

Finally, let E_1, E_2 be Hilbert C^*-modules over A, let F_1, F_2 be Hilbert C^*-modules over B with representations $\pi_i \colon A \to \mathcal{L}_B(F_i)$ $(i = 1, 2)$, and let $S \in \mathcal{L}_A(E_1, E_2)$, $T \in \mathcal{L}_B(F_1, F_2)$ such that $T\pi_1(a) = \pi_2(a)T$ for all $a \in A$. Then there exists a unique operator $S \otimes T \in \mathcal{L}_B(E_1 \otimes F_1, E_2 \otimes F_2)$ such that $(S \otimes T)(\eta \otimes \xi) = S\eta \otimes T\xi$ for all $\eta \in E_1$, $\xi \in F_1$, and $(S \otimes T)^* = S^* \otimes T^*$ [8, Proposition 1.34].

2. C^*-pseudo-multiplicative unitaries

Recall that a multiplicative unitary on a Hilbert space H is a unitary $V \colon H \otimes H \to H \otimes H$ that satisfies the *pentagon equation* $V_{12}V_{13}V_{23} = V_{23}V_{12}$ (see [3]). Here, V_{12}, V_{13}, V_{23} are operators on $H \otimes H \otimes H$ defined by $V_{12} = V \otimes \mathrm{id}$, $V_{23} = \mathrm{id} \otimes V$, $V_{13} = (\Sigma \otimes \mathrm{id})V_{23}(\Sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \Sigma)V_{12}(\mathrm{id} \otimes \Sigma)$, where $\Sigma \in \mathcal{L}(H \otimes H)$ denotes the flip $\eta \otimes \xi \mapsto \xi \otimes \eta$. If G is a locally compact group with left Haar measure λ, then the formula

$$(Vf)(x, y) = f(x, x^{-1}y) \tag{2}$$

defines a linear bijection of $C_c(G \times G)$ which extends to a unitary on $L^2(G \times G, \lambda \otimes \lambda) \cong L^2(G, \lambda) \otimes L^2(G, \lambda)$. This unitary is multiplicative, and the pentagon equation amounts to associativity of the multiplication in G.

We shall generalize the notion of a multiplicative unitary so that it covers the example above if we replace the group G by a locally compact Hausdorff groupoid G. In that case, formula (2) defines a linear bijection from $C_c(G_s \times_r G)$ to $C_c(G_r \times_r G)$. If G is finite, that bijection is a unitary from $l^2(G_s \times_r G)$ to $l^2(G_r \times_r G)$, and these two Hilbert spaces can be identified with tensor products of $l^2(G)$ with $l^2(G)$ relative to the algebra $C(G^0)$. For a general groupoid, the algebraic tensor product of modules has to be replaced by a refined version. In the measurable setting, the appropriate substitute is the tensor product of Hilbert modules relative to a von Neumann algebra also known as Connes' fusion, see [33]. To take the topology of G into account, we shall work in the setting of C^*-algebras and use the relative tensor product of C^*-modules over C^*-bases introduced in [25].

2.1. **The relative tensor product of C^*-modules over C^*-bases.** Fundamental to the definition of a C^*-pseudo-multiplicative unitary is the relative tensor product of C^*-modules over C^*-bases introduced in [25]. We briefly recall this construction; for further details, see [25]. An example will appear in subsection 2.3.

C^*-modules over C^*-bases. A C^*-*base* is a triple $(\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ consisting of a Hilbert space $\mathfrak{K}$ and two commuting nondegenerate C^*-algebras $\mathfrak{B}, \mathfrak{B}^\dagger \subseteq \mathcal{L}(\mathfrak{K})$. A C^*-base should be thought of as a C^*-algebraic counterpart to pairs consisting of a von Neumann algebra and its commutant. As an example, one can associate to every faithful KMS-state μ on a C^*-algebra B the C^*-base (H_μ, B, B^{op}), where H_μ is the GNS-space for μ and B and B^{op} act on $H_\mu = H_{\mu^{op}}$ via the GNS-representations [25, Example 2.9]. The *opposite* of a C^*-base $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ is the C^*-base $\mathfrak{b}^\dagger := (\mathfrak{K}, \mathfrak{B}^\dagger, \mathfrak{B})$.

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. A C^*-$\mathfrak{b}$-*module* is a pair $H_\alpha = (H, \alpha)$, where H is a Hilbert space and $\alpha \subseteq \mathcal{L}(\mathfrak{K}, H)$ is a closed subspace satisfying $[\alpha\mathfrak{K}] = H$, $[\alpha\mathfrak{B}] = \alpha$, and $[\alpha^*\alpha] = \mathfrak{B} \subseteq \mathcal{L}(\mathfrak{K})$. If H_α is a C^*-$\mathfrak{b}$-module, then α is a Hilbert C^*-module over $\mathfrak{B}$ with inner product $(\xi, \xi') \mapsto \xi^*\xi'$ and there exist isomorphisms

$$\alpha \ominus \mathfrak{K} \to H,\ \xi \ominus \zeta \mapsto \xi\zeta, \qquad \mathfrak{K} \oslash \alpha \to H,\ \zeta \oslash \xi \mapsto \xi\zeta, \tag{3}$$

and a nondegenerate representation

$$\rho_\alpha \colon \mathfrak{B}^\dagger \to \mathcal{L}(H), \quad \rho_\alpha(b^\dagger)(\xi\zeta) = \xi b^\dagger \zeta \quad \text{for all } b^\dagger \in \mathfrak{B}^\dagger, \xi \in \alpha, \zeta \in \mathfrak{K}.$$

A *morphism* between C^*-$\mathfrak{b}$-modules H_α and K_β is an operator $T \in \mathcal{L}(H, K)$ satisfying $T\alpha \subseteq \beta$ and $T^*\beta \subseteq \alpha$. We denote the set of all morphisms by $\mathcal{L}(H_\alpha, K_\beta)$. If $T \in \mathcal{L}(H_\alpha, K_\beta)$, then $T\rho_\alpha(b^\dagger) = \rho_\beta(b^\dagger)T$ for all $b^\dagger \in \mathfrak{B}^\dagger$, and left multiplication by T defines an operator in $\mathcal{L}_{\mathfrak{B}}(\alpha, \beta)$ which we again denote by T.

Let $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ be C^*-bases, where $\mathfrak{b}_i = (\mathfrak{K}_i, \mathfrak{B}_i, \mathfrak{B}_i^\dagger)$ for each i. A C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-*module* is a tuple $(H, \alpha_1, \ldots, \alpha_n)$, where H is a Hilbert space and (H, α_i) is a C^*-$\mathfrak{b}_i$-module for each i such that $[\rho_{\alpha_i}(\mathfrak{B}_i^\dagger)\alpha_j] = \alpha_j$ whenever $i \neq j$. In the case $n = 2$, we abbreviate ${}_\alpha H_\beta := (H, \alpha, \beta)$. We note that if $(H, \alpha_1, \ldots, \alpha_n)$ is a C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-module, then $[\rho_{\alpha_i}(\mathfrak{B}_i^\dagger), \rho_{\alpha_j}(\mathfrak{B}_j^\dagger)] = 0$ whenever $i \neq j$. The set of *morphisms* between C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-modules $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_n)$, $\mathcal{K} = (K, \gamma_1, \ldots, \gamma_n)$ is the set $\mathcal{L}(\mathcal{H}, \mathcal{K}) := \bigcap_{i=1}^n \mathcal{L}(H_{\alpha_i}, K_{\gamma_i}) \subseteq \mathcal{L}(H, K)$.

The relative tensor product. Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, H_β C^*-$\mathfrak{b}$-module, and K_γ a C^*-$\mathfrak{b}^\dagger$-module. The *relative tensor product* of H_β and K_γ is the Hilbert space

$$H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K := \beta \ominus \mathfrak{K} \oslash \gamma.$$

It is spanned by elements $\xi \ominus \zeta \oslash \eta$, where $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, and

$$\langle \xi \ominus \zeta \oslash \eta | \xi' \ominus \zeta' \oslash \eta' \rangle = \langle \zeta | \xi^*\xi'\eta^*\eta'\zeta' \rangle = \langle \zeta | \eta^*\eta'\xi^*\xi'\zeta' \rangle$$

for all $\xi, \xi' \in \beta$, $\zeta, \zeta' \in \mathfrak{K}$, $\eta, \eta' \in \gamma$. The formula $\xi \ominus \zeta \oslash \eta \mapsto \eta \ominus \zeta \oslash \xi$ obviously defines a unitary *flip* $\Sigma \colon H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K \to K_\gamma \underset{\mathfrak{b}^\dagger}{\otimes}_\beta H$. Using the unitaries in (3) on H_β and K_γ, respectively, we shall make the following identifications without further notice:

$$H_{\rho_\beta} \oslash \gamma \cong H_\beta \underset{\mathfrak{b}}{\otimes}_\gamma K \cong \beta \ominus_{\rho_\gamma} K, \quad \xi\zeta \oslash \eta \equiv \xi \ominus \zeta \oslash \eta \equiv \xi \ominus \eta\zeta.$$

For all $S \in \rho_\beta(\mathfrak{B}^\dagger)'$ and $T \in \rho_\gamma(\mathfrak{B})'$, we have operators

$$S \otimes \mathrm{id} \in \mathcal{L}(H_{\rho_\beta} \otimes \gamma) = \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K), \qquad \mathrm{id} \otimes T \in \mathcal{L}(\beta \otimes_{\rho_\gamma} K) = \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K).$$

If $S \in \mathcal{L}(H_\beta)$ or $T \in \mathcal{L}(K_\gamma)$, then $(S \otimes \mathrm{id})(\xi \otimes \eta\zeta) = S\xi \otimes \eta\zeta$ or $(\mathrm{id} \otimes T)(\xi\zeta \otimes \eta) = \xi\zeta \otimes T\eta$, respectively, for all $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, so that we can define

$$S \underset{\mathfrak{b}}{\otimes} T := (S \otimes \mathrm{id})(\mathrm{id} \otimes T) = (\mathrm{id} \otimes T)(S \otimes \mathrm{id}) \in \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$$

for all (S,T) in $\mathcal{L}(H_\beta) \times \rho_\gamma(\mathfrak{B})'$ or $\rho_\beta(\mathfrak{B}^\dagger)' \times \mathcal{L}(K_\gamma)$.
For each $\xi \in \beta$ and $\eta \in \gamma$, there exist bounded linear operators

$$|\xi\rangle_1 \colon K \to H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K,\ \omega \mapsto \xi \otimes \omega, \qquad |\eta\rangle_2 \colon H \to H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K,\ \omega \mapsto \omega \otimes \eta,$$

whose adjoints $\langle\xi|_1 := |\xi\rangle_1^*$ and $\langle\eta|_2 := |\eta\rangle_2^*$ are given by $\xi' \otimes \omega \mapsto \rho_\gamma(\xi^*\xi')\omega$ and $\omega \otimes \eta' \mapsto \rho_\beta(\eta^*\eta')\omega$, respectively. We put $|\beta\rangle_1 := \{|\xi\rangle_1 \,|\, \xi \in \beta\} \subseteq \mathcal{L}(K, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$ and similarly define $\langle\beta|_1$, $|\gamma\rangle_2$, $\langle\gamma|_2$.
Let $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_m, \beta)$ be a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{b})$-module and $\mathcal{K} = (K, \gamma, \delta_1, \ldots, \delta_n)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, where $\mathfrak{a}_i = (\mathfrak{H}_i, \mathfrak{A}_i, \mathfrak{A}_i^\dagger)$ and $\mathfrak{c}_j = (\mathfrak{L}_j, \mathfrak{C}_j, \mathfrak{C}_j^\dagger)$ for all i, j. We put

$$\alpha_i \triangleleft \gamma := [|\gamma\rangle_2 \alpha_i] \subseteq \mathcal{L}(\mathfrak{H}_i, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K), \qquad \beta \triangleright \delta_j := [|\beta\rangle_1 \delta_j] \subseteq \mathcal{L}(\mathfrak{L}_j, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$$

for all i, j. Then $(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K, \alpha_1 \triangleleft \gamma, \ldots, \alpha_m \triangleleft \gamma, \beta \triangleright \delta_1, \ldots, \beta \triangleright \delta_n)$ is a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m,$ $\mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, called the *relative tensor product* of $\mathcal{H}$ and $\mathcal{K}$ and denoted by $\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}$.
For all i, j and $a^\dagger \in \mathfrak{A}_i^\dagger$, $c^\dagger \in \mathfrak{C}_j^\dagger$,

$$\rho_{(\alpha_i \triangleleft \gamma)}(a^\dagger) = \rho_{\alpha_i}(a^\dagger) \underset{\mathfrak{b}}{\otimes} \mathrm{id}, \qquad \rho_{(\beta \triangleright \delta_j)}(c^\dagger) = \mathrm{id} \underset{\mathfrak{b}}{\otimes} \rho_{\delta_j}(c^\dagger).$$

The relative tensor product is functorial, unital and associative in the following sense. Let $\tilde{\mathcal{H}} = (\tilde{H}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, \tilde{\beta})$ be a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{b})$-module, $\tilde{\mathcal{K}} = (\tilde{K}, \tilde{\gamma}, \tilde{\delta}_1, \ldots, \tilde{\delta}_n)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, and $S \in \mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}})$, $T \in \mathcal{L}(\mathcal{K}, \tilde{\mathcal{K}})$. Then there exists a unique operator $S \underset{\mathfrak{b}}{\otimes} T \in \mathcal{L}(\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}, \tilde{\mathcal{H}} \underset{\mathfrak{b}}{\otimes} \tilde{\mathcal{K}})$ satisfying $(S \underset{\mathfrak{b}}{\otimes} T)(\xi \otimes \zeta \otimes \eta) = S\xi \otimes \zeta \otimes T\eta$ for all $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$. Next, the triple $\mathcal{U} := (\mathfrak{K}, \mathfrak{B}^\dagger, \mathfrak{B})$ is a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-module and the maps

$$l_{\mathcal{H}} \colon H_\beta \underset{\mathfrak{b}}{\otimes}{}_{\mathfrak{B}^\dagger} \mathfrak{K} \to H,\ \xi \otimes \zeta \otimes b^\dagger \mapsto \xi b^\dagger \zeta, \quad r_{\mathcal{K}} \colon \mathfrak{K}_{\mathfrak{B}} \underset{\mathfrak{b}}{\otimes}{}_\gamma K \to K,\ b \otimes \zeta \otimes \eta \mapsto \eta b \zeta, \tag{4}$$

are isomorphisms of C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{b})$-modules and C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-modules $\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{U} \to \mathcal{H}$ and $\mathcal{U} \underset{\mathfrak{b}}{\otimes} \mathcal{K} \to \mathcal{K}$, respectively, natural in $\mathcal{H}$ and $\mathcal{K}$. Finally, let $\mathfrak{d}, \mathfrak{e}_1, \ldots, \mathfrak{e}_l$ be C^*-bases, $\hat{\mathcal{K}} = (K, \gamma, \delta_1, \ldots, \delta_n, \epsilon)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n, \mathfrak{d})$-module and $\mathcal{L} = (L, \phi, \psi_1, \ldots, \psi_l)$ a C^*-$(\mathfrak{d}^\dagger, \mathfrak{e}_1, \ldots, \mathfrak{e}_l)$-module. Then there exists a canonical isomorphism

$$a_{\mathcal{H}, \hat{\mathcal{K}}, \mathcal{L}} \colon (H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)_{\beta \triangleright \epsilon} \underset{\mathfrak{d}}{\otimes}{}_\phi L \to \beta \otimes_{\rho_\gamma} K_{\rho_\epsilon} \otimes \phi \to H_\beta \underset{\mathfrak{b}}{\otimes}{}_{\gamma \triangleleft \phi} (K_\epsilon \underset{\mathfrak{d}}{\otimes}{}_\phi L)$$

which is an isomorphism of C^*-$(\mathfrak{a}_1,\dots,\mathfrak{a}_m,\mathfrak{c}_1,\dots,\mathfrak{c}_n,\mathfrak{e}_1,\dots,\mathfrak{e}_l)$-modules $(\mathcal{H}\underset{\mathfrak{b}}{\otimes}\hat{\mathcal{K}})\underset{\mathfrak{d}}{\otimes}\mathcal{L}\to\mathcal{H}\underset{\mathfrak{b}}{\otimes}(\hat{\mathcal{K}}\underset{\mathfrak{d}}{\otimes}\mathcal{L})$. We identify the Hilbert spaces above and denote them by $H_{\beta}\underset{\mathfrak{b}}{\otimes}{}_{\gamma}K_{\epsilon}\underset{\mathfrak{d}}{\otimes}{}_{\phi}L$.

2.2. **The definition of C^*-pseudo-multiplicative unitaries.** Let $\mathfrak{b}=(\mathfrak{K},\mathfrak{B},\mathfrak{B}^\dagger)$ be a C^*-base, $(H,\hat{\beta},\alpha,\beta)$ a C^*-$(\mathfrak{b}^\dagger,\mathfrak{b},\mathfrak{b}^\dagger)$-module, and $V\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H\to H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H$ a unitary satisfying

$$V(\alpha\triangleleft\alpha)=\alpha\triangleright\alpha,\quad V(\widehat{\beta}\triangleright\beta)=\widehat{\beta}\triangleleft\beta,\quad V(\widehat{\beta}\triangleright\widehat{\beta})=\alpha\triangleright\widehat{\beta},\quad V(\beta\triangleleft\alpha)=\beta\triangleleft\beta \tag{5}$$

in $\mathcal{L}(\mathfrak{K},H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H)$. Then all operators in the following diagram are well defined,

$$\begin{tikzcd}
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V_{12}"] \arrow[d, "V_{23}"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V_{23}"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H, \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{(\alpha\triangleright\alpha)}(H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H) \arrow[d, "\Sigma_{23}"] & & (H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H)_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "V_{12}"'] \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H_{\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[rr, "V_{12}"] & & \big(H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H\big)_{(\widehat{\beta}\triangleleft\beta)}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[u, "\Sigma'_{23}"']
\end{tikzcd} \tag{6}$$

where we adopted the leg notation [3] and wrote

$$V_{12}\text{ for }V\underset{\mathfrak{b}^\dagger}{\otimes}\mathrm{id},\ V\underset{\mathfrak{b}}{\otimes}\mathrm{id};\qquad V_{23}\text{ for }V\underset{\mathfrak{b}}{\otimes}\mathrm{id},\ V\underset{\mathfrak{b}^\dagger}{\otimes}\mathrm{id};\qquad \Sigma_{23}\text{ for }\mathrm{id}\underset{\mathfrak{b}^\dagger}{\otimes}\Sigma,$$

and where Σ'_{23} denotes the isomorphism

$$\big(H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H\big)_{(\widehat{\beta}\triangleleft\beta)}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H\cong(H_{\rho_\alpha}\circledcirc\beta)_{\rho_{(\widehat{\beta}\triangleleft\beta)}}\circledcirc\alpha\xrightarrow{\cong}(H_{\rho_{\widehat{\beta}}}\circledcirc\alpha)_{\rho_{(\alpha\triangleleft\alpha)}}\circledcirc\beta\cong(H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H)_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H$$

given by $(\zeta\circledcirc\xi)\circledcirc\eta\mapsto(\zeta\circledcirc\eta)\circledcirc\xi$. We furthermore write V_{13} for $\Sigma'_{23}(V\underset{\mathfrak{b}^\dagger}{\otimes}\mathrm{id})\Sigma_{23}$.

Definition 2.1. *A* C^*-pseudo-multiplicative unitary *is a tuple* $(\mathfrak{b},H,\widehat{\beta},\alpha,\beta,V)$ *consisting of a C^*-base $\mathfrak{b}$, a C^*-$(\mathfrak{b}^\dagger,\mathfrak{b},\mathfrak{b}^\dagger)$-module $(H,\widehat{\beta},\alpha,\beta)$, and a unitary $V\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H\to H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H$ such that equation* (5) *holds and diagram* (6) *commutes. We frequently call just V a C^*-pseudo-multiplicative unitary.*

Remarks and Examples 2.2.

- i) If $\mathfrak{b}$ is the trivial C^*-base $(\mathbb{C},\mathbb{C},\mathbb{C})$, then $H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H\cong H\otimes H\cong H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H$, and V is a multiplicative unitary.
- ii) If we consider $\rho_{\widehat{\beta}}$ and ρ_β as representations $\rho_{\widehat{\beta}},\rho_\beta\colon\mathfrak{B}\to\mathcal{L}(H_\alpha)\cong\mathcal{L}_{\mathfrak{B}}(\alpha)$, then the map $\alpha_{\rho_{\widehat{\beta}}}\circledcirc\alpha\cong\alpha\triangleleft\alpha\to\alpha\triangleright\alpha\cong\alpha\circledcirc_{\rho_\beta}\alpha$ given by $\omega\mapsto V\omega$ is a pseudo-multiplicative unitary on C^*-modules in the sense of [27].
- iii) Assume that $\mathfrak{b}=\mathfrak{b}^\dagger$; then $\mathfrak{B}=\mathfrak{B}^\dagger$ is commutative. If $\widehat{\beta}=\alpha$, then the pseudo-multiplicative unitary in ii) is a pseudo-multiplicative unitary in the sense of O'uchi [19]. If additionally $\widehat{\beta}=\alpha=\beta$, then the unitary in ii) is a continuous field of multiplicative unitaries in the sense of Blanchard [4].

iv) Assume that $\mathfrak{b}$ is the C^*-base associated to a faithful proper KMS-weight μ on a C^*-algebra B (see [25, Example 2.9]). Then μ extends to a n.s.f. weight $\tilde{\mu}$ on $[\![\mathfrak{B}]\!]$, and with respect to the canonical isomorphisms $H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \cong H_{\rho_{\widehat{\beta}}} \underset{\tilde{\mu}^{op}}{\bar{\otimes}} {}_{\rho_\alpha} H$ and $H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H \cong H_{\rho_\alpha} \underset{\tilde{\mu}}{\bar{\otimes}} {}_{\rho_\beta} H$ (see [25, Corollary 2.21]), V is a pseudo-multiplicative unitary on Hilbert spaces in the sense of Vallin [33].

v) In [31], a C^*-pseudo-multiplicative unitary is associated to every compact C^*-quantum groupoid.

vi) The *opposite* of a C^*-pseudo-multiplicative unitary $(\mathfrak{b}, H, \hat{\beta}, \alpha, \beta, V)$ is the tuple $(\mathfrak{b}, H, \beta, \alpha, \hat{\beta}, V^{op})$, where V^{op} denotes the composition $\Sigma V^* \Sigma \colon H_\beta \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \xrightarrow{\Sigma} H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H \xrightarrow{V^*} H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \xrightarrow{\Sigma} H_\alpha \underset{\mathfrak{b}}{\otimes} {}_{\widehat{\beta}} H$. A tedious but straightforward calculation shows that this is a C^*-pseudo-multiplicative unitary.

2.3. **The C^*-pseudo-multiplicative unitary of a groupoid.** Let G be a locally compact, Hausdorff, second countable groupoid with left Haar system λ and associated right Haar system λ^{-1}, and let μ be a measure on G^0 with full support. We associate to this data a C^*-pseudo-multiplicative unitary such that the underlying pseudo-multiplicative unitary and the associated unitary on C^*-modules are the ones introduced by Vallin [33] and O'uchi [19, 27], respectively. We focus on the aspects that are new in the present setting.

Define measures ν, ν^{-1} on G by

$$\int_G f \,\mathrm{d}\nu := \int_{G^0} \int_{G^u} f(x) \,\mathrm{d}\lambda^u(x) \,\mathrm{d}\mu(u), \qquad \int_G f d\nu^{-1} = \int_{G^0} \int_{G_u} f(x) \,\mathrm{d}\lambda_u^{-1}(x) \,\mathrm{d}\mu(u)$$

for all $f \in C_c(G)$. Thus, $\nu^{-1} = i_*\nu$, where $i \colon G \to G$ is given by $x \mapsto x^{-1}$. We assume that μ is quasi-invariant in the sense that ν and ν^{-1} are equivalent, and denote by $D := \mathrm{d}\nu / \mathrm{d}\nu^{-1}$ the Radon-Nikodym derivative.

We identify functions in $C_b(G^0)$ and $C_b(G)$ with multiplication operators on the Hilbert spaces $L^2(G^0, \mu)$ and $L^2(G, \nu)$, respectively, and let

$$\mathfrak{K} := L^2(G^0, \mu), \quad \mathfrak{B} = \mathfrak{B}^\dagger := C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K}), \quad \mathfrak{b} := (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger), \quad H := L^2(G, \nu).$$

Pulling functions on G^0 back to G along r or s, we obtain representations

$$r^* \colon C_0(G^0) \to C_b(G) \hookrightarrow \mathcal{L}(H), \qquad s^* \colon C_0(G^0) \to C_b(G) \hookrightarrow \mathcal{L}(H).$$

We define Hilbert C^*-modules $L^2(G, \lambda)$ and $L^2(G, \lambda^{-1})$ over $C_0(G^0)$ as the respective completions of the pre-C^*-module $C_c(G)$, the structure maps being given by

$$\langle \xi' | \xi \rangle(u) = \int_{G^u} \overline{\xi'(x)} \xi(x) \,\mathrm{d}\lambda^u(x), \qquad \xi f = r^*(f)\xi \qquad \text{in the case of } L^2(G, \lambda),$$
$$\langle \xi' | \xi \rangle(u) = \int_{G_u} \overline{\xi'(x)} \xi(x) \,\mathrm{d}\lambda_u^{-1}(x), \qquad \xi f = s^*(f)\xi \qquad \text{in the case of } L^2(G, \lambda^{-1})$$

respectively, for all $\xi, \xi' \in C_c(G)$, $u \in G^0$, $f \in C_0(G^0)$.

Lemma 2.3. *There exist embeddings $j\colon L^2(G,\lambda) \to \mathcal{L}(\mathfrak{K}, H)$ and $\hat{j}\colon L^2(G,\lambda^{-1}) \to \mathcal{L}\big(\mathfrak{K}, H\big)$ such that for all $\xi \in C_c(G)$, $\zeta \in C_c(G^0)$*

$$\big(j(\xi)\zeta\big)(x) = \xi(x)\zeta(r(x)), \qquad \big(\hat{j}(\xi)\zeta\big)(x) = \xi(x)D^{-1/2}(x)\zeta(s(x)).$$

Proof. Let $E := L^2(G,\lambda)$, $\hat{E} := L^2(G,\lambda^{-1})$, and $\xi,\xi' \in C_c(G)$, $\zeta,\zeta' \in C_c(G^0)$. Then

$$\begin{aligned}
\big\langle j(\xi')\zeta' \big| j(\xi)\zeta\big\rangle &= \int_{G^0}\int_{G^u} \overline{\xi'(x)\zeta'(r(x))}\xi(x)\zeta(r(x))\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u) = \big\langle \zeta' \big| \langle\xi'|\xi\rangle_E \zeta\big\rangle, \\
\big\langle \hat{j}(\xi')\zeta' \big| \hat{j}(\xi)\zeta\big\rangle &= \int_G \overline{\xi'(x)\zeta'(s(x))}\xi(x)\zeta(s(x))\,\underbrace{D^{-1}(x)\,\mathrm{d}\nu(x)}_{=\,\mathrm{d}\nu^{-1}(x)} \\
&= \int_{G^0}\int_{G_u} \overline{\zeta'(u)\xi'(x)}\xi(x)\zeta(u)\,\mathrm{d}\lambda_u^{-1}(x)\,\mathrm{d}\mu(u) = \big\langle \zeta' \big| \langle\xi'|\xi\rangle_{\hat{E}} \zeta\big\rangle. \qquad \square
\end{aligned}$$

Let $\alpha := \beta := j(L^2(G,\lambda))$ and $\widehat{\beta} := \hat{j}(L^2(G,\lambda^{-1}))$. Easy calculations show:

Lemma 2.4. *$(H, \widehat{\beta}, \alpha, \beta)$ is a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module, $\rho_\alpha = \rho_\beta = r^*$ and $\rho_{\widehat{\beta}} = s^*$, and j and $\hat{j}$ are unitary maps of Hilbert C^*-modules over $C_0(G^0) \cong \mathfrak{B}$.* $\square$

We define measures $\nu^2_{s,r}$ on $G_s{\times}_rG$ and $\nu^2_{r,r}$ on $G_r{\times}_rG$ by

$$\begin{aligned}
\int_{G_s\times_rG} f\,\mathrm{d}\nu^2_{s,r} &:= \int_{G^0}\int_{G^u}\int_{G^{s(x)}} f(x,y)\,\mathrm{d}\lambda^{s(x)}(y)\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u), \\
\int_{G_r\times_rG} g\,\mathrm{d}\nu^2_{r,r} &:= \int_{G^0}\int_{G^u}\int_{G^u} g(x,y)\,\mathrm{d}\lambda^u(y)\,\mathrm{d}\lambda^u(x)\,\mathrm{d}\mu(u)
\end{aligned}$$

for all $f \in C_c(G_s{\times}_rG)$, $g \in C_c(G_r{\times}_rG)$. Routine calculations show that there exist isomorphisms $\Phi_{\widehat{\beta},\alpha}\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to L^2(G_s{\times}_rG, \nu^2_{s,r})$ and $\Phi_{\alpha,\beta}\colon H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H \to L^2(G_r{\times}_rG, \nu^2_{r,r})$ such that for all $\eta,\xi \in C_c(G)$ and $\zeta \in C_c(G^0)$,

$$\begin{aligned}
\Phi_{\widehat{\beta},\alpha}\big(\hat{j}(\eta) \otimes \zeta \otimes j(\xi)\big)(x,y) &= \eta(x)D^{-1/2}(x)\zeta(s(x))\xi(y), \\
\Phi_{\alpha,\beta}\big(j(\eta) \otimes \zeta \otimes j(\xi)\big)(x,y) &= \eta(x)\zeta(r(x))\xi(y).
\end{aligned}$$

We shall use these isomorphisms to identify the spaces above without further notice.

Theorem 2.5. *There exists a C^*-pseudo-multiplicative unitary $(\mathfrak{b}, H, \hat{\beta}, \alpha, \beta, V)$ such that $(V\omega)(x,y) = \omega(x, x^{-1}y)$ for all $\omega \in C_c(G_s{\times}_rG)$ and $(x,y) \in G_r{\times}_rG$.*

Proof. Straightforward calculations show that $(H, \widehat{\beta}, \alpha, \beta)$ is a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module. Using left-invariance of λ, one finds that the bijection $V_0\colon C_c(G_s{\times}_rG) \to C_c(G_r{\times}_rG)$ given by $(V_0\omega)(x,y) = \omega(x,x^{-1}y)$ for all $\omega \in C_c(G_s{\times}_rG)$ and $(x,y) \in G_r{\times}_rG$ extends to a unitary $V\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \cong L^2(G_s{\times}_rG) \to L^2(G_r{\times}_rG) \cong H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H$. We claim that $V(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}$. For each $\xi,\xi' \in C_c(G)$, $\zeta \in C_c(G^0)$, and $(x,y) \in G_s{\times}_rG$,

$$\begin{aligned}
\big(V|\hat{j}(\xi)\rangle_1\hat{j}(\xi')\zeta\big)(x,y) &= \big(|\hat{j}(\xi)\rangle_1\hat{j}(\xi')\zeta\big)(x,x^{-1}y) \\
&= \xi(x)\xi'(x^{-1}y)D^{-1/2}(x)D^{-1/2}(x^{-1}y)\zeta(s(y)), \\
\big(|j(\xi)\rangle_1\hat{j}(\xi')\zeta\big)(x,y) &= \xi(x)\xi'(y)D^{-1/2}(y)\zeta(s(y)).
\end{aligned}$$

Using standard approximation arguments and the fact that $D(x)D(x^{-1}y) = D(y)$ for $\nu^2_{r,r}$-almost all $(x,y) \in G_r{\times}_r G$ (see [12] or [20, p. 89]), we find that $V(\widehat{\beta} \triangleright \widehat{\beta}) = [T(C_c(G_r{\times}_r G))] = \alpha \triangleright \widehat{\beta}$, where for each $\omega \in C_c(G_r{\times}_r G)$,

$$(T(\omega)\zeta)(x,y) = \omega(x,y)D^{-1/2}(y)\zeta(s(y)) \quad \text{for all } \zeta \in C_c(G^0),\ (x,y) \in G_r{\times}_r G.$$

Similar calculations show that the remaining relations in (5) hold. Tedious but straightforward calculations show that diagram (6) commutes; see also [33]. Therefore, V is a C^*-pseudo-multiplicative unitary. □

3. The legs of a C^*-pseudo multiplicative unitary

To every regular multiplicative unitary V on a Hilbert space H, Baaj and Skandalis associate two Hopf C^*-algebras $(\widehat{A}_V, \widehat{\Delta}_V)$ and (A_V, Δ_V) as follows [3]. The C^*-algebras $\widehat{A}_V$ and A_V are the norm closures of the subspaces $\widehat{A}^0_V$ and A^0_V of $\mathcal{L}(H)$ given by

$$(7) \qquad \widehat{A}^0_V := \{(\mathrm{id} \mathbin{\bar{\otimes}} \omega)(V) \mid \omega \in \mathcal{L}(H)_*\}, \qquad A^0_V := \{(\upsilon \mathbin{\bar{\otimes}} \mathrm{id})(V) \mid \upsilon \in \mathcal{L}(H)_*\},$$

and the $*$-homomorphisms $\widehat{\Delta}_V \colon \widehat{A}_V \to M(\widehat{A}_V \otimes \widehat{A}_V) \subseteq \mathcal{L}(H \otimes H)$ and $\Delta_V \colon A_V \to M(A_V \otimes A_V) \subseteq \mathcal{L}(H \otimes H)$ are given by

$$(8) \qquad \widehat{\Delta}_V \colon \widehat{a} \mapsto V^*(1 \otimes \widehat{a})V, \qquad \Delta_V \colon a \mapsto V(a \otimes 1)V^*,$$

respectively. Applied to the multiplicative unitary of a locally compact group G, this construction yields the C^*-algebras $C_0(G)$ and $C^*_r(G)$, and $\widehat{\Delta} \colon C_0(G) \to M(C_0(G) \otimes C_0(G)) \cong C_b(G \times G)$ and $\Delta \colon C^*_r(G) \to M(C^*_r(G) \otimes C^*_r(G))$ are given by

$$(9) \qquad \widehat{\Delta}(f)(x,y) = f(xy) \text{ for all } f \in C_0(G), \qquad \Delta(U_x) = U_x \otimes U_x \text{ for all } x \in G,$$

where $U \colon G \to M(C^*_r(G)), x \mapsto U_x$, is the canonical embedding.
To adapt these constructions to C^*-pseudo-multiplicative unitaries, we have to generalize the notion of a Hopf C^*-algebra and identify the targets of the comultiplications $\widehat{\Delta}_V$ and Δ_V. For the C^*-pseudo-multiplicative unitary of a groupoid G, we expect to obtain the C^*-algebras $\widehat{A}_V = C_0(G)$ and $A_V = C^*_r(G)$ with $*$-homomorphisms $\widehat{\Delta}$ and Δ given by the same formulas as in (9). Then the target of $\widehat{\Delta}$ would be $M(C_0(G_s{\times}_r G))$, and $C_0(G_s{\times}_r G)$ can be identified with the relative tensor product $C_0(G)_{s^*} \underset{C_0(G^0)}{\otimes} {}_{r^*}C_0(G)$ of $C_0(G^0)$-algebras [4]. But the target of Δ can not be described in a similar way, and in general, we need to replace the balanced tensor product by a fiber product relative to some base. In the setting of von Neumann algebras, the targets of the comultiplications can be described using Sauvageot's fiber product [24, 32]. The appropriate construction in the setting of C^*-algebras is given below.

3.1. **The fiber product and Hopf C^*-bimodules.** Fundamental to the notion of a Hopf C^*-bimodule is the fiber product of C^*-algebras over C^*-bases introduced in [25]. We briefly recall this construction and subsequently introduce Hopf C^*-bimodules; for additional motivation and details, see [25]. Two examples can be found in subsection 3.4.

Let $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ be C^*-bases, where $\mathfrak{b}_i = (\mathfrak{K}_i, \mathfrak{B}_i, \mathfrak{B}_i^\dagger)$ for each i. A *(nondegenerate) C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-algebra* consists of a C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-module $(H, \alpha_1, \ldots, \alpha_n)$ and a (nondegenerate) C^*-algebra $A \subseteq \mathcal{L}(H)$ such that $\rho_{\alpha_i}(\mathfrak{B}_i^\dagger)A$ is contained in A for each i. We are interested in the cases $n = 1, 2$ and abbreviate $A_H^\alpha := (H_\alpha, A)$, $A_H^{\alpha,\beta} := ({}_\alpha H_\beta, A)$.
Let $\mathcal{A} = (\mathcal{H}, A)$ and $\mathcal{C} = (\mathcal{K}, C)$ be C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-algebras, where $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_n)$ and $\mathcal{K} = (K, \gamma_1, \ldots, \gamma_n)$. A *morphism* from $\mathcal{A}$ to $\mathcal{C}$ is a $*$-homomorphism $\pi\colon A \to C$ satisfying $[\mathcal{L}^\pi(\mathcal{H}, \mathcal{K})\alpha_i] = \gamma_i$ for each i, where $\mathcal{L}^\pi(\mathcal{H}, \mathcal{K}) = \mathcal{L}^\pi(H, K) \cap \mathcal{L}(\mathcal{H}, \mathcal{K})$. One easily verifies that every morphism π between C^*-$\mathfrak{b}$-algebras A_H^α and C_K^γ satisfies $\pi(\rho_\alpha(b^\dagger)) = \rho_\gamma(b^\dagger)$ for all $b^\dagger \in \mathfrak{B}^\dagger$.
Let $\mathfrak{b}$ be a C^*-base, A_H^β a C^*-$\mathfrak{b}$-algebra, and B_K^γ a C^*-$\mathfrak{b}^\dagger$-algebra. The *fiber product* of A_H^β and B_K^γ is the C^*-algebra

$$A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B := \big\{x \in \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K) \,\big|\, x|\beta\rangle_1, x^*|\beta\rangle_1 \subseteq [|\beta\rangle_1 B] \text{ as subsets of } \mathcal{L}(K, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K),$$
$$x|\gamma\rangle_2, x^*|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A] \text{ as subsets of } \mathcal{L}(H, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)\big\}.$$

If A and B are unital, so is $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$, but otherwise, $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$ may be degenerate. Clearly, conjugation by the flip $\Sigma\colon H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K \to K_\gamma \underset{\mathfrak{b}^\dagger}{\otimes}{}_\beta H$ yields an isomorphism $\mathrm{Ad}_\Sigma\colon A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \to B_\gamma \underset{\mathfrak{b}^\dagger}{*}{}_\beta A$. If $\mathfrak{a}, \mathfrak{c}$ are C^*-bases, $A_H^{\alpha,\beta}$ is a C^*-$(\mathfrak{a}, \mathfrak{b})$-algebra and $B_K^{\gamma,\delta}$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebra, then

$$A_H^{\alpha,\beta} \underset{\mathfrak{b}}{*} B_K^{\gamma,\delta} = ({}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta,\ A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)$$

is a C^*-$(\mathfrak{a}, \mathfrak{c})$-algebra, called the *fiber product* of $A_H^{\alpha,\beta}$ and $B_K^{\gamma,\delta}$.
Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be C^*-bases, ϕ a morphism of C^*-$(\mathfrak{a}, \mathfrak{b})$-algebras $\mathcal{A} = A_H^{\alpha,\beta}$ and $\mathcal{C} = C_L^{\kappa,\lambda}$, and ψ a morphism of C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebras $\mathcal{B} = B_K^{\gamma,\delta}$ and $\mathcal{D} = D_M^{\mu,\nu}$. Then there exists a unique morphism of C^*-$(\mathfrak{a}, \mathfrak{c})$-algebras $\phi \underset{\mathfrak{b}}{*} \psi\colon \mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{B} \to \mathcal{C} \underset{\mathfrak{b}}{*} \mathcal{D}$ such that

$$(\phi \underset{\mathfrak{b}}{*} \psi)(x)R = Rx \quad \text{for all } x \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \text{ and } R \in I_M J_H + J_L I_K,$$

where $I_X = \mathcal{L}^\phi(H, L) \underset{\mathfrak{b}}{\otimes} \mathrm{id}_X$, $J_Y = \mathrm{id}_Y \underset{\mathfrak{b}}{\otimes} \mathcal{L}^\psi(K, M)$ for $X \in \{K, M\}, Y \in \{H, L\}$.
The fiber product need not be associative, but whenever it appears as the target of a comultiplication, coassociativity will compensate the non-associativity.

Definition 3.1. *A* comultiplication *on a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-algebra $A_H^{\beta,\alpha}$ is a morphism Δ from $A_H^{\beta,\alpha}$ to $A_H^{\beta,\alpha} \underset{\mathfrak{b}}{*} A_H^{\beta,\alpha}$ that is coassociative in the sense that $(\Delta \underset{\mathfrak{b}}{*} \mathrm{id}) \circ \Delta = (\mathrm{id} \underset{\mathfrak{b}}{*} \Delta) \circ \Delta$ as maps from A to $\mathcal{L}(H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H)$. A* Hopf C^*-bimodule over $\mathfrak{b}$ *is a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-algebra with a comultiplication. A* morphism *of Hopf C^*-bimodules $(A_H^{\beta,\alpha}, \Delta_A)$, $(B_K^{\delta,\gamma}, \Delta_B)$ over $\mathfrak{b}$ is a morphism π from $A_H^{\beta,\alpha}$ to $B_K^{\delta,\gamma}$ satisfying $\Delta_B \circ \pi = (\pi \underset{\mathfrak{b}}{*} \pi) \circ \Delta_A$.*

3.2. **The Hopf C^*-bimodules of a C^*-pseudo-multiplicative unitary.** Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, $(H, \widehat{\beta}, \alpha, \beta)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module and $V\colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H$

a C^*-pseudo-multiplicative unitary. We associate to V two algebras and, if V is well behaved, two Hopf C^*-bimodules as follows. Let

$$\widehat{A}_V := [\langle\beta|_2 V|\alpha\rangle_2] \subseteq \mathcal{L}(H), \qquad A_V := [\langle\alpha|_1 V|\widehat{\beta}\rangle_1] \subseteq \mathcal{L}(H), \tag{10}$$

where $|\alpha\rangle_2, |\widehat{\beta}\rangle_1 \subseteq \mathcal{L}(H, H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H)$ and $\langle\beta|_2, \langle\alpha|_1 \subseteq \mathcal{L}(H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H, H)$ are defined as in Subsection 2.1.

Proposition 3.2. i) $\widehat{A}_{V^{op}} = A_V^*$, $[\widehat{A}_V\widehat{A}_V] = \widehat{A}_V$, $[\widehat{A}_V H] = H = [\widehat{A}_V^* H]$, $[\widehat{A}_V\beta] = \beta = [\widehat{A}_V^*\beta]$, *and* $[\widehat{A}_V\rho_{\widehat{\beta}}(\mathfrak{B})] = [\rho_{\widehat{\beta}}(\mathfrak{B})\widehat{A}_V] = \widehat{A}_V = [\widehat{A}_V\rho_\alpha(\mathfrak{B}^\dagger)] = [\rho_\alpha(\mathfrak{B}^\dagger)\widehat{A}_V]$.

ii) $A_{V^{op}} = \widehat{A}_V^*$, $[A_V A_V] = A_V$, $[A_V H] = H = [A_V^* H]$, $[A_V\widehat{\beta}] = \widehat{\beta} = [A_V^*\widehat{\beta}]$, *and* $[A_V\rho_\beta(\mathfrak{B})] = [\rho_\beta(\mathfrak{B})A_V] = A_V = [A_V\rho_\alpha(\mathfrak{B}^\dagger)] = [\rho_\alpha(\mathfrak{B}^\dagger)A_V]$.

We shall prove some of the equations above using commutative diagrams, where the vertices are labelled by Hilbert spaces, the arrows are labelled by single operators or closed spaces of operators, and the composition is given by the closed linear span of all possible compositions of operators.

Proof. i) First, $\widehat{A}_{V^{op}} = [\langle\widehat{\beta}|_2\Sigma V^*\Sigma|\alpha\rangle_2] = [\langle\widehat{\beta}|_1 V^*|\alpha\rangle_1] = A_V^*$.
Next, $[\widehat{A}_V\widehat{A}_V] = [\langle\beta|_2\langle\alpha|_3 V_{12}|\alpha\rangle_3|\alpha\rangle_2]$ because the diagram below commutes:

$$\begin{tikzcd}
 & H \arrow[dl, "|\alpha\rangle_2"'] \arrow[d, "|\alpha\rangle_2"] \arrow[rr, "\widehat{A}_V"] & & H \arrow[dr, "|\alpha\rangle_2"] \arrow[rr, "\widehat{A}_V"] & & H & \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[ddr, "|\alpha\rangle_3"'] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V"] \arrow[d, "|\alpha\rangle_2"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[ur, "\langle\beta|_2"] \arrow[d, "|\alpha\rangle_2"] & \text{(C)} & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\beta|_2"'] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[ul, "\langle\beta|_2"'] \\
 & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{(\alpha\triangleright\alpha)}(H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H) \arrow[r, "V_{13}"'] \arrow[d, "V_{23}^*"] & (H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H)_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[urr, "\langle\beta|_3"] \arrow[rrr, "V_{12}"'] & \text{(P)} & & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\beta|_3"'] & \\
 & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[rrrr, "V_{12}"] & & & & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[u, "V_{23}"'] \arrow[uur, "\langle\alpha|_3"'] &
\end{tikzcd}$$

Indeed, cell (C) commutes because for all $\xi \in \alpha$, $\eta, \eta' \in \beta$, $\zeta \in H$,

$$|\xi\rangle_2\langle\eta'|_2(\zeta \olessthan \eta) = \rho_\alpha\big(\eta'^*\eta\big)\zeta \olessthan \xi = \rho_{(\alpha\triangleleft\alpha)}\big(\eta'^*\eta\big)(\zeta \olessthan \xi) = \langle\eta'|_3|\xi\rangle_2(\zeta \olessthan \eta), \tag{11}$$

cell (P) is diagram (6), and the other cells commute by definition of $\widehat{A}_V$ and because of (5). Now, $[\langle\beta|_2\langle\alpha|_3 V_{12}|\alpha\rangle_3|\alpha\rangle_2] = \widehat{A}_V$ because the following diagram commutes:

$$\begin{tikzcd}
 & H \arrow[dl, "|\alpha\rangle_2"'] \arrow[d, "|\alpha\rangle_2"] \arrow[r, "\widehat{A}_V"] & H \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "\rho_{(\widehat{\beta}\triangleright\widehat{\beta})}(\mathfrak{B})"] \arrow[dr, "|\alpha\rangle_3"'] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\beta|_2"'] \\
 & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[u, "\langle\alpha|_3"'] \arrow[r, "V_{12}"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[u, "\langle\alpha|_3"']
\end{tikzcd}$$

Finally, we prove some of the remaining equations; the other ones follow similarly:

$$\begin{aligned}
[\widehat{A}_V\beta] &= [\langle\beta|_2V|\alpha\rangle_2\beta] = [\langle\beta|_2|\beta\rangle_2\beta] = [\rho_\alpha(\mathfrak{B}^\dagger)\beta] = \beta,\\
[\widehat{A}_V\rho_{\widehat{\beta}}(\mathfrak{B})] &= [\langle\beta|_2V|\alpha\rangle_2\rho_{\widehat{\beta}}(\mathfrak{B})] = [\langle\beta|_2V|\alpha\mathfrak{B}\rangle_2] = \widehat{A}_V,\\
[\rho_{\widehat{\beta}}(\mathfrak{B})\widehat{A}_V] &= [\rho_{\widehat{\beta}}(\mathfrak{B})\langle\beta|_2V|\alpha\rangle_2] = [\langle\beta|_2(\rho_{\widehat{\beta}}(\mathfrak{B}) \underset{\mathfrak{b}}{\otimes} \mathrm{id})V|\alpha\rangle_2]\\
&= [\langle\beta|_2V(\mathrm{id}\underset{\mathfrak{b}^\dagger}{\otimes}\rho_\beta(\mathfrak{B}))|\alpha\rangle_2] = [\langle\beta|_2V|\rho_\beta(\mathfrak{B})\alpha\rangle_2] = \widehat{A}_V.
\end{aligned}$$

ii) This follows from i) after replacing V by V^{op}. □

Define $\widehat{\Delta}_V\colon \rho_\beta(\mathfrak{B})' \to \mathcal{L}(H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H)$ and $\Delta_V\colon \rho_{\widehat{\beta}}(\mathfrak{B})' \to \mathcal{L}(H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H)$ by

$$\widehat{\Delta}_V\colon y \mapsto V^*(\mathrm{id}\underset{\mathfrak{b}}{\otimes}y)V, \qquad\qquad \Delta_V\colon z\mapsto V(z\underset{\mathfrak{b}^\dagger}{\otimes}\mathrm{id})V^*.$$

Evidently, $\widehat{\Delta}_{V^{op}} = \mathrm{Ad}_\Sigma\circ\Delta_V$ and $\Delta_{V^{op}} = \mathrm{Ad}_\Sigma\circ\widehat{\Delta}_V$. Moreover, if $\eta\in\beta$ and $\xi\in\alpha$, then $\widehat{a} := \langle\eta|_2V|\xi\rangle_2$ lies in $\mathcal{L}(H_\beta)\subseteq\rho_\beta(\mathfrak{B})'$ by Proposition 3.2 and

$$\widehat{\Delta}_V(\langle\eta|_2V|\xi\rangle_2) = \widehat{\Delta}_V(\widehat{a}) = V^*(1\underset{\mathfrak{b}}{\otimes}\widehat{a})V = \langle\eta|_3V_{12}^*V_{23}V_{12}|\xi\rangle_3 = \langle\eta|_3V_{13}V_{23}|\xi\rangle_3. \tag{12}$$

Similarly, if $\eta\in\alpha$ and $\xi\in\widehat{\beta}$, then $a := \langle\eta|_1V|\xi\rangle_1$ lies in $\rho_{\widehat{\beta}}(\mathfrak{B})'$ and

$$\Delta_V(\langle\eta|_1V|\xi\rangle_1) = \Delta_V(a) = V(a\underset{\mathfrak{b}^\dagger}{\otimes}1)V^* = \langle\eta|_1V_{23}V_{12}V_{23}^*|\xi\rangle_1 = \langle\eta|_1V_{12}V_{13}|\xi\rangle_1. \tag{13}$$

Lemma 3.3. *The map $\widehat{\Delta}_V$ is a morphism of the C^*-$(\mathfrak{b},\mathfrak{b}^\dagger)$-algebras $(\rho_\beta(\mathfrak{B})')_H^{\alpha,\widehat{\beta}}$ and $\big((\rho_\beta(\mathfrak{B})_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha\rho_\beta(\mathfrak{B}))'\big)^{(\alpha\triangleleft\alpha),(\widehat{\beta}\triangleright\widehat{\beta})}_{H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H}$, and Δ_V is a morphism of the C^*-$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebras $(\rho_{\widehat{\beta}}(\mathfrak{B})')_H^{\beta,\alpha}$ and $\big((\rho_{\widehat{\beta}}(\mathfrak{B})_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta\rho_{\widehat{\beta}}(\mathfrak{B}))'\big)^{(\beta\triangleleft\beta),(\alpha\triangleright\alpha)}_{H_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H}$.*

Proof. We only prove the assertions concerning $\widehat{\Delta}_V$. First, $\widehat{\Delta}_V(\rho_\beta(\mathfrak{B})')\subseteq(\rho_\beta(\mathfrak{B})\underset{\mathfrak{b}^\dagger}{\otimes}\rho_\beta(\mathfrak{B}))'$ because $V(\rho_\beta(\mathfrak{B})\underset{\mathfrak{b}^\dagger}{\otimes}\rho_\beta(\mathfrak{B}))V^* = \rho_\beta(\mathfrak{B})\underset{\mathfrak{b}}{\otimes}\rho_{\widehat{\beta}}(\mathfrak{B})\subseteq\mathrm{id}\underset{\mathfrak{b}}{\otimes}\rho_\beta(\mathfrak{B})'$ by (5). Next, $V^*|\alpha\rangle_1\subseteq\mathcal{L}^{\widehat{\Delta}_V}\big(H, H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H\big)$ because $\widehat{\Delta}(y)V^*|\xi\rangle_1 = V^*(\mathrm{id}\underset{\mathfrak{b}}{\otimes}y)|\xi\rangle_1 = V^*|\xi\rangle_1y$ for all $y\in\rho_{\widehat{\beta}}(\mathfrak{B})'$, $\xi\in\alpha$, and $\alpha\triangleleft\alpha = [V^*|\alpha\rangle_1\alpha]$ and $\widehat{\beta}\triangleright\widehat{\beta} = [V^*|\alpha\rangle_1\widehat{\beta}]$ by (5). □

Theorem 3.4. *If $\widehat{A}_V = \widehat{A}_V^*$, then $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}},\widehat{\Delta}_V)$ is a Hopf C^*-bimodule. Similarly, if $A_V = A_V^*$, then $((A_V)_H^{\beta,\alpha},\Delta_V)$ is a Hopf C^*-bimodule.*

Proof. We only prove the first assertion, the second one follows by replacing V by V^{op}. Write $\widehat{A} = \widehat{A}_V$, $\widehat{\Delta} = \widehat{\Delta}_V$, and assume that $\widehat{A} = \widehat{A}^*$. By Proposition 3.2, $\widehat{\mathcal{A}} := \widehat{A}_H^{\alpha,\widehat{\beta}}$ is a C^*-$(\mathfrak{b},\mathfrak{b}^\dagger)$-algebra and $\widehat{A}\subseteq\mathcal{L}(H_\beta)\subseteq\rho_\beta(\mathfrak{B})'$. We claim that $\widehat{\Delta}(\widehat{A})\subseteq\widehat{A}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{*}{}_\alpha\widehat{A}$. By equation (12), $\widehat{\Delta}(\widehat{A}) = [\langle\beta|_3V_{13}V_{23}|\alpha\rangle_3]$, and the following commutative diagram shows that $[\widehat{\Delta}(\widehat{A})|\alpha\rangle_2] = [\langle\beta|_3 V_{13}V_{23}|\alpha\rangle_3|\alpha\rangle_2] \subseteq [|\alpha\rangle_2\langle\beta|_2 V|\alpha\rangle_2] = [|\alpha\rangle_2\widehat{A}]$:

$$\begin{tikzcd}
H \arrow[r, "|\alpha\rangle_2"] \arrow[d, "|\alpha\rangle_2"] & H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[r, "V"] \arrow[d, "|\alpha\rangle_3"] & H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[r, "\langle\beta|_2"] \arrow[d, "|\alpha\rangle_3"] & H \arrow[d, "|\alpha\rangle_2"] \\
H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \arrow[r, "V_{23}|\alpha\rangle_3"] & H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_{(\alpha \triangleright \alpha)} (H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H) \arrow[r, "V_{13}"] & (H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H)_{(\alpha \triangleleft \alpha)} \underset{\mathfrak{b}}{\otimes}{}_\beta H \arrow[r, "\langle\beta|_3"] & H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H
\end{tikzcd}$$

Similarly, one proves that $[\widehat{\Delta}(\widehat{A})|\widehat{\beta}\rangle_1] = [|\widehat{\beta}\rangle_1\widehat{A}]$, and the claim follows. By Lemma 3.3, $\widehat{\Delta}$ is a morphism of the C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-algebras $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}} \underset{\mathfrak{b}}{*} \widehat{\mathcal{A}}$. It only remains to show that $\widehat{\Delta}$ is coassociative. Let $\widehat{a} \in \widehat{A}$. Then

$$(\widehat{\Delta} \underset{\mathfrak{b}^\dagger}{*} \mathrm{id})(\widehat{\Delta}(\widehat{a})) = V_{12}^*(\mathrm{id} \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\widehat{a}))V_{12} = V_{12}^*V_{23}^*(\mathrm{id} \underset{\mathfrak{b}}{\otimes} \mathrm{id} \underset{\mathfrak{b}}{\otimes} \widehat{a})V_{23}V_{12}.$$

Here, we can replace $V_{23}V_{12}$ by $V_{12}^*V_{23}V_{12} = V_{13}V_{23}$. Therefore, $(\widehat{\Delta} \underset{\mathfrak{b}^\dagger}{*} \mathrm{id})(\widehat{\Delta}(\widehat{a}))$ equals

$$V_{23}V_{13}^*(\mathrm{id} \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id} \underset{\mathfrak{b}}{\otimes} \widehat{a})V_{13}V_{23} = V_{23}\Sigma_{23}(\widehat{\Delta}(\widehat{a}) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})\Sigma_{23}V_{23} = (\mathrm{id} \underset{\mathfrak{b}^\dagger}{*} \widehat{\Delta})(\widehat{\Delta}(\widehat{a})). \qquad \square$$

3.3. **The Fourier algebras of a C^*-pseudo-multiplicative unitary.** We first introduce certain spaces of maps on C^*-algebras and slice maps on fiber products, and then associate to every Hopf C^*-bimodule several convolution algebras and to every C^*-pseudo-multiplicative unitary two Fourier algebras.

Let $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ and $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be C^*-bases, H a Hilbert space, H_α a C^*-$\mathfrak{a}$-module, H_β a C^*-$\mathfrak{b}$-module, and $A \subseteq \mathcal{L}(H)$ a closed subspace. We denote by α^∞ the space of all sequences $\eta = (\eta_k)_{k\in\mathbb{N}}$ in α for which the sum $\sum_k \eta_k^*\eta_k$ converges in norm, and put $\|\eta\| := \|\sum_k \eta_k^*\eta_k\|^{1/2}$ for each $\eta \in \alpha^\infty$. Similarly, we define β^∞. Standard arguments show that for all $\eta \in \beta^\infty, \eta' \in \alpha^\infty$, there exists a bounded linear map

$$\omega_{\eta,\eta'} \colon A \to \mathcal{L}(\mathfrak{H}, \mathfrak{K}), \quad T \mapsto \sum_{k\in\mathbb{N}} \eta_k^* T \eta_k',$$

where the sum converges in norm and $\|\omega_{\eta,\eta'}\| \leq \|\eta\|\|\eta'\|$. We put

$$\Omega_{\beta,\alpha}(A) := \{\omega_{\eta,\eta'} \mid \eta \in \beta^\infty, \eta' \in \alpha^\infty\} \subseteq L(A, \mathcal{L}(\mathfrak{H}, \mathfrak{K}),$$

where $L(A, \mathcal{L}(\mathfrak{H}, \mathfrak{K}))$ denotes the space of bounded linear maps from A to $\mathcal{L}(\mathfrak{H}, \mathfrak{K})$. If $\beta = \alpha$, we abbreviate $\Omega_\beta(A) := \Omega_{\beta,\alpha}(A)$. It is easy to see that $\Omega_{\beta,\alpha}(A)$ is a subspace of $L(A, \mathcal{L}(\mathfrak{H}, \mathfrak{K}))$ and that the following formula defines a norm on $\Omega_{\beta,\alpha}(A)$:

$$\|\omega\| := \inf\left\{\|\eta\|\|\eta'\| \middle| \eta \in \beta^\infty, \eta' \in \alpha^\infty, \omega = \omega_{\eta,\eta'}\right\} \text{ for all } \omega \in \Omega_{\beta,\alpha}(A).$$

Standard arguments show that $\Omega_{\beta,\alpha}(A)$ is a Banach space. Moreover, if $A = A^*$, then there exists an anti-linear isometry $\Omega_{\beta,\alpha}(A) \to \Omega_{\alpha,\beta}(A)$, $\omega \mapsto \omega^*$, such that $\omega^*(a) = \omega(a^*)^*$ for all $a \in A$ and $(\omega_{\eta,\eta'})^* = \omega_{\eta',\eta}$ for all $\eta \in \beta^\infty, \eta' \in \alpha^\infty$.

Proposition 3.5. i) *Let π be a morphism of C^*-$\mathfrak{b}$-algebras A_H^α and B_K^γ. Then there exists a linear contraction $\pi^* \colon \Omega_\gamma(B) \to \Omega_\alpha(A)$ given by $\omega \mapsto \omega \circ \pi$.*

ii) *Let π be a morphism of C^*-$(\mathfrak{a}, \mathfrak{b})$-algebras $A_H^{\alpha,\beta}$ and $B_K^{\gamma,\delta}$. Then there exists a linear contraction $\pi^* \colon \Omega_{\delta,\gamma}(B) \to \Omega_{\beta,\alpha}(A)$ given by $\omega \mapsto \omega \circ \pi$.*

Proof. We only prove ii). Let $I := \mathcal{L}^\pi({}_\alpha H_\beta, {}_\gamma K_\delta)$ and $\eta \in \delta^\infty$, $\eta' \in \gamma^\infty$. Then there exists a closed separable subspace $I_0 \subseteq I$ such that $\eta_n \in [I_0\beta]$ and $\eta'_n \in [I_0\alpha]$ for all $n \in \mathbb{N}$. We may assume that $I_0 I_0^* I_0 \subseteq I_0$, and then $[I_0 I_0^*]$ is a σ-unital C^*-algebra and has a bounded sequential approximate unit $(u_k)_k$ of the form $u_k = \sum_{l=1}^k T_l T_l^*$, where $(T_l)_l$ is a sequence in I_0 [15, Proposition 6.7]. We choose a bijection $i\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ and let $\xi_{i(l,n)} := T_l^*\eta_n \in \beta$ and $\xi'_{i(l,n)} := T_l^*\eta'_n \in \alpha$ for all $l, n \in \mathbb{N}$. Then the sum $\sum_l \xi^*_{i(l,n)}\xi_{i(l,n)} = \sum_l \eta_n^* T_l T_l^* \eta_n$ converges to $\eta_n^*\eta_n$ for each $n \in \mathbb{N}$ in norm because $\eta_n \in [I_0\beta]$. Therefore, $\xi \in \beta^\infty$ and $\|\xi\| = \|\eta\|$, and a similar argument shows that $\xi' \in \alpha^\infty$ and $\|\xi'\| = \|\eta'\|$. Finally,

$$\omega_{\xi,\xi'}(a) = \sum_{l,n} \eta_n^* T_l a T_l^* \eta'_n = \sum_{l,n} \eta_n^* \pi(a) T_l T_l^* \eta'_n = \sum_n \eta_n^* \pi(a)\eta'_n = \omega_{\eta,\eta'}(\pi(a))$$

for each $a \in A$, where the sum converges in norm, and hence $\omega_{\eta,\eta'} \circ \pi = \omega_{\xi,\xi'} \in \Omega_{\beta,\alpha}(A)$ and $\|\omega_{\eta,\eta'} \circ \pi\| \leq \|\xi\|\|\xi'\| = \|\eta\|\|\eta'\|$. $\square$

For each map of the form considered above, we can form a slice map as follows [25, Proposition 3.30]. Let H_β be a C^*-$\mathfrak{b}$-module, let K_γ be a C^*-$\mathfrak{b}^\dagger$-module, let $A \subseteq \mathcal{L}(H)$ and $B \subseteq \mathcal{L}(K)$ be closed subspaces, and let

$$A {}_\beta\underset{\mathfrak{b}}{\bar{*}}{}_\gamma B = \{x \in \mathcal{L}(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K) \mid \langle\beta|_1 x|\beta\rangle_1 \subseteq B, \langle\gamma|_2 x|\gamma\rangle_2 \subseteq A\}.$$

Proposition 3.6. i) *There exists a linear contraction $\Omega_\beta(A) \to \Omega_{|\beta\rangle_1}(A {}_\beta\underset{\mathfrak{b}}{\bar{*}}{}_\gamma B)$, $\phi \mapsto \phi * \mathrm{id}$, such that $\omega_{\xi,\xi'} * \mathrm{id} = \omega_{\tilde{\xi},\tilde{\xi}'}$ for all $\xi, \xi' \in \beta^\infty$, where $\tilde{\xi}_n = |\xi_n\rangle_1$ and $\tilde{\xi}'_n = |\xi'_n\rangle_1$ for all $n \in \mathbb{N}$.*

ii) *There exists a linear contraction $\Omega_\gamma(B) \to \Omega_{|\gamma\rangle_2}(A {}_\beta\underset{\mathfrak{b}}{\bar{*}}{}_\gamma B)$, $\psi \mapsto \mathrm{id} * \psi$, such that $\mathrm{id} * \omega_{\eta,\eta'} = \omega_{\tilde{\eta},\tilde{\eta}'}$ for all $\eta, \eta' \in \gamma^\infty$, where $\tilde{\eta}_n = |\eta_n\rangle_2$ and $\tilde{\eta}'_n = |\eta'_n\rangle_2$ for all $n \in \mathbb{N}$.*

iii) *We have $\psi \circ (\phi * \mathrm{id}) = \phi \circ (\mathrm{id} * \psi)$ for all $\phi \in \Omega_\beta(A)$ and $\psi \in \Omega_\gamma(B)$.* $\square$

Assume that ${}_\alpha H_\beta$ is a C^*-$(\mathfrak{a}, \mathfrak{b})$-module and ${}_\gamma K_\delta$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-module. Denote by "$\hat{\otimes}$" the projective tensor product of Banach spaces. Clearly, there exist linear contractions

$$\Omega_\alpha(A) \hat{\otimes} \Omega_\gamma(B) \to \Omega_{(\alpha \triangleleft \gamma)}(A {}_\beta\underset{\mathfrak{b}}{\bar{*}}{}_\gamma B),\ \omega \otimes \omega' \mapsto \omega \boxtimes \omega' := \omega \circ (\mathrm{id} * \omega'),$$

$$\Omega_\beta(A) \hat{\otimes} \Omega_\delta(B) \to \Omega_{(\beta \triangleright \delta)}(A {}_\beta\underset{\mathfrak{b}}{\bar{*}}{}_\gamma B),\ \omega \otimes \omega' \mapsto \omega \boxtimes \omega' := \omega' \circ (\omega * \mathrm{id}).$$

Proposition 3.7. *There exist linear contractions*

$$\Omega_{\alpha,\beta}(A) \hat{\otimes} \Omega_{\gamma,\delta}(B) \to \Omega_{(\alpha \triangleleft \gamma),(\beta \triangleright \delta)}(A {}_\beta\underset{\mathfrak{b}}{\bar{*}}{}_\gamma B),\ \omega \otimes \omega' \mapsto \omega \boxtimes \omega',$$

$$\Omega_{\beta,\alpha}(A) \hat{\otimes} \Omega_{\delta,\gamma}(B) \to \Omega_{(\beta \triangleright \delta),(\alpha \triangleleft \gamma)}(A {}_\beta\underset{\mathfrak{b}}{\bar{*}}{}_\gamma B),\ \omega \otimes \omega' \mapsto \omega \boxtimes \omega',$$

such that for all $\xi \in \alpha^\infty$, $\xi' \in \beta^\infty$, $\eta \in \gamma^\infty$, $\eta' \in \delta^\infty$ and each bijection $i\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we have $\omega_{\xi,\xi'} \boxtimes \omega_{\eta,\eta'} = \omega_{\theta,\theta'}$ and $\omega_{\xi',\xi} \boxtimes \omega_{\eta',\eta} = \omega_{\theta',\theta}$ where

$$\theta_{i(m,n)} = |\eta_n\rangle_2 \xi_m \in \alpha \triangleleft \gamma, \qquad \theta'_{i(m,n)} = |\xi'_m\rangle_1 \eta'_n \in \beta \triangleright \delta \qquad \text{for all } m, n \in \mathbb{N}.$$

Proof. We only prove the existence of the first contraction. Let $\xi, \xi', \eta, \eta', i, \theta, \theta'$ be as above. Then $\theta \in (\alpha \triangleleft \gamma)^\infty$ and $\|\theta\| \leq \|\xi\|\|\eta\|$ because

$$\sum_k \theta_k^*\theta_k = \sum_{m,n} \xi_m^* \langle\eta_n|_2|\eta_n\rangle_2 \xi_m = \sum_{m,n} \xi_m^* \rho_\beta(\eta_n^*\eta_n)\xi_m \leq \|\eta\|^2 \sum_m \xi_m^*\xi_m \leq \|\eta\|^2\|\xi\|^2,$$

and similarly $\theta' \in (\beta \triangleright \delta)^\infty$ and $\|\theta'\| \leq \|\xi'\|\|\eta'\|$. Next, we show that $\omega_{\theta,\theta'}$ does not depend on ξ and ξ' but only on $\omega_{\xi,\xi'} \in \Omega_{\alpha,\beta}(A)$. Let $\zeta' \in \mathfrak{K}$ and $x \in A_\beta \underset{\mathfrak{b}}{\bar{*}}_\gamma B$. Then

$$\omega_{\theta,\theta'}(x)\zeta' = \sum_{m,n\in\mathbb{N}} \xi_m^* \langle \eta_n|_2 x|\xi'_m\rangle_1 \eta'_n \zeta',$$

where the sum converges in norm. Fix any $n \in \mathbb{N}$. Then we find a sequence $(k_r)_r$ in $\mathbb{N}$ and $\eta''_{r,1}, \ldots, \eta''_{r,k_r} \in \gamma$, $\zeta''_{r,1}, \ldots, \zeta''_{r,k_r} \in \mathfrak{K}$ such that the sum $\sum_{l=1}^{k_r} \eta''_{r,l}\zeta''_{r,l}$ converges in norm to $\eta'_n\zeta'$ as r tends to infinity. But then

$$\begin{aligned}
\sum_m \xi_m^* \langle \eta_n|_2 x|\xi'_m\rangle_1 \eta'_n \zeta' &= \lim_{r\to\infty} \sum_m \sum_{l=1}^{k_r} \xi_m^* \langle \eta_n|_2 x|\xi'_m\rangle_1 \eta''_{r,l}\zeta''_{r,l} \\
&= \lim_{r\to\infty} \sum_{l=1}^{k_r} \sum_m \xi_m^* \langle \eta_n|_2 x|\eta''_{r,l}\rangle_2 \xi'_m \zeta''_{r,l} \\
&= \lim_{r\to\infty} \sum_{l=1}^{k_r} \omega_{\xi,\xi'}(\langle \eta_n|_2 x|\eta''_{r,l}\rangle_2)\zeta''_{r,l}.
\end{aligned}$$

Note here that $\langle \eta_n|x|\eta''_{r,l}\rangle_2 \in A$. Therefore, the sum on the left hand side only depends on $\omega_{\xi,\xi'} \in \Omega_{\alpha,\beta}(A)$ but not on ξ, ξ', and since $n \in \mathbb{N}$ was arbitrary, the same is true for $\omega_{\theta,\theta'}(x)\zeta'$. A similar argument shows that $\omega_{\theta,\theta'}(x)^*\zeta$ depends on $\omega_{\eta,\eta'} \in \Omega_{\gamma,\delta}(B)$ but not on η, η' for each $\zeta \in \mathfrak{K}$. □

Proposition 3.8. *Let $(A_H^{\beta,\alpha}, \Delta)$ be a Hopf C^*-bimodule over $\mathfrak{b}$. Then each of the spaces $\Omega = \Omega_\alpha(A), \Omega_\beta(A), \Omega_{\alpha,\beta}(A), \Omega_{\beta,\alpha}(A)$ is a Banach algebra with respect to the multiplication $\Omega \times \Omega \to \Omega$ given by $(\omega, \omega') \mapsto \omega * \omega' := (\omega \boxtimes \omega') \circ \Delta$.*

Proof. The multiplication is well defined by Propositions 3.5 and 3.7, and associative because Δ is coassociative. □

Now, let $(H, \widehat{\beta}, \alpha, \beta)$ be a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module, $V\colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}_\alpha H \to H_\alpha \underset{\mathfrak{b}}{\otimes}_\beta H$ a C^*-pseudo-multiplicative unitary, and $\tilde{\Omega}_{\beta,\alpha} := \Omega_{\beta,\alpha}(\rho_{\widehat{\beta}}(\mathfrak{B})')$, $\tilde{\Omega}_{\alpha,\widehat{\beta}} := \Omega_{\alpha,\widehat{\beta}}(\rho_\beta(\mathfrak{B})')$. Using Lemma 3.3, the inclusions

$$(\rho_{\widehat{\beta}}(\mathfrak{B})_\alpha \underset{\mathfrak{b}}{\otimes}_\beta \rho_{\widehat{\beta}}(\mathfrak{B}))' \subseteq \rho_{\widehat{\beta}}(\mathfrak{B})'_\alpha \underset{\mathfrak{b}}{\bar{*}}_\beta \rho_{\widehat{\beta}}(\mathfrak{B})', \quad (\rho_\beta(\mathfrak{B})_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}_\alpha \rho_\beta(\mathfrak{B}))' \subseteq \rho_\beta(\mathfrak{B})_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\bar{*}}_\alpha \rho_\beta(\mathfrak{B})',$$

and Proposition 3.7, we define maps

$$\begin{aligned}
\tilde{\Omega}_{\beta,\alpha} \times \tilde{\Omega}_{\beta,\alpha} \to \tilde{\Omega}_{\beta,\alpha}, &\quad (\omega, \omega') \mapsto \omega * \omega' := (\omega \boxtimes \omega') \circ \Delta_V, \\
\tilde{\Omega}_{\alpha,\widehat{\beta}} \times \tilde{\Omega}_{\alpha,\widehat{\beta}} \to \tilde{\Omega}_{\alpha,\widehat{\beta}}, &\quad (\omega, \omega') \mapsto \omega * \omega' := (\omega \boxtimes \omega') \circ \widehat{\Delta}_V,
\end{aligned}$$

Theorem 3.9. i) *The maps above turn $\tilde{\Omega}_{\beta,\alpha}$ and $\tilde{\Omega}_{\alpha,\widehat{\beta}}$ into Banach algebras.*

ii) *There exist contractive homomorphisms $\widehat{\pi}_V\colon \tilde{\Omega}_{\beta,\alpha} \to \widehat{A}_V$ and $\pi_V\colon \tilde{\Omega}_{\alpha,\widehat{\beta}} \to A_V$ such that $\widehat{\pi}_V(\omega_{\xi,\eta}) = \sum_n \langle \xi_n|_2 V|\eta_n\rangle_2$ and $\pi_V(\omega_{\eta,\zeta}) = \sum_n \langle \eta_n|_1 V|\zeta_n\rangle_1$ for all $\xi \in \beta^\infty$, $\eta \in \alpha^\infty$, $\zeta \in \widehat{\beta}^\infty$.*

Proof. We only prove the assertions concerning $\tilde{\Omega}_{\beta,\alpha}$.
i) One only needs to show that the multiplication on $\tilde{\Omega}_{\beta,\alpha}$ is associative. Let $\omega, \omega', \omega'' \in \tilde{\Omega}_{\beta,\alpha}$, where $\omega = \omega_{\eta,\xi}$, $\omega' = \omega_{\eta',\xi'}$, $\omega'' = \omega_{\eta'',\xi''}$ and $\eta, \eta', \eta'' \in \beta^\infty$, $\xi, \xi', \xi'' \in \alpha^\infty$. Then a short calculation shows that for all $x \in \rho_{\widehat{\beta}}(\mathfrak{B})'$,

$$((\omega * \omega') * \omega'')(x) = \sum_{k,l,m} \eta_k^* \langle \eta_l'|_2 \langle \eta_m''|_3 V_{12} V_{13} (x \underset{\mathfrak{b}^\dagger}{\otimes} (1 \underset{\mathfrak{b}}{\otimes} 1)) V_{13}^* V_{12}^* |\xi_k\rangle_1 |\xi_l'\rangle_1 \xi_m'',$$

$$(\omega * (\omega' * \omega''))(x) = \sum_{k,l,m} \eta_k^* \langle \eta_l'|_2 \langle \eta_m''|_3 V_{23} V_{12} (x \underset{\mathfrak{b}^\dagger}{\otimes} 1 \underset{\mathfrak{b}^\dagger}{\otimes} 1) V_{12}^* V_{23}^* |\xi_k\rangle_1 |\xi_l'\rangle_1 \xi_m'',$$

and by (6), the right hand sides coincide.
ii) The map $\widehat{\pi}_V$ is well-defined because $\eta^* \widehat{\pi}_V(\omega)\xi = \omega(\langle \eta|_1 V |\xi\rangle_1)$ and $\langle \eta|_1 V|\xi\rangle_1 \in A_V \subseteq \rho_{\widehat{\beta}}(\mathfrak{B})'$ for all $\eta \in \alpha$, $\xi \in \widehat{\beta}$, $\omega \in \tilde{\Omega}_{\beta,\alpha}$, contractive because V is unitary, and a homomorphism because for all $\omega, \omega' \in \tilde{\Omega}_{\beta,\alpha}$, $\eta \in \alpha, \xi \in \widehat{\beta}$,

$$\begin{aligned}
\eta^* \widehat{\pi}_V(\omega)\widehat{\pi}_V(\omega')\xi &= (\omega \boxtimes \omega')(\langle \eta|_1 V_{12} V_{13} |\xi\rangle_1) \\
&= (\omega \boxtimes \omega')(\langle \eta|_1 V_{23} V_{12} V_{23}^* |\xi\rangle_1) \\
&= (\omega \boxtimes \omega')\big(V(\langle \eta|_1 V|\xi\rangle_1 {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha \mathrm{id})V^*\big) \\
&= (\omega * \omega')(\langle \eta|_1 V|\xi\rangle_1) = \eta^* \widehat{\pi}_V(\omega * \omega')\xi. \qquad \square
\end{aligned}$$

Definition 3.10. *We call the algebras* $\widehat{A}_V^0 := \widehat{\pi}_V(\tilde{\Omega}_{\beta,\alpha}) \subseteq \widehat{A}_V$ *and* $A_V^0 := \pi_V(\tilde{\Omega}_{\alpha,\widehat{\beta}}) \subseteq A_V$*, equipped with the quotient norms coming from the surjections* $\widehat{\pi}_V$ *and* π_V*, the* Fourier algebra *and the* dual Fourier algebra *of* V*, respectively.*

The pairs $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$ and $((A_V)_H^{\beta,\alpha}, \Delta_V)$ stand in a generalized Pontrjagin duality which is captured by the following pairing.

Proposition 3.11. i) *There exists a bilinear map* $(\cdot|\cdot)\colon \widehat{A}_V^0 \times A_V^0 \to \mathcal{L}(\mathfrak{K})$ *such that* $\omega(\pi_V(\upsilon)) = \big(\widehat{\pi}_V(\omega)\big|\pi_V(\upsilon)\big) = \upsilon(\widehat{\pi}_V(\omega))$ *for all* $\omega \in \tilde{\Omega}_{\beta,\alpha}, \upsilon \in \tilde{\Omega}_{\alpha,\widehat{\beta}}$*. This map is nondegenerate in the sense that for each* $\widehat{a} \in \widehat{A}_V^0$ *and* $a \in A_V^0$*, there exist* $\widehat{a}' \in \widehat{A}_V^0$ *and* $a' \in A_V^0$ *such that* $(\widehat{a}|a') \neq 0$ *and* $(\widehat{a}'|a) \neq 0$.

ii) $(\widehat{\pi}_V(\omega)\widehat{\pi}_V(\omega')|a) = (\omega \boxtimes \omega')(\Delta_V(a))$ *and* $(\widehat{a}|\pi_V(\upsilon)\pi_V(\upsilon')) = (\upsilon \boxtimes \upsilon')(\widehat{\Delta}_V(\widehat{a}))$ *for all* $\omega, \omega' \in \tilde{\Omega}_{\beta,\alpha}$*,* $a \in A_V^0$*,* $\upsilon, \upsilon' \in \tilde{\Omega}_{\alpha,\widehat{\beta}}$*,* $\widehat{a} \in \widehat{A}_V^0$.

Proof. i) If $\omega = \omega_{\xi,\xi'}$ and $\upsilon = \omega_{\eta,\eta'}$, where $\xi \in \beta^\infty$, $\xi', \eta \in \alpha^\infty$, $\eta' \in \widehat{\beta}^\infty$, then

$$\omega(\pi_V(\upsilon)) = \sum_{m,n} \xi_m^* \langle \eta_n|_1 V|\eta_n'\rangle_1 \xi_m' = \sum_{m,n} \eta_n^* \langle \xi_m|_2 V|\xi_m'\rangle_2 \eta_n' = \upsilon(\widehat{\pi}_V(\omega)).$$

iii) For all ω, ω', a as above, $(\widehat{\pi}_V(\omega)\widehat{\pi}_V(\omega')|a) = (\widehat{\pi}_V(\omega * \omega')|a) = (\omega * \omega')(a) = (\omega \boxtimes \omega)(\Delta_V(a))$. The second equation follows similarly. $\square$

Part i) of the preceding result implies the following relation between the Fourier algebras $\widehat{A}_V^0$ and A_V^0 and the convolution algebras constructed in Proposition 3.8.

Corollary 3.12. *If* $((A_V)_H^{\beta,\alpha}, \widehat{\Delta}_V)$ *or* $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \Delta_V)$ *is a Hopf* C^**-bimodule, then there exists an isometric isomorphism of Banach algebras* $\widehat{\pi}\colon \Omega_{\beta,\alpha}(A_V) \to \widehat{A}_V^0$ *or* $\pi\colon \Omega_{\alpha,\widehat{\beta}}(\widehat{A}_V) \to A_V^0$*, respectively, whose composition with the quotient map* $\tilde{\Omega}_{\beta,\alpha} \to \Omega_{\beta,\alpha}(A_V)$ *or* $\tilde{\Omega}_{\alpha,\widehat{\beta}} \to \Omega_{\alpha,\widehat{\beta}}(\widehat{A}_V)$ *is equal to* $\widehat{\pi}_V$ *or* π_V*, respectively.* □

3.4. **The legs of the unitary of a groupoid.** Let G be a locally compact, Hausdorff, second countable groupoid G as in subsection 2.3. We keep the notation introduced there and determine the legs of the C^*-pseudo-multiplicative unitary V associated to G. Denote by $m\colon C_0(G) \to \mathcal{L}(H)$ the representation given by multiplication operators, and by $L^1(G,\lambda)$ the completion of $C_c(G)$ with respect to the norm given by $\|f\| := \sup_{u \in G^0} \int_{G^u} |f(u)| \,\mathrm{d}\lambda^u(x)$ for all $f \in C_c(G)$. Then $L^1(G,\lambda)$ is a Banach algebra with respect to the convolution product

$$(f * g)(y) = \int_{G^{r(y)}} g(x) f(x^{-1}y) \,\mathrm{d}\lambda^{r(y)}(x) \quad \text{for all } f, g \in L^1(G,\lambda), y \in G,$$

and there exists a contractive algebra homomorphism $L\colon L^1(G,\lambda) \to \mathcal{L}(H)$ such that

$$\big(L(f)\xi\big)(y) = \int_{G^{r(y)}} f(x) D^{-1/2}(x) \xi(x^{-1}y) \,\mathrm{d}\lambda^{r(y)}(x) \quad \text{for all } f, \xi \in C_c(G), y \in G.$$

Routine arguments show that there exists a unique continuous map

$$L^2(G,\lambda) \times L^2(G,\lambda) \to C_0(G), \ (\xi, \xi') \mapsto \overline{\xi} * \xi'^*,$$

such that $(\overline{\xi} * \xi'^*)(x) = \int_{G^{r(x)}} \overline{\xi(y)} \xi'(x^{-1}y) \,\mathrm{d}\lambda^{r(x)}(y)$ for all $\xi, \xi' \in C_c(G), x \in G$.

Lemma 3.13. *Let* $\widehat{a}_{\xi,\xi'} := \langle j(\xi)|_2 V | j(\xi')\rangle_2$ *and* $a_{\eta,\eta'} := \langle j(\eta)|_1 V | \hat{j}(\eta')\rangle_1 \in A_V^0$*, where* $\xi, \xi' \in L^2(G,\lambda)$ *and* $\eta, \eta' \in C_c(G)$*. Then* $\widehat{a}_{\xi,\xi'} = m(\overline{\xi} * \xi'^*)$ *and* $a_{\eta,\eta'} = L(\overline{\eta}\eta')$*.*

Proof. By continuity, we may assume $\xi, \xi' \in C_c(G)$. Then for all $\zeta, \zeta' \in C_c(G)$,

$$\begin{aligned}
\langle \zeta | \widehat{a}_{\xi,\xi'} \zeta' \rangle &= \langle \zeta \otimes j(\xi) | V(\zeta' \otimes j(\xi')) \rangle \\
&= \int_G \int_{G^{r(x)}} \overline{\zeta(x)\xi(y)} \zeta'(x) \xi'(x^{-1}y) \,\mathrm{d}\lambda^{r(x)}(y) \,\mathrm{d}\nu(x) \\
\langle \zeta | a_{\eta,\eta'} \zeta' \rangle &= \langle j(\eta) \otimes \zeta | V(\hat{j}(\eta') \otimes \zeta') \rangle \\
&= \int_G \int_{G^{r(y)}} \overline{\eta(x)\zeta(y)} \eta'(x) D^{-1/2}(x) \zeta'(x^{-1}y) \,\mathrm{d}\lambda^{r(y)}(x) \,\mathrm{d}\nu(y).
\end{aligned}$$ □

The algebra $\widehat{A}_V^0$ can be considered as a continuous Fourier algebra of the locally compact groupoid G. A Fourier algebra for measured groupoids was defined and studied by Renault [23], and for measured quantum groupoids by Vallin [32].

Remark 3.14. A Fourier algebra $A(G)$ for locally compact groupoids was defined by Paterson in [21] as follows. He constructs a Fourier-Stieltjes algebra $B(G) \subseteq C(G)$ and defines $A(G)$ to be the norm-closed subalgebra of $B(G)$ generated by the set $A_{cf}(G) := \{\widehat{a}_{\xi,\xi'} \mid \xi, \xi' \in L^2(G,\lambda)\}$. The definition of $B(G)$ in immediately implies that $\|\widehat{\pi}_V(\omega_{\xi,\xi'})\|_{B(G)} \leq \|\xi\|\|\xi'\|$ for all $\xi \in \alpha^\infty, \xi' \in \beta^\infty$ with finitely many non-zero components. Therefore, the identity on $A_{cf}(G)$ extends to a contractive homomorphism from $\widehat{A}_V^0$ to $A(G)$.

Remark 3.15. Another Fourier space $\widetilde{\mathcal{A}}(G)$ considered in [21, Note after Proposition 13] is defined as follows. For each $\eta \in L^2(G,\lambda)$ and $u \in G^0$, write $\|\xi_n(u)\| := \langle \xi_n|\xi_n\rangle(u)^{1/2}$. Denote by M the set of all pairs (ξ,ξ') of sequences in $L^2(G,\lambda)$ such that the supremum $|(\xi,\xi')|_M := \sup_{u,v\in G^0} \sum_n \|\xi_n(u)\|\|\xi'_n(v)\|$ is finite, and denote by $\widetilde{\mathcal{A}}(G)$ the completion of the linear span of $A_{cf}(G)$ with respect to the norm defined by

$$\|\widehat{a}\|_{\widetilde{\mathcal{A}}(G)} = \inf \left\{ |(\xi,\xi')|_M \,\middle|\, \widehat{a} = \sum_n \widehat{a}_{\xi_n,\xi'_n} \right\}.$$

The identity on $A_{cf}(G)$ extends to a linear contraction from $\widehat{A}^0_V$ to $\widetilde{\mathcal{A}}(G)$ because

$$\|\xi\|^2 = \sup_{u\in G^0} \sum_n \langle \xi_n|\xi_n\rangle(u) = \sup_{u\in G^0} \sum_n \|\xi_n(u)\|^2, \qquad \|\xi'\|^2 = \sup_{v\in G^0} \sum_n \|\xi_n(v)\|^2,$$

for all $\xi,\xi' \in L^2(G,\lambda)^\infty$ and hence $|(\xi,\xi')|_M = \sup_{u,v\in G^0} \sum_n \|\xi_n(u)\|\|\xi'_n(v)\| \le \|\xi\|\|\xi'\|$.

Recall that the reduced groupoid C^*-algebra $C^*_r(G)$ is the closed linear span of all operators of the $L(g)$, where $g \in L^1(G,\lambda)$ [22].

Theorem 3.16. *Let V be the C^*-pseudo-multiplicative unitary of a locally compact groupoid G. Then $((\widehat{A}_V)^{\beta,\alpha}_H, \widehat{\Delta}_V)$ and $((A_V)^{\alpha,\widehat{\beta}}_H, \Delta_V)$ are Hopf C^*-bimodules and*

$$\begin{aligned} \widehat{A}_V &= m(C_0(G)), \quad & \big(\widehat{\Delta}_V(m(f))\omega\big)(x,y) &= f(xy)\omega(x,y), \\ A_V &= C^*_r(G), \quad & \big(\Delta_V(L(g))\omega'\big)(x',y') &= \int_{G^{u'}} g(z) D^{-1/2}(z)\omega'(z^{-1}x', z^{-1}y')\,\mathrm{d}\lambda^{u'}(z) \end{aligned}$$

for all $f \in C_0(G)$, $\omega \in H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H$, $(x,y) \in G_s{\times}_r G$ and $g \in C_c(G)$, $\omega' \in H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H$, $(x',y') \in G_r{\times}_r G$, where $u' = r(x') = r(y')$.

Proof. The first assertion will follow from Example 4.3 and Theorem 4.5 in subsection 4.1. The equations concerning $\widehat{A}_V$, A_V and $\widehat{\Delta}_V$, Δ_V follow from Lemma 3.13 and straightforward calculations. □

4. Regular, proper and étale C^*-pseudo-multiplicative unitaries

Let $V\colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H$ be a C^*-pseudo-multiplicative unitary as before.

4.1. Regularity.

In [3], Baaj and Skandalis showed that the pairs $(\widehat{A}_V, \widehat{\Delta}_V)$ and (A_V, Δ_V) associated to a multiplicative unitary V on a Hilbert space H form Hopf C^*-algebras if the unitary satisfies the regularity condition $[\langle H|_2 V|H\rangle_1] = \mathcal{K}(H)$. This condition was generalized by Baaj in [1, 2] and extended to pseudo-multiplicative unitaries by Enock [9]. To adapt it to C^*-pseudo-multiplicative unitaries, we consider the space

$$C_V := [\langle \alpha|_1 V|\alpha\rangle_2] \subseteq \mathcal{L}(H).$$

Proposition 4.1. $[C_V C_V] = C_V$, $C_{V^{op}} = C^*_V$, $[C_V\alpha] = \alpha$, *and* $[C_V \rho_\beta(\mathfrak{B})] = [\rho_\beta(\mathfrak{B})C_V] = C_V = [C_V\rho_{\widehat{\beta}}(\mathfrak{B})] = [\rho_{\widehat{\beta}}(\mathfrak{B})C_V]$.

Proof. The proof is completely analogous to the proof of Proposition 3.2; for example, the first equation follows from the commutativity of the following two diagrams:

$$\begin{tikzcd}
 & H \arrow[rr, "C_V"] \arrow[dl, "|\alpha\rangle_2"'] \arrow[d, "|\alpha\rangle_2"] & & H \arrow[rr, "C_V"] \arrow[dr, "|\alpha\rangle_2"] & & H & \\
H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[ddr, "|\alpha\rangle_2"'] & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[r, "V"] \arrow[d, "|\alpha\rangle_3"] & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[ur, "\langle\alpha|_1"] \arrow[dr, "|\alpha\rangle_3"] & & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[r, "V"] & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[u, "\langle\alpha|_1"'] & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[ul, "\langle\alpha|_1"'] \\
 & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[rr, "V_{12}"] \arrow[d, "V_{23}"] & & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[ur, "\langle\alpha|_1"'] \arrow[rr, "V_{23}"] & & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[u, "\langle\alpha|_1"'] & \\
 & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{(\alpha\triangleright\alpha)} (H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H) \arrow[rrrr, "V_{13}"] & & & & (H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H) {}_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[u, "V_{12}"'] \arrow[uur, "\langle\alpha|_2"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & H \arrow[r, "C_V"] \arrow[dl, "|\alpha\rangle_2"'] \arrow[d, "|\alpha\rangle_2"] & H \\
H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[r, "\rho_{(\widehat{\beta}\triangleright\beta)}(\mathfrak{B})"] \arrow[dr, "|\alpha\rangle_2"'] & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[r, "V"] & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[u, "\langle\alpha|_1"'] \\
 & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{(\alpha\triangleright\alpha)} (H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H) \arrow[u, "\langle\alpha|_2"] \arrow[r, "V_{13}"] & (H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H) {}_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[u, "\langle\alpha|_2"']
\end{tikzcd}$$

□

Definition 4.2. *A C^*-pseudo-multiplicative unitary $(\mathfrak{b}, H, \widehat{\beta}, \alpha, \beta, V)$ is* semi-regular *if $C_V \supseteq [\alpha\alpha^*]$, and* regular *if $C_V = [\alpha\alpha^*]$.*

Examples 4.3. i) V is (semi-)regular if and only if V^{op} is (semi-)regular.

ii) The C^*-pseudo-multiplicative unitary of a locally compact Hausdorff groupoid G (see Theorem 2.5) is regular. To prove this assertion, we use the notation introduced in subsection 2.3 and calculate that for each $\xi, \xi' \in C_c(G)$, $\zeta \in C_c(G) \subseteq L^2(G, \nu)$, $y \in G$,

$$\big(\langle j(\xi')|_1 V |j(\xi)\rangle_2 \zeta\big)(y) = \int_{G^{r(y)}} \overline{\xi'(x)}\zeta(x)\xi(x^{-1}y)\,\mathrm{d}\lambda^{r(y)}(x),$$
$$\big(j(\xi')j(\xi)^*\zeta\big)(y) = \xi'(y)\int_{G^{r(y)}} \overline{\xi(x)}\zeta(x)\,\mathrm{d}\lambda^{r(y)}(x).$$

Using standard approximation arguments, we find $[\langle\alpha|_1 V|\alpha\rangle_2] = [S(C_c(G_r{\times_r}G))] = [\alpha\alpha^*]$, where for each $\omega \in C_c(G_r{\times_r}G)$, the operator $S(\omega)$ is given by

$$(S(\omega)\zeta)(y) = \int_{G^{r(y)}} \omega(x,y)\zeta(x)d\lambda^{r(y)}(x) \quad \text{for all } \zeta \in C_c(G),\ y \in G.$$

iii) In [31], we introduce compact C^*-quantum groupoids and construct for each such quantum groupoid a C^*-pseudo-multiplicative unitary that turns out to be regular.

We now deduce several properties of semi-regular and regular C^*-pseudo-multiplicative unitaries, using commutative diagrams as in Subsection 3.2.

Proposition 4.4. *If V is semi-regular, then C_V is a C^*-algebra.*

Proof. Assume that V is regular. Then the following two diagrams commute, whence $[C_V C_V^*] = [\langle\alpha|_1 \langle\alpha|_1 V_{23} |\alpha\rangle_1 |\alpha\rangle_2] = C_V$:

$$
\begin{tikzcd}
 & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[dr, "|\alpha\rangle_1"] & & \\
H \arrow[ur, "|\alpha\rangle_2"] \arrow[r, "|\alpha\rangle_1"'] \arrow[d, "C_V^*"'] & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[r, "|\alpha\rangle_3"'] \arrow[d, "V^*"] & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[d, "V_{12}^*"] \arrow[dddr, "V_{23}"] & \\
H \arrow[d, equal] & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[l, "\langle\alpha|_2"] \arrow[r, "|\alpha\rangle_3"] \arrow[dr, phantom, "(R)"] & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[d, "V_{23}"] & \\
H \arrow[r, "|\alpha\rangle_2"] \arrow[d, "C_V"'] & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[d, "V"] & H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{(\alpha \triangleright \alpha)} (H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H) \arrow[l, "\langle\alpha|_2"] \arrow[d, "V_{13}"] & \\
H & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[l, "\langle\alpha|_1"] & (H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H)_{(\alpha \triangleleft \alpha)} \underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[l, "\langle\alpha|_2"] & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[l, "V_{12}^*"] \arrow[dll, "\langle\alpha|_1"] \\
 & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[ul, "\langle\alpha|_1"] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
H \arrow[r, "C_V"] \arrow[d, "|\alpha\rangle_2"] & H & \\
H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[r, "V"] \arrow[d, "|\alpha\rangle_1"] & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[u, "\langle\alpha|_1"'] \arrow[r, "\rho_{(\beta \triangleleft \beta)}(\mathfrak{B})"] \arrow[d, "\langle\alpha|_1"] & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[ul, "|\alpha\rangle_1"'] \\
H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H {}_{\widehat{\beta}}\underset{\mathfrak{b}^{\dagger}}{\otimes}{}_{\alpha} H \arrow[r, "V_{23}"] & H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H {}_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[ur, "|\alpha\rangle_1"] &
\end{tikzcd}
$$

Now, assume that V is semi-regular. Then cell (R) in the first diagram need not commute, but still $[|\alpha\rangle_2 \langle\alpha|_2] \subseteq [\langle\alpha|_2 V_{23} |\alpha\rangle_3]$ and hence $[C_V C_V^*] \subseteq [\langle\alpha|_1 \langle\alpha|_1 V_{23} |\alpha\rangle_1 |\alpha\rangle_2] = C_V$. A similar argument shows that also $[C_V^* C_V] \subseteq C_V$, and from Proposition 3.2 and [1, Lemme 3.3], it follows that C_V is a C^*-algebra. □

Theorem 4.5. *If $C_V = C_V^*$, then $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$ and $((A_V)_H^{\beta,\alpha}, \Delta_V)$ are Hopf C^*-bimodules. In particular, this is the case if V is semi-regular.*

The key step in the proof is the following lemma:

Lemma 4.6. $[V(1 \underset{\mathfrak{b}^{\dagger}}{\otimes} C_V)V^* |\beta\rangle_2] = [|\beta\rangle_2 \widehat{A}_V^*]$.

Proof. The following diagram commutes and shows that one has $[V(1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V)V^*|\beta\rangle_2] = [|\beta\rangle_2\langle\alpha|_2V^*|\beta\rangle_2] = [|\beta\rangle_2\widehat{A}_V^*]$:

$$\begin{tikzcd}
H \arrow[rrr, "|\beta\rangle_2"] \arrow[d, "|\beta\rangle_2"] & & & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[dl, "|\beta\rangle_3"'] \arrow[ddd, "V^*"] \\
H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "|\alpha\rangle_3"] \arrow[d, "V^*"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V_{23}"] \arrow[d, "V_{12}^*"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[dd, "V_{12}^*"] & \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "|\alpha\rangle_3"] \arrow[d, "1\underset{\mathfrak{b}^\dagger}{\otimes}C_V"] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[d, "V_{23}"] \arrow[r, phantom, "\text{(P)}"] & {} & \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[d, "V"] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{(\alpha\triangleright\alpha)}(H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H) \arrow[l, "\langle\alpha|_2"] \arrow[r, "V_{13}"] & (H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H)_{(\alpha\triangleleft\alpha)}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[dll, "\langle\alpha|_2"] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[l, "|\beta\rangle_3"'] \arrow[d, "\langle\alpha|_2"'] \\
H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H & & & H \arrow[lll, "|\beta\rangle_2"]
\end{tikzcd}$$

Indeed, cell (P) commutes by (6), and the remaining cells by (5) or by inspection. □

Proof of Theorem 4.5. By Theorem 3.4, it suffices to show that $\widehat{A}_V = \widehat{A}_V^*$. But by Proposition 3.2 and Lemma 4.6, $\widehat{A}_V^* = [\rho_\alpha(\mathfrak{B}^\dagger)\widehat{A}_V^*] = [\langle\beta|_2|\beta\rangle_2\widehat{A}_V^*] = [\langle\beta|_2V(1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V)V^*|\beta\rangle_2]$. Replacing V by V^{op}, we obtain the assertion concerning A_V. □

Remark 4.7. If V is regular, then $[V|\alpha\rangle_2\widehat{A}_V] = [|\beta\rangle_2\widehat{A}_V]$ and $[V|\widehat{\beta}\rangle_1A_V] = [|\alpha\rangle_1A_V]$. Indeed, using Lemma 4.6 and the relation $\widehat{A}_V = \widehat{A}_V^*$ (Theorem 4.5), we find that $[V|\alpha\rangle_2\widehat{A}_V] = [V|\alpha\rangle_2\langle\alpha|_2V^*|\beta\rangle_2] = [V(1 \underset{\mathfrak{b}^\dagger}{\otimes} C_V)V^*|\beta\rangle_2] = [|\beta\rangle_2\widehat{A}_V]$, and replacing V by V^{op}, we obtain the second equation.

4.2. **Proper and étale C^*-pseudo-multiplicative unitaries.** In [3], Baaj and Skandalis characterized multiplicative unitaries that correspond to compact or discrete quantum groups by the existence of fixed or cofixed vectors, respectively, and showed that from such vectors, one can construct a Haar state and a counit on the associated legs. We adapt some of their constructions to C^*-pseudo-multiplicative unitaries as follows. Given a C^*-$\mathfrak{b}^{(\dagger)}$-module K_γ, let $M(\gamma) = \{T \in \mathcal{L}(\mathfrak{K}, K) \mid T\mathfrak{B}^{(\dagger)} \subseteq \gamma, T^*\gamma \subseteq \mathfrak{B}^{(\dagger)}\}$.

Definition 4.8. *A* fixed element *for V is an $\eta \in M(\widehat{\beta}) \cap M(\alpha) \subseteq \mathcal{L}(\mathfrak{K}, H)$ satisfying $V|\eta\rangle_1 = |\eta\rangle_1$. A* cofixed element *for V is a $\xi \in M(\alpha) \cap M(\beta) \subseteq \mathcal{L}(\mathfrak{K}, H)$ satisfying $V|\xi\rangle_2 = |\xi\rangle_2$. We denote the set of all fixed/cofixed elements for V by* $\mathrm{Fix}(V)/\mathrm{Cofix}(V)$.

Example 4.9. Let V be the C^*-pseudo-multiplicative unitary of a groupoid G; see subsection 2.3. Identify $M(L^2(G, \lambda))$ in the natural way with the completion of the space

$$\left\{f \in C(G) \,\middle|\, r\colon \operatorname{supp} f \to G^0 \text{ is proper, } \sup_{u\in G^0} \textstyle\int_{G^u} |f(x)|^2 \,\mathrm{d}\lambda^u(x) \text{ is finite}\right\}$$

with respect to the norm $f \mapsto \sup_{u\in G^0} \left(\int_{G^u} |f(x)|^2 \,\mathrm{d}\lambda^u(x)\right)^{1/2}$. Similarly as in [27, Lemma 7.11], one finds that

i) $\eta_0 \in M(L^2(G,\lambda))$ is a fixed element if and only if for each $u \in G^0$, $\eta_0|_{G^u\setminus\{u\}} = 0$ almost everywhere with respect to λ^u;
ii) $\xi_0 \in M(L^2(G,\lambda))$ is a cofixed element if and only if $\xi_0(x) = \xi_0(s(x))$ for all $x \in G$.

Remarks 4.10. i) $\mathrm{Fix}(V) = \mathrm{Cofix}(V^{op})$ and $\mathrm{Cofix}(V) = \mathrm{Fix}(V^{op})$.
ii) $\mathrm{Fix}(V)^*\mathrm{Fix}(V)$ and $\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)$ are contained in $M(\mathfrak{B}) \cap M(\mathfrak{B}^\dagger)$.
iii) $\rho_\alpha(\mathfrak{B}^\dagger)\mathrm{Fix}(V) = \mathrm{Fix}(V)\mathfrak{B}^\dagger \subseteq \widehat{\beta}$ and $\rho_{\widehat{\beta}}(\mathfrak{B})\mathrm{Fix}(V) = \mathrm{Fix}(V)\mathfrak{B} \subseteq \alpha$ because $\mathrm{Fix}(V) \subseteq M(\widehat{\beta}) \cap M(\alpha)$, and similarly $\rho_\beta(\mathfrak{B})\mathrm{Cofix}(V) \subseteq \alpha$ and $\rho_\alpha(\mathfrak{B}^\dagger)\mathrm{Cofix}(V) \subseteq \beta$.

Lemma 4.11. i) $\langle\xi|_2V|\xi'\rangle_2 = \rho_\alpha(\xi^*\xi') = \rho_{\widehat{\beta}}(\xi^*\xi')$ *for all* $\xi, \xi' \in \mathrm{Cofix}(V)$.
ii) $\langle\eta|_1V|\eta'\rangle_1 = \rho_\beta(\eta^*\eta') = \rho_\alpha(\eta^*\eta')$ *for all* $\eta, \eta' \in \mathrm{Fix}(V)$.

Proof. Let $\zeta \in H$ and $\xi, \xi' \in \mathrm{Cofix}(V)$. Then $\langle\xi|_2V|\xi'\rangle_2\zeta = \langle\xi|_2|\xi'\rangle_2\zeta = \rho_\alpha(\xi^*\xi')\zeta$ and $(\langle\xi|_2V|\xi'\rangle_2)^*\zeta = \langle\xi'|_2|\xi\rangle_2\zeta = \rho_{\widehat{\beta}}((\xi')^*\xi)\zeta$. The proof of ii) is similar. □

Proposition 4.12. i) $\rho_{\widehat{\beta}}(M(\mathfrak{B}))\mathrm{Cofix}(V) \subseteq \mathrm{Cofix}(V)$ *and* $\rho_\beta(\mathfrak{B})\mathrm{Fix}(V) \subseteq \mathrm{Fix}(V)$.
ii) $[\mathrm{Cofix}(V)\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)] = \mathrm{Cofix}(V)$ *and* $[\mathrm{Fix}(V)\mathrm{Fix}(V)^*\mathrm{Fix}(V)] = \mathrm{Fix}(V)$.
iii) $[\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)]$ *and* $[\mathrm{Fix}(V)^*\mathrm{Fix}(V)]$ *are C^*-subalgebras of* $M(\mathfrak{B}) \cap M(\mathfrak{B}^\dagger)$*; in particular, they are commutative.*

Proof. We only prove the assertions concerning $\mathrm{Cofix}(V)$.
i) Let $T \in M(\mathfrak{B})$ and $\xi \in \mathrm{Cofix}(V)$. Then $\rho_{\widehat{\beta}}(T)\xi \subseteq M(\beta) \cap M(\alpha)$ because $\rho_{\widehat{\beta}}(\mathfrak{B})\beta \subseteq \beta$ and $\rho_{\widehat{\beta}}(\mathfrak{B})\alpha \subseteq \alpha$. The relation $V(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}$ furthermore implies

$$V|\rho_{\widehat{\beta}}(T)\xi\rangle_2 = V\rho_{(\widehat{\beta}\triangleright\widehat{\beta})}(T)|\xi\rangle_2 = \rho_{(\alpha\triangleright\widehat{\beta})}(T)V|\xi\rangle_2 = \rho_{(\alpha\triangleright\widehat{\beta})}(T)|\xi\rangle_2 = |\rho_{\widehat{\beta}}(T)\xi\rangle_2.$$

ii) Using i) and the relation $\mathrm{Cofix}(V)^*\mathrm{Cofix}(V) \subseteq M(\mathfrak{B}^\dagger)$, we find that

$$[\mathrm{Cofix}(V)\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)] \subseteq [\mathrm{Cofix}(V)M(\mathfrak{B}^\dagger)] = [\rho_{\widehat{\beta}}(M(\mathfrak{B}))\mathrm{Cofix}(V)] \subseteq \mathrm{Cofix}(V).$$

Therefore, $[\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)]$ is a C^*-algebra and $\mathrm{Cofix}(V)$ is a Hilbert C^*-module over $[\mathrm{Cofix}(V)^*\mathrm{Cofix}(V)]$. Now, [15, p. 5] implies that the inclusion above is an equality.
iii) This follows from ii) and Remark 4.10 ii). □

Definition 4.13. *The C^*-pseudo-multiplicative unitary V is* étale *if $\eta^*\eta = \mathrm{id}_{\mathfrak{K}}$ for some $\eta \in \mathrm{Fix}(V)$,* proper *if $\xi^*\xi = \mathrm{id}_{\mathfrak{K}}$ for some $\xi \in \mathrm{Cofix}(V)$, and* compact *if it is proper and $\mathfrak{B}, \mathfrak{B}^\dagger$ are unital.*

Example 4.14. The C^*-pseudo-multiplicative unitary of a groupoid G (subsection 2.3) is étale/proper/compact if and only if G is étale/proper/compact. This follows from similar arguments as in [27, Theorem 7.12].

Remarks 4.15. i) By Remark 4.10, V is étale/proper if and only if V^{op} is proper/étale.
ii) If V is proper and $\xi \in \mathrm{Cofix}(V)$, $\xi^*\xi = \mathrm{id}_{\mathfrak{K}}$, then

$$[\rho_{\widehat{\beta}}(\mathfrak{B})\rho_\alpha(\mathfrak{B}^\dagger)] = [\rho_\alpha(\mathfrak{B}^\dagger)\langle\xi|_2V|\xi\rangle_2\rho_{\widehat{\beta}}(\mathfrak{B})] = [\langle\xi\mathfrak{B}^\dagger|_2V|\xi\mathfrak{B}\rangle_2] \subseteq [\langle\beta|_2V|\alpha\rangle_2] = \widehat{A}_V.$$

Similarly, if V is étale, then $[\rho_\beta(\mathfrak{B})\rho_\alpha(\mathfrak{B}^\dagger)] \in A_V$.

Fixed and cofixed vectors give rise to invariant operator-valued weights and counits on the legs of V as follows.

Definition 4.16. *Let $(A_H^{\beta,\alpha}, \Delta)$ be a Hopf C^*-bimodule over $\mathfrak{b}$.*
A bounded left Haar weight *for $(A_H^{\beta,\alpha}, \Delta)$ is a completely positive contraction $\phi\colon A \to \mathfrak{B}$ satisfying $\phi(a\rho_\beta(b)) = \phi(a)b$ and $\phi(\langle\xi|_1\Delta(a)|\xi'\rangle_1) = \xi^*\rho_\beta(\phi(a))\xi'$ for all $a \in A$, $b \in \mathfrak{B}$, $\xi, \xi' \in \alpha$. We call ϕ* normal *if $\phi \in \Omega_{M(\beta)}(A)$.*
Similarly, a bounded right Haar weight *for $(A_H^{\beta,\alpha}, \Delta)$ is a completely positive contraction $\psi\colon A \to \mathfrak{B}^\dagger$ satisfying $\psi(a\rho_\alpha(b^\dagger)) = \psi(a)b^\dagger$ and $\psi(\langle\eta|_2\Delta(a)|\eta'\rangle_2) = \eta^*\rho_\alpha(\psi(a))\eta'$ for all $a \in A$, $b^\dagger \in \mathfrak{B}^\dagger$, $\eta, \eta' \in \beta$. We call ψ* normal *if $\psi \in \Omega_{M(\alpha)}(A)$.*
A bounded (left/right) counit *for $(A_H^{\beta,\alpha}, \Delta)$ is a morphism of C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-algebras $\epsilon\colon A_H^{\beta,\alpha} \to \mathcal{L}(\mathfrak{K})_{\mathfrak{K}}^{\mathfrak{B}^\dagger,\mathfrak{B}}$ that makes the (left/right one of the) following two diagrams commute,*

$$\begin{tikzcd} A_\alpha \underset{\mathfrak{b}}{*}{}_\beta A \arrow[d, "\epsilon \underset{\mathfrak{b}}{*} \mathrm{id}"'] & & A \arrow[ll, "\Delta"'] \arrow[d] \\ \mathcal{L}(\mathfrak{K})_{\mathfrak{B}} \underset{\mathfrak{b}}{*}{}_\beta A \arrow[r] & \mathcal{L}(\mathfrak{K}_{\mathfrak{B}} \underset{\mathfrak{b}}{\otimes}{}_\beta H) \arrow[r, "\cong"] & \mathcal{L}(H), \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\Delta"] \arrow[d] & & A_\alpha \underset{\mathfrak{b}}{*}{}_\beta A \arrow[d, "\mathrm{id} \underset{\mathfrak{b}}{*} \epsilon"] \\ \mathcal{L}(H) & \mathcal{L}(H_\alpha \underset{\mathfrak{b}}{\otimes}{}_{\mathfrak{B}^\dagger} \mathfrak{K}) \arrow[l, "\cong"'] & A_\alpha \underset{\mathfrak{b}}{*}{}_{\mathfrak{B}^\dagger} \mathcal{L}(\mathfrak{K}), \arrow[l] \end{tikzcd}$$

where the isomorphisms $\mathcal{L}(\mathfrak{K}_{\mathfrak{B}} \underset{\mathfrak{b}}{\otimes}{}_\beta H) \cong \mathcal{L}(H) \cong \mathcal{L}(H_\alpha \underset{\mathfrak{b}}{\otimes}{}_{\mathfrak{B}^\dagger} \mathfrak{K})$ are induced by the isomorphisms (4).

Remark 4.17. Let $(A_H^{\beta,\alpha}, \Delta)$ be a Hopf C^*-bimodule over $\mathfrak{b}$. Evidently, a completely positive contraction $\phi\colon A \to \mathfrak{B}$ is a normal bounded left Haar weight for $(A_H^{\beta,\alpha}, \Delta)$ if and only if $\phi \in \Omega_{M(\beta)}(A)$ and $(\mathrm{id} * \phi) \circ \Delta = \rho_\beta \circ \phi$. A similar remark applies to right Haar weights.

Theorem 4.18. *Let V be an étale C^*-pseudo-multiplicative unitary.*

i) *There exists a contractive homomorphism $\widehat{\epsilon}\colon \widehat{A}_V \to \mathcal{L}(\mathfrak{K})$ such that $\widehat{\epsilon}(\langle\eta|_2 V|\xi\rangle_2) = \eta^*\xi$ for all $\eta \in \beta, \xi \in \alpha$.*
ii) *Assume that V is regular and let $D := [\beta^*\alpha]$. Then $D_{\mathfrak{K}}^{\mathfrak{B},\mathfrak{B}^\dagger}$ is a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-algebra and $\widehat{\epsilon}$ is a morphism from $(\widehat{A}_V)_H^{\alpha,\widehat{\beta}}$ to $D_{\mathfrak{K}}^{\mathfrak{B},\mathfrak{B}^\dagger}$ and a bounded counit for $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$.*

Proof. Choose an $\eta_0 \in \mathrm{Fix}(V)$ with $\eta_0^*\eta_0 = \mathrm{id}_{\mathfrak{K}}$ and define $\widehat{\epsilon}\colon \widehat{A}_V \to \mathcal{L}(\mathfrak{K})$ by $\hat{a} \mapsto \eta_0^*\hat{a}\eta_0$. Then $\widehat{\epsilon}$ is contractive. For all $\xi \in \alpha, \eta \in \beta, \zeta \in \mathfrak{K}$,

$$\langle\eta|_2 V|\xi\rangle_2\eta_0\zeta = \langle\eta|_2 V(\eta_0 \otimes \xi\zeta) = \langle\eta|_2(\eta_0 \otimes \xi\zeta) = \eta_0(\eta^*\xi)\zeta,$$

and hence $\widehat{a}\eta_0 = \eta_0\widehat{\epsilon}(\widehat{a})$ and $\widehat{\epsilon}(\widehat{b}\widehat{a}) = \eta_0^*\widehat{b}\widehat{a}\eta_0 = \eta_0^*\widehat{b}\eta_0\widehat{\epsilon}(\widehat{a}) = \widehat{\epsilon}(\widehat{b})\widehat{\epsilon}(\widehat{a})$ for all $\widehat{a}, \widehat{b} \in \widehat{A}_V$.
Assume that V is regular. Then D is a C^*-algebra and $\widehat{\epsilon}$ is a morphism because by construction, $\widehat{\epsilon}$ is a $*$-homomorphism, $D = \widehat{\epsilon}(\widehat{A}_V)'$, $\eta_0^* \in \mathcal{L}^{\widehat{\epsilon}}({}_\alpha H_{\widehat{\beta}}, {}_{\mathfrak{B}}\mathfrak{K}_{\mathfrak{B}^\dagger})$, and $[\eta_0^*\alpha] \supseteq [\eta_0^*\eta_0\mathfrak{B}] = \mathfrak{B}$ and $[\eta_0^*\widehat{\beta}] \supseteq [\eta_0^*\eta_0\mathfrak{B}^\dagger] = \mathfrak{B}^\dagger$. Let $\hat{a} \in \widehat{A}_V$. Then

$$\begin{aligned}(\widehat{\epsilon} \underset{\mathfrak{b}^\dagger}{*} \mathrm{id})(\widehat{\Delta}_V(\hat{a})) &= \langle\eta_0|_1\widehat{\Delta}_V(\hat{a})|\eta_0\rangle_1 \\ &= \langle\eta_0|_1 V^*(1 \underset{\mathfrak{b}}{\otimes} \hat{a})V|\eta_0\rangle_1 = \langle\eta_0|_1(1 \underset{\mathfrak{b}}{\otimes} \hat{a})|\eta_0\rangle_1 = \rho_\beta(\eta_0^*\eta_0)\hat{a} = \hat{a},\end{aligned}$$

and if $\widehat{a} = \langle \eta|_2 V|\xi\rangle_2$ for some $\eta \in \beta$, $\xi \in \alpha$, then $(\mathrm{id} * \widehat{\epsilon})(\widehat{\Delta}_V(\widehat{a})) = \langle \eta_0|_2 \widehat{\Delta}_V(\widehat{a})|\eta_0\rangle_2 = \widehat{a}$ because the following diagram commutes:

$$
\begin{tikzcd}
H \arrow[r, "|\xi\rangle_2"] \arrow[d, "|\eta_0\rangle_2"'] & H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[r, "V"] \arrow[d, "|\eta_0\rangle_2"'] \arrow[dr, phantom, "(*)"] & H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H \arrow[rr, "\mathrm{id}"] \arrow[dr, "|\eta_0\rangle_2"'] & & H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H \arrow[r, "\langle\eta|_2"] & H \\
H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[r, "|\xi\rangle_3"] \arrow[rrrrr, bend right, "\widehat{\Delta}_V(\widehat{a})"'] & H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[rr, "V_{13}V_{23}"] & & (H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H)_{(\alpha \triangleleft \alpha)} \underset{\mathfrak{b}}{\otimes} {}_\beta H \arrow[ur, "\langle\eta_0|_2"'] \arrow[rr, "\langle\eta|_3"] & & H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \arrow[u, "\langle\eta_0|_2"']
\end{tikzcd}
$$

Indeed, the lower cell commutes by equation (12), cell (*) commutes because $V_{23}|\eta_0\rangle_2 = |\eta_0\rangle_2$, and the other cells commute as well. $\square$

Theorem 4.19. *Let V be a proper regular C^*-pseudo-multiplicative unitary. Then there exists a normal bounded left Haar weight ϕ for $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$.*

Proof. Choose $\xi_0 \in \mathrm{Cofix}(V)$ with $\xi_0^*\xi_0 = \mathrm{id}_{\mathfrak{K}}$. By Proposition 3.2 and Remark 4.10 i), $[\xi_0^*\widehat{A}_V\xi_0] = [\xi_0^*\rho_\alpha(\mathfrak{B}^\dagger)\widehat{A}_V\rho_\alpha(\mathfrak{B}^\dagger)\xi_0] \subseteq [\beta^*\widehat{A}_V\beta] \subseteq \mathfrak{B}^\dagger$. Hence, we can define a completely positive map $\phi\colon \widehat{A}_V \to \mathfrak{B}^\dagger$ by $\hat{a} \mapsto \xi_0^*\hat{a}\xi_0$, and $\phi \in \Omega_{M(\alpha)}(\widehat{A}_V)$. For all $\hat{a} \in \widehat{A}_V$, $(\mathrm{id} * \phi)(\widehat{\Delta}_V(\hat{a})) = \langle \xi_0|_2 V^*(\mathrm{id} \underset{\mathfrak{b}}{\otimes} \hat{a})V|\xi_0\rangle_2 = \langle\xi_0|_2(\mathrm{id} \underset{\mathfrak{b}}{\otimes} \hat{a})|\xi_0\rangle_2 = \rho_\alpha(\xi_0^*\hat{a}\xi_0)$. $\square$

As an example, consider the C^*-pseudo-multiplicative unitary $V\colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_\alpha H \to H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H$ associated to a locally compact, Hausdorff, second countable groupoid G as in subsection 2.3.

Proposition 4.20. i) *Let G be étale. Then V is étale, $\widehat{A}_V \cong C_0(G)$, $\widehat{\epsilon}(\widehat{A}_V) \cong C_0(G^0)$, and $\widehat{\epsilon}$ is given by the restriction of functions on G to functions on G^0.*

ii) *Let G be proper. Then V^{op} is étale, $A_V = \widehat{A}_{V^{op}} = C_r^*(G)$, and for each $f \in C_c(G)$, the operator $\widehat{\epsilon}(L(f)) \in \mathcal{L}(L^2(G^0,\mu))$ is given by*

$$(\widehat{\epsilon}(L(f))\zeta)(u) = \int_{G^u} f(x)D^{-1/2}(x)\zeta(s(x))\,\mathrm{d}\lambda^u(x) \quad \textit{for all } \zeta \in L^2(G^0,\mu),\ x \in G.$$

Proof. For all $\xi, \xi' \in C_c(G)$, $\zeta \in L^2(G^0,\mu)$ and $u \in G^0$, we have by Lemma 3.13

$$
\begin{aligned}
(\widehat{\epsilon}(m(\bar{\xi} * \xi'^*))\zeta)(u) = (\widehat{\epsilon}(\widehat{a}_{\xi,\xi'})\zeta)(u) &= (j(\xi)^*j(\xi')\zeta)(u) \\
&= \int_{G^u} \overline{\xi(x)}\xi'(x)\zeta(u)\,\mathrm{d}\lambda^u(x) = (\bar{\xi} * \xi'^*)(u)\zeta(u), \\
(\widehat{\epsilon}(L(\bar{\xi}\xi'))\zeta)(u) = (\widehat{\epsilon}(a_{\xi,\xi'})\zeta)(u) &= (j(\xi)^*\hat{j}(\xi')\zeta)(u) \\
&= \int_{G^u} \overline{\xi(x)}\xi'(x)D^{-1/2}(x)\zeta(s(x))\,\mathrm{d}\lambda^u(x).
\end{aligned}
$$
$\square$

Proposition 4.21. i) *Let G be proper. Then V is proper, $\widehat{A}_V \cong C_0(G)$, and the map $\phi\colon \widehat{A}_V \to C_0(G^0)$ given by $(\phi(f))(u) = \int_{G^u} f(x)\,\mathrm{d}\lambda^u(x)$ is a normal bounded left Haar weight for $((\widehat{A}_V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$.*

ii) *Let G be étale. Then V^{op} is proper and there exists a normal bounded left and right Haar weight ϕ for $((A_V)^{\beta,\alpha}_H, \Delta_V)$ given by $L(f) \mapsto f|_{G^0}$ for all $f \in C_c(G)$.*

Proof. This follows from Theorem 4.19 and similar calculations as in 4.20. □

References

[1] S. Baaj. Représentation régulière du groupe quantique $E_\mu(2)$ de Woronowicz. *C. R. Acad. Sci. Paris Sér. I Math.*, 314(13):1021–1026, 1992.

[2] S. Baaj. Représentation régulière du groupe quantique des déplacements de Woronowicz. *Astérisque*, (232):11–48, 1995. Recent advances in operator algebras (Orléans, 1992).

[3] S. Baaj and G. Skandalis. Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres. *Ann. Sci. École Norm. Sup. (4)*, 26(4):425–488, 1993.

[4] É. Blanchard. Déformations de C^*-algèbres de Hopf. *Bull. Soc. Math. France*, 124(1):141–215, 1996.

[5] G. Böhm, F. Nill, and K. Szlachányi. Weak Hopf algebras. I. Integral theory and C^*-structure. *J. Algebra*, 221(2):385–438, 1999.

[6] G. Böhm and K. Szlachányi. Weak Hopf algebras. II. Representation theory, dimensions, and the Markov trace. *J. Algebra*, 233(1):156–212, 2000.

[7] G. Böhm and K. Szlachányi. Weak C^*-Hopf algebras and multiplicative isometries. *J. Operator Theory*, 45(2):357–376, 2001.

[8] S. Echterhoff, S. Kaliszewski, J. Quigg, and I. Raeburn. A categorical approach to imprimitivity theorems for C^*-dynamical systems. *Mem. Amer. Math. Soc.*, 180(850):viii+169, 2006.

[9] M. Enock. Quantum groupoids of compact type. *J. Inst. Math. Jussieu*, 4(1):29–133, 2005.

[10] M. Enock and J.-M. Vallin. Inclusions of von Neumann algebras, and quantum groupoids. *J. Funct. Anal.*, 172(2):249–300, 2000.

[11] M. Enock and J.-M. Vallin. Inclusions of von Neumann algebras and quantum groupoids. II. *J. Funct. Anal.*, 178(1):156–225, 2000.

[12] P. Hahn. Haar measure for measure groupoids. *Trans. Amer. Math. Soc.*, 242:1–33, 1978.

[13] J. Kustermans and S. Vaes. Locally compact quantum groups. *Ann. Sci. École Norm. Sup. (4)*, 33(6):837–934, 2000.

[14] J. Kustermans and S. Vaes. Locally compact quantum groups in the von Neumann algebraic setting. *Math. Scand.*, 92(1):68–92, 2003.

[15] E. C. Lance. *Hilbert C^*-modules. A toolkit for operator algebraists*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995.

[16] F. Lesieur. Measured quantum groupoids. *Mém. Soc. Math. Fr. (N.S.)*, (109):iv+158 pp. (2008), 2007.

[17] T. Masuda, Y. Nakagami, and S. L. Woronowicz. A C^*-algebraic framework for quantum groups. *Internat. J. Math.*, 14(9):903–1001, 2003.

[18] D. Nikshych. A duality theorem for quantum groupoids. In *New trends in Hopf algebra theory (La Falda, 1999)*, volume 267 of *Contemp. Math.*, pages 237–243. Amer. Math. Soc., Providence, RI, 2000.

[19] M. O'uchi. Pseudo-multiplicative unitaries on Hilbert C^*-modules. *Far East J. Math. Sci. (FJMS)*, Special Volume, Part II:229–249, 2001.

[20] A. L. T. Paterson. *Groupoids, inverse semigroups, and their operator algebras*, volume 170 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999.

[21] A. L. T. Paterson. The Fourier algebra for locally compact groupoids. *Canad. J. Math.*, 56(6):1259–1289, 2004.

[22] J. Renault. *A groupoid approach to C^*-algebras*, volume 793 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[23] J. Renault. The Fourier algebra of a measured groupoid and its multipliers. *J. Funct. Anal.*, 145(2):455–490, 1997.

[24] J.-L. Sauvageot. Produits tensoriels de Z-modules et applications. In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, volume 1132 of *Lecture Notes in Mathematics*, pages 468–485. Springer, Berlin, 1985.

[25] T. Timmermann. The relative tensor product and a minimal fiber product in the setting of C^*-algebras. *J. Operator Theory* 68(2):101-140, 2012.

[26] T. Timmermann. From Hopf C^*-families to concrete Hopf C^*-bimodules. Technical report, SFB 478, WW-University Muenster, December 2007.

[27] T. Timmermann. Pseudo-multiplicative unitaries on C^*-modules and Hopf C^*-families. *J. Noncomm. Geom.*, 1:497–542, 2007.

[28] T. Timmermann. *An invitation to quantum groups and duality.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[29] T. Timmermann. C^*-pseudo-multiplicative unitaries and Hopf C^*-bimodules. *arXiv:0908.1850*, 2009.

[30] T. Timmermann. Coactions of Hopf C^*-bimodules. *J. Operator Theory* 68(1):19-66, 2012.

[31] T. Timmermann. A definition of compact C^*-quantum groupoids. In *Operator Structures and Dynamical Systems, Leiden 2008*, *Contemp. Math.* 503:267-290, 2009.

[32] J.-M. Vallin. Bimodules de Hopf et poids opératoriels de Haar. *J. Operator Theory*, 35(1):39–65, 1996.

[33] J.-M. Vallin. Unitaire pseudo-multiplicatif associé à un groupoïde. Applications à la moyennabilité. *J. Operator Theory*, 44(2):347–368, 2000.

T. TIMMERMANN, University of Muenster,, Einsteinstr. 62, 48149 Muenster, Germany
E-mail address: `timmermt@math.uni-muenster.de`

APPENDIX I.3

A DEFINITION OF COMPACT C^*-QUANTUM GROUPOIDS

THOMAS TIMMERMANN

Contemp. Math. *503:267-290, 2009.*

Abstract. We propose a definition of compact quantum groupoids in the setting of C^*-algebras and associate to every such quantum groupoid a fundamental unitary. These two notions are based on a new approach to relative tensor products of Hilbert modules and to fiber products of C^*-algebras. Using the fundamental unitary, we associate to every compact quantum groupoid a dual Hopf C^*-bimodule and a measurable quantum groupoid in the sense of Enock and Lesieur. Examples related to compact groupoids, r-discrete groupoids, and center-valued traces are outlined.

Contents

2000 *Mathematics Subject Classification.* Primary 46L55; Secondary 16W30, 46L08, 57T05.

supported by the SFB 478 "Geometrische Strukturen in der Mathematik" of the DFG (Deutsche Forschungsgemeinschaft).

1. Introduction

1.1. **Overview.** In the setting of von Neumann algebras, measurable quantum groupoids — in particular compact ones — were studied by Enock and Lesieur [6, 5, 8], building on Vallin's Hopf-von Neumann bimodules and pseudo-multiplicative unitaries [20, 21] and Haagerup's operator-valued weights.

In this article, we propose a definition of compact quantum groupoids in the setting of C^*-algebras — briefly called compact C^*-quantum groupoids — building on the notion of a Hopf-C^*-bimodule and a C^*-pseudo-multiplicative unitary [15, 16, 17]. To each compact C^*-quantum groupoid, we associate a regular C^*-pseudo-multiplicative unitary, a dual Hopf C^*-bimodule, and a measurable quantum groupoid. To illustrate the general theory, we outline several examples of compact C^*-quantum groupoids: the C^*-algebra of continuous functions on a compact groupoid, the reduced C^*-algebra of an étale groupoid with compact base, and principal compact C^*-quantum groupoids.

Further results on the dual Hopf C^*-bimodule of a compact quantum groupoid and a detailed discussion of the examples listed above can be found in [19]. An article on the general framework of Hopf C^*-bimodules and C^*-pseudo-multiplicative unitaries is in preparation [15].

1.2. **Plan.** This article is organized as follows. The definition a compact quantum groupoid in the setting of C^*-algebras and the necessary preliminaries are introduced in Sections 2–4. Recall that a measured compact groupoid consists of a base space G^0, a total space G, range and source maps $r, s\colon G \to G^0$, a multiplication $G_s{\times}_r G \to G$, a left and a right Haar system, and a quasi-invariant measure on G^0. Roughly, the corresponding ingredients of a compact C^*-quantum groupoid are unital C^*-algebras B and A, representations $r, s\colon B^{(op)} \to A$, a comultiplication $\Delta\colon A \to A * A$, a left and a right Haar weight $\phi, \psi\colon A \to B^{(op)}$, and a KMS-state on B, subject to several axioms. We introduce these ingredients in several steps. First, we focus on the tuple (B, A, r, ϕ, s, ψ), which can be considered as a compact C^*-quantum graph, and review some related constructions (Section 2). Next, we construct the fiber product $A * A$ and the underlying relative tensor product of Hilbert modules [15, 16, 17] (Section 3). Finally, we give the definition of a compact C^*-quantum groupoid and establish first properties like uniqueness of the Haar weights up to scaling (Section 4).

In Sections 5–7, we study further properties of compact C^*-quantum groupoids and give some examples. First, we associate to every compact C^*-quantum groupoid a fundamental unitary and, using that unitary, a dual Hopf C^*-bimodule and a measurable quantum groupoid (Section 5). The fundamental unitary generalizes the multiplicative unitaries of Baaj and Skandalis [1] and can be considered as a particular pseudo-multiplicative unitary in the sense of Vallin [21]. Second, we sketch examples of compact C^*-quantum groupoids related to center-valued traces on C^*-algebras and to compact or étale groupoids (Section 6, 7).

1.3. **Preliminaries.** Let us fix some general notation and terminology.

Given a subset Y of a normed space X, we denote by $[Y] \subseteq X$ the closed linear span of Y. Given a Hilbert space H and a subset $X \subseteq \mathcal{L}(H)$, we denote by X' the commutant of X. Given a C^*-algebra A and a C^*-subalgebra $B \subseteq M(A)$, we denote by $A \cap B'$ the relative commutant $\{a \in A \mid ab = ba \text{ for all } b \in B\}$. All sesquilinear maps like inner products of Hilbert spaces are assumed to be conjugate-linear in the first component and linear in the second one.

We shall make extensive use of (right) Hilbert C^*-modules [7].

Let A and B be C^*-algebras. Given Hilbert C^*-modules E and F over B, we denote the space of all adjointable operators from E to F by $\mathcal{L}_B(E,F)$. Let E and F be C^*-modules over A and B, respectively, and let $\pi\colon A \to \mathcal{L}_B(F)$ be a $*$-homomorphism. Recall that the internal tensor product $E \otimes_\pi F$ is a Hilbert C^*-module over B [7, §4] and the closed linear span of elements $\eta \otimes_\pi \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \eta \otimes_\pi \xi | \eta' \otimes_\pi \xi' \rangle = \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\eta \otimes_\pi \xi)b = \eta \otimes_\pi \xi b$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$. We denote the internal tensor product by "$\varobar$" and drop the index π if the representation is understood; thus, for example, $E \varobar F = E \varobar_\pi F = E \otimes_\pi F$.

We also define a *flipped internal tensor product* $F {}_\pi\!\varoslash E$ as follows. We equip the algebraic tensor product $F \odot E$ with the structure maps $\langle \xi \odot \eta | \xi' \odot \eta' \rangle := \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$, $(\xi \odot \eta)b := \xi b \odot \eta$, form the separated completion, and obtain a Hilbert C^*-B-module $F {}_\pi\!\varoslash E$ which is the closed linear span of elements $\xi {}_\pi\!\varoslash \eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \xi {}_\pi\!\varoslash \eta | \xi' {}_\pi\!\varoslash \eta' \rangle = \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\xi {}_\pi\!\varoslash \eta)b = \xi b {}_\pi\!\varoslash \eta$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$. As above, we drop the index π and simply write "$\varoslash$" instead of "${}_\pi\!\varoslash$" if the representation π is understood. Evidently, the usual and the flipped internal tensor product are related by a unitary $\Sigma\colon F \varobar E \xrightarrow{\cong} E \varoslash F$, $\eta \varobar \xi \mapsto \xi \varoslash \eta$.

Finally, let E_1, E_2 be Hilbert C^*-modules over A, let F_1, F_2 be Hilbert C^*-modules over B with completely positive maps $\pi_i\colon A \to \mathcal{L}_B(F_i)$ $(i = 1, 2)$, and let $S \in \mathcal{L}_A(E_1, E_2)$, $T \in \mathcal{L}_B(F_1, F_2)$ such that $T\pi_1(a) = \pi_2(a)T$ for all $a \in A$. Then there exists a unique operator $S \varobar T \in \mathcal{L}_B(E_1 \varobar F_1, E_2 \varobar F_2)$ such that $(S \varobar T)(\eta \varobar \xi) = S\eta \varobar T\xi$ for all $\eta \in E_1$, $\xi \in F_1$, and $(S \varobar T)^* = S^* \varobar T^*$ [3, Proposition 1.34]

2. Compact C^*-quantum graphs

The first ingredient in the definition of a compact C^*-quantum groupoid is a compact C^*-quantum graph with a coinvolution. Roughly, the latter consists of a C^*-algebra B with a faithful KMS-state, a C^*-algebra A, two compatible module structures consisting of representations $B, B^{op} \to A$ and conditional expectations $A \to B, B^{op}$, and a $*$-antiautomorphism of A that intertwines these module structures. Thinking of the underlying graph of a groupoid, these objects correspond to the space of units with a quasi-invariant measure, the total space of arrows, the range and the source map, the left and the right Haar weight, and the inversion of the groupoid.

Before we can define compact C^*-quantum graphs and coinvolutions, we have to recall KMS-states on C^*-algebras, introduce module structures on C^*-algebras with respect to such states, and present an associated GNS-Rieffel-construction.

2.1. **KMS-states on C^*-algebras and associated GNS-constructions.** Let μ be a faithful KMS-state on a C^*-algebra B [11, §8.12]. We denote by σ^μ the modular automorphism group, by H_μ the GNS-space, by $\Lambda_\mu\colon B \to H_\mu$ the GNS-map, by $\zeta_\mu = \Lambda_\mu(1_B)$ the cyclic vector, and by $J_\mu\colon H_\mu \to H_\mu$ the modular conjugation associated to μ. Recall that

$$J_\mu\Lambda_\mu(b) = \Lambda_\mu(\sigma^\mu_{i/2}(b)^*) \quad \text{for all } b \in \mathrm{Dom}(\sigma^\mu_{i/2}). \tag{1}$$

We omit explicit mentioning of the GNS-representation $\pi_\mu\colon B \to \mathcal{L}(H_\mu)$ and identify B with $\pi_\mu(B)$; thus, $\Lambda_\mu(b) = b\zeta_\mu$ for all $b \in B$.

We denote by B^{op} the opposite C^*-algebra of B, which coincides with B as a Banach space with involution but has the reversed multiplication, by $B \to B^{op}$, $b \mapsto b^{op}$, the canonical antiisomorphism, and by $\mu^{op}\colon B^{op} \to \mathbb{C}$, $b^{op} \mapsto \mu(b)$, the opposite state of μ. Using formula (1), one easily verifies that μ^{op} is a KMS-state, that the modular automorphism group $\sigma^{\mu^{op}}$ is given by $\sigma^{\mu^{op}}_t(b^{op}) = \sigma^\mu_{-t}(b)^{op}$ for all $b \in B$, $t \in \mathbb{R}$, and that one can always choose the GNS-space and GNS-map for μ^{op} such that $H_{\mu^{op}} = H_\mu$ and $\Lambda_{\mu^{op}}(b^{op}) = J_\mu\Lambda_\mu(b^*)$ for all $b \in B$. Then $\zeta_{\mu^{op}} = \zeta_\mu$, $J_{\mu^{op}} = J_\mu$, $\pi_{\mu^{op}}(b) = J_\mu\pi_\mu(b)^*J_\mu$ for all $b \in B$, and for all $b \in \mathrm{Dom}(\sigma^\mu_{-i/2})$, $x \in B$,

$$\Lambda_{\mu^{op}}(b^{op}) = \Lambda_\mu(\sigma^\mu_{-i/2}(b)), \qquad\qquad b^{op}\Lambda_\mu(x) = \Lambda_\mu(x\sigma^\mu_{-i/2}(b)).$$

For later application, we note the extension to von Neumann algebras: The state $\tilde{\mu}$ on $N := B'' \subseteq \mathcal{L}(H_\mu)$ given by $y \mapsto \langle\zeta_\mu|y\zeta_\mu\rangle$ is the unique normal extension of μ, it is faithful because ζ_μ is cyclic for $\pi_{\mu^{op}}(B^{op}) \subseteq N'$, and the Hilbert space H_μ and the map $\Lambda_{\tilde{\mu}}\colon N \to H_{\tilde{\mu}}$, $y \mapsto y\zeta_\mu$, form a GNS-representation for $\tilde{\mu}$.

2.2. **Module structures and associated Rieffel constructions.** We shall use the following kind of module structures on C^*-algebras relative to KMS-states:

Definition 2.1. *Let μ be a faithful KMS-state on a unital C^*-algebra B. A μ-*module structure *on a unital C^*-algebra A consists of a unital embedding $r\colon B \to A$ and a faithful completely positive map $\phi\colon A \to B$ such that $r \circ \phi\colon A \to r(B)$ is a unital conditional expectation, $\nu := \mu \circ \phi$ is a KMS-state, and $\sigma^\nu_t \circ r = r \circ \sigma^\mu_t$, $\sigma^\mu_t \circ \phi = \phi \circ \sigma^\nu_t$ for all $t \in \mathbb{R}$.*

Given a module structure as above, we can form a GNS-Rieffel-construction:

Lemma 2.2. *Let μ be a faithful KMS-state on a unital C^*-algebra B, let (r, ϕ) be a μ-module structure on a unital C^*-algebra A, and put $\nu := \mu \circ \phi$.*

i) *There exists a unique isometry $\zeta_\phi\colon H_\mu \hookrightarrow H_\nu$ such that $\zeta_\phi\Lambda_\mu(b) = \Lambda_\nu(r(b))$ for all $b \in B$.*
ii) *$\zeta_\phi J_\mu = J_\nu\zeta_\phi$, $\zeta_\phi b = r(b)\zeta_\phi$, $\zeta^*_\phi\Lambda_\nu(a) = \Lambda_\mu(\phi(a))$, $\zeta^*_\phi a = \phi(a)\zeta^*_\phi$ for all $b \in B$, $a \in A$.*
iii) *There exists a μ^{op}-module structure (r^{op}, ϕ^{op}) on A^{op} such that $r^{op}(b^{op}) = r(b)^{op}$ and $\phi^{op}(a^{op}) = \phi(a)^{op}$ for all $b \in B$, $a \in A$. For all $b \in B$, $\zeta_\phi\Lambda_{\mu^{op}}(b^{op}) = \Lambda_{\nu^{op}}(r^{op}(b^{op}))$.*

iv) *Put $N := B'' \subseteq \mathcal{L}(H_\mu)$, $M := A'' \subseteq \mathcal{L}(H_\nu)$. Then r extends uniquely to a normal embedding $\tilde{r}\colon N \to M$, and ϕ extends uniquely to a faithful normal completely positive map $\tilde{\phi}\colon M \to N$. Moreover, $\tilde{\nu} = \tilde{\mu} \circ \tilde{\phi}$, $\zeta_\phi y = \tilde{r}(y)\zeta_\phi$, $\zeta_\phi^* x = \tilde{\phi}(x)\zeta_\phi^*$, $\tilde{\phi}(x\tilde{r}(y)) = \tilde{\phi}(x)\tilde{r}(y)$ for all $x \in M$, $y \in N$.*

Proof. (1)–(3) The proof of assertion (1) is straightforward, and $\zeta_\phi J_\mu = J_\nu \zeta_\phi$ because $\mathrm{Dom}(\sigma^\mu_{i/2})$ is dense in B and because $\zeta_\phi J_\mu \Lambda_\mu(b) = \zeta_\phi \Lambda_\mu(\sigma^\mu_{i/2}(b)^*) = \Lambda_\nu(r(\sigma^\mu_{i/2}(b)^*)) = \Lambda_\nu(\sigma^\nu_{i/2}(r(b))^*) = J_\nu \zeta_\phi \Lambda_\mu(b)$ for all $b \in \mathrm{Dom}(\sigma^\mu_{i/2})$. The proof of the remaining assertions is routine.

(4) Since $r(b)a^{op}\zeta_\phi = a^{op}\zeta_\phi b$ for all $a \in A$, $b \in B$ and $[A^{op}\zeta_\phi H_\mu] = H_\nu$, r is continuous with respect to the σ-weak topologies on $\mathcal{L}(H_\mu)$ and $\mathcal{L}(H_\nu)$ and extends uniquely to $\tilde{r}\colon M \to N$ as claimed. The map $\tilde{\phi}$ is uniquely determined by $\tilde{\phi}(x) = \zeta_\phi^* x \zeta_\phi$ for all $x \in M$, and $\tilde{\nu}(x) = \langle \zeta_\nu | x \zeta_\nu \rangle = \langle \zeta_\mu | \zeta_\phi^* x \zeta_\phi \zeta_\mu \rangle = (\tilde{\mu} \circ \tilde{\phi})(x)$ for all $x \in M$. Since ϕ is faithful, so are ν, $\tilde{\nu}$ and necessarily also $\tilde{\phi}$. The proof of the remaining assertions is routine again. □

2.3. **Compact C^*-quantum graphs.** We need the following simple variant of a Radon-Nikodym derivative for KMS-states:

Lemma 2.3. *Let A be a unital C^*-algebra with a faithful KMS-state ν and a positive invertible element δ such that $\nu(\delta) = 1$ and $\sigma^\nu_t(\delta) = \delta$ for all $t \in \mathbb{R}$.*

i) *The state ν_δ on A given by $\nu_\delta(a) = \nu(\delta^{1/2} a \delta^{1/2})$ for all $a \in A$ is a faithful KMS-state and $\sigma^{\nu_\delta}_t = \mathrm{Ad}_{\delta^{it}} \circ \sigma^\nu_t = \sigma^\nu_t \circ \mathrm{Ad}_{\delta^{it}}$ for all $t \in \mathbb{R}$.*
ii) *The map $\Lambda_{\nu_\delta}\colon A \to H_\nu$, $a \mapsto \Lambda_\nu(a\delta^{1/2})$, is a GNS-map for ν_δ, and the associated modular conjugation J_{ν_δ} is equal to J_ν.*
iii) *If $\tilde{\delta} \in A$ is another positive invertible element satisfying $\nu(\tilde{\delta}) = 1$, $\sigma^\nu_t(\tilde{\delta}) = \delta$ for all $t \in \mathbb{R}$, and $\nu_{\tilde{\delta}} = \nu_\delta$, then $\delta = \tilde{\delta}$.* □

Definition 2.4. *A* compact C^*-quantum graph *is a tuple $\mathcal{G} = (B, \mu, A, r, \phi, s, \psi, \delta)$ that consists of*

i) *a unital C^*-algebra B with a faithful KMS-state μ and a unital C^*-algebra A,*
ii) *a μ-module structure (r, ϕ) and a μ^{op}-module structure (s, ψ) on A, respectively, such that $r(B)$ and $s(B^{op})$ commute,*
iii) *a positive, invertible, σ^ν-invariant element $\delta \in A \cap r(B)' \cap s(B^{op})'$ satisfying $\nu(\delta) = 1$ and $\mu^{op} \circ \psi = (\mu \circ \phi)_\delta$.*

Given such a compact C^*-quantum graph, we put $\nu := \mu \circ \phi$, $\nu^{-1} := \mu^{op} \circ \psi$ and denote by $\zeta_\phi, \zeta_\psi\colon H_\mu \to H_\nu$ the isometries defined in Lemma 2.2.

Till the end of this section, let $\mathcal{G}$ be a compact C^*-quantum graph as above. Since $\psi(r(b))c^{op} = \psi(r(b)s(c^{op})) = \psi(s(c^{op})r(b)) = c^{op}\psi(r(b))$ and $\phi(s(b^{op}))c = c\phi(s(b^{op}))$ for all $b, c \in B$, we can define completely positive maps

$$\tau := \psi \circ r\colon B \to Z(B^{op}) \qquad \text{and} \qquad \tau^\dagger := \phi \circ s\colon B^{op} \to Z(B). \tag{2}$$

We identify $Z(B)$ and $Z(B^{op})$ with $B \cap B^{op} \subseteq \mathcal{L}(H_\mu)$ in the natural way.

Clearly, $\nu \circ r = \mu \circ \phi \circ r = \mu$ and $\nu^{-1} \circ s = \mu^{op} \circ \psi \circ s = \mu^{op}$. The compositions $\nu \circ s = \mu \circ \tau^\dagger$ and $\nu^{-1} \circ r = \mu^{op} \circ \tau$ are related to μ^{op} and μ, respectively, as follows.

Lemma 2.5. i) *$\phi(\delta) \in B$ and $\psi(\delta^{-1}) \in B^{op}$ are positive, invertible, central, invariant with respect to σ^{μ} and $\sigma^{\mu^{op}}$, respectively, and $\mu(\phi(\delta)) = 1 = \mu^{op}(\psi(\delta^{-1}))$.*
ii) *$\nu^{-1} \circ r = \mu^{op} \circ \tau = \mu_{\phi(\delta)}$ and $\nu \circ s = \mu \circ \tau^{\dagger} = \mu^{op}_{\psi(\delta^{-1})}$.*

Proof. (1) We only prove the assertions concerning $\phi(\delta)$. Since δ is positive and invertible, there exists an $\epsilon > 0$ such that $\delta > \epsilon 1_A$, and since ϕ is positive, we can conclude $\phi(\delta) > \epsilon\phi(1_A) = \epsilon 1_B$. Therefore, $\phi(\delta)$ is positive and invertible. It is central because $b\phi(\delta) = \phi(r(b)\delta) = \phi(\delta r(b)) = \phi(\delta)b$ for all $b \in B$, and invariant under σ^{μ} because $\sigma^{\mu}_t(\phi(\delta)) = \phi(\sigma^{\nu}_t(\delta)) = \phi(\delta)$ for all $t \in \mathbb{R}$.

(2) The first relation holds because $\nu^{-1}(r(b)) = \mu(\phi(\delta^{1/2}r(b)\delta^{1/2})) = \mu(b\phi(\delta)) = \mu(\phi(\delta)^{1/2}b\phi(\delta)^{1/2})$ for all $b \in B$. The second relation follows similarly. □

2.4. **Coinvolutions.** The unitary antipode of a compact C^*-quantum groupoid will be a coinvolution of the underlying compact C^*-quantum graph.

Definition 2.6. *A* coinvolution *for $\mathcal{G}$ is a $*$-antiautomorphism $R\colon A \to A$ satisfying $R \circ R = \mathrm{id}_A$ and $R(r(b)) = s(b^{op})$, $\phi(R(a)) = \psi(a)^{op}$ for all $b \in B$, $a \in A$.*

Lemma 2.7. *Let R be a coinvolution for $\mathcal{G}$.*
i) *$R(\delta) = \delta^{-1}$, $\phi(\delta) = \psi(\delta^{-1})^{op}$, $\nu \circ R = \nu^{-1}$, $\sigma^{\nu}_t \circ R = R \circ \sigma^{\nu^{-1}}_{-t}$ for all $t \in \mathbb{R}$.*
ii) *$\tau(b) = \tau^{\dagger}(b^{op})$ for all $b \in B$.*
iii) *There exists a unique antiunitary $I\colon H_\nu \to H_\nu$, $\Lambda_{\nu^{-1}}(a) \mapsto \Lambda_\nu(R(a)^*)$, and $I\Lambda_\nu(a) = \Lambda_\nu(R(a\delta^{1/2})^*)$, $Ia^*I = R(a)$ for all $a \in A$, $I^2 = \mathrm{id}_H$, $I\zeta_\psi J_\mu = \zeta_\phi$, $IJ_\nu = J_\nu I$.*

Proof. (1) The last equation follows from the fact that R is an antiautomorphism and that $\nu \circ R = \nu^{-1}$. Lemma 2.3 (3) implies that the element $\delta' = R(\delta^{-1})$ is equal to δ because $\nu(\delta') = \nu^{-1}(\delta^{-1}) = \nu(1) = 1$, $\sigma^{\nu}_t(\delta') = R(\sigma^{\nu^{-1}}_{-t}(\delta^{-1})) = R(\delta^{-1}) = \delta'$ and $\nu_\delta(a) = \nu^{-1}(a) = \nu(R(a)) = \nu^{-1}(\delta^{-1/2}R(a)\delta^{-1/2}) = \nu(\delta'^{1/2}a\delta'^{1/2})$ for all $a \in A$. Finally, $\phi(\delta) = (\psi \circ R)(R(\delta^{-1}))^{op} = \psi(\delta^{-1})^{op}$.

(2) $(\phi \circ s)(b^{op}) = (\phi \circ R \circ R \circ s)(b^{op}) = (\psi \circ r)(b)^{op}$ for all $b \in B$.

(3) The formula for I defines an antiunitary since $\nu(R(a)R(a)^*) = (\nu \circ R)(a^*a) = \nu^{-1}(a^*a)$ for all $a \in A$. The first two equations given in (3) follow immediately. The remaining equations follow from the fact that for all $a \in A$, $b \in B$, $c \in \mathrm{Dom}(\sigma^{\nu^{-1}}_{i/2})$,

$$I^2\Lambda_\nu(a) = \Lambda_\nu(R(R(a\delta^{1/2})^*\delta^{1/2})^*) = \Lambda_\nu(a\delta^{1/2}\delta^{-1/2}) = \Lambda_\nu(a),$$
$$I\zeta_\psi J_\mu \Lambda_\mu(b^*) = I\zeta_\psi \Lambda_{\mu^{op}}(b^{op}) = I\Lambda_{\nu^{-1}}(s(b^{op})) = \Lambda_\nu(r(b)^*) = \zeta_\phi \Lambda_\mu(b^*),$$
$$J_\nu I \Lambda_{\nu^{-1}}(a) = \Lambda_\nu(\sigma^{\nu}_{i/2}(R(a)^*)^*) = \Lambda_\nu(R(\sigma^{\nu^{-1}}_{i/2}(a)^*)^*) = IJ_{\nu^{-1}}\Lambda_{\nu^{-1}}(a).$$ □

3. The relative tensor product and the fiber product

Fundamental to our definition of a compact C^*-quantum groupoid are C^*-modules and C^*-algebras over KMS-states, the relative tensor product of such C^*-modules, and the fiber product of such C^*-algebras. The fiber product is needed to define the target of the comultiplication, and the relative tensor product is needed to define this fiber product and the domain and the range of the fundamental unitary.

For proofs and further details, we refer to [15, 16, 17].

3.1. **C^*-modules and C^*-algebras over KMS-states.** We adopt the framework of C^*-modules and C^*-algebras over C^*-bases [15, 16, 17], but restrict to C^*-bases associated to KMS-states. A *C^*-base* is a triple $(H, B, B^\dagger)$ consisting of a Hilbert space H and two commuting nondegenerate C^*-algebras $B, B^\dagger \subseteq \mathcal{L}(H)$. Let μ be a faithful KMS-state on a C^*-algebra B. Then (H_μ, B, B^{op}) is a C^*-base, where H_μ is the GNS-space for μ and B and B^{op} act on $H_\mu = H_{\mu^{op}}$ via the GNS-representations. Thus, we can reformulate the theory developed in [17] for concrete KMS-states instead of general C^*-bases.

Definition 3.1. *A C^*-μ-*module *is a pair $H_\alpha = (H, \alpha)$, where H is a Hilbert space and $\alpha \subseteq \mathcal{L}(H_\mu, H)$ is a closed subspace satisfying $[\alpha H_\mu] = H$, $[\alpha B] = \alpha$, and $[\alpha^*\alpha] = B \subseteq \mathcal{L}(H_\mu)$. A* morphism *between C^*-μ-modules H_α and K_β is an operator $T \in \mathcal{L}(H, K)$ satisfying $T\alpha \subseteq \beta$, $T^*\beta \subseteq \alpha$. We denote the set of all such morphisms by $\mathcal{L}(H_\alpha, K_\beta)$.*

Lemma 3.2. *Let H_α be a C^*-μ-module.*

i) *α is a Hilbert C^*-B-module with inner product $(\xi, \xi') \mapsto \xi^*\xi'$.*
ii) *There exist isomorphisms $\alpha \otimes H_\mu \to H$, $\xi \otimes \zeta \mapsto \xi\zeta$, and $H_\mu \otimes \alpha \to H$, $\zeta \otimes \xi \mapsto \xi\zeta$.*
iii) *There exists a faithful, nondegenerate representation $\rho_\alpha \colon B^{op} \to \mathcal{L}(H)$ such that $\rho_\alpha(b^{op})(\xi\zeta) = \xi b^{op}\zeta$ for all $b \in B$, $\xi \in \alpha$, $\zeta \in H_\mu$.*
iv) *Let K_β be a C^*-μ-module and $T \in \mathcal{L}(H_\alpha, K_\beta)$. Then $T\rho_\alpha(b^{op}) = \rho_\beta(b^{op})T$ for all $b \in B$, and left multiplication by T defines an operator in $\mathcal{L}_B(\alpha, \beta)$, again denoted by T.* □

Let $\mu_1, \ldots, \mu_n$ be faithful KMS-states on C^*-algebras $B_1, \ldots, B_n$.

Definition 3.3. *A C^*-$(\mu_1, \ldots, \mu_n)$-*module *is a tuple $(H, \alpha_1, \ldots, \alpha_n)$, where H is a Hilbert space and (H, α_i) is a C^*-μ_i-module for each $i = 1, \ldots, n$ such that $[\rho_{\alpha_i}(B_i^{op})\alpha_j] = \alpha_j$ whenever $i \neq j$. In the case $n = 2$, we abbreviate ${}_\alpha H_\beta := (H, \alpha, \beta)$. The set of morphisms between C^*-$(\mu_1, \ldots, \mu_n)$-modules $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_n)$, $\mathcal{K} = (K, \gamma_1, \ldots, \gamma_n)$ is $\mathcal{L}(\mathcal{H}, \mathcal{K}) := \bigcap_{i=1}^n \mathcal{L}(H_{\alpha_i}, K_{\gamma_i}) \subseteq \mathcal{L}(H, K)$.*

Remark 3.4. If $(H, \alpha_1, \ldots, \alpha_n)$ is a C^*-$(\mu_1, \ldots, \mu_n)$-module, then $\rho_{\alpha_i}(B_i^{op}) \subseteq \mathcal{L}(H_{\alpha_j})$ and, in particular, $[\rho_{\alpha_i}(B_i^{op}), \rho_{\alpha_j}(B_j^{op})] = 0$ whenever $i \neq j$.

Definition 3.5. *A C^*-$(\mu_1, \ldots, \mu_n)$-*algebra *consists of a C^*-$(\mu_1, \ldots, \mu_n)$-module $(H, \alpha_1, \ldots, \alpha_n)$ and a nondegenerate C^*-algebra $A \subseteq \mathcal{L}(H)$ such that $\rho_{\alpha_i}(B_i^{op})A$ is contained in A for each $i = 1, \ldots, n$. In the cases $n = 1, 2$, we abbreviate $A_H^\alpha := (H_\alpha, A)$, $A_H^{\alpha,\beta} := ({}_\alpha H_\beta, A)$. A* morphism *of C^*-$(\mu_1, \ldots, \mu_n)$-algebras $\mathcal{A} = ((H, \alpha_1, \ldots, \alpha_n), A)$ and $\mathcal{C} = ((K, \gamma_1, \ldots, \gamma_n), C)$ is a nondegenerate $*$-homomorphism $\phi \colon A \to M(C)$ such that $[I_{\phi,i}\alpha_i] = \gamma_i$ for each $i = 1, \ldots, n$, where $I_{\phi,i} := \{T \in \mathcal{L}(H_{\alpha_i}, K_{\gamma_i}) \mid Ta = \phi(a)T$ for all $a \in A\}$. We denote the set of all such morphisms by* $\mathrm{Mor}(\mathcal{A}, \mathcal{C})$.

Remark 3.6. If ϕ is a morphism between C^*-μ-algebras A_H^α and C_K^γ, then $\phi(\rho_\alpha(b^{op})) = \rho_\gamma(b^{op})$ for all $b \in B$ [16, Lemma 2.2].

3.2. **The C^*-module of a compact C^*-quantum graph.** For every compact C^*-quantum graph, the GNS-Rieffel-construction in Lemma 2.2 yields a C^*-module as follows. Let μ be a faithful KMS-state on a unital C^*-algebra B again.

Lemma 3.7. *Let (r, ϕ) be a μ-module structure on a unital C^*-algebra A. Put $\nu := \mu\circ\phi$, $H := H_\nu$, $\widehat{\alpha} := [A\zeta_\phi]$, $\beta := [A^{op}\zeta_\phi]$, where ζ_ϕ is as in Lemma 2.2.*

i) ${}_{\widehat{\alpha}}H_\beta$ *is a C^*-(μ, μ^{op})-module and $\rho_{\widehat{\alpha}} = r^{op}$, $\rho_\beta = r$.*
ii) A^β_H *is a C^*-μ^{op}-algebra.*
iii) $a^{op}\zeta_\phi = \sigma^\nu_{-i/2}(a)\zeta_\phi$ *for all $a \in \mathrm{Dom}(\sigma^\nu_{-i/2}) \cap r(B)'$.*
iv) $A + (A \cap r(B)')^{op} \subseteq \mathcal{L}(H_{\widehat{\alpha}})$ *and $A^{op} + (A \cap r(B)') \subseteq \mathcal{L}(H_\beta)$.*

Proof. (1) Lemma 2.2 implies that $H_{\widehat{\alpha}}$ is a C^*-μ-module and H_β a C^*-μ^{op}-module. The equations for $\rho_{\widehat{\alpha}}$ and ρ_β follow from the fact that by Lemma 2.2, $\rho_{\widehat{\alpha}}(b^{op})a\zeta_\phi = a\zeta_\phi b^{op} = ar(b)^{op}\zeta_\phi = r(b)^{op}a\zeta_\phi$ and $\rho_\beta(b)a^{op}\zeta_\phi = a^{op}\zeta_\phi b = a^{op}r(b)\zeta_\phi = r(b)a^{op}\zeta_\phi$ for all $b \in B$, $a \in A$. Hence, $[\rho_{\widehat{\alpha}}(B^{op})\beta] = [r(B)^{op}A^{op}\zeta_\phi] = \beta$ and $[\rho_\beta(B)\widehat{\alpha}] = [r(B)A\zeta_\phi] = \widehat{\alpha}$, so ${}_{\widehat{\alpha}}H_\beta$ is a C^*-(μ, μ^{op})-module.

(2) By (1), $[\rho_\beta(B)A] = [r(B)A] = A$.

(3) Since $\sigma^\nu_t(r(B)) \subseteq r(B)$ for all $t \in \mathbb{R}$, σ^ν restricts to a one-parameter group of automorphisms of $A \cap r(B)'$; in particular, $a \in \mathrm{Dom}(\sigma^\nu_{-i/2}) \cap r(B)'$ is dense in $A \cap r(B)'$. Now, the claim follows from the fact that $a^{op}\zeta_\phi\Lambda_\mu(b) = \Lambda_\nu(r(b)\sigma^\nu_{-i/2}(a)) = \Lambda_\nu(\sigma^\nu_{-i/2}(a)r(b)) = \sigma^\nu_{-i/2}(a)\zeta_\phi\Lambda_\mu(b)$ for all $a \in r(B)' \cap \mathrm{Dom}(\sigma^\nu_{-i/2})$ and $b \in B$.

(4) We only prove the first inclusion, the second one follows similarly. Clearly, $[A\widehat{\alpha}] = \widehat{\alpha}$, and by (3), $[(A \cap r(B)')^{op}\widehat{\alpha}] = [A(A \cap r(B)')^{op}\zeta_\phi] \subseteq [A\zeta_\phi] = \widehat{\alpha}$. □

For a compact C^*-quantum graph, Lemmas 2.7 and 3.7 imply:

Proposition 3.8. *Let $\mathcal{G} = (B, \mu, A, r, \phi, s, \psi, \delta)$ be a compact C^*-quantum graph. Put $\nu := \mu \circ \phi$, $\nu^{-1} := \mu^{op} \circ \psi = \nu_\delta$ and*

$$H := H_\nu, \qquad \widehat{\alpha} := [A\zeta_\phi], \qquad \beta := [A^{op}\zeta_\phi], \qquad \widehat{\beta} := [A\zeta_\psi], \qquad \alpha := [A^{op}\zeta_\psi]. \tag{3}$$

i) $(H, \widehat{\alpha}, \beta, \widehat{\beta}, \alpha)$ *is a C^*-$(\mu, \mu^{op}, \mu^{op}, \mu)$-module.*
ii) $\rho_{\widehat{\alpha}} = r^{op}$, $\rho_\beta = r$, $\rho_{\widehat{\beta}} = s^{op}$, $\rho_\alpha = s$.
iii) $A^{\beta,\alpha}_H$ *is a C^*-(μ^{op}, μ)-algebra.*
iv) *Let R be a coinvolution for $\mathcal{G}$ and define $I \colon H_\nu \to H_\nu$ by $\Lambda_{\nu^{-1}}(a) \mapsto \Lambda_\nu(R(a)^*)$. Then $I\widehat{\beta}J_\mu = \widehat{\alpha}$ and $I\beta J_\mu = \alpha$.* □

3.3. **The relative tensor product of C^*-modules.** The relative tensor product of C^*-modules over KMS-states is a C^*-algebraic analogue of the relative tensor product of Hilbert spaces over a von Neumann algebra. We summarize the definition and main properties; for proofs and further details, see [15, 16, 17].

Let μ be a faithful KMS-state on a C^*-algebra B and let H_β, K_γ be a C^*-μ-module and a C^*-μ^{op}-module, respectively. The *relative tensor product* of H_β and K_γ is the Hilbert space $H_\beta \underset{\mu}{\otimes}{}_\gamma K := \beta \otimes H_\mu \otimes \gamma$. It is spanned by elements $\xi \otimes \zeta \otimes \eta$, where $\xi \in \beta$, $\zeta \in H_\mu$, $\eta \in \gamma$, and the inner product is given by $\langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta'\rangle = \langle \zeta | \xi^*\xi'\eta^*\eta'\zeta'\rangle = \langle \zeta | \eta^*\eta'\xi^*\xi'\zeta'\rangle$ for all $\xi, \xi' \in \beta$, $\zeta, \zeta' \in H_\mu$, $\eta, \eta' \in \gamma$.

Obviously, there exists a *flip* isomorphism

$$\Sigma \colon H_\beta \underset{\mu}{\otimes}{}_\gamma K \to K_\gamma \underset{\mu^{op}}{\otimes}{}_\beta H, \quad \xi \otimes \zeta \otimes \eta \mapsto \eta \otimes \zeta \otimes \xi.$$

The isomorphisms $\beta \ominus H_\mu \cong H$, $\xi \ominus \zeta \equiv \xi\zeta$, and $H_\mu \oslash \gamma \cong K$, $\zeta \oslash \eta \equiv \eta\zeta$, of Lemma 3.2 induce the following isomorphisms, which we use without further notice:

$$H_{\rho_\beta} \oslash \gamma \cong H_\beta \underset{\mu}{\otimes}_\gamma K \cong \beta \ominus_{\rho_\gamma} K, \quad \xi\zeta \oslash \eta \equiv \xi \ominus \zeta \oslash \eta \equiv \xi \ominus \eta\zeta.$$

Using these isomorphisms, we define the following tensor products of operators:

$$S_\beta \underset{\mu}{\otimes}_\gamma T := S \ominus T \in \mathcal{L}(\beta \ominus_{\rho_\gamma} K) = \mathcal{L}(H_\beta \underset{\mu}{\otimes}_\gamma K) \text{ for all } S \in \mathcal{L}(H_\beta),\, T \in \rho_\gamma(B)',$$

$$S_\beta \underset{\mu}{\otimes}_\gamma T := S \oslash T \in \mathcal{L}(H_{\rho_\beta} \oslash \gamma) = \mathcal{L}(H_\beta \underset{\mu}{\otimes}_\gamma K) \text{ for all } S \in \rho_\beta(B^{op})',\, T \in \mathcal{L}(K_\gamma).$$

Note that $S \ominus T = S \ominus \mathrm{id} \oslash T = S \oslash T$ for all $S \in \mathcal{L}(H_\beta)$, $T \in \mathcal{L}(K_\gamma)$.

For each $\xi \in \beta$, $\eta \in \gamma$, there exist bounded linear operators

$$|\xi\rangle_1 \colon K \to H_\beta \underset{\mu}{\otimes}_\gamma K,\ \omega \mapsto \xi \ominus \omega, \qquad \langle\xi|_1 := |\xi\rangle_1^* \colon \xi' \ominus \omega \mapsto \rho_\gamma(\xi^*\xi')\omega,$$

$$|\eta\rangle_2 \colon H \to H_\beta \underset{\mu}{\otimes}_\gamma K,\ \omega \mapsto \omega \oslash \eta, \qquad \langle\eta|_2 := |\eta\rangle_2^* \colon \omega \oslash \eta' \mapsto \rho_\beta(\eta^*\eta')\omega.$$

We put $|\beta\rangle_1 := \{|\xi\rangle_1 \,|\, \xi \in \beta\}$ and similarly define $\langle\beta|_1$, $|\gamma\rangle_2$, $\langle\gamma|_2$.

Assume that $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_m, \beta)$ is a C^*-$(\sigma_1, \ldots, \sigma_m, \mu)$-module and that $\mathcal{K} = (K, \gamma, \delta_1, \ldots, \delta_n)$ is a C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n)$-module, where $\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n$ are faithful KMS-states on C^*-algebras $A_1, \ldots, A_m, C_1, \ldots, C_n$. We put

$$\alpha_i \triangleleft \gamma := [|\gamma\rangle_2 \alpha_i] \subseteq \mathcal{L}(H_{\sigma_i}, H_\beta \underset{\mu}{\otimes}_\gamma K), \qquad \beta \triangleright \delta_j := [|\beta\rangle_1 \delta_j] \subseteq \mathcal{L}(H_{\tau_j}, H_\beta \underset{\mu}{\otimes}_\gamma K)$$

for all $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then $(H_\beta \underset{\mu}{\otimes}_\gamma K, \alpha_1 \triangleleft \gamma, \ldots, \alpha_m \triangleleft \gamma, \beta \triangleright \delta_1, \ldots, \beta \triangleright \delta_n)$ is a C^*-$(\sigma_1, \ldots, \sigma_m,\, \tau_1, \ldots, \tau_n)$-module, called the *relative tensor product* of $\mathcal{H}$ and $\mathcal{K}$ and denoted by $\mathcal{H} \underset{\mu}{\otimes} \mathcal{K}$. For all $i = 1, \ldots, m$, $a \in A_i$ and $j = 1, \ldots, n$, $c \in C_j$,

$$\rho_{(\alpha_i \triangleleft \gamma)}(a^{op}) = \rho_{\alpha_i}(a^{op})_\beta \underset{\mu}{\otimes}_\gamma \mathrm{id}, \qquad \rho_{(\beta \triangleright \delta_j)}(c^{op}) = \mathrm{id}_\beta \underset{\mu}{\otimes}_\gamma \rho_{\delta_j}(c^{op}).$$

The relative tensor product has nice categorical properties:

Bifunctoriality. If $\tilde{\mathcal{H}} = (\tilde{H}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, \tilde{\beta})$ is a C^*-$(\sigma_1, \ldots, \sigma_m, \mu)$-module, $\tilde{\mathcal{K}} = (\tilde{K}, \tilde{\gamma}, \tilde{\delta}_1, \ldots, \tilde{\delta}_n)$ a C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n)$-module, and $S \in \mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}})$, $T \in \mathcal{L}(\mathcal{K}, \tilde{\mathcal{K}})$, then there exists a unique operator $S \underset{\mu}{\otimes} T \in \mathcal{L}(\mathcal{H} \underset{\mu}{\otimes} \mathcal{K}, \tilde{\mathcal{H}} \underset{\mu}{\otimes} \tilde{\mathcal{K}})$ satisfying

$$(S \underset{\mu}{\otimes} T)(\xi \ominus \zeta \oslash \eta) = S\xi \ominus \zeta \oslash T\eta \quad \text{for all } \xi \in \beta,\, \zeta \in H_\mu,\, \eta \in \gamma.$$

We need the following straightforward analogue not mentioned in [17]:

Lemma 3.9. *Let H_β, $\tilde{K}_{\tilde{\gamma}}$ be C^*-μ-modules, K_γ, $\tilde{H}_{\tilde{\beta}}$ C^*-μ^{op}-modules, and $I \colon H \to \tilde{H}$, $J \colon K \to \tilde{K}$ antiunitaries such that $I\beta J_\mu = \tilde{\beta}$ and $J\gamma J_\mu = \tilde{\gamma}$.*

i) *There exists an antiunitary $I_\beta \underset{J_\mu}{\otimes}_\gamma J \colon H_\beta \underset{\mu}{\otimes}_\gamma K \to \tilde{H}_{\tilde{\beta}} \underset{\mu^{op}}{\otimes}_{\tilde{\gamma}} \tilde{K}$ such that $(I_\beta \underset{J_\mu}{\otimes}_\gamma J)(\xi \ominus \zeta \oslash \eta) = I\xi J_\mu \ominus J_\mu \zeta \oslash J\eta J_\mu$ for all $\xi \in \beta$, $\zeta \in H_\mu$, $\eta \in \gamma$.*

ii) *$(I_\beta \underset{J_\mu}{\otimes}_\gamma J)|\xi\rangle_1 = |I\xi J_\mu\rangle_1 J$ and $(I_\beta \underset{J_\mu}{\otimes}_\gamma J)|\eta\rangle_2 = |J\eta J_\mu\rangle_2 I$ for all $\xi \in \beta$, $\eta \in \gamma$.*

iii) *$(I_\beta \underset{J_\mu}{\otimes}_\gamma J)(S_\beta \underset{\mu}{\otimes}_\gamma T) = (ISI^*{}_{\tilde{\beta}} \underset{\mu^{op}}{\otimes}_{\tilde{\gamma}} JTJ^*)(I_\beta \underset{J_\mu}{\otimes}_\gamma J)$ for all $S \in \mathcal{L}(H_\beta)$, $T \in \mathcal{L}(K_\gamma)$.*

□

Unitality. If we embed B, B^{op} into $\mathcal{L}(H_\mu)$ via the GNS-representations, then $\mathcal{U} := (H_\mu, B^{op}, B)$ is a C^*-(μ^{op}, μ)-module and the maps

$$H_\beta \underset{\mu}{\otimes}_{B^{op}} H_\mu \to H,\ \xi \otimes \zeta \otimes b^{op} \mapsto \xi b^{op} \zeta, \quad H_\mu\, {}_B\underset{\mu}{\otimes}_\gamma K \to K,\ b \otimes \zeta \otimes \eta \mapsto \eta b \zeta,$$

are isomorphisms of C^*-$(\sigma_1, \ldots, \sigma_m, \mu)$-modules and C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n)$-modules $\mathcal{H} \underset{\mu}{\otimes} \mathcal{U} \cong \mathcal{H}$ and $\mathcal{U} \underset{\mu}{\otimes} \mathcal{K} \cong \mathcal{K}$, respectively, natural in $\mathcal{H}$ and $\mathcal{K}$.

Associativity. Assume that $\nu, \rho_1, \ldots, \rho_l$ are faithful KMS-states on some C^*-algebras and that $\hat{\mathcal{K}} = (K, \gamma, \delta_1, \ldots, \delta_n, \epsilon)$ is a C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n, \nu)$-module and $\mathcal{L} = (L, \phi, \psi_1, \ldots, \psi_l)$ a C^*-$(\nu^{op}, \rho_1, \ldots, \rho_l)$-module. Then the isomorphisms

$$(H_\beta \underset{\mu}{\otimes}_\gamma K)_{\beta \triangleright \epsilon} \underset{\nu}{\otimes}_\phi L \cong \beta \otimes_{\rho_\gamma} K_{\rho_\epsilon} \otimes \phi \cong H_\beta \underset{\mu}{\otimes}_{\gamma \triangleleft \phi} (K_\epsilon \underset{\nu}{\otimes}_\phi L)$$

are isomorphisms of C^*-$(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n, \rho_1, \ldots, \rho_l)$-modules $(\mathcal{H} \underset{\mu}{\otimes} \hat{\mathcal{K}}) \underset{\nu}{\otimes} \mathcal{L} \cong \mathcal{H} \underset{\mu}{\otimes} (\hat{\mathcal{K}} \underset{\nu}{\otimes} \mathcal{L})$. From now on, we identify the Hilbert spaces above and denote them by $H_\beta \underset{\mu}{\otimes}_\gamma K_\epsilon \underset{\nu}{\otimes}_\phi L$.

3.4. **The fiber product of C^*-algebras.** The fiber product of C^*-algebras over KMS-states is an analogue of the fiber product of von Neumann algebras. We summarize the definition and main properties; for proofs and further details, see [15, 16, 17].

Let μ be a faithful KMS-state on a C^*-algebra B, let A_H^β be a C^*-μ-algebra, and let C_K^γ be a C^*-μ^{op}-algebra. The *fiber product* of A_H^β and C_K^γ is the C^*-algebra

$$A_\beta \underset{\mu}{*}_\gamma C := \{x \in \mathcal{L}(H_\beta \underset{\mu}{\otimes}_\gamma K) \mid x|\beta\rangle_1, x^*|\beta\rangle_1 \subseteq [|\beta\rangle_1 C] \text{ and } x|\gamma\rangle_2, x^*|\gamma\rangle_2 \subseteq [|\gamma\rangle_2 A]\}.$$

If A and C are unital, so is $A_\beta \underset{\mu}{*}_\gamma C$, but otherwise, $A_\beta \underset{\mu}{*}_\gamma C$ may be degenerate.

Clearly, conjugation by the flip $\Sigma \colon H_\beta \underset{\mu}{\otimes}_\gamma K \to K_\gamma \underset{\mu^{op}}{\otimes}{}_\beta H$ yields an isomorphism

$$\mathrm{Ad}_\Sigma \colon A_\beta \underset{\mu}{*}_\gamma C \to C_\gamma \underset{\mu^{op}}{*}{}_\beta A.$$

Assume that $\mathcal{A} = (\mathcal{H}, A)$ is a C^*-$(\sigma_1, \ldots, \sigma_m, \mu)$-algebra and $\mathcal{C} = (\mathcal{K}, C)$ a C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n)$-algebra, where $\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n$ are faithful KMS-states on some C^*-algebras and $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_m, \beta)$, $\mathcal{K} = (K, \gamma, \delta_1, \ldots, \delta_n)$. If $A_\beta \underset{\mu}{*}_\gamma C$ is nondegenerate, then $(\mathcal{H} \underset{\mu}{\otimes} \mathcal{K}, A_\beta \underset{\mu}{*}_\gamma C)$ is a C^*-$(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n)$-algebra, called the *fiber product* of $\mathcal{A}$ and $\mathcal{C}$ and denoted by $\mathcal{A} \underset{\mu}{*} \mathcal{C}$.

Assume furthermore that $\tilde{\mathcal{A}} = (\tilde{\mathcal{H}}, \tilde{A})$ is a C^*-$(\sigma_1, \ldots, \sigma_m, \mu)$-algebra and $\tilde{\mathcal{C}} = (\tilde{\mathcal{K}}, \tilde{C})$ is a C^*-$(\mu^{op}, \tau_1, \ldots, \tau_n)$-algebra, where $\tilde{\mathcal{H}} = (\tilde{H}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, \tilde{\beta})$, $\tilde{\mathcal{K}} = (\tilde{K}, \tilde{\gamma}, \tilde{\delta}_1, \ldots, \tilde{\delta}_n)$. Then for each $\phi \in \mathrm{Mor}(\mathcal{A}, \tilde{\mathcal{A}})$, $\psi \in \mathrm{Mor}(\mathcal{C}, \tilde{\mathcal{C}})$, there exists a unique morphism $\phi \underset{\mu}{*} \psi \in \mathrm{Mor}(\mathcal{A} \underset{\mu}{*} \mathcal{C}, \tilde{\mathcal{A}} \underset{\mu}{*} \tilde{\mathcal{C}})$ such that for all $x \in A_\beta \underset{\mu}{*}_\gamma C$, $S \in \mathcal{L}(H_\beta, \tilde{H}_{\tilde{\beta}})$, $T \in \mathcal{L}(K_\gamma, \tilde{K}_{\tilde{\gamma}})$ satisfying $Sa = \phi(a)S$, $Tc = \psi(c)T$ for all $a \in A$, $c \in C$,

$$(\phi \underset{\mu}{*} \psi)(x)(S_\beta \underset{\mu}{\otimes}_\gamma T) = (S_\beta \underset{\mu}{\otimes}_\gamma T)x.$$

We shall need the following simple construction:

Lemma 3.10. *Let A_H^{β}, $\tilde{C}_{\tilde{K}}^{\tilde{\gamma}}$ be C^*-μ-algebras, $\tilde{A}_{\tilde{H}}^{\tilde{\beta}}$, C_K^{γ} C^*-μ^{op}-algebras, and $R\colon A \to \tilde{A}^{op}$, $S\colon C \to \tilde{C}^{op}$ $*$-homomorphisms. Assume that $I\colon H \to \tilde{H}$ and $J\colon K \to \tilde{K}$ are antiunitaries such that $I\beta J_\mu = \tilde{\beta}$, $J\gamma J_\mu = \tilde{\gamma}$ and $R(a) = I^*a^*I$, $S(c) = J^*c^*J$ for all $a \in A$, $c \in C$. Then there exists a $*$-homomorphism $R_{\beta} \underset{J_\mu}{*} {}_{\gamma}S\colon A_{\beta} \underset{\mu}{*} {}_{\gamma}C \to (\tilde{A}_{\tilde{\beta}} \underset{\mu^{op}}{*} {}_{\tilde{\gamma}}\tilde{C})^{op}$ such that $(R_{\beta} \underset{J_\mu}{*} {}_{\gamma}S)(x) = (I_{\tilde{\beta}} \underset{J_\mu}{\otimes} {}_{\tilde{\gamma}}J)^* x^* (I_{\tilde{\beta}} \underset{J_\mu}{\otimes} {}_{\tilde{\gamma}}J)$ for all $x \in A_{\beta} \underset{\mu}{*} {}_{\gamma}C$. This $*$-homomorphism does not depend on the choice of I, J.*

Proof. Evidently, the formula defines a $*$-homomorphism $R_{\beta} \underset{J_\mu}{*} {}_{\gamma}S$. The definition does not depend on the choice of J because $\langle \xi|_1 (R_{\beta} \underset{J_\mu}{*} {}_{\gamma}S)(x)|\xi'\rangle_1 = J^*\langle I\xi J_\mu|_1 x^*|I\xi' J_\mu\rangle_1 J = S(\langle I\xi' J_\mu|_1 x|I\xi J_\mu\rangle_1)$ for all $x \in A_{\beta} \underset{\mu}{*} {}_{\gamma}C$ by Lemma 3.9 (2), and a similar argument shows that it does not depend on the choice of I. □

Unfortunately, the fiber product need not be associative, but in our applications, it will only appear as the target of a comultiplication whose coassociativity will compensate the non-associativity of the fiber product.

4. Compact C^*-quantum groupoids

A compact C^*-quantum groupoid consists of a compact C^*-quantum graph with a coinvolution and a comultiplication satisfying several relations, most importantly, left- and right-invariance of the Haar weights and a strong invariance condition relating the coinvolution to the Haar weights and the comultiplication.

Before we give the precise definition, we recall the underlying notion of a Hopf C^*-bimodule and the left- and right-invariance conditions; afterwards, we prove some elementary properties of compact C^*-quantum groupoids.

4.1. **Hopf C^*-bimodules over KMS-states.** Let μ be a faithful KMS-state on a C^*-algebra B.

Definition 4.1 ([17]). *A* comultiplication *on a C^*-(μ^{op}, μ)-algebra $A_H^{\beta,\alpha}$ is a morphism $\Delta \in \mathrm{Mor}(A_H^{\beta,\alpha}, A_H^{\beta,\alpha} \underset{\mu}{*} A_H^{\beta,\alpha})$ that makes the following diagram commute:*

$$\begin{tikzcd}
A \arrow[rr, "\Delta"] \arrow[dd, "\Delta"] & & A_{\alpha} \underset{\mu}{*} {}_{\beta}A \arrow[d, "\mathrm{id} \underset{\mu}{*} \Delta"] \\
& & A_{\alpha} \underset{\mu}{*} {}_{\beta \triangleleft \beta}(A_{\alpha} \underset{\mu}{*} {}_{\beta}A) \arrow[d, hook] \\
A_{\alpha} \underset{\mu}{*} {}_{\beta}A \arrow[r, "\Delta \underset{\mu}{*} \mathrm{id}"] & (A_{\alpha} \underset{\mu}{*} {}_{\beta}A)_{\alpha \triangleright \alpha} \underset{\mu}{*} {}_{\beta}A \arrow[r, hook] & \mathcal{L}(H_{\alpha} \underset{\mu}{*} {}_{\beta}H_{\alpha} \underset{\mu}{*} {}_{\beta}H).
\end{tikzcd}$$

A Hopf C^*-bimodule over μ *is a C^*-(μ^{op}, μ)-algebra with a comultiplication.*

Let $(A_H^{\beta,\alpha}, \Delta)$ be a Hopf C^*-bimodule over μ. A bounded left Haar weight *for $(A_H^{\beta,\alpha}, \Delta)$ is a non-zero completely positive contraction $\phi\colon A \to B$ satisfying*

i) $\phi(a\rho_\beta(b)) = \phi(a)b$ *for all* $a \in A$, $b \in B$, *and*
ii) $\phi(\langle\xi|_1\Delta(a)|\xi'\rangle_1) = \xi^*\rho_\beta(\phi(a))\xi'$ *for all* $a \in A$ *and* $\xi, \xi' \in \alpha$.

A bounded right Haar weight *for* $(A_H^{\beta,\alpha}, \Delta)$ *is a non-zero completely positive contraction* $\psi\colon A \to B^{op}$ *satisfying*

(1)' $\psi(a\rho_\alpha(b^{op})) = \psi(a)b^{op}$ *for all* $a \in A$, $b \in B$, *and*
(2)' $\psi(\langle\eta|_2\Delta(a)|\eta'\rangle_2) = \eta^*\rho_\alpha(\psi(a))\eta'$ *for all* $a \in A$ *and* $\eta, \eta' \in \beta$.

Remarks 4.2. Let $(A_H^{\beta,\alpha}, \Delta)$ be a Hopf C^*-bimodule over μ.

i) $\Delta(\rho_\alpha(b^{op})\rho_\beta(c)) = \rho_\beta(c) \underset{\mu}{\otimes} \rho_\alpha(b^{op})$ for all $b, c \in B$ by Remark 3.6.

Let $\phi\colon A \to B$ be a completely positive contraction.

ii) If condition (1) above holds, then $\rho_\beta \circ \phi\colon A \to \rho_\beta(B)$ is a conditional expectation.
iii) If condition (2) above holds and $[\langle\alpha|_1\Delta(A)|\alpha\rangle_1] = A$, then also (1) holds because $\phi(\langle\xi|_1\Delta(a)|\xi'\rangle_1\rho_\beta(b)) = \phi(\langle\xi|_1\Delta(a)|\xi' b\rangle_1) = \xi^*\rho_\beta(\phi(a))\xi' b = \phi(\langle\xi|_1\Delta(a)|\xi'\rangle_1)b$ for all $a \in A$, $b \in B$, $\xi, \xi' \in \alpha$.

Similar remarks apply to conditions (1)' and (2)'.

4.2. **Definition of compact C^*-quantum groupoids.** Given a compact C^*-quantum graph $(B, \mu, A, r, \phi, s, \psi, \delta)$ with coinvolution R, we use the notation of Proposition 3.8, put $\nu := \mu \circ \phi$, $\nu^{-1} := \mu^{op} \circ \psi = \nu_\delta$, $J := J_\nu = J_{\nu^{-1}}$,

$$H := H_\nu, \qquad \widehat{\alpha} := [A\zeta_\phi], \qquad \beta := [A^{op}\zeta_\phi], \qquad \widehat{\beta} := [A\zeta_\psi], \qquad \alpha := [A^{op}\zeta_\psi],$$

and define an antiunitary $I\colon H \to H$ by $I\Lambda_{\nu^{-1}}(a) = \Lambda_\nu(R(a)^*)$ for all $a \in A$ and a $*$-antihomomorphism $R_\alpha \underset{J_\mu}{*} {}_\beta R\colon A_\alpha \underset{\mu}{*} {}_\beta A \to A_\beta \underset{\mu^{op}}{*} {}_\alpha A$ by $x \mapsto (I_\beta \underset{J_\mu}{\otimes} {}_\alpha I)^* x^* (I_\beta \underset{J_\mu}{\otimes} {}_\alpha I)$ (see Lemma 3.10).

Definition 4.3. *A* compact C^*-quantum groupoid *is a compact C^*-quantum graph* $(B, \mu, A, r, \phi, s, \psi, \delta)$ *with a coinvolution R and a comultiplication Δ for $A_H^{\beta,\alpha}$ such that*

i) $[\Delta(A)|\alpha\rangle_1] = [|\alpha\rangle_1 A] = [\Delta(A)|\zeta_\psi\rangle_1 A]$ *and* $[\Delta(A)|\beta\rangle_2] = [|\beta\rangle_2 A] = [\Delta(A)|\zeta_\phi\rangle_2 A]$;
ii) ϕ *is a bounded left and* ψ *a bounded right Haar weight for* $(A_H^{\beta,\alpha}, \Delta)$;
iii) $R(\langle\zeta_\psi|_1\Delta(a)(d^{op}{}_\alpha \underset{\mu}{\otimes} {}_\beta 1)|\zeta_\psi\rangle_1) = \langle\zeta_\psi|_1(a^{op}{}_\alpha \underset{\mu}{\otimes} {}_\beta 1)\Delta(d)|\zeta_\psi\rangle_1$ *for all* $a, d \in A$.

Let $(B, \mu, A, r, \phi, s, \psi, \delta, R, \Delta)$ be a compact C^*-quantum groupoid.

Lemma 4.4. $\{a \in A \cap r(B)' \mid \Delta(a) = 1_\alpha \underset{\mu}{\otimes} {}_\beta a\} = s(B^{op})$ *and* $\{a \in A \cap s(B^{op})' \mid \Delta(a) = a_\alpha \underset{\mu}{\otimes} {}_\beta 1\} = r(B)$.

Proof. We only prove the first equation. Clearly, the right hand side is contained in the left hand side. Conversely, if $a \in A \cap r(B)'$ and $\Delta(a) = 1_\alpha \underset{\mu}{\otimes} {}_\beta a$, then $a = \langle\zeta_\psi|_1\Delta(a)|\zeta_\psi\rangle_1 = s(\psi(a))$ by right-invariance of ψ. □

4.3. **The conditional expectation onto the C^*-algebra of orbits.** We study the maps $\tau := \psi \circ r\colon B \to Z(B^{op})$, $\tau^\dagger := \phi \circ s\colon B^{op} \to Z(B)$ introduced in (2). As before, we identify $Z(B^{op})$ and $Z(B)$ with $B \cap B^{op} \subseteq \mathcal{L}(H_\mu)$.

Proposition 4.5. i) *τ and $\tau^\dagger$ are conditional expectations onto a C^*-subalgebra of $B \cap B^{op}$, and $\tau(b) = \tau^\dagger(b^{op})$ for all $b \in B$. In particular, $\nu \circ s = \mu \circ \tau^\dagger = \mu^{op} \circ \tau = \nu^{-1} \circ r$.*

ii) *$s \circ \tau = r \circ \tau$ and $\tau \circ \phi = \tau^\dagger \circ \psi$.*

iii) *$\sigma^\mu_t \circ \tau = \tau = \tau \circ \sigma^\mu_t$ for all $t \in \mathbb{R}$.*

iv) *$\tau(b\sigma^\mu_{-i/2}(d)) = \tau(d\sigma^\mu_{-i/2}(d))$ for all $b, d \in \mathrm{Dom}(\sigma^\mu_{-i/2})$.*

Lemma 4.6. *Let $b, c, e \in B$, $d \in \mathrm{Dom}(\sigma^\mu_{-i/2})$ and $x = r(b)s(c^{op})$, $y = r(d)s(e^{op})$. Then $\langle\zeta_\psi|_1\Delta(x)(y^{op}\,{}_\alpha\underset{\mu}{\otimes}{}_\beta 1)|\zeta_\psi\rangle_1 = r(\tau(b\sigma^\mu_{-i/2}(d)))r(e)s(c^{op})$.*

Proof. By Lemma 3.7, $\zeta_\psi^* r(b)r(d)^{op}\zeta_\psi = \zeta_\psi^* r(b\sigma^\mu_{-i/2}(d))\zeta_\psi = \tau(b\sigma^\mu_{-i/2}(d))$, so

$$\begin{aligned}\langle\zeta_\psi|_1\Delta(x)(y^{op}\,{}_\alpha\underset{\mu}{\otimes}{}_\beta 1)|\zeta_\psi\rangle_1 &= \langle\zeta_\psi|_1(r(b)y^{op}\,{}_\alpha\underset{\mu}{\otimes}{}_\beta s(c^{op}))|\zeta_\psi\rangle_1 \\ &= \rho_\beta(\zeta_\psi^* r(b)r(d)^{op}s(e^{op})^{op}\zeta_\psi)s(c^{op}) \\ &= r(\zeta_\psi^* r(b)r(d)^{op}\zeta_\psi e)s(c^{op}) \\ &= r(\tau(b\sigma^\mu_{-i/2}(d)))r(e)s(c^{op}).\end{aligned}$$

□

Proof of Proposition 4.5. (1), (2) Left- and right-invariance of ϕ, ψ imply

$$\begin{aligned}\phi(s(\psi(a))) = \zeta_\phi^* s(\psi(a))\zeta_\phi &= \zeta_\psi^*\langle\zeta_\phi|_2\Delta(a)|\zeta_\phi\rangle_2\zeta_\psi \\ &= \zeta_\phi^*\langle\zeta_\psi|_1\Delta(a)|\zeta_\psi\rangle_1\zeta_\phi^* = \zeta_\psi^* r(\phi(a))\zeta_\psi = \psi(r(\phi(a)))\end{aligned}$$

for all $a \in A$. Therefore, $\tau^\dagger \circ \psi = \tau \circ \phi$ and $\tau \circ \tau = \tau^\dagger \circ \tau = \tau^\dagger \circ (\psi \circ r) = \tau \circ \phi \circ r = \tau$. Next, $s \circ \tau = r \circ \tau$ because $s(\tau(b)) = s(\psi(r(b))) = \langle\zeta_\psi|_1\Delta(r(b))|\zeta_\psi\rangle_1 = r(\tau(b))$ for all $b \in B$ by Lemma 4.6. In particular, for all $b, c, d \in B$,

$$\begin{aligned}\tau(b)\tau(c)\tau(d) = \tau(b)\psi(r(c))\tau(d) &= \psi(s(\tau(b))r(c)s(\tau(d))) \\ &= \psi(r(\tau(b)c\tau(d))) = \tau(\tau(b)c\tau(d)).\end{aligned}$$

(3), (4) Let $t \in \mathbb{R}$. Then $\sigma^\mu_t(\tau(B)) \subseteq \tau(B)$ because $\sigma^\mu_t \circ \tau = \sigma^{\mu^{op}}_{-t} \circ \psi \circ r = \psi \circ \sigma^{\nu^{-1}}_{-t} \circ r = \psi \circ \sigma^\nu_t \circ r = \psi \circ r \circ \sigma^\mu_t$. Therefore, σ^μ restricts to the modular automorphism group for the trace $\upsilon := \mu|_{\tau(B)}$, which is $\mathrm{id}_{\tau(B)}$, and hence $\sigma^\mu_t \circ \tau = \tau \circ \sigma^\upsilon_t = \tau = \tau \circ \sigma^\mu_t$. Let $b, d \in \mathrm{Dom}(\sigma^\mu_{-i/2})$. By Lemma 4.6 and Definition 4.3 (3),

$$\begin{aligned}r(\tau(b\sigma^\mu_{-i/2}(d))) &= \langle\zeta_\psi|_1\Delta(r(b))(r(d)^{op}\,{}_\alpha\underset{\mu}{\otimes}{}_\beta 1)|\zeta_\psi\rangle_1 \\ &= R(\langle\zeta_\psi|_1\Delta(r(d))(r(b)^{op}\,{}_\alpha\underset{\mu}{\otimes}{}_\beta 1)|\zeta_\psi\rangle_1) = s(\tau(d\sigma^\mu_{-i/2}(b))).\end{aligned}$$

Since $s \circ \tau = r \circ \tau$ and r is injective, we can conclude $\tau(b\sigma^\mu_{-i/2}(d)) = \tau(d\sigma^\mu_{-i/2}(b))$. □

4.4. **The modular element.** The modular element of a compact C^*-quantum groupoid can be described in terms of the element $\theta := \phi(\delta) = \psi(\delta^{-1}) \in B \cap B^{op}$ (see Lemmas 2.5, 2.7) as follows.

Proposition 4.7. *$\delta = r(\theta)s(\theta)^{-1}$ and $\Delta(\delta) = \delta\,{}_\alpha\underset{\mu}{\otimes}{}_\beta\delta$.*

Proof. By Lemma 2.5 (1), $\tilde{\delta} := r(\theta)s(\theta)^{-1}$ is positive, invertible, and invariant with respect to σ^{ν}. Moreover, $\nu^{-1}(a) = \nu(\tilde{\delta}^{1/2}a\tilde{\delta}^{1/2})$ for all $a \in A$ because

$$\begin{aligned}\nu^{-1}(s(\theta)^{1/2}as(\theta)^{1/2}) &= \mu^{op}(\theta^{1/2}\psi(a)\theta^{1/2}) = (\nu \circ s \circ \psi)(a) \\ &= (\nu^{-1} \circ r \circ \phi)(a) = \mu(\theta^{1/2}\phi(a)\theta^{1/2}) = \nu(r(\theta)^{1/2}ar(\theta)^{1/2})\end{aligned}$$

for all $a \in A$ by Proposition 4.5 and Lemma 2.5. By Lemma 2.3, $\delta = \tilde{\delta}$, and $\Delta(\delta) = r(\theta)\,{}_{\alpha}\underset{\mu}{\otimes}{}_{\beta}\,s(\theta)^{-1} = r(\theta)\rho_{\alpha}(\theta^{-1})\,{}_{\alpha}\underset{\mu}{\otimes}{}_{\beta}\,\rho_{\beta}(\theta)s(\theta)^{-1} = \delta\,{}_{\alpha}\underset{\mu}{\otimes}{}_{\beta}\,\delta$ because $\theta \in B \cap B^{op}$. □

An important consequence of the preceding result is that for every compact C^*-quantum groupoid, there exists a faithful invariant KMS-state on the basis:

Corollary 4.8. $\mu_{\theta} \circ \phi = (\mu_{\theta})^{op} \circ \psi$.

Proof. We get

$$\mu(\theta^{1/2}\phi(a)\theta^{1/2}) = \nu(r(\theta)^{1/2}ar(\theta)^{1/2}) = \nu^{-1}(s(\theta)^{1/2}as(\theta)^{1/2}) = \mu^{op}(\theta^{1/2}\psi(a)\theta^{1/2})$$

for all $a \in A$. □

Therefore, we could in principle restrict to compact C^*-quantum groupoids with trivial modular element $\delta = 1_A$.

The KMS-state μ can be factorized into a state υ on the commutative C^*-algebra $\tau(B) \subseteq Z(B)$ and a perturbation of τ as follows. We define maps

$$\tau_{\theta^{-1}} \colon B \to \tau(B),\ b \mapsto \tau(\theta^{-1/2}b\theta^{-1/2}), \qquad \upsilon = \mu_{\theta}|_{\tau(B)} \colon \tau(B) \to \mathbb{C},\ b \mapsto \mu(\theta^{1/2}b\theta^{1/2}).$$

Note that $\tau(\theta^{-1}) = 1$ because $\theta = \phi(\delta) = \phi(r(\theta)s(\theta)^{-1}) = \theta\tau(\theta^{-1})$.

Proposition 4.9. $\mu = \upsilon \circ \tau_{\theta^{-1}}$.

Proof. By Propositions 4.5 and 4.7,

$$\mu(b) = \nu(r(b)) = \nu^{-1}(\delta^{-1/2}r(b)\delta^{-1/2}) = \mu^{op}(\theta^{1/2}\psi(r(\theta^{-1/2}b\theta^{-1/2}))\theta^{1/2}) = (\upsilon \circ \tau_{\theta^{-1}})(b)$$

for all $b \in B$. □

4.5. **Uniqueness of the Haar weights.** The Haar weights of a compact C^*-quantum groupoid are not unique but can be rescaled by elements of B as follows. For every positive $\gamma \in B^{op}$, the map $\phi_{s(\gamma)} \colon A \to B$ given by $a \mapsto \phi(s(\gamma)^{1/2}as(\gamma)^{1/2})$ is a bounded left Haar weight for $(A_H^{\beta,\alpha}, \Delta)$ because

$$\begin{aligned}\phi_{s(\gamma)}(\langle\xi|_1\Delta(a)|\xi'\rangle_1) &= \phi(\langle\xi|_1(1\,{}_{\alpha}\underset{\mu}{\otimes}{}_{\beta}\,s(\gamma)^{1/2})\Delta(a)(1\,{}_{\alpha}\underset{\mu}{\otimes}{}_{\beta}\,s(\gamma)^{1/2})|\xi'\rangle_1) \\ &= \phi(\langle\xi|_1\Delta(s(\gamma)^{1/2}as(\gamma^{1/2}))|\xi'\rangle_1) = \xi^*\phi_{s(\gamma)}(a)\xi'\end{aligned}$$

for all $a \in A$, $\xi, \xi' \in \alpha$. Similarly, for every positive $\gamma \in B$, the map $\psi_{r(\gamma)} \colon A \to B^{op}$ given by $a \mapsto \psi(r(\gamma)^{1/2}ar(\gamma)^{1/2})$ is a bounded right Haar weight for $(A_H^{\beta,\alpha}, \Delta)$.

Theorem 4.10. i) *Let $(B, \mu, A, r, \tilde{\phi}, s, \tilde{\psi}, \tilde{\delta})$ be a compact C^*-quantum graph and $\tilde{\phi}$ a bounded left Haar weight for $(A_H^{\beta,\alpha}, \Delta)$. Then $\tilde{\phi} = \phi_{\gamma}$, where $\gamma = \tilde{\psi}(\tilde{\delta}^{-1})\theta^{-1}$.*

ii) *Let $(B, \mu, A, r, \tilde{\phi}, s, \tilde{\psi}, \tilde{\delta})$ be a compact C^*-quantum graph and $\tilde{\psi}$ a bounded right Haar weight for $(A_H^{\beta,\alpha}, \Delta)$. Then $\tilde{\psi} = \psi_{\gamma}$, where $\gamma = \tilde{\phi}(\tilde{\delta})\theta^{-1}$.*

Proof. We only prove (1), the proof of (2) is similar. Put $\tilde{\nu} := \mu \circ \tilde{\phi}$, $\tilde{\nu}^{-1} := \mu^{op} \circ \tilde{\psi}$, $\tilde{\theta} := \tilde{\psi}(\tilde{\delta}^{-1})$. Let $a \in A$. Then

$$\tilde{\phi}(s(\psi(a))) = \tilde{\phi}(\langle\zeta_\psi|_1\Delta(a)|\zeta_\psi\rangle_1) = \psi(r(\tilde{\phi}(a))). \tag{4}$$

We apply μ to the left hand side and find, using Lemma 2.5 (2),

$$\tilde{\nu}(s(\psi(a))) = \mu^{op}_{\tilde{\theta}}(\psi(a)) = \nu^{-1}(s(\tilde{\theta})^{1/2}as(\tilde{\theta})^{1/2}) = \nu(\delta^{1/2}s(\tilde{\theta})^{1/2}as(\tilde{\theta})^{1/2}\delta^{1/2}).$$

Next, we apply μ to the right hand side of (4) and find $\nu^{-1}(r(\tilde{\phi}(a))) = \mu_\theta(\tilde{\phi}(a)) = \tilde{\nu}(r(\theta)^{1/2}ar(\theta)^{1/2})$. Since $\delta = r(\theta)s(\theta)^{-1}$, we can conclude $\nu(s(\gamma)^{1/2}as(\gamma)^{1/2}) = \tilde{\nu}(a)$ for all $a \in A$ and in particular

$$\mu(b^*\tilde{\phi}(a)) = \tilde{\nu}(r(b)^*a) = \nu(s(\gamma)^{1/2}r(b)^*as(\gamma)^{1/2}) = \mu(b^*\phi(s(\gamma)^{1/2}as(\gamma)^{1/2}))$$

for all $b \in B$, $a \in A$. Since μ is faithful, $\tilde{\phi}(a) = \phi(s(\gamma)^{1/2}as(\gamma)^{1/2})$ for all $a \in A$. □

5. The fundamental unitary

In the theory of locally compact quantum groups, a fundamental rôle is played by the associated multiplicative unitaries, whose theory was developed by Baaj, Skandalis [1] and Woronowicz [22]. We shall associate to every compact C^*-quantum groupoid a C^*-pseudo-multiplicative unitary [17] that can be considered as a generalized multiplicative unitary. This unitary will be used to prove that the coinvolution of a compact C^*-quantum groupoid reverses the comultiplication, to construct a generalized Pontrjagin dual of the compact C^*-quantum groupoid in form of a Hopf C^*-bimodule, and to associate to every compact C^*-quantum groupoid a measured quantum groupoid in the sense of Enock and Lesieur [4, 8].

5.1. **C^*-pseudo-multiplicative unitaries.** The notion of a C^*-pseudo-multiplicative unitary extends the notion of a multiplicative unitary [1], of a continuous field of multiplicative unitaries [2] and of a pseudo-multiplicative unitary on C^*-modules [9, 18], and is closely related to pseudo-multiplicative unitaries on Hilbert spaces [21]; see [17, Section 4.1]. We give the precise definition and the main properties; for proofs and details, see [15, 17]. Let μ be a faithful KMS-state on a C^*-algebra B.

Definition 5.1. *A* C^*-pseudo-multiplicative unitary over μ *consists of a* C^*-(μ^{op},μ,μ^{op})-*module* $(H,\widehat{\beta},\alpha,\beta)$ *and a unitary* $V\colon H_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{\alpha}H \to H_{\alpha}\underset{\mu}{\otimes}{}_{\beta}H$ *such that*

$$V(\alpha \triangleleft \alpha) = \alpha \triangleright \alpha, \quad V(\widehat{\beta} \triangleright \beta) = \widehat{\beta} \triangleleft \beta, \quad V(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}, \quad V(\beta \triangleleft \alpha) = \beta \triangleleft \beta$$

and the following diagram commutes:

$$\begin{tikzcd}
H {}_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{\alpha} H {}_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{\alpha} H \arrow[r, "V \underset{\mu^{op}}{\otimes} \mathrm{id}"] \arrow[d, "\mathrm{id} \underset{\mu^{op}}{\otimes} V"'] & H {}_{\alpha}\underset{\mu}{\otimes}{}_{\beta} H {}_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{\alpha} H \arrow[r, "\mathrm{id} \underset{\mu}{\otimes} V"] & H {}_{\alpha}\underset{\mu}{\otimes}{}_{\beta} H {}_{\alpha}\underset{\mu}{\otimes}{}_{\beta} H, \\
H {}_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{\alpha \triangleright \alpha} (H {}_{\alpha}\underset{\mu}{\otimes}{}_{\beta} H) \arrow[d, "\mathrm{id} \underset{\mu^{op}}{\otimes} \Sigma"'] & & (H {}_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{\alpha} H) {}_{\alpha \triangleleft \alpha}\underset{\mu}{\otimes}{}_{\beta} H \arrow[u, "V \underset{\mu}{\otimes} \mathrm{id}"'] \\
H {}_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{\alpha} H {}_{\beta}\underset{\mu^{op}}{\otimes}{}_{\alpha} H \arrow[rr, "V \underset{\mu^{op}}{\otimes} \mathrm{id}"] & & \big(H {}_{\alpha}\underset{\mu}{\otimes}{}_{\beta} H\big) {}_{\widehat{\beta} \triangleleft \beta}\underset{\mu^{op}}{\otimes}{}_{\alpha} H \arrow[u, "\Sigma_{23}"']
\end{tikzcd}$$

where Σ_{23} *is given by* $(H_{\rho_\alpha} \olessthan \beta)_{\rho_{\widehat{\beta} \triangleleft \beta}} \olessthan \alpha \xrightarrow{\cong} (H_{\rho_{\widehat{\beta}}} \olessthan \alpha)_{\rho_{\alpha \triangleleft \alpha}} \olessthan \beta$, $(\zeta \olessthan \xi) \olessthan \eta \mapsto (\zeta \olessthan \eta) \olessthan \xi$.

Let $((H, \widehat{\beta}, \alpha, \beta), V)$ be a C^*-pseudo-multiplicative unitary. We abbreviate the operators $V \underset{\mu^{op}}{\otimes} \mathrm{id}$ and $V \underset{\mu}{\otimes} \mathrm{id}$ by V_{12}, the operators $\mathrm{id} \underset{\mu}{\otimes} V$ and $\mathrm{id} \underset{\mu}{\otimes} V$ by V_{23}, and $(\mathrm{id} \underset{\mu^{op}}{\otimes} \Sigma) V_{12} \Sigma_{23}$ by V_{13}. Thus, the indices indicate those positions in a relative tensor product where the operator acts like V. We put

$$\widehat{A}(V) := \big[\langle \beta|_2 V |\alpha\rangle_2\big] \subseteq \mathcal{L}(H), \qquad A(V) := \big[\langle \alpha|_1 V |\widehat{\beta}\rangle_1\big] \subseteq \mathcal{L}(H).$$

The assumptions on V imply $\widehat{A}(V) \subseteq \mathcal{L}(H_\beta)$, $A(V) \subseteq \mathcal{L}(H_{\widehat{\beta}})$, so that we can define

$$\begin{aligned}
\widehat{\Delta}_V \colon \widehat{A} &\to \mathcal{L}(H {}_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{\alpha} H), & \Delta_V \colon A &\to \mathcal{L}(H {}_{\alpha}\underset{\mu}{\otimes}{}_{\beta} H), \\
\widehat{a} &\mapsto V^*(\mathrm{id} {}_{\alpha}\underset{\mu}{\otimes}{}_{\beta} \widehat{a})V, & a &\mapsto V(a {}_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{\alpha} \mathrm{id})V^*.
\end{aligned}$$

Definition 5.2. $((H, \widehat{\beta}, \alpha, \beta), V)$ *is* regular *if* $[\langle \alpha|_1 V |\alpha\rangle_2] = [\alpha\alpha^*]$, *and* well-behaved *if* $(\widehat{A}(V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$ *and* $(A(V)_H^{\beta,\alpha}, \Delta_V)$ *are Hopf* C^**-bimodules over* μ^{op} *and* μ*, respectively.*

We cite the following result [17, Theorem 4.14]:

Theorem 5.3. *Every regular* C^**-pseudo-multiplicative unitary is well-behaved.*

5.2. **The fundamental unitary of a compact C^*-quantum groupoid.** Throughout this section, let $(B, \mu, A, r, \phi, s, \psi, \delta, R, \Delta)$ be a compact C^*-quantum groupoid. We use the same notation as in the preceding section.

Theorem 5.4. *There exists a regular* C^**-pseudo-multiplicative unitary* $((H, \widehat{\beta}, \alpha, \beta), V)$ *over* μ *such that* $V|a\zeta_\psi\rangle_1 = \Delta(a)|\zeta_\psi\rangle_1$ *for all* $a \in A$.

Uniqueness is evident. The proof of existence proceeds in several steps.

Proposition 5.5. i) *There exists a unitary* $V \colon H {}_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{\alpha} H \to H {}_{\alpha}\underset{\mu}{\otimes}{}_{\beta} H$ *such that* $V|a\zeta_\psi\rangle_1 = \Delta(a)|\zeta_\psi\rangle_1$ *for all* $a \in A$.

ii) $V(a\zeta_\nu \olessthan d^{op}\zeta_\psi) = \Delta(a)(\zeta_\nu \olessthan d^{op}\zeta_\phi)$ *for all* $a, d \in A$.

iii) $V(\widehat{\beta} \triangleright \beta) = \widehat{\beta} \triangleleft \beta$, $V(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}$, $V(\widehat{\beta} \triangleright \widehat{\alpha}) = \alpha \triangleright \widehat{\alpha}$, $V(\widehat{\alpha} \triangleleft \alpha) = \widehat{\alpha} \triangleleft \beta$.

Proof. (1) Let $a, a' \in A$, $\omega, \omega' \in H$. Since ψ is a right-invariant,

$$\begin{aligned}\langle \Delta(a)(\zeta_\psi \ominus \omega)|\Delta(a')(\zeta_\psi \ominus \omega')\rangle_{(H_\alpha \underset{\mu}{\otimes}{}_\beta H)} &= \langle \omega|\langle \zeta_\psi|_1 \Delta(a^*a')|\zeta_\psi\rangle_1 \omega'\rangle \\ &= \langle \omega|\rho_\alpha(\zeta_\psi^* a^* a' \zeta_\psi)\omega'\rangle \\ &= \langle a\zeta_\psi \ominus \omega|a'\zeta_\psi \ominus \omega'\rangle_{(H_{\hat\beta} \underset{\mu^{op}}{\otimes}{}_\alpha H)}.\end{aligned}$$

Therefore, there exists an isometry $V\colon H_{\hat\beta} \underset{\mu^{op}}{\otimes}{}_\alpha H \to H_\alpha \underset{\mu}{\otimes}{}_\beta H$ satisfying $V|a\zeta_\psi\rangle_1 = \Delta(a)|\zeta_\psi\rangle_1$ for all $a \in A$. The relation $[\Delta(A)|\beta\rangle_2] = [|\beta\rangle_2 A]$ implies $V(\widehat{\beta} \triangleleft \beta) = [V|A\zeta_\psi\rangle_1\beta] = [\Delta(A)|\zeta_\psi\rangle_1\beta] = [\Delta(A)|\beta\rangle_2\zeta_\psi] = [|\beta\rangle_2 A\zeta_\psi] = \widehat{\beta} \triangleleft \beta$, whence V is surjective and unitary.

(2) By Proposition 4.7, we have for all $a, d \in A$

$$\begin{aligned}V(a\zeta_\nu \oslash d^{op}\zeta_\psi) &= V(a\delta^{-1/2}\zeta_\psi\zeta_\mu \oslash d^{op}\zeta_\psi) \\ &= V(a\delta^{-1/2}\zeta_\psi \ominus d^{op}\zeta_{\nu^{-1}}) \\ &= \Delta(a\delta^{-1/2})(\zeta_\psi \ominus d^{op}\delta^{1/2}\zeta_\nu) \\ &= \Delta(a)(\delta^{-1/2}{}_\alpha \underset{\mu}{\otimes}{}_\beta \delta^{-1/2})(\zeta_{\nu^{-1}} \oslash d^{op}\delta^{1/2}\zeta_\phi) = \Delta(a)(\zeta_\nu \oslash d^{op}\zeta_\phi).\end{aligned}$$

(3) The first relation was already proven above. Since $[\Delta(A)|\zeta_\psi\rangle_1 A] = [|\alpha\rangle_1 A]$, $V(\widehat{\beta} \triangleright \widehat{\beta}) = [V|A\zeta_\psi\rangle_1 A\zeta_\psi] = [\Delta(A)|\zeta_\psi\rangle_1 A\zeta_\psi] = [|\alpha\rangle_1 A\zeta_\psi] = \alpha \triangleright \widehat{\beta}$ and similarly $V(\widehat{\beta} \triangleright \widehat{\alpha}) = \alpha \triangleright \widehat{\alpha}$. Finally, by (2), for all $b \in B$ and $a, d \in A$

$$\begin{aligned}V|a^{op}\zeta_\psi\rangle_2 d\zeta_\phi b\zeta_\mu = V(dr(b)\zeta_\nu \oslash a^{op}\zeta_\psi) &= \Delta(dr(b))(\zeta_\nu \oslash a^{op}\zeta_\phi) \\ &= \Delta(d)(r(b)\zeta_\nu \oslash a^{op}\zeta_\phi) = \Delta(d)|a^{op}\zeta_\phi\rangle_2\zeta_\phi b\zeta_\mu\end{aligned}$$

and hence $[V|\alpha\rangle_2\widehat{\alpha}] = [V|A^{op}\zeta_\psi\rangle_2 A\zeta_\phi] = [\Delta(A)|A^{op}\zeta_\phi\rangle_2\zeta_\phi] = [\Delta(A)|\beta\rangle_2\zeta_\phi] = [|\beta\rangle_2 A\zeta_\phi] = \widehat{\alpha} \triangleleft \beta$. □

Condition (3) of Definition 4.3 yields the following inversion formula for V:

Theorem 5.6. $V^* = (J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)V(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)$.

Proof. Put $\tilde{V} := (J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)V(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)$. Then for all $a, b, c, d \in A$

$$\begin{aligned}\langle a\zeta_\psi \ominus b^{op}\zeta_{\nu^{-1}}|V^*(c^{op}\zeta_{\nu^{-1}} \oslash d^{op}\zeta_\phi)\rangle &= \langle \Delta(a)(\zeta_\psi \ominus b^{op}\zeta_{\nu^{-1}})|c^{op}\zeta_\psi \ominus d^{op}\zeta_\nu\rangle \\ &= \langle \zeta_\psi \ominus \zeta_{\nu^{-1}}|\Delta(a^*)(c^{op}{}_\alpha \underset{\mu}{\otimes}{}_\beta 1)(\zeta_\psi \ominus (b^{op})^* d^{op}\zeta_\nu)\rangle \\ &= \langle \zeta_{\nu^{-1}}|\langle \zeta_\psi|_1 \Delta(a^*)(c^{op}{}_\alpha \underset{\mu}{\otimes}{}_\beta 1)|\zeta_\psi\rangle_1 (db^*)^{op}\zeta_\nu\rangle,\end{aligned}$$

$$\begin{aligned}\langle a\zeta_\psi \otimes b^{op}\zeta_{\nu^{-1}}|\tilde{V}(c^{op}\zeta_{\nu^{-1}} \otimes d^{op}\zeta_\phi)\rangle &= \overline{\langle (a^*)^{op}\zeta_\psi \otimes Ib^{op}\zeta_{\nu^{-1}}|V(c^*\zeta_{\nu^{-1}} \otimes Id^{op}\zeta_\phi J_\mu)\rangle} \\ &= \overline{\langle \zeta_\psi \otimes Ib^{op}\zeta_{\nu^{-1}}|(a^{op}{}_\alpha\underset{\mu}{\otimes}{}_\beta 1)\Delta(c^*)(\zeta_\psi \otimes Id^{op}\zeta_\nu)\rangle} \\ &= \overline{\langle \zeta_\psi \otimes I\zeta_{\nu^{-1}}|(a^{op}{}_\alpha\underset{\mu}{\otimes}{}_\beta 1)\Delta(c^*)(\zeta_\psi \otimes I(b^{op})^*d^{op}\zeta_\nu)\rangle} \\ &= \overline{\langle I\zeta_{\nu^{-1}}|\langle\zeta_\psi|_1(a^{op}{}_\alpha\underset{\mu}{\otimes}{}_\beta 1)\Delta(c^*)|\zeta_\psi\rangle_1 I(db^*)^{op}\zeta_\nu\rangle} \\ &= \langle \zeta_{\nu^{-1}}|I\langle\zeta_\psi|_1(a^{op}{}_\alpha\underset{\mu}{\otimes}{}_\beta 1)\Delta(c^*)|\zeta_\psi\rangle_1 I(db^*)^{op}\zeta_\nu\rangle.\end{aligned}$$

Now, the claim follows from condition (3) in Definition 4.3. □

Proof of Theorem 5.4. By Lemma 3.9 and Propositions 3.8, 5.5, multiplication by the composition $(J_{\widehat{\beta}}\underset{J_\mu}{\otimes}{}_\alpha I)V^*(J_{\widehat{\beta}}\underset{J_\mu}{\otimes}{}_\alpha I)$ acts on subspaces of $\mathcal{L}(H_\mu, H_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_\alpha H)$ as follows:

$$\begin{aligned}
&[|\alpha\rangle_2\alpha] \xrightarrow{(J_{\widehat{\beta}}\underset{J_\mu}{\otimes}{}_\alpha I)} [|\beta\rangle_2\widehat{\beta}] \xrightarrow{V^*} [|\widehat{\beta}\rangle_1\beta] \xrightarrow{(J_{\widehat{\beta}}\underset{J_\mu}{\otimes}{}_\alpha I)} [|\alpha\rangle_1 I\beta J_\mu] = [|\alpha\rangle_1\alpha],\\
&[|\alpha\rangle_2\beta] \xrightarrow{(J_{\widehat{\beta}}\underset{J_\mu}{\otimes}{}_\alpha I)} [|\beta\rangle_2\widehat{\alpha}] \xrightarrow{V^*} [|\alpha\rangle_2\widehat{\alpha}] \xrightarrow{(J_{\widehat{\beta}}\underset{J_\mu}{\otimes}{}_\alpha I)} [|\beta\rangle_2 J\widehat{\alpha}J_\mu] = [|\beta\rangle_2\beta],\\
&[|\widehat{\beta}\rangle_1\widehat{\beta}] \xrightarrow{(J_{\widehat{\beta}}\underset{J_\mu}{\otimes}{}_\alpha I)} [|\alpha\rangle_1\widehat{\alpha}] \xrightarrow{V^*} [|\widehat{\beta}\rangle_1\widehat{\alpha}] \xrightarrow{(J_{\widehat{\beta}}\underset{J_\mu}{\otimes}{}_\alpha I)} [|\alpha\rangle_1 I\widehat{\alpha}J_\mu] = [|\alpha\rangle_1\widehat{\beta}].
\end{aligned}$$

Theorem 5.6 implies $V(\alpha \triangleleft \alpha) = \alpha \triangleright \alpha$, $V(\beta \triangleleft \alpha) = \beta \triangleleft \beta$, $V(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}$.

Next, we prove that $V_{23}V_{12} = V_{12}V_{13}V_{23}$. Let $a, d \in A$, $\omega \in H$ and $\Delta^{(2)} = (\Delta \underset{\mu}{*} \mathrm{id})\circ\Delta = (\mathrm{id} \underset{\mu}{*} \Delta) \circ \Delta$, $\Delta_{13}(a) = \Sigma_{23}(\Delta(a)_{\widehat{\beta}\triangleleft\beta}\underset{\mu^{op}}{\otimes}{}_\alpha \mathrm{id})\Sigma_{23}$. Then

$$\begin{aligned}V_{23}V_{12}(a\zeta_\psi \otimes d\zeta_\psi \otimes \omega) &= V_{23}(\Delta(a)_{\alpha\triangleright\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_\alpha \mathrm{id})(\zeta_\psi \otimes d\zeta_\psi \otimes \omega) \\ &= \Delta^{(2)}(a)(\zeta_\psi \otimes \Delta(d)(\zeta_\psi \otimes \omega)) \\ &= V_{12}\Delta_{13}(a)(\zeta_\psi \otimes \Delta(d)(\zeta_\psi \otimes \omega)) \\ &= V_{12}V_{13}(a\zeta_\psi \otimes \Delta(d)(\zeta_\psi \otimes \omega)) = V_{12}V_{13}V_{23}(a\zeta_\psi \otimes d\zeta_\psi \otimes \omega).\end{aligned}$$

Finally, V is regular since by Theorem 5.6, Lemma 3.9 and Proposition 3.8,

$$\begin{aligned}[\langle\alpha|_1 V|\alpha\rangle_2] &= [\langle\alpha|_1(J_{\widehat{\beta}}\underset{J_\mu}{\otimes}{}_\alpha I)V^*(J_{\widehat{\beta}}\underset{J_\mu}{\otimes}{}_\alpha I)|\alpha\rangle_2] \\ &= [I\langle\widehat{\beta}|_1 V^*|\beta\rangle_2 J] \\ &= [I\langle\zeta_\psi|_1\Delta(A)|\beta\rangle_2 J] \\ &= [I\langle\zeta_\psi|_1|\beta\rangle_2 AJ] = [I\beta J_\mu \cdot J_\mu\zeta_\psi^* AJ] = [\alpha\alpha^*].\end{aligned}$$

□

By Theorem 5.3, the regular C^*-pseudo-multiplicative unitary $((H, \widehat{\beta}, \alpha, \beta), V)$ yields two Hopf C^*-bimodules $(A(V)_H^{\beta,\alpha}, \Delta_V)$ and $(\widehat{A}(V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$.

Proposition 5.7. $(A(V)_H^{\beta,\alpha}, \Delta_V) = (A_H^{\beta,\alpha}, \Delta)$.

Proof. We have $A(V) = [\langle\alpha|_1 V|\widehat{\beta}\rangle_1] = [\langle\alpha|_1\Delta(A)|\zeta_\psi\rangle_1] = [A\langle\alpha|_1|\zeta_\psi\rangle_1] = [A\rho_\alpha(\alpha^*\zeta_\psi)] = [As(B^{op})] = A$, $\Delta_V(a) = V(a_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_\alpha 1)V^* = \Delta(a)$ for all $a \in A$. □

The Hopf C^*-bimodule $(\widehat{A}(V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$ will be discussed in the next subsection.

Our first application of the unitary V will be to prove that the coinvolution reverses the comultiplication. We need the following lemma:

Lemma 5.8. i) $\Delta(\langle\xi|_1 V|\xi'\rangle_1) = \langle\xi|_1 V_{12}V_{13}|\xi'\rangle_1$ *for all* $\xi \in \alpha, \xi' \in \widehat{\beta}$.

ii) $R(\langle\xi|_1 V|\xi'\rangle_1) = \langle J\xi' J_\mu|_1 V|J\xi J_\mu\rangle_1$ *for all* $\xi \in \alpha, \xi' \in \widehat{\beta}$.

Proof. (1) For all $\xi \in \alpha$, $\xi' \in \widehat{\beta}$,

$$\Delta(\langle\xi|_1 V|\xi'\rangle_1) = V((\langle\xi|_1 V|\xi'\rangle_1)_{\widehat{\beta}} \underset{\mu^{op}}{\otimes} {}_{\alpha}1)V^* = \langle\xi|_1 V_{23}V_{12}V_{23}^*|\xi'\rangle_1 = \langle\xi|_1 V_{12}V_{13}|\xi'\rangle_1;$$

see also [17, Lemma 4.13].

(2) Lemma 3.9 and Theorem 5.6 imply that for all $\xi \in \alpha$, $\xi' \in \widehat{\beta}$, $R(\langle\xi|_1 V|\xi'\rangle_1) = I\langle\xi'|_1 V^*|\xi\rangle_1 I = \langle J\xi' J_\mu|_1 (J_\alpha \underset{J_\mu}{\otimes} {}_\beta I)^* V^* (J_\alpha \underset{J_\mu}{\otimes} {}_\beta I)^* |J\xi J_\mu\rangle_1 = \langle J\xi' J_\mu|_1 V|J\xi J_\mu\rangle_1$. ☐

Theorem 5.9. $(R_\alpha \underset{J_\mu}{*} {}_\beta R) \circ \Delta = \mathrm{Ad}_\Sigma \circ \Delta \circ R$.

Proof. Let $\xi \in \alpha$ and $\xi' \in \widehat{\beta}$. By Lemma 5.8,

$$\begin{aligned}(\mathrm{Ad}_\Sigma \circ (R_\alpha \underset{J_\mu}{*} {}_\beta R) \circ \Delta)(\langle\xi|_1 V|\xi'\rangle_1) &= (\mathrm{Ad}_\Sigma \circ (R_\alpha \underset{J_\mu}{*} {}_\beta R))(\langle\xi|_1 V_{12}V_{13}|\xi'\rangle_1) \\ &= \mathrm{Ad}_\Sigma((I_\alpha \underset{J_\mu}{\otimes} {}_\beta I)^* \langle\xi'|_1 V_{13}^* V_{12}^*|\xi\rangle_1 (I_\alpha \underset{J_\mu}{\otimes} {}_\beta I)).\end{aligned}$$

By Lemma 3.9 (2), we can rewrite this expression in the form

$$\mathrm{Ad}_\Sigma(\langle J\xi' J_\mu|_1 (J_{\widehat{\beta}} \underset{J_\mu}{\otimes} {}_{\alpha \triangleright \alpha} (I_\alpha \underset{J_\mu}{\otimes} {}_\beta I)) V_{13}^* V_{12}^* (J_\alpha \underset{J_\mu}{\otimes} {}_\beta I_\alpha \underset{J_\mu}{\otimes} {}_\beta I)^* |J\xi J_\mu\rangle_1).$$

By Lemmas 3.9, 5.8 and Theorem 5.6 this expression is equal to

$$\begin{aligned}\mathrm{Ad}_\Sigma(\langle J\xi' J_\mu|_1 V_{13}V_{12}|J\xi J_\mu\rangle_1) &= \langle J\xi' J_\mu|_1 V_{12}V_{13}|J\xi J_\mu\rangle_1 \\ &= \Delta(\langle J\xi' J_\mu|_1 V|J\xi J_\mu\rangle_1) = \Delta(R(\langle\xi|_1 V|\xi'\rangle_1)).\end{aligned}$$ ☐

Remarks 5.10. i) One can prove the existence of a regular C^*-pseudo-multiplicative unitary $((H, \alpha, \beta, \widehat{\alpha}), W)$ satisfying $W^*|a\zeta_\phi\rangle_2 = \Delta(a)|\zeta_\phi\rangle_2$ for all $a \in A$ and express this unitary in terms of V as follows: $W = \Sigma(I_{\widehat{\beta}} \underset{J_\mu}{\otimes} {}_\alpha I)V^*(I_\beta \underset{J_\mu}{\otimes} {}_\alpha I)\Sigma$; see [19, Theorem 5.10].

ii) Using Theorem 5.9, one can prove the following analogue of condition (3) in Definition 4.3; see [19, Lemma 4.7]:

$$R\big(\langle\zeta_\phi|_2 \Delta(a)(1_\alpha \underset{\mu}{\otimes} {}_\beta d^{op})|\zeta_\phi\rangle_2\big) = \langle\zeta_{\phi|_2}(1_\alpha \underset{\mu}{\otimes} {}_\beta a^{op})\Delta(d)|\zeta_\phi\rangle_2$$

for all $a, d \in A$. If we would replace the former condition by the latter, we could develop the same theory using W instead of V and finally conclude that also the former condition holds.

5.3. **The dual Hopf C^*-bimodule.** The Hopf C^*-bimodule $(\widehat{A}(V)_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$ obtained from $((H, \widehat{\beta}, \alpha, \beta), V)$ can be considered as the generalized Pontrjagin dual of our initial compact C^*-quantum groupoid. Let us describe $\widehat{A} := \widehat{A}(V)$.

Proposition 5.11. i) *For each $a \in A$, there exists an operator $\lambda(a) \in \mathcal{L}(H)$ such that $\lambda(a)\Lambda_\nu(d) = \Lambda_\nu(\langle\zeta_\phi|_2\Delta(d)|a^{op}\zeta_\phi\rangle_2)$ for all $d \in A$, and $\lambda(a)^* = J\lambda(R(a))J$.*
ii) *$\langle x^{op}\zeta_\phi|_2V|y^{op}\zeta_\psi\rangle_2 = \lambda(yx^*)$ for all $x, y \in A$.*
iii) $\widehat{A} = [\lambda(A)]$.
iv) *There exists a $*$-antiautomorphism $\widehat{R}\colon \widehat{A} \to \widehat{A}$, $\widehat{a} \mapsto J\widehat{a}^*J$.*

Proof. By definition, the space $\widehat{A}$ is the closed linear span of all operators of the form $\langle x^{op}\zeta_\phi|_2V|y^{op}\zeta_\psi\rangle_2$, where $x, y \in A$. Let $x, y, d \in A$ and put $a = yx^*$. Then

$$\begin{aligned}\langle x^{op}\zeta_\phi|_2V|y^{op}\zeta_\psi\rangle_2 d\zeta_\nu &= \langle x^{op}\zeta_\phi|_2V(d\zeta_\nu \otimes y^{op}\zeta_\psi)\\ &= \langle x^{op}\zeta_\phi|_2\Delta(d)(\zeta_\nu \otimes y^{op}\zeta_\phi) = \Lambda_\nu(\langle\zeta_\phi|_2\Delta(d)|a^{op}\zeta_\phi\rangle_2).\end{aligned}$$

This calculation proves the existence of the operators $\lambda(a)$ for all $a \in A$ and that $\widehat{A} = [\lambda(A)]$. By Theorem 5.6, Lemma 3.9 and Proposition 3.8,

$$\begin{aligned}\lambda(a)^* &= (\langle x^{op}\zeta_\phi|_2V|y^{op}\zeta_\psi\rangle_2)^*\\ &= \langle y^{op}\zeta_\psi|_2(J_{\alpha}\underset{J_\mu}{\otimes}{}_{\beta}I)V(J_{\alpha}\underset{J_\mu}{\otimes}{}_{\beta}I)|x^{op}\zeta_\phi\rangle_2\\ &= J\langle Iy^{op}\zeta_\psi J_\mu|_2V|Ix^{op}\zeta_\phi J_\mu\rangle_2 J\\ &= J\langle R(y^*)^{op}\zeta_\phi|_2V|R(x^*)^{op}\zeta_\psi\rangle_2 J = J\lambda(R(x)^*R(y))J = J\lambda(R(a))J.\end{aligned}$$
□

Remarks 5.12. i) Using Theorem 5.6, one can show that $\widehat{R}$ is a coinvolution of $(\widehat{A}_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V)$ in the sense that it reverses the comultiplication: $\widehat{\Delta}_V \circ \widehat{R} = \mathrm{Ad}_\Sigma \circ (\widehat{R}_{\widehat{\beta}}\underset{J_\mu}{*}{}_{\alpha}\widehat{R}) \circ \widehat{\Delta}_V$; see [19, Corollary 7.6].

ii) Put $K := H_\mu$. There exists a morphism $\widehat{\epsilon} \in \mathrm{Mor}\big(\widehat{A}_H^{\alpha,\widehat{\beta}}, \mathcal{L}(K)_K^{B,B^{op}}\big)$, given by $\lambda(a) \mapsto \zeta_\psi^*\lambda(a)\zeta_\psi$ for all $a \in A$, which is a counit for $\big(\widehat{A}_H^{\alpha,\widehat{\beta}}, \widehat{\Delta}_V\big)$ in the sense that the maps

$$\begin{aligned}(\widehat{\epsilon} \underset{\mu^{op}}{*} \mathrm{id}) \circ \widehat{\Delta}_V &\colon \widehat{A} \to \mathcal{L}(K)_{B^{op}}\underset{\mu^{op}}{*}{}_{\alpha}\widehat{A} \subseteq \mathcal{L}(K_{B^{op}}\underset{\mu^{op}}{\otimes}{}_{\alpha}H) \cong \mathcal{L}(H),\\ (\mathrm{id} \underset{\mu^{op}}{*} \widehat{\epsilon}) \circ \widehat{\Delta}_V &\colon \widehat{A} \to \widehat{A}_{\widehat{\beta}}\underset{\mu^{op}}{*}{}_{B}\mathcal{L}(K) \subseteq \mathcal{L}(H_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{B}K) \cong \mathcal{L}(H)\end{aligned}$$

are equal to the embedding $A \hookrightarrow \mathcal{L}(H)$; see [19, Proposition 7.7].

5.4. **The passage to measurable quantum groupoids.** The compact C^*-quantum groupoid $(B, \mu, A, r, \phi, s, \psi, \delta, R, \Delta)$ can be completed to a measurable quantum groupoid in the sense of Enock and Lesieur [4, 8] as follows.

Put $N := B'' \subseteq \mathcal{L}(H_\mu)$, $N^{op} := (B^{op})'' = N' \subseteq \mathcal{L}(H_\mu)$, $M := A'' \subseteq \mathcal{L}(H)$ and denote by $\tilde{\mu}, \tilde{r}, \tilde{\phi}, \tilde{s}, \tilde{\psi}$ the normal extensions of μ, r, ϕ, s, ψ; see Lemma 2.2. The comultiplication $\Delta = \Delta_V$ extends uniquely to a normal $*$-homomorphism $\tilde{\Delta}\colon M \to \mathcal{L}(H_{\alpha}\underset{\mu}{\otimes}{}_{\beta}H)$ via $x \mapsto V(x_{\widehat{\beta}}\underset{\mu^{op}}{\otimes}{}_{\alpha}1)V^*$. To obtain a Hopf-von Neumann bimodule [20] $(N, M, \tilde{r}, \tilde{s}, \tilde{\Delta})$, we need to identify $\tilde{\Delta}(M)$ with a subalgebra of the fiber product $M {}_{\tilde{s}}\bar{*}_{\tilde{r}}^{\tilde{\mu}} M$ [13] which acts on the relative tensor product $H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H$ [14].

Let us recall the definition of $H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H$ and of $M {}_{\tilde{s}}\underset{\tilde{\mu}}{\bar{*}}{}_{\tilde{r}} M$. Put

$$D({}_{\tilde{r}}H;\tilde{\mu}) := \{\eta \in H \mid \exists C > 0 \forall y \in N : \|\tilde{r}(y)\eta\| \leqslant C\|y\zeta_\mu\|\}.$$

Thus, an element $\eta \in H$ belongs to $D({}_{\tilde{r}}H;\tilde{\mu})$ if and only if the map $N\zeta_\mu \to H$, $y\zeta_\mu \mapsto \tilde{r}(y)\eta$, extends to an operator $R_{\tilde{\mu}}(\eta) \in \mathcal{L}(H_\mu, H)$. Clearly, $R_{\tilde{\mu}}(\eta)^* R_{\tilde{\mu}}(\eta') \in N'$ for all $\eta, \eta' \in D({}_{\tilde{r}}H;\tilde{\mu})$. The space $H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H$ is the separated completion of the algebraic tensor product $H \odot D({}_{\tilde{r}}H;\tilde{\mu})$ with respect to the sesquilinear form defined by $\langle \omega \odot \eta | \omega' \odot \eta' \rangle = \langle \omega | \tilde{s}(R_{\tilde{\mu}}(\eta)^* R_{\tilde{\mu}}(\eta'))\omega' \rangle$ for all $\omega, \omega' \in H$, $\eta, \eta' \in D({}_{\tilde{r}}H;\tilde{\mu})$. We denote the image of an element $\omega \odot \eta$ in $H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H$ by $\omega {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} \eta$. Clearly, $\tilde{r}(N)' D({}_{\tilde{r}}H;\tilde{\mu}) \subseteq D({}_{\tilde{r}}H;\tilde{\mu})$, and for each $x, y \in M'$, there exists a well-defined operator $x {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} y \subseteq \mathcal{L}(H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H)$ such that $(x {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} y)(\omega {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} \eta) = x\omega {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} y\eta$ for all $\omega \in H$, $\eta \in D({}_{\tilde{r}}H;\tilde{\mu})$. Now, $M {}_{\tilde{s}}\underset{\tilde{\mu}}{\bar{*}}{}_{\tilde{r}} M = (M' {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} M')' \subseteq \mathcal{L}(H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H)$.

Lemma 5.13. i) $a^{op}\zeta_\nu \in D({}_{\tilde{r}}H;\tilde{\mu})$, $R_{\tilde{\mu}}(a^{op}\zeta_\nu) = a^{op}\zeta_\phi \in \beta$ *for all* $a \in A$.

ii) *There exist inverse isomorphisms*

$$H {}_\alpha\underset{\mu}{\otimes}{}_\beta H \cong H_{\rho_\alpha} \otimes \beta \xrightarrow{\Phi} H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H \xrightarrow{\Psi} \alpha \otimes_{\rho_\beta} H \cong H {}_\alpha\underset{\mu}{\otimes}{}_\beta H$$

such that $\Phi(\omega \otimes a^{op}\zeta_\phi) = \omega {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} a^{op}\zeta_\nu$, $\Psi(\xi\zeta {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} \eta) = \xi \otimes R_{\tilde{\mu}}(\eta)\zeta$ *for all* $\omega \in H$, $a \in A$, $\xi \in \alpha$, $\eta \in D({}_{\tilde{r}}H, \tilde{\mu})$, $\zeta \in H_\mu$.

Proof. (1) For all $a \in A$, $y \in N$, $\tilde{r}(y)a^{op}\zeta_\nu = a^{op}\tilde{r}(y)\zeta_\phi\zeta_\mu = a^{op}\zeta_\phi y\zeta_\mu$.

(2) Φ and Ψ are well-defined inverse isometries because

$$\begin{aligned}
\|\omega \otimes a^{op}\zeta_\phi\|^2 &= \langle \omega | \rho_\alpha(\zeta_\phi^*(a^{op})^* a^{op}\zeta_\phi)\omega \rangle \\
&= \langle \omega | \tilde{s}(R_{\tilde{\mu}}(a^{op}\zeta_\nu)^* R_{\tilde{\mu}}(a^{op}\zeta_\nu))\omega \rangle = \|\omega {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} a^{op}\zeta_\nu\|^2, \\
\|\xi\zeta {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} \eta\|^2 &= \langle \xi\zeta | \tilde{s}(R_{\tilde{\mu}}(\eta)^* R_{\tilde{\mu}}(\eta))\xi\zeta \rangle \\
&= \langle \zeta | \xi^*\xi R_{\tilde{\mu}}(\eta)^* R_{\tilde{\mu}}(\eta)\zeta \rangle \\
&= \langle \zeta | R_{\tilde{\mu}}(\eta)^* \rho_\beta(\xi^*\xi) R_{\tilde{\mu}}(\eta)\zeta \rangle = \|\xi \otimes R_{\tilde{\mu}}(\eta)\zeta\|^2, \\
(\Psi \circ \Phi)(\xi\zeta \otimes a^{op}\zeta_\phi) &= \xi \otimes R_{\tilde{\mu}}(a^{op}\zeta_\nu)\zeta = \xi \otimes a^{op}\zeta_\phi\zeta \equiv \xi\zeta \otimes a^{op}\zeta_\phi
\end{aligned}$$

for all $\omega, a, \xi, \eta, \zeta$ as above. □

We identify $H {}_\alpha\underset{\mu}{\otimes}{}_\beta H$ with $H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H$ via Φ, Ψ and consider $\tilde{\Delta}$ as a map $M \to \mathcal{L}(H {}_{\tilde{s}}\underset{\tilde{\mu}}{\otimes}{}_{\tilde{r}} H)$.

Theorem 5.14. $(N, M, \tilde{r}, \tilde{s}, \tilde{\Delta}, \tilde{\phi}, \tilde{\psi}, \tilde{\mu})$ *is a measurable quantum groupoid.*

Proof. First, the relation $\Delta(A) \subseteq A_{\alpha}\underset{\mu}{*}{}_{\beta}A \subseteq (A'{}_{\alpha}\underset{\mu}{\otimes}{}_{\beta}\,\mathrm{id}) \cap (\mathrm{id}\,{}_{\alpha}\underset{\mu}{\otimes}{}_{\beta}A')$ [17, Lemma 3.8] implies $\tilde{\Delta}(M) \subseteq M_{\tilde{s}}\underset{\tilde{\mu}}{\bar{*}}{}_{\tilde{r}}M$, and the definition of $\tilde{\Delta}$ and the fact that V is a C^*-pseudo-multiplicative unitary imply that $(N, M, \tilde{r}, \tilde{s}, \tilde{\Delta})$ is a Hopf-von Neumann bimodule.

Second, one has to check that $\tilde{\phi}$ and $\tilde{\psi}$ are left- and right-invariant, respectively. This follows from the fact that these maps are normal extensions of ϕ and ψ, which are left- and right-invariant, respectively.

Finally, one has to check that the modular automorphism groups of $\tilde{\nu} = \tilde{\mu} \circ \tilde{\phi}$ and $\tilde{\nu}^{-1} = \tilde{\mu}^{op} \circ \tilde{\psi}$ commute, but this follows from the fact that $\tilde{\nu}^{-1} = \tilde{\nu}_{\delta^{1/2}}$. ☐

6. Principal compact C^*-quantum groupoids

Principal compact C^*-quantum groupoids are particularly simple examples of compact C^*-quantum groupoids. We give the definition and discuss some of the main properties. For proofs and further details, see [19].

Recall that a compact groupoid G is principal if the map $G \to G^0 \times G^0$ given by $x \mapsto (r(x), s(x))$ is injective or, equivalently, if $C(G) = [r^*(C(G^0))s^*(C(G^0))]$. This condition can be carried over to compact C^*-quantum groupoids as follows:

Definition 6.1. *A compact C^*-quantum groupoid $(B, \mu, A, r, \phi, s, \psi, \delta, R, \Delta)$ is* principal *if $A = [r(B)s(B^{op})]$.*

To simplify the following discussion, we only consider the case where $\delta = 1_A$, which is not a serious restriction; see Corollary 4.8.

Essentially, a principal compact C^*-quantum groupoid is completely determined by the conditional expectation $\tau\colon B \to \tau(B) \subseteq Z(B)$ introduced in Subsection 4.3. The first result in this direction is the following proposition:

Proposition 6.2. *Let $(B, \mu, A, r, \phi, s, \psi, 1_A)$ be a compact C^*-quantum graph such that $A = [r(B)s(B^{op})]$. Then the following two conditions are equivalent:*

- i) *There exist R, Δ such that $(B, \mu, A, r, \phi, s, \psi, 1_A, R, \Delta)$ is a compact C^*-quantum groupoid.*
- ii) *$\tau(b) = \tau^{\dagger}(b^{op})$ for all $b \in B$, $\tau\colon B \to \tau(B)$ is a conditional expectation, $\mu \circ \tau = \mu$, $r \circ \tau = s \circ \tau$, and $\tau(b\sigma^{\mu}_{-i/2}(d)) = \tau(d\sigma^{\mu}_{-i/2}(b))$ for all $b, d \in \mathrm{Dom}(\sigma^{\mu}_{-i/2})$.* ☐

To every compact groupoid G, we can associate a principal compact groupoid whose total space is $\{(r(x), s(x)) \mid x \in G\}$. Likewise, we can associate to every compact C^*-quantum groupoid a principal one:

Corollary 6.3. *Let $(B, \mu, A, r, \phi, s, \psi, 1_A, R, \Delta)$ be a compact C^*-quantum groupoid and put $\tilde{A} = [r(B)s(B^{op})]$, $\tilde{\phi} = \phi|_{\tilde{A}}$, $\tilde{\psi} = \psi|_{\tilde{A}}$, $\tilde{R} = R|_{\tilde{A}}$. Then there exists a unique $*$-homomorphism $\tilde{\Delta}$ such that $(B, \mu, \tilde{A}, r, \tilde{\phi}, s, \tilde{\psi}, 1_{\tilde{A}}, \tilde{R}, \tilde{\Delta})$ is a principal compact C^*-quantum groupoid.* ☐

A principal compact C^*-quantum groupoid can be reconstructed from the conditional expectation τ as follows. Assume that

- C is a commutative unital C^*-algebra with a faithful state υ,
- B is a unital C^*-algebra with a υ-module structure (ι, τ) such that $\iota(C) \subseteq Z(B)$.

We put $\mu := \upsilon \circ \tau$, identify C with $\iota(C)$ via ι, define an isometry $\zeta_\tau \colon H_\upsilon \to H_\mu$ as in Lemma 3.7, and put $\gamma := [B\zeta_\tau] \subseteq \mathcal{L}(H_\upsilon, H_\mu)$, $\gamma^{op} := [B^{op}\zeta_\tau] \subseteq \mathcal{L}(H_\upsilon, H_\mu)$.

Theorem 6.4. *There exists a unique principal compact C^*-quantum groupoid $(B, \mu, A, r, \phi, s, \psi, 1_A, R, \Delta)$ such that $A = B_{\gamma^{op}} \underset{\upsilon}{\otimes}_\gamma B^{op} \subseteq \mathcal{L}((H_\mu)_{\gamma^{op}} \underset{\upsilon}{\otimes}_\gamma (H_\mu))$ and $r(b) = b_{\gamma^{op}} \underset{\upsilon}{\otimes}_\gamma 1^{op}$, $\phi(b_{\gamma^{op}} \underset{\upsilon}{\otimes}_\gamma c^{op}) = b\tau(c)$, $s(c^{op}) = 1_{\gamma^{op}} \underset{\upsilon}{\otimes}_\gamma c^{op}$, $\psi(b_{\gamma^{op}} \underset{\upsilon}{\otimes}_\gamma c^{op}) = \tau(b)c^{op}$ for all $b, c \in B$.* □

Essentially, every principal compact C^*-quantum groupoid is of the form above:

Proposition 6.5. *Let $(B, \mu, A, r, \phi, s, \psi, 1_A, R, \Delta)$ be a principal compact C^*-quantum groupoid.*

i) *$C := \tau(B)$ is a commutative unital C^*-algebra, $\upsilon := \mu|_C$ is a faithful state on C, (id, τ) is a υ-module structure on B, and $\mu = \upsilon \circ \tau$.*

Denote by $\zeta_\tau \colon H_\upsilon \to H_\mu$ the isometry $c\zeta_\upsilon \mapsto c\zeta_\mu$. Put $\gamma := [B\zeta_\tau]$, $\gamma^{op} := [B^{op}\zeta_\tau]$.

ii) *There exists a unitary $\Xi \colon H_\nu \to (H_\mu)_{\gamma^{op}} \underset{\upsilon}{\otimes}_\gamma (H_\mu)$ such that for all $b, c \in B$, $\Xi(r(b)^{op}s(c^{op})^{op}\zeta_\nu) = b^{op}\zeta_\tau \otimes \zeta_\upsilon \otimes c\zeta_\tau$, $\Xi(r(b)s(c^{op})\zeta_\nu) = b\zeta_\tau \otimes \zeta_\upsilon \otimes c^{op}\zeta_\tau$.*

iii) *Ad_Ξ restricts to an isomorphism $A \to B_{\gamma^{op}} \underset{\upsilon}{\otimes}_\gamma B^{op}$ such that $r(b)s(c^{op}) \mapsto b_{\gamma^{op}} \underset{\upsilon}{\otimes}_\gamma c^{op}$ for all $b, c \in B$.* □

7. Examples related to groupoids

Prototypical examples of compact C^*-quantum groupoids are the function algebra of a compact groupoid and the reduced groupoid C^*-algebra of an r-discrete groupoid with compact space of units. We outline these examples; for proofs and further details, see [19]. For some background on groupoids, see [10, 12].

Let G be a locally compact, Hausdorff, second countable groupoid with unit space G^0, range and source maps $r_G, s_G \colon G \to G^0$, left Haar system λ, and associated right Haar system λ^{-1}. For each $u \in G^0$, put $G^u := r_G^{-1}(u)$, $G_u := s_G^{-1}(u)$. Let μ_G be a probability measure on G^0 with full support and define measures ν_G, ν_G^{-1} on G such that for all $f \in C_c(G)$,

$$\int_G f\, d\nu_G = \int_{G^0} \int_{G^u} f(x)\, d\lambda^u(x)\, d\mu_G(u), \quad \int_G f d\nu_G^{-1} = \int_{G^0} \int_{G_u} f(x) d\lambda_u^{-1}(x)\, d\mu_G(u).$$

We impose the following assumptions:

i) *the space of units G^0 is compact;*
ii) *μ_G is quasi-invariant in the sense that ν_G and ν_G^{-1} are equivalent;*
iii) *the Radon-Nikodym derivative $D = d\nu_G/d\nu_G^{-1}$ is continuous.*

To equip the function algebra and the reduced groupoid C^*-algebra of G with the structure of Hopf C^*-bimodules, we use a C^*-pseudo-multiplicative unitary naturally associated to G [15, 17]. This unitary is constructed as follows.

Denote by μ the trace on $C(G^0)$ given by $f \mapsto \int_{G^0} f d\mu_G$, put $H := L^2(G, \nu_G)$, and define Hilbert C^*-$C(G^0)$-modules $L^2(G, \lambda)$, $L^2(G, \lambda^{-1})$ to be the completions of the pre-C^*-module $C_c(G)$, where for all $\xi, \xi' \in C_c(G)$, $u \in G^0$, $f \in C(G^0)$, $x \in G$,

$$\langle \xi' | \xi \rangle(u) = \int_{G^u} \overline{\xi'(x)} \xi(x) d\lambda^u(x), \quad (\xi f)(x) = \xi(x) f(r_G(x)) \quad \text{in case of } L^2(G, \lambda),$$
$$\langle \xi' | \xi \rangle(u) = \int_{G_u} \overline{\xi'(x)} \xi(x) d\lambda^{-1}_u(x), \quad (\xi f)(x) = \xi(x) f(s_G(x)) \quad \text{in case of } L^2(G, \lambda^{-1}).$$

There exist embeddings $j \colon L^2(G, \lambda) \to \mathcal{L}(H_\mu, H)$ and $\hat{j} \colon L^2(G, \lambda^{-1}) \to \mathcal{L}(H_\mu, H)$ such that $(j(\xi)\zeta)(x) = \xi(x)\zeta(r_G(x))$ and $(\hat{j}(\xi)\zeta)(x) = \xi(x)D(x)^{-1/2}\zeta(s_G(x))$ for all $\xi \in C_c(G)$, $\zeta \in C(G^0)$, $x \in G$, and with $\rho := j(L^2(G, \lambda))$ and $\sigma := \hat{j}(L^2(G, \lambda^{-1}))$, the tuple (H, σ, ρ, ρ) is a C^*-$(\mu^{op}, \mu, \mu^{op})$-module.

The relative tensor products $H_\sigma \underset{\mu^{op}}{\otimes} {}_\rho H$ and $H_\rho \underset{\mu}{\otimes} {}_\rho H$ can be described as follows. Define measures $\nu^2_{s,r}$ on $G^2_{s,r} := \{(x, y) \in G \times G \mid s_G(x) = r_G(y)\}$ and $\nu^2_{r,r}$ on $G^2_{r,r} := \{(x, y) \in G \times G \mid r_G(x) = r_G(y)\}$ such that for all $f \in C_c(G^2_{s,r})$, $g \in C_c(G^2_{r,r})$

$$\int_{G^2_{s,r}} f \, d\nu^2_{s,r} := \int_{G^0} \int_{G^u} \int_{G^{s_G(x)}} f(x, y) \, d\lambda^{s_G(x)}(y) \, d\lambda^u(x) \, d\mu_G(u),$$
$$\int_{G^2_{r,r}} g \, d\nu^2_{r,r} := \int_{G^0} \int_{G^u} \int_{G^u} g(x, y) \, d\lambda^u(y) \, d\lambda^u(x) \, d\mu_G(u).$$

Then there exist isomorphisms

$$\Phi \colon H_\sigma \underset{\mu^{op}}{\otimes} {}_\rho H \to L^2(G^2_{s,r}, \nu^2_{s,r}), \qquad \Psi \colon H_\rho \underset{\mu}{\otimes} {}_\rho H \to L^2(G^2_{r,r}, \nu^2_{r,r})$$

such that for all $\eta, \xi \in C_c(G)$, $\zeta \in C_c(G^0)$, $(x, y) \in G^2_{s,r}$, $(x', y') \in G^2_{r,r}$,

$$\Phi(\hat{j}(\eta) \otimes \zeta \otimes j(\xi))(x, y) = \eta(x) D(x)^{-1/2} \zeta(s_G(x)) \xi(y),$$
$$\Psi(j(\eta) \otimes \zeta \otimes j(\xi))(x', y') = \eta(x') \zeta(r_G(x')) \xi(y').$$

We identify $H_\sigma \underset{\mu^{op}}{\otimes} {}_\rho H$ with $L^2(G^2_{s,r}, \nu^2_{s,r})$ and $H_\rho \underset{\mu}{\otimes} {}_\rho H$ with $L^2(G^2_{r,r}, \nu^2_{r,r})$ via Φ, Ψ.

Theorem 7.1. *There exists a regular C^*-pseudo-multiplicative unitary $((H, \sigma, \rho, \rho), V)$ such that $(V\omega)(x, y) = \omega(x, x^{-1}y)$ for all $\omega \in C_c(G^2_{s,r})$, $(x, y) \in G^2_{r,r}$, $\xi \in C_c(G)$, $z \in G$.* □

The Hopf C^*-bimodules $(A(V)^{\rho,\rho}_H, \Delta_V)$ and $(\widehat{A}(V)^{\rho,\sigma}_H, \widehat{\Delta}_V)$ can be described as follows. Embed $C_0(G)$ into $\mathcal{L}(H)$ via the representation given by multiplication operators, and denote by $C^*_r(G)$ the reduced groupoid C^*-algebra of G, that is, the closed linear span of all operators $L(g) \in \mathcal{L}(H)$, where $g \in C_c(G)$ and

$$(L(g)f)(x) = \int_{G^{r_G(x)}} g(z) D(z)^{-1/2} f(z^{-1}x) d\lambda^{r_G(x)}(z) \quad \text{for all } f \in C_c(G), x \in G.$$

Proposition 7.2. i) *$A(V) = C^*_r(G)$ and*

$$(\Delta_V(L(g))\omega)(x, y) = \int_{G^{r_G(x)}} g(z) D(z)^{-1/2} \omega(z^{-1}x, z^{-1}y) d\lambda^{r_G(x)}(z)$$

for all $g \in C_c(G)$, $\omega \in C_c(G^2_{r,r})$, $(x, y) \in G^2_{r,r}$.

ii) $\widehat{A}(V) = C_0(G)$ *and* $(\widehat{\Delta}_V(f)\omega)(x,y) = f(xy)\omega(x,y)$ *for all* $f \in C_0(G)$, $\omega \in C_c(G^2_{s,r})$, $(x,y) \in G^2_{s,r}$. □

Using the preceding result, it is not difficult to prove the following theorems:

Theorem 7.3. *If G is compact, there exists a compact C^*-quantum groupoid*

$$(C(G^0), \mu, C(G), r, \phi, s, \psi, D^{-1}, \widehat{\Delta}_V, R)$$

such that $r(f) = r^*_G(f)$, $s(f) = s^*_G(f)$,

$$(\phi(g))(u) = \int_{G^u} g(y)d\lambda^u(y), \qquad (\psi(g))(u) = \int_{G_u} g(y)d\lambda^{-1}_u(y)$$

and $(R(g))(x) = g(x^{-1})$ *for all* $f \in C(G^0)$, $g \in C(G)$, $u \in G^0$, $x \in G$. □

Theorem 7.4. *Let G be r-discrete and let λ be the family of counting measures. Embed $C(G^0)$ into $C_c(G)$ by extending functions outside of G^0 by 0. Then there exists a compact C^*-quantum groupoid $(C(G^0), \mu, C^*_r(G), \iota, \phi, \iota, \phi, L(1_{G^0}), R, \Delta_V)$ such that $\iota(f) = L(f)$, $\phi(L(g)) = g|_{G^0}$, $R(L(g)) = L(g^\dagger)$ for all $f \in C(G^0)$, $g \in C_c(G)$, where $g^\dagger(x) = g(x^{-1})$ for all $x \in G$.* □

References

[1] Saad Baaj and Georges Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 4, 425–488.

[2] Étienne Blanchard, *Déformations de C^*-algèbres de Hopf*, Bull. Soc. Math. France **124** (1996), no. 1, 141–215.

[3] Siegfried Echterhoff, S. Kaliszewski, John Quigg, and Iain Raeburn, *A categorical approach to imprimitivity theorems for C^*-dynamical systems*, Mem. Amer. Math. Soc. **180** (2006), no. 850, viii+169.

[4] Michel Enock, *Continuous fields of C*-algebras and Lesieur's Quantum Groupoids*, preprint, September 2006, arXiv:OA/0605512.

[5] ———, *Quantum groupoids of compact type*, J. Inst. Math. Jussieu **4** (2005), no. 1, 29–133.

[6] Michel Enock and Jean-Michel Vallin, *Inclusions of von Neumann algebras, and quantum groupoids*, J. Funct. Anal. **172** (2000), no. 2, 249–300.

[7] E. C. Lance, *Hilbert C^*-modules. a toolkit for operator algebraists*, London Mathematical Society Lecture Note Series, vol. 210, Cambridge University Press, Cambridge, 1995.

[8] Franck Lesieur, *Measured quantum groupoids*, Mémoires de la SMF **109** (2007).

[9] Moto O'uchi, *Pseudo-multiplicative unitaries on Hilbert C^*-modules*, Far East J. Math. Sci. (FJMS), Special Volume, Part II (2001), 229–249.

[10] Alan L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Progress in Mathematics, vol. 170, Birkhäuser Boston Inc., Boston, MA, 1999.

[11] Gert K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs, vol. 14, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1979.

[12] Jean Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, vol. 793, Springer, Berlin, 1980.

[13] Jean-Luc Sauvageot, *Produits tensoriels de Z-modules et applications*, Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983), Lecture Notes in Mathematics, vol. 1132, Springer, Berlin, 1985, pp. 468–485.

[14] Masamichi Takesaki, *Theory of operator algebras. II*, Encyclopaedia of Mathematical Sciences, vol. 125, Springer-Verlag, Berlin, 2003.

[15] Thomas Timmermann, *C^*-pseudo-multiplicative unitaries, Hopf C^*-bimodules and their Fourier algebras*, J. Inst. Math. Jussieu **11** (2012), no. 1, 189-220.

[16] ———, *Coactions of Hopf C^*-bimodules*, J. Operator Theory **68** (2012), no. 1, 19-66.

[17] ———, *C^*-pseudo-multiplicative unitaries*, Tech. Report 481, SFB 478, WW-University Muenster, September 2007, arXiv:0709.2995.

[18] ———, *Pseudo-multiplicative unitaries on C^*-modules and Hopf C^*-families*, J. Noncomm. Geom. **1** (2007), 497–542.

[19] ——— *A definition of compact C^*-quantum groupoids*, In Operator Structures and Dynamical Systems, Leiden 2008, Contemp. Math. **503** (2009), 267-290.

[20] Jean-Michel Vallin, *Bimodules de Hopf et poids opératoriels de Haar*, J. Operator Theory **35** (1996), no. 1, 39–65.

[21] ———, *Unitaire pseudo-multiplicatif associé à un groupoïde. Applications à la moyennabilité*, J. Operator Theory **44** (2000), no. 2, 347–368.

[22] Stanislaw L. Woronowicz, *From multiplicative unitaries to quantum groups*, Internat. J. Math. **7** (1996), no. 1, 127–149.

T. TIMMERMANN, University of Muenster,, Einsteinstr. 62, 48149 Muenster, Germany
E-mail address: `timmermt@math.uni-muenster.de`

APPENDIX I.4

COACTIONS OF HOPF C^*-BIMODULES

THOMAS TIMMERMANN

J. Operator Theory *68(1):19-66, 2012.*

Abstract. Coactions of Hopf C^*-bimodules simultaneously generalize coactions of Hopf C^*-algebras and actions of groupoids. Following an approach of Baaj and Skandalis, we construct reduced crossed products and establish a duality for fine coactions. Examples of coactions arise from Fell bundles on groupoids and actions of a groupoid on bundles of C^*-algebras. Continuous Fell bundles on an étale groupoid correspond to coactions of the reduced groupoid algebra, and actions of a groupoid on a continuous bundle of C^*-algebras correspond to coactions of the function algebra.

Contents

2000 *Mathematics Subject Classification.* Primary 46L55; Secondary 22A22, 20G42.
Key words and phrases. quantum groupoid, groupoid, duality, coaction, crossed product.

1. Introduction and preliminaries

Actions of quantum groupoids that simultaneously generalize actions of quantum groups and actions of groupoids have been studied in various settings, including that of weak Hopf algebras or finite quantum groupoids [24], [25], Hopf algebroids or algebraic quantum groupoids [7], [13], and Hopf-von Neumann bimodules or measured quantum groupoids [10], [11], [30]. In this article, we introduce and investigate coactions of Hopf C^*-bimodules or reduced locally compact quantum groupoids within the framework developed in [27], [29].

In the first part of this article, we construct reduced crossed products and dual coactions, and show that the bidual of a fine coaction is Morita equivalent to the initial coaction. These constructions apply to pairs of Hopf C^*-bimodules that appear as the left and the right leg of a (weak) C^*-pseudo-Kac system, which consists of a C^*-pseudo-multiplicative unitary [29] and an additional symmetry. We associate such a C^*-pseudo-Kac system to every groupoid and to every compact C^*-quantum groupoid and expect that the same can be done for every reduced locally compact quantum groupoid once this concept has been defined properly. The constructions in this part generalize corresponding constructions of Baaj and Skandalis [3] for coactions of Hopf C^*-algebras.

Coactions of the Hopf C^*-bimodules associated to a locally compact Hausdorff groupoid — the function algebra on one side and the reduced groupoid algebra on the other — are studied in detail in the second part of this article. We show that actions of the groupoid on continuous bundles of C^*-algebras correspond to coactions of the first Hopf C^*-bimodule, and that continuous Fell bundles on G naturally yield coactions of the second Hopf C^*-bimodule. Generalizing results of Quigg [22] and Baaj and Skandalis [2] from groups to groupoids, we show that if the groupoid is étale, every coaction of the reduced groupoid algebra arises from a Fell bundle.

This article is organized as follows. The first part is concerned with coactions of Hopf C^*-bimodules and associated reduced crossed products.

Section 2 summarizes the relative tensor product of C^*-modules and the fiber product of C^*-algebras over C^*-bases [27] which are fundamental to everything that follows, and introduces coactions of Hopf C^*-bimodules.

Section 3 is concerned with C^*-pseudo-Kac systems. Every C^*-pseudo-Kac system gives rise to two Hopf C^*-bimodules, called the legs of the system, which are dual to each other in a suitable sense. Coactions of these legs on C^*-algebras, associated reduced crossed products, dual coactions and a duality theorem concerning iterated crossed products are discussed in Section 4.

Section 5 gives the construction of the C^*-pseudo-Kac system of a locally compact Hausdorff groupoid G. The associated Hopf C^*-bimodules are the function algebra on one side and the reduced groupoid C^*-algebra of G on the other side. The second part of the article relates coactions of these Hopf C^*-bimodules to well-known notions.

Section 6 shows that actions of a groupoid G on continuous bundles of C^*-algebras correspond to certain fine coactions of the function algebra of G.

Section 7 contains preliminaries on Fell bundles, their morphisms and multipliers.

Section 8 shows that continuous Fell bundles on G give rise to coactions of the reduced groupoid C^*-algebra of G, and section 9 gives a reverse construction that associates to every sufficiently nice coaction of the groupoid algebra a Fell bundle provided that the groupoid G is étale.

We use the following notation. Given a subset Y of a normed space X, we denote by $[Y] \subset X$ the closed linear span of Y. All sesquilinear maps like inner products of Hilbert spaces are assumed to be conjugate-linear in the first component and linear in the second one. Given a Hilbert space H, we use the ket-bra notation and define for each $\xi \in H$ operators $|\xi\rangle\colon \mathbb{C} \to H$, $\lambda \mapsto \lambda\xi$, and $\langle\xi| = |\xi\rangle^*\colon H \to \mathbb{C}$, $\xi' \mapsto \langle\xi|\xi'\rangle$. Given a C^*-algebra A and a subspace $B \subset A$, we denote by $A \cap B'$ the relative commutant $\{a \in A \mid [a, B] = 0\}$.

We shall make extensive use of (right) Hilbert C^*-modules; see [16]. In particular, we use the internal tensor product and the KSGNS-construction. Let E be a Hilbert C^*-module over a C^*-algebra A, let F be a Hilbert C^*-module over a C^*-algebra B, and let $\phi\colon A \to \mathcal{L}(F)$ be a completely positive map. We denote by $E \ominus_\phi F$ the Hilbert C^*-module over B which is the closed linear span of elements $\eta \ominus_\phi \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \eta \ominus_\phi \xi|\eta' \ominus_\phi \xi'\rangle = \langle\xi|\phi(\langle\eta|\eta'\rangle)\xi'\rangle$ and $(\eta \ominus_\phi \xi)b = \eta \ominus_\phi \xi b$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. If ϕ is a $*$-homomorphism, this is the usual internal tensor product; if $F = B$, this is the KSGNS-construction. If $S \in \mathcal{L}(E)$ and $T \in \mathcal{L}(F) \cap \phi(A)'$, then there exists a unique operator $S \ominus_\phi T \in \mathcal{L}(E \ominus_\phi E)$ such that $(S \ominus_\phi T)(\eta \ominus_\phi \xi) = S\eta \ominus_\phi T\xi$ for all $\eta \in E$, $\xi \in F$; see Proposition 1.34 in [9]. We sloppily write "$\ominus_A$" or "$\ominus$" instead of "$\ominus_\phi$" if no confusion may arise. We also define a flipped product $F {}_\phi\!\oslash E$ as follows. We equip the algebraic tensor product $F \odot E$ with the structure maps $\langle \xi \odot \eta|\xi' \odot \eta'\rangle := \langle\xi|\phi(\langle\eta|\eta'\rangle)\xi'\rangle$, $(\xi \odot \eta)b := \xi b \odot \eta$, form the separated completion, and obtain a Hilbert C^*-module $F {}_\phi\!\oslash E$ over B which is the closed linear span of elements $\xi {}_\phi\!\oslash \eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \xi {}_\phi\!\oslash \eta|\xi' {}_\phi\!\oslash \eta'\rangle = \langle\xi|\phi(\langle\eta|\eta'\rangle)\xi'\rangle$ and $(\xi {}_\phi\!\oslash \eta)b = \xi b {}_\phi\!\oslash \eta$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. Again, we sloppily write "${}_A\!\oslash$" or "$\oslash$" instead of "${}_\phi\!\oslash$" if no confusion may arise. Evidently, there exists a unitary $\Sigma\colon F \ominus E \xrightarrow{\cong} E \oslash F$, $\eta \ominus \xi \mapsto \xi \oslash \eta$.

2. Hopf C^*-bimodules and coactions

A groupoid differs from a group in the non-triviality of its unit space. In almost every approach to quantum groupoids, the unit space is replaced by a nontrivial algebra, and a relative tensor product of modules and a fiber product of algebras over that algebra become fundamentally important. We shall use the corresponding constructions for C^*-algebras introduced in [27] and briefly summarize the main definitions and results below. For additional details and motivation, see [27], [29].

2.1. **The relative tensor product.** A C^**-base* is a triple $(\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ consisting of a Hilbert space $\mathfrak{K}$ and two commuting nondegenerate C^*-algebras $\mathfrak{B}, \mathfrak{B}^\dagger \subseteq \mathcal{L}(\mathfrak{K})$. It should be thought of as a C^*-algebraic counterpart to pairs consisting of a von Neumann algebra and its commutant. Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. Its *opposite* is the C^*-base $\mathfrak{b}^\dagger := (\mathfrak{K}, \mathfrak{B}^\dagger, \mathfrak{B})$.

A C^*-$\mathfrak{b}$*-module* is a pair $H_\alpha = (H, \alpha)$, where H is a Hilbert space and $\alpha \subseteq \mathcal{L}(\mathfrak{K}, H)$ is a closed subspace satisfying $[\alpha\mathfrak{K}] = H$, $[\alpha\mathfrak{B}] = \alpha$, and $[\alpha^*\alpha] = \mathfrak{B} \subseteq \mathcal{L}(\mathfrak{K})$. If H_α is a C^*-$\mathfrak{b}$-module, then α is a Hilbert C^*-module over B with inner product $(\xi, \xi') \mapsto \xi^*\xi'$ and there exist isomorphisms

$$\alpha \otimes \mathfrak{K} \to H,\ \xi \otimes \zeta \mapsto \xi\zeta, \qquad \mathfrak{K} \otimes \alpha \to H,\ \zeta \otimes \xi \mapsto \xi\zeta, \tag{2.1}$$

and a nondegenerate representation

$$\rho_\alpha \colon \mathfrak{B}^\dagger \to \mathcal{L}(H), \quad \rho_\alpha(b^\dagger)(\xi\zeta) = \xi b^\dagger\zeta \quad \text{for all } b^\dagger \in \mathfrak{B}^\dagger, \xi \in \alpha, \zeta \in \mathfrak{K}.$$

A *semi-morphism* between C^*-$\mathfrak{b}$-modules H_α and K_β is an operator $T \in \mathcal{L}(H, K)$ satisfying $T\alpha \subseteq \beta$. If additionally $T^*\beta \subseteq \alpha$, we call T a *morphism.* We denote the set of all (semi-)morphisms by $\mathcal{L}_{(s)}(H_\alpha, K_\beta)$. If $T \in \mathcal{L}_s(H_\alpha, K_\beta)$, then $T\rho_\alpha(b^\dagger) = \rho_\beta(b^\dagger)T$ for all $b^\dagger \in \mathfrak{B}^\dagger$, and if additionally $T \in \mathcal{L}(H_\alpha, K_\beta)$, then left multiplication by T defines an operator in $\mathcal{L}(\alpha, \beta)$ which we again denote by T.

We shall use the following notion of C^*-bi- and C^*-n-modules. Let $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ be C^*-bases, where $\mathfrak{b}_i = (\mathfrak{K}_i, \mathfrak{B}_i, \mathfrak{B}_i^\dagger)$ for each i. A C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$*-module* is a tuple $(H, \alpha_1, \ldots, \alpha_n)$, where H is a Hilbert space and (H, α_i) is a C^*-$\mathfrak{b}_i$-module for each i such that $[\rho_{\alpha_i}(\mathfrak{B}_i^\dagger)\alpha_j] = \alpha_j$ whenever $i \neq j$. In the case $n = 2$, we abbreviate ${}_\alpha H_\beta := (H, \alpha, \beta)$. If $(H, \alpha_1, \ldots, \alpha_n)$ is a C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-module, then $[\rho_{\alpha_i}(\mathfrak{B}_i^\dagger), \rho_{\alpha_j}(\mathfrak{B}_j^\dagger)] = 0$ whenever $i \neq j$. The set of *(semi-)morphisms* between C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-modules $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_n)$ and $\mathcal{K} = (K, \beta_1, \ldots, \beta_n)$ is $\mathcal{L}_{(s)}(\mathcal{H}, \mathcal{K}) := \bigcap_{i=1}^n \mathcal{L}_{(s)}(H_{\alpha_i}, K_{\beta_i}) \subseteq \mathcal{L}(H, K)$.

Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base, H_β C^*-$\mathfrak{b}$-module, and K_γ a C^*-$\mathfrak{b}^\dagger$-module. The *relative tensor product* of H_β and K_γ is the Hilbert space

$$H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K := \beta \otimes \mathfrak{K} \otimes \gamma.$$

It is spanned by elements $\xi \otimes \zeta \otimes \eta$, where $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, and the inner product is given by $\langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta' \rangle = \langle \zeta | \xi^*\xi'\eta^*\eta'\zeta' \rangle = \langle \zeta | \eta^*\eta'\xi^*\xi'\zeta' \rangle$ for all $\xi, \xi' \in \beta$, $\zeta, \zeta' \in \mathfrak{K}$, $\eta, \eta' \in \gamma$. Obviously, there exists a unitary *flip*

$$\Sigma \colon H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K \to K {}_\gamma\underset{\mathfrak{b}^\dagger}{\otimes}{}_\beta H, \quad \xi \otimes \zeta \otimes \eta \mapsto \eta \otimes \zeta \otimes \xi.$$

Using the unitaries in (2.1) on H_β and K_γ, respectively, we shall make the following identifications without further notice:

$$H {}_{\rho_\beta}\!\otimes \gamma \cong H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K \cong \beta \otimes_{\rho_\gamma} K, \quad \xi\zeta \otimes \eta \equiv \xi \otimes \zeta \otimes \eta \equiv \xi \otimes \eta\zeta.$$

For all $S \in \rho_\beta(\mathfrak{B}^\dagger)'$ and $T \in \rho_\gamma(\mathfrak{B})'$, we have operators

$$S \otimes \mathrm{id} \in \mathcal{L}(H {}_{\rho_\beta}\!\otimes \gamma) = \mathcal{L}(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K), \qquad \mathrm{id} \otimes T \in \mathcal{L}(\beta \otimes_{\rho_\gamma} K) = \mathcal{L}(H {}_\beta\underset{\mathfrak{b}}{\otimes}{}_\gamma K).$$

If $S \in \mathcal{L}_s(H_\beta)$ or $T \in \mathcal{L}_s(K_\gamma)$, then $(S \otimes \mathrm{id})(\xi \otimes \eta\zeta) = S\xi \otimes \eta\zeta$ or $(\mathrm{id} \otimes T)(\xi\zeta \otimes \eta) = \xi\zeta \otimes T\eta$, respectively, for all $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$, so that we can define

$$S \underset{\mathfrak{b}}{\otimes} T := (S \otimes \mathrm{id})(\mathrm{id} \otimes T) = (\mathrm{id} \otimes T)(S \otimes \mathrm{id}) \in \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$$
$$\text{for all } (S,T) \in \big(\mathcal{L}_s(H_\beta) \times \rho_\gamma(\mathfrak{B})'\big) \cup \big(\rho_\beta(\mathfrak{B}^\dagger)' \times \mathcal{L}_s(K_\gamma)\big).$$

For each $\xi \in \beta$ and $\eta \in \gamma$, there exist bounded linear operators

$$|\xi\rangle_1 \colon K \to H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K,\ \omega \mapsto \xi \otimes \omega, \qquad |\eta\rangle_2 \colon H \to H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K,\ \omega \mapsto \omega \otimes \eta,$$

whose adjoints $\langle\xi|_1 := |\xi\rangle_1^*$ and $\langle\eta|_2 := |\eta\rangle_2^*$ are given by

$$\langle\xi|_1 \colon \xi' \otimes \omega \mapsto \rho_\gamma(\xi^*\xi')\omega, \qquad \langle\eta|_2 \colon \omega \otimes \eta' \mapsto \rho_\beta(\eta^*\eta')\omega.$$

We write $|\beta\rangle_1 := \{|\xi\rangle_1 \mid \xi \in \beta\} \subseteq \mathcal{L}(K, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$ and similarly define $\langle\beta|_1$, $|\gamma\rangle_2$, and $\langle\gamma|_2$.

Let $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_m, \beta)$ be a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{b})$-module and $\mathcal{K} = (K, \gamma, \delta_1, \ldots, \delta_n)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, where $\mathfrak{a}_i = (\mathfrak{H}_i, \mathfrak{A}_i, \mathfrak{A}_i^\dagger)$ and $\mathfrak{c}_j = (\mathfrak{L}_j, \mathfrak{C}_j, \mathfrak{C}_j^\dagger)$ are C^*-bases for all i, j. We define

$$\alpha_i \triangleleft \gamma := [|\gamma\rangle_2 \alpha_i] \subseteq \mathcal{L}(\mathfrak{H}_i, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K), \qquad \beta \triangleright \delta_j := [|\beta\rangle_1 \delta_j] \subseteq \mathcal{L}(\mathfrak{L}_j, H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)$$

for all i, j. Then $(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K, \alpha_1 \triangleleft \gamma, \ldots, \alpha_m \triangleleft \gamma, \beta \triangleright \delta_1, \ldots, \beta \triangleright \delta_n)$ is a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, called the *relative tensor product* of $\mathcal{H}$ and $\mathcal{K}$ and denoted by $\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}$. For all i, j and $a^\dagger \in \mathfrak{A}_i^\dagger$, $c^\dagger \in \mathfrak{C}_j^\dagger$,

$$\rho_{(\alpha_i \triangleleft \gamma)}(a^\dagger) = \rho_{\alpha_i}(a^\dagger) \underset{\mathfrak{b}}{\otimes} \mathrm{id}, \qquad \rho_{(\beta \triangleright \delta_j)}(c^\dagger) = \mathrm{id} \underset{\mathfrak{b}}{\otimes} \rho_{\delta_j}(c^\dagger).$$

The relative tensor product is functorial in the following sense. Let $\tilde{\mathcal{H}} = (\tilde{H}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, \tilde{\beta})$ be a C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{b})$-module, let $\tilde{\mathcal{K}} = (\tilde{K}, \tilde{\gamma}, \tilde{\delta}_1, \ldots, \tilde{\delta}_n)$ be a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$-module, and let $S \in \mathcal{L}_{(s)}(\mathcal{H}, \tilde{\mathcal{H}})$, $T \in \mathcal{L}_{(s)}(\mathcal{K}, \tilde{\mathcal{K}})$. Then there exists a unique operator $S \underset{\mathfrak{b}}{\otimes} T \in \mathcal{L}_{(s)}(\mathcal{H} \underset{\mathfrak{b}}{\otimes} \mathcal{K}, \tilde{\mathcal{H}} \underset{\mathfrak{b}}{\otimes} \tilde{\mathcal{K}})$ satisfying $(S \underset{\mathfrak{b}}{\otimes} T)(\xi \otimes \zeta \otimes \eta) = S\xi \otimes \zeta \otimes T\eta$ for all $\xi \in \beta$, $\zeta \in \mathfrak{K}$, $\eta \in \gamma$.

Finally, the relative tensor product is associative in the following sense. Let $\mathfrak{d}, \mathfrak{e}_1, \ldots, \mathfrak{e}_l$ be C^*-bases, $\hat{\mathcal{K}} = (K, \gamma, \delta_1, \ldots, \delta_n, \epsilon)$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c}_1, \ldots, \mathfrak{c}_n, \mathfrak{d})$-module and $\mathcal{L} = (L, \phi, \psi_1, \ldots, \psi_l)$ a C^*-$(\mathfrak{d}^\dagger, \mathfrak{e}_1, \ldots, \mathfrak{e}_l)$-module. Then there exists a canonical isomorphism

$$a_{\mathcal{H},\mathcal{K},\mathcal{L}} \colon (H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K)_{\beta \triangleright \epsilon} \underset{\mathfrak{d}}{\otimes}{}_\phi L \to \beta \otimes_{\rho_\gamma} K_{\rho_\epsilon} \otimes \phi \to H_\beta \underset{\mathfrak{b}}{\otimes}{}_{\gamma \triangleleft \phi} (K_\epsilon \underset{\mathfrak{d}}{\otimes}{}_\phi L) \tag{2.2}$$

which is an isomorphism of the C^*-$(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{c}_1, \ldots, \mathfrak{c}_n, \mathfrak{e}_1, \ldots, \mathfrak{e}_l)$-modules $(\mathcal{H} \underset{\mathfrak{b}}{\otimes} \hat{\mathcal{K}}) \underset{\mathfrak{d}}{\otimes} \mathcal{L}$ and $\mathcal{H} \underset{\mathfrak{b}}{\otimes} (\hat{\mathcal{K}} \underset{\mathfrak{d}}{\otimes} \mathcal{L})$. From now on, we identify the Hilbert spaces in (2.2) and denote them by $H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K_\epsilon \underset{\mathfrak{d}}{\otimes}{}_\phi L$.

2.2. **The fiber product of C^*-algebras.** Let $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ be C^*-bases, where $\mathfrak{b}_i = (\mathfrak{K}_i, \mathfrak{B}_i, \mathfrak{B}_i^\dagger)$ for each i. A *(nondegenerate) C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-algebra* consists of a C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-module $(H, \alpha_1, \ldots, \alpha_n)$ and a (nondegenerate) C^*-algebra $A \subseteq \mathcal{L}(H)$ such that $\rho_{\alpha_i}(\mathfrak{B}_i^\dagger)A$ is contained in A for each i. We shall only be interested in the cases $n = 1, 2$, where we abbreviate $A_H^\alpha := (H_\alpha, A)$, $A_H^{\alpha,\beta} := ({}_\alpha H_\beta, A)$. Given a C^*-$(b_1, \ldots, b_n)$-algebra $\mathcal{A} = ((H, \alpha_1, \ldots, \alpha_n), A)$, we identify $M(A)$ with a C^*-subalgebra of $\mathcal{L}([AH]) \subseteq \mathcal{L}(H)$ and obtain C^*-$(b_1, \ldots, b_n)$-algebra $M(\mathcal{A}) = ((H, \alpha_1, \ldots, \alpha_n), M(A))$.

We need several notions of a morphism. Let $\mathcal{A} = (\mathcal{H}, A)$ and $\mathcal{C} = (\mathcal{K}, C)$ be C^*-$(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$-algebras, where $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_n)$ and $\mathcal{K} = (K, \gamma_1, \ldots, \gamma_n)$. A $*$-homomorphism $\pi\colon A \to C$ is called a *jointly (semi-)normal morphism* or briefly *(semi-)morphism* from $\mathcal{A}$ to $\mathcal{C}$ if $[\mathcal{L}^\pi_{(s)}(\mathcal{H}, \mathcal{K})\alpha_i] = \gamma_i$ for each i, where

$$\mathcal{L}^\pi_{(s)}(\mathcal{H}, \mathcal{K}) = \{T \in \mathcal{L}_{(s)}(\mathcal{H}, \mathcal{K}) \mid Ta = \pi(a)T \text{ for all } a \in A\}.$$

One easily verifies that every (semi-)morphism π between C^*-$\mathfrak{b}$-algebras A_H^α and C_K^γ satisfies $\pi(\rho_\alpha(b^\dagger)) = \rho_\gamma(b^\dagger)$ for all $b^\dagger \in \mathfrak{B}^\dagger$.

We construct a fiber product of C^*-algebras over C^*-bases as follows. Given Hilbert spaces H, K, a closed subspace $E \subseteq \mathcal{L}(H, K)$, and a C^*-algebra $A \subseteq \mathcal{L}(H)$, we define a C^*-algebra

$$\operatorname{Ind}_E(A) := \{T \in \mathcal{L}(K) \mid TE \subseteq [EA] \text{ and } T^*E \subseteq [EA]\} \subseteq \mathcal{L}(K).$$

Let $\mathfrak{b}$ be a C^*-base, A_H^β a C^*-$\mathfrak{b}$-algebra, and B_K^γ a C^*-$\mathfrak{b}^\dagger$-algebra. The *fiber product* of A_H^β and B_K^γ is the C^*-algebra

$$A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B := \operatorname{Ind}_{|\beta\rangle_1}(B) \cap \operatorname{Ind}_{|\gamma\rangle_2}(A) \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K).$$

To define coactions, we also need to consider the C^*-algebra

$$A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B := \operatorname{Ind}_{[|\beta\rangle_1 B]}(B) \cap \operatorname{Ind}_{|\gamma\rangle_2}(A) \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K),$$

which evidently contains $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$. If A and B are unital, so is $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$, but otherwise, $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B$ and $A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B$ may be degenerate. Clearly, conjugation by the flip $\Sigma\colon H_\beta \underset{\mathfrak{b}}{\otimes}{}_\gamma K \to K_\gamma \underset{\mathfrak{b}^\dagger}{\otimes}{}_\beta H$ yields an isomorphism

$$\operatorname{Ad}_\Sigma\colon A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \to B_\gamma \underset{\mathfrak{b}^\dagger}{*}{}_\beta A.$$

If $\mathfrak{a}, \mathfrak{c}$ are C^*-bases, $A_H^{\alpha,\beta}$ is a C^*-$(\mathfrak{a}, \mathfrak{b})$-algebra and $B_K^{\gamma,\delta}$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{c})$-algebra, then

$$A_H^{\alpha,\beta} \underset{\mathfrak{b}}{*} B_K^{\gamma,\delta} := ({}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta,\ A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B)$$

is a C^*-$(\mathfrak{a}, \mathfrak{c})$-algebra, called the *fiber product* of $A_H^{\alpha,\beta}$ and $B_K^{\gamma,\delta}$; see Proposition 3.15 in [27]. Likewise, $({}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta,\ A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B)$ is a C^*-$(\mathfrak{a}, \mathfrak{c})$-algebra.

The fiber product need not be associative, but in this article, it will only appear as the target of a comultiplication or coaction whose coassociativity will compensate the non-associativity of the fiber product.

More importantly, the fiber product is functorial in the following sense. Let ϕ be a (semi-)morphism of C^*-$(\mathfrak{a}, \mathfrak{b})$-algebras $\mathcal{A} = A_H^{\alpha,\beta}$ and $\mathcal{C} = C_L^{\kappa,\lambda}$, and ψ a (semi-)morphism of C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-algebras $\mathcal{B} = B_K^{\gamma,\delta}$ and $\mathcal{D} = D_M^{\mu,\nu}$. Then there exists a unique (semi-)morphism of C^*-$(\mathfrak{a},\mathfrak{c})$-algebras $\phi * \psi$ from $({}_\alpha H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta,\ A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B)$ to $({}_\kappa L_\lambda \underset{\mathfrak{b}}{\otimes} {}_\mu M_\nu,\ C_\lambda \underset{\mathfrak{b}}{*}{}^\mu D)$ such that

$$(\phi * \psi)(x)R = Rx \quad \text{for all } x \in A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \text{ and } R \in I_M J_H + J_L I_K,$$

where $I_X = \mathcal{L}^\phi(H,L) \underset{\mathfrak{b}}{\otimes} \mathrm{id}_X$, $J_Y = \mathrm{id}_Y \underset{\mathfrak{b}}{\otimes} \mathcal{L}^\psi(K,M)$ for $X \in \{K,M\}, Y \in \{H,L\}$, and $\phi * \psi$ restricts to a (semi-)morphism from $A_H^{\alpha,\beta} \underset{\mathfrak{b}}{*} B_K^{\gamma,\delta}$ to $C_L^{\kappa,\lambda} \underset{\mathfrak{b}}{*} D_M^{\mu,\nu}$; see Theorem 3.19 in [27]. The proof uses the following result, which essentially is Lemma 3.18 in [27].

Lemma 2.1. *Let $\mathfrak{c}$ be a C^*-base, π a semi-morphism of C^*-$\mathfrak{b}$-algebras A_H^β, C_L^λ, and ${}_\gamma K_\delta$ a C^*-$(\mathfrak{b}^\dagger,\mathfrak{c})$-module. Let $I := \mathcal{L}_s^\pi(H_\beta, L_\lambda) \underset{\mathfrak{b}}{\otimes} \mathrm{id} \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K, L_\lambda \underset{\mathfrak{b}}{\otimes} {}_\gamma K)$ and*

$$X := (I^*I)' \subseteq \mathcal{L}(H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K), \qquad Y := (II^*)' \subseteq \mathcal{L}(L_\lambda \underset{\mathfrak{b}}{\otimes} {}_\gamma K).$$

(i) *$\mathcal{X} := (H_\beta \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta, X)$ and $\mathcal{Y} := (L_\lambda \underset{\mathfrak{b}}{\otimes} {}_\gamma K_\delta, Y)$ are C^*-$\mathfrak{c}$-algebras.*

(ii) *There exists a semi-morphism $\mathrm{Ind}_{|\gamma\rangle_2}(\pi) \colon \mathcal{X} \to \mathcal{Y}$ such that $(\mathrm{Ind}_{|\gamma\rangle_2}(\pi))(x)z = zx$ for all $x \in X$ and $z \in I$.*

(iii) *If B_K^γ is a C^*-$\mathfrak{b}^\dagger$-algebra, then $A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B \subseteq A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B \subseteq X$, $(\mathrm{Ind}_{|\gamma\rangle_2}(\pi))(A_\beta \underset{\mathfrak{b}}{*}{}_\gamma B) \subseteq C_\lambda \underset{\mathfrak{b}}{*}{}_\gamma B$ and $(\mathrm{Ind}_{|\gamma\rangle_2}(\pi))(A_\beta \underset{\mathfrak{b}}{*}{}^\gamma B) \subseteq C_\lambda \underset{\mathfrak{b}}{*}{}^\gamma B$.*

(iv) *$[|\gamma\rangle_2 A \langle\gamma|_2] \subseteq X$ and $(\mathrm{Ind}_{|\gamma\rangle_2}(\pi))([|\gamma\rangle_2 A\langle\gamma|_2]) = [|\gamma\rangle_2 \pi(A)\langle\gamma|_2]$.*

2.3. **Hopf C^*-bimodules and coactions.** The notion of a Hopf C^*-bimodule was introduced in [29].

Definition 2.2. *Let $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be a C^*-base. A* Hopf C^*-bimodule over $\mathfrak{b}$ *is a C^*-$(\mathfrak{b}^\dagger,\mathfrak{b})$-algebra $A_H^{\beta,\alpha}$ with a morphism Δ from $A_H^{\beta,\alpha}$ to $A_H^{\beta,\alpha} \underset{\mathfrak{b}}{*} A_H^{\beta,\alpha}$ satisfying $(\delta * \mathrm{id}) \circ \delta = (\mathrm{id} * \Delta) \circ \delta$ as maps from A to $\mathcal{L}(H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H)$.*

Let $(\mathcal{A}, \Delta)$ be a Hopf C^*-bimodule, where $\mathcal{A} = A_H^{\beta,\alpha}$.

A coaction *of $(\mathcal{A},\Delta)$ consists of a C^*-$\mathfrak{b}$-algebra C_K^γ and a semi-morphism δ from (K_γ, C) to $(K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H_\alpha, C_\gamma \underset{\mathfrak{b}}{*}{}^\beta A)$ such that $(\delta * \mathrm{id}) \circ \delta = (\mathrm{id} * \Delta) \circ \delta$ as maps from C to $\mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes} {}_\beta H)$. We call such a coaction (C_K^γ, δ)*

(i) left-full *if $[\delta(C)|\gamma\rangle_1 A] = [|\gamma\rangle_1 A]$, and* right-full *if $[\delta(C)|\beta\rangle_2] = [|\beta\rangle_2 C]$;*

(ii) fine *if δ is injective, a morphism, and right-full, and if $[\rho_\gamma(\mathfrak{B}^\dagger)C] = C$;*

(iii) very fine *if it is fine and if $\delta^{-1} \colon \delta(C) \to C$ is a morphism of C^*-$\mathfrak{b}$-algebras from $(K_\gamma \underset{\mathfrak{b}}{\otimes} {}_\beta H_\alpha, \delta(C))$ to (K_γ, C).*

A morphism *between coactions (C_K^γ, δ_C) and (D_L^ϵ, δ_D) is a semi-morphism ρ from C_K^γ to $M(D)_L^\epsilon$ satisfying $[\rho(C)D] = D$ and $\delta_D(d) \cdot (\rho * \mathrm{id})(\delta_C(c)) = \delta_D(d\rho(c))$ for all $d \in D$, $c \in C$. We denote the category of all coactions of $(\mathcal{A},\Delta)$ by $\mathbf{Coact}_{(\mathcal{A},\Delta)}$.*

Examples of Hopf C^*-bimodules and coactions will be discussed in detail in Sections 5, 6, and 8.

3. Weak C^*-pseudo-Kac systems

To form a reduced crossed product for a coaction of a Hopf C^*-bimodule $(\mathcal{A}, \Delta)$ and to equip this reduced crossed product with a dual coaction, one needs a second Hopf C^*-bimodule $(\widehat{\mathcal{A}}, \widehat{\Delta})$ that is dual to $(\mathcal{A}, \Delta)$ in a suitable sense. We shall see that a good notion of duality is that $(\mathcal{A}, \Delta)$ and $(\widehat{\mathcal{A}}, \widehat{\Delta})$ are the legs of a weak C^*-pseudo-Kac system, which is a generalization of the balanced multiplicative unitaries and Kac systems introduced by Baaj and Skandalis [1, 3].

3.1. **C^*-pseudo-multiplicative unitaries.** A weak C^*-pseudo-Kac system is a well-behaved C^*-pseudo-multiplicative unitary V together with a symmetry U satisfying a number of axioms. Before we state these axioms, we recall the notion of a C^*-pseudo-multiplicative unitary and the construction of the associated Hopf C^*-bimodules from [29].

Let $\mathfrak{b}$ be a C^*-base. A *C^*-pseudo-multiplicative unitary* over $\mathfrak{b}$ consists of a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module $(H, \widehat{\beta}, \alpha, \beta)$ and a unitary $V\colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H \to H_{\alpha} \underset{\mathfrak{b}}{\otimes}{}_{\beta} H$ such that

$$V(\alpha \triangleleft \alpha) = \alpha \triangleright \alpha, \quad V(\widehat{\beta} \triangleright \beta) = \widehat{\beta} \triangleleft \beta, \quad V(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}, \quad V(\beta \triangleleft \alpha) = \beta \triangleleft \beta \tag{3.1}$$

in $\mathcal{L}(\mathfrak{K}, H_{\alpha} \underset{\mathfrak{b}}{\otimes}{}_{\beta} H)$ and $V_{12}V_{13}V_{23} = V_{23}V_{12}$ in the sense that the following diagram

$$\begin{tikzcd}
H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H \arrow[r, "V_{12}"] \arrow[d, "V_{23}"] & H_{\alpha} \underset{\mathfrak{b}}{\otimes}{}_{\beta} H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H \arrow[r, "V_{23}"] & H_{\alpha} \underset{\mathfrak{b}}{\otimes}{}_{\beta} H_{\alpha} \underset{\mathfrak{b}}{\otimes}{}_{\beta} H, \\
H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha \triangleright \alpha} (H_{\alpha} \underset{\mathfrak{b}}{\otimes}{}_{\beta} H) \arrow[rr, "V_{13}"] & & (H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H)_{\alpha \triangleleft \alpha} \underset{\mathfrak{b}}{\otimes}{}_{\beta} H \arrow[u, "V_{12}"]
\end{tikzcd} \tag{3.2}$$

commutes, where V_{ij} is the leg notation for the operator that acts like V on the ith and jth factor in the relative tensor product; see [29].

Let V be a C^*-pseudo-multiplicative unitary as above, let

$$\widehat{A} = \widehat{A}_V = [\langle \beta|_2 V |\alpha\rangle_2] \subseteq \mathcal{L}(H), \quad \widehat{\Delta} = \widehat{\Delta}_V\colon \widehat{A} \to \mathcal{L}(H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H), \ \hat{a} \mapsto V^*(1 \underset{\mathfrak{b}}{\otimes} \hat{a})V,$$

$$A = A_V = [\langle \alpha|_1 V |\widehat{\beta}\rangle_1] \subseteq \mathcal{L}(H), \quad \Delta = \Delta_V\colon A \to \mathcal{L}(H_{\alpha} \underset{\mathfrak{b}}{\otimes}{}_{\beta} H), \ a \mapsto V(a \underset{\mathfrak{b}^\dagger}{\otimes} 1)V^*,$$

and let $\widehat{\mathcal{A}} = \widehat{A}_H^{\alpha,\widehat{\beta}}$ and $\mathcal{A} = A_H^{\beta,\alpha}$. We call V *well-behaved* if $(\widehat{\mathcal{A}}, \widehat{\Delta})$ and $(\mathcal{A}, \Delta)$ are Hopf C^*-bimodules. This happens for example if V is *regular* in the sense that $[\langle \alpha|_1 V |\alpha\rangle_2] = [\alpha\alpha^*] \subseteq \mathcal{L}(H)$; see Theorem 4.5 in [29].

The *opposite* of V is the C^*-pseudo-multiplicative unitary

$$V^{op} := \Sigma V^* \Sigma\colon H_{\beta} \underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H \xrightarrow{\Sigma} H_{\alpha} \underset{\mathfrak{b}}{\otimes}{}_{\beta} H \xrightarrow{V^*} H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha} H \xrightarrow{\Sigma} H_{\alpha} \underset{\mathfrak{b}}{\otimes}{}_{\widehat{\beta}} H.$$

If V is well-behaved or regular, then the same is true for V^{op}, and then

$$\widehat{A}_{V^{op}} = A_V, \qquad \widehat{\Delta}_{V^{op}} = \mathrm{Ad}_\Sigma \circ \Delta_V, \qquad A_{V^{op}} = \widehat{A}_V, \qquad \Delta_{V^{op}} = \mathrm{Ad}_\Sigma \circ \widehat{\Delta}_V. \tag{3.3}$$

Let $(H, \widehat{\alpha}, \widehat{\beta}, \alpha, \beta)$ be a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module and $U \in \mathcal{L}({}_{\widehat{\alpha}}H_{\widehat{\beta}}, {}_{\alpha}H_{\beta})$ a symmetry, that is, $U = U^* = U^{-1}$. Then $U\widehat{\alpha} = \alpha$, $U\widehat{\beta} = \beta$, and the diagram

$$
\begin{tikzcd}
H_{\widehat{\alpha}} \underset{\mathfrak{b}}{\otimes} {}_{\widehat{\beta}}H \arrow[rr, leftrightarrow, "(1 \underset{\mathfrak{b}^{(\dagger)}}{\otimes} U)\Sigma"] \arrow[d, leftrightarrow, "(U \underset{\mathfrak{b}^{(\dagger)}}{\otimes} 1)\Sigma"'] \arrow[rrd, leftrightarrow] & & H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{\alpha}H \arrow[d, leftrightarrow, "(1 \underset{\mathfrak{b}^{(\dagger)}}{\otimes} U)\Sigma"] \arrow[lld, leftrightarrow] \\
H_{\beta} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{\widehat{\alpha}}H \arrow[rr, leftrightarrow, "(U \underset{\mathfrak{b}^{(\dagger)}}{\otimes} 1)\Sigma"'] & & H_{\alpha} \underset{\mathfrak{b}}{\otimes} {}_{\beta}H
\end{tikzcd}
$$

commutes, where each arrow can be read in both directions and the diagonal maps are $U \underset{\mathfrak{b}^{(\dagger)}}{\otimes} U$. We use the leg notation and write U_1 for $U \underset{\mathfrak{b}^{(\dagger)}}{\otimes} 1$ and U_2 for $1 \underset{\mathfrak{b}^{(\dagger)}}{\otimes} U$.

For each $T \in \mathcal{L}(H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{\alpha}H, H_{\alpha} \underset{\mathfrak{b}}{\otimes} {}_{\beta}H)$, let

$$
\begin{aligned}
\check{T} &:= \Sigma(1 \underset{\mathfrak{b}}{\otimes} U)T(1 \underset{\mathfrak{b}^\dagger}{\otimes} U)\Sigma \colon H_{\widehat{\alpha}} \underset{\mathfrak{b}}{\otimes} {}_{\widehat{\beta}}H \to H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{\alpha}H, \\
\widehat{T} &:= \Sigma(U \underset{\mathfrak{b}}{\otimes} 1)T(U \underset{\mathfrak{b}^\dagger}{\otimes} 1)\Sigma \colon H_{\alpha} \underset{\mathfrak{b}}{\otimes} {}_{\beta}H \to H_{\beta} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{\widehat{\alpha}}H.
\end{aligned}
$$

Switching from $(\mathfrak{b}, H, \widehat{\alpha}, \widehat{\beta}, \alpha, \beta)$ to $(\mathfrak{b}^\dagger, H, \beta, \widehat{\alpha}, \widehat{\beta}, \alpha)$ or to $(\mathfrak{b}^\dagger, H, \widehat{\beta}, \alpha, \beta, \widehat{\alpha})$, respectively, we can iterate the assignments $T \mapsto \check{T}$ and $T \mapsto \widehat{T}$, and obtain

$$
\check{\check{\check{T}}} = \widehat{T}, \qquad \check{\check{T}} = (U \underset{\mathfrak{b}}{\otimes} U)T(U \underset{\mathfrak{b}^\dagger}{\otimes} U) = \widehat{\widehat{T}}, \qquad \check{T} = \widehat{\widehat{\widehat{T}}}. \tag{3.4}
$$

Definition 3.1. *A* balanced C^*-pseudo-multiplicative unitary (V, U) *on a* C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-*module* $(H, \widehat{\alpha}, \widehat{\beta}, \alpha, \beta)$ *consists of a symmetry* $U \in \mathcal{L}({}_{\widehat{\alpha}}H_{\widehat{\beta}}, {}_{\alpha}H_{\beta})$ *and a* C^**-pseudo-multiplicative unitary* $V \colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes} {}_{\alpha}H \to H_{\alpha} \underset{\mathfrak{b}}{\otimes} {}_{\beta}H$ *such that* $\check{V}$ *and* $\widehat{V}$ *are* C^**-pseudo-multiplicative unitaries again.*

Note that in this definition, $(\check{V}, U)$ is a C^*-pseudo-multiplicative unitary if and only if $(\widehat{V}, U)$ is one because $\check{V} = (U \underset{\mathfrak{b}^\dagger}{\otimes} U)\widehat{V}(U \underset{\mathfrak{b}}{\otimes} U)$.

Let (V, U) be a balanced C^*-pseudo-multiplicative unitary as above.

Remark 3.2. (i) *One easily verifies that* $(\check{V}, U)$, $(\widehat{V}, U)$, (V^{op}, U) *are balanced* C^**-pseudo-multiplicative unitaries again. We call them the* predual, dual, *and* opposite *of* (V, U), *respectively.*

(ii) *The relations* (3.1) *for the unitaries* $\check{V}$, $\widehat{V}$ *read as follows:*

$$
\begin{aligned}
&\widehat{\beta} \triangleleft \widehat{\beta} \xrightarrow{\check{V}} \widehat{\beta} \triangleright \widehat{\beta}, \qquad \widehat{\alpha} \triangleright \alpha \xrightarrow{\check{V}} \widehat{\alpha} \triangleleft \alpha, \qquad \widehat{\alpha} \triangleright \widehat{\alpha} \xrightarrow{\check{V}} \widehat{\beta} \triangleright \widehat{\alpha}, \qquad \alpha \triangleleft \widehat{\beta} \xrightarrow{\check{V}} \alpha \triangleleft \alpha, \\
&\beta \triangleleft \beta \xrightarrow{\widehat{V}} \beta \triangleright \beta, \qquad \alpha \triangleright \widehat{\alpha} \xrightarrow{\widehat{V}} \alpha \triangleleft \widehat{\alpha}, \qquad \alpha \triangleright \alpha \xrightarrow{\widehat{V}} \beta \triangleright \alpha, \qquad \widehat{\alpha} \triangleleft \beta \xrightarrow{\widehat{V}} \widehat{\alpha} \triangleleft \widehat{\alpha},
\end{aligned}
$$

where $X \xrightarrow{W} Y$ *means* $WX = Y$. *They furthermore imply*

$$\hat\beta \triangleright \hat\alpha \xrightarrow{V} \alpha \triangleright \hat\alpha, \qquad \hat\alpha \triangleright \beta \xrightarrow{\check V} \hat\beta \triangleright \beta, \qquad \alpha \triangleright \hat\beta \xrightarrow{\hat V} \beta \triangleright \hat\beta,$$
$$\hat\alpha \triangleleft \alpha \xrightarrow{V} \hat\alpha \triangleleft \beta, \qquad \beta \triangleleft \hat\beta \xrightarrow{\check V} \beta \triangleleft \alpha, \qquad \hat\beta \triangleleft \beta \xrightarrow{\hat V} \hat\beta \triangleleft \hat\alpha.$$

(iii) *The spaces* $\hat A$ *and* A *are contained in* $\mathcal{L}(H_{\hat\alpha})$ *since* $[\hat A\hat\alpha] = [\langle\beta|_2 V|\alpha\rangle_2\hat\alpha] = [\langle\beta|_2|\beta\rangle_2\hat\alpha] = [\rho_\alpha(\mathfrak{B}^\dagger)\hat\alpha] = \hat\alpha$ *and similarly* $[A\hat\alpha] = [\langle\alpha|_1 V|\hat\beta\rangle_1\hat\alpha] = \hat\alpha$.

Lemma 3.3. $V_{13}V_{23}\check V_{12} = \check V_{12}V_{13}$ *and* $\hat V_{23}V_{12}V_{13} = V_{13}\hat V_{23}$, *that is, the diagrams*

$$\begin{tikzcd}
(H_{\hat\alpha}\underset{\mathfrak b}{\otimes}{}_{\hat\beta}H)_{\hat\beta\triangleleft\hat\beta}\underset{\mathfrak b^\dagger}{\otimes}{}_\alpha H \arrow[r, "V_{13}"] \arrow[d, "\check V_{12}"'] & (H_{\hat\alpha}\underset{\mathfrak b}{\otimes}{}_{\hat\beta}H)_{\alpha\triangleleft\hat\beta}\underset{\mathfrak b}{\otimes}{}_\beta H \arrow[r, "\check V_{12}"] & (H_{\hat\beta}\underset{\mathfrak b^\dagger}{\otimes}{}_\alpha H)_{\alpha\triangleleft\alpha}\underset{\mathfrak b}{\otimes}{}_\beta H, \\
H_{\hat\beta}\underset{\mathfrak b^\dagger}{\otimes}{}_\alpha H_{\hat\beta}\underset{\mathfrak b^\dagger}{\otimes}{}_\alpha H \arrow[rr, "V_{23}"] & & H_{\hat\beta}\underset{\mathfrak b^\dagger}{\otimes}{}_{\alpha\triangleright\alpha}(H_\alpha\underset{\mathfrak b}{\otimes}{}_\beta H) \arrow[u, "V_{13}"']
\end{tikzcd} \tag{3.5}$$

$$\begin{tikzcd}
H_{\hat\beta}\underset{\mathfrak b^\dagger}{\otimes}{}_{\alpha\triangleright\alpha}(H_\alpha\underset{\mathfrak b}{\otimes}{}_\beta H) \arrow[r, "\hat V_{23}"] \arrow[d, "V_{13}"'] & H_{\hat\beta}\underset{\mathfrak b^\dagger}{\otimes}{}_{\beta\triangleright\alpha}(H_\beta\underset{\mathfrak b^\dagger}{\otimes}{}_{\hat\alpha}H) \arrow[r, "V_{13}"] & H_\alpha\underset{\mathfrak b}{\otimes}{}_{\beta\triangleright\beta}(H_\beta\underset{\mathfrak b^\dagger}{\otimes}{}_{\hat\alpha}H) \\
(H_{\hat\beta}\underset{\mathfrak b^\dagger}{\otimes}{}_\alpha H)_{\alpha\triangleleft\alpha}\underset{\mathfrak b}{\otimes}{}_\beta H \arrow[rr, "V_{12}"] & & H_\alpha\underset{\mathfrak b}{\otimes}{}_\beta H_\alpha\underset{\mathfrak b}{\otimes}{}_\beta H \arrow[u, "\hat V_{23}"']
\end{tikzcd} \tag{3.6}$$

commute.

Proof. Let $W := \Sigma V\Sigma$. We insert the relation $\check V = U_1WU_1$ into the equation $\check V_{12}\check V_{13}\check V_{23} = \check V_{23}\check V_{12}$ and obtain $U_1W_{12}U_1 \cdot U_1W_{13}U_1 \cdot \check V_{23} = \check V_{23} \cdot U_1W_{12}U_1$ and hence $W_{12}W_{13}\check V_{23} = \check V_{23}W_{12}$. Renumbering the legs of the operators according to the permutation $(1,2,3) \mapsto (2,3,1)$, we find $V_{13}V_{23}\check V_{12} = \check V_{12}V_{13}$. A similar calculation shows that $\hat V_{23}V_{12}V_{13} = V_{13}\hat V_{23}$. □

Proposition 3.4. $\hat A_{\check V} = UA_VU$, $\hat\Delta_{\check V} = \mathrm{Ad}_{(U\underset{\mathfrak b}{\otimes}U)} \circ \Delta_V \circ \mathrm{Ad}_U$, $A_{\check V} = \hat A_V$, $\Delta_{\check V} = \hat\Delta_V$ *and* $A_{\hat V} = U\hat A_VU$, $\Delta_{\hat V} = \mathrm{Ad}_{(U\underset{\mathfrak b^\dagger}{\otimes}U)} \circ \hat\Delta_V \circ \mathrm{Ad}_U$, $\hat A_{\hat V} = A_V$, $\hat\Delta_{\hat V} = \Delta_V$.

Proof. By definition,

$$A_{\check V} = [\langle\hat\beta|_1\Sigma U_2VU_2\Sigma|\hat\alpha\rangle_1] = [\langle U\hat\beta|_2V|U\hat\alpha\rangle_2] = [\langle\beta|_2V|\alpha\rangle_2] = \hat A_V.$$

Let $\widehat{a} = \langle\xi'|_2 V|\xi\rangle_2 \in \widehat{A}_V$, where $\xi' \in \beta, \xi \in \alpha$. Then $\Delta_{\check{V}}(\widehat{a}) = \check{V}(\widehat{a}\underset{\mathfrak{b}}{\otimes}1)\check{V}^* = V^*(1\underset{\mathfrak{b}}{\otimes}\widehat{a})V = \widehat{\Delta}_V(\widehat{a})$ because the diagram

$$\begin{tikzcd}
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "\check{V}^*"'] \arrow[dd, "|\xi\rangle_3"] & H_{\widehat{\alpha}}\underset{\mathfrak{b}}{\otimes}{}_{\widehat{\beta}}H \arrow[r, "\widehat{a}\underset{\mathfrak{b}}{\otimes}1"] \arrow[d, "|\xi\rangle_3"] & H_{\widehat{\alpha}}\underset{\mathfrak{b}}{\otimes}{}_{\widehat{\beta}}H \arrow[r, "\check{V}"'] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \\
 & (H_{\widehat{\alpha}}\underset{\mathfrak{b}}{\otimes}{}_{\widehat{\beta}}H)_{\widehat{\beta}\triangleleft\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V_{13}"'] & (H_{\widehat{\alpha}}\underset{\mathfrak{b}}{\otimes}{}_{\widehat{\beta}}H)_{\alpha\triangleleft\widehat{\beta}}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[u, "\langle\xi'|_3"] \arrow[dr, "\check{V}_{12}"] & \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[ur, "\check{V}_{12}^*"] \arrow[rrr, "V_{13}V_{23}"] \arrow[dr, "V_{12}"'] & & & (H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H)_{\alpha\triangleleft\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[uu, "\langle\xi'|_3"'] \arrow[dd, "\langle\xi'|_3"] \\
 & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V_{23}"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[ur, "V_{12}^*"] \arrow[d, "\langle\xi'|_3"] & \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V"] \arrow[uu, "|\xi\rangle_3"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "1\underset{\mathfrak{b}}{\otimes}\widehat{a}"] \arrow[u, "|\xi\rangle_3"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "V^*"] & H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H
\end{tikzcd}$$

commutes. Since elements of the form like $\widehat{a}$ are dense in $\widehat{A}_V$, we can conclude $\Delta_{\check{V}} = \widehat{\Delta}_{\check{V}}$. The proof of the remaining assertions is similar. □

Corollary 3.5. *If V is well-behaved, then also $\check{V}$ and $\widehat{V}$ are well-behaved.*

Weak C^*-pseudo-Kac systems. Let (V, U) as above.

Lemma 3.6. *For each $\widehat{a} \in \widehat{A}$ and $a \in A$, we have equivalences*

$$(1 \underset{\mathfrak{b}^\dagger}{\otimes} \widehat{a})\widehat{V} = \widehat{V}(1 \underset{\mathfrak{b}}{\otimes} \widehat{a}) \quad\Leftrightarrow\quad (U\widehat{a}U \underset{\mathfrak{b}}{\otimes} 1)V = V(U\widehat{a}U \underset{\mathfrak{b}^\dagger}{\otimes} 1) \quad\Leftrightarrow\quad [U\widehat{a}U, \widehat{A}] = 0,$$

$$(a \underset{\mathfrak{b}^\dagger}{\otimes} 1)\check{V} = \check{V}(a \underset{\mathfrak{b}}{\otimes} 1) \quad\Leftrightarrow\quad (1 \underset{\mathfrak{b}}{\otimes} UaU)V = V(1 \underset{\mathfrak{b}^\dagger}{\otimes} UaU) \quad\Leftrightarrow\quad [UaU, A] = 0.$$

These equivalent conditions hold for all $\widehat{a} \in \widehat{A}$ and $a \in A$ if and only if $V_{23}\widehat{V}_{12} = \widehat{V}_{12}V_{23}$ and $\check{V}_{23}V_{12} = V_{12}\check{V}_{23}$ in the sense that the following diagrams commute:

$$\begin{tikzcd}
H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "\widehat{V}_{12}"'] \arrow[d, "V_{23}"] & H_{\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\widehat{\alpha}}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[d, "V_{23}"] \\
H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \arrow[r, "\widehat{V}_{12}"] & H_{\beta}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\widehat{\alpha}}H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H,
\end{tikzcd}
\qquad
\begin{tikzcd}
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H_{\widehat{\alpha}}\underset{\mathfrak{b}}{\otimes}{}_{\widehat{\beta}}H \arrow[r, "V_{12}"'] \arrow[d, "\check{V}_{23}"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\alpha}}\underset{\mathfrak{b}}{\otimes}{}_{\widehat{\beta}}H \arrow[d, "\check{V}_{23}"] \\
H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \arrow[r, "V_{12}"] & H_{\alpha}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H.
\end{tikzcd}$$

Proof. This is straightforward, for example, $V_{23}\widehat{V}_{12} = \widehat{V}_{12}V_{23}$ holds if and only if we have $\langle\xi'|_3 V_{23}\widehat{V}_{12}|\xi\rangle_3 = \langle\xi'|_3\widehat{V}_{12}V_{23}|\xi\rangle_3$ for all $\xi \in \alpha, \xi' \in \beta$ and hence if and only if $(1 \underset{\mathfrak{b}^\dagger}{\otimes} \widehat{a})\widehat{V} = \widehat{V}(1 \underset{\mathfrak{b}}{\otimes} \widehat{a})$ for all $\widehat{a} \in \widehat{A}$. □

Definition 3.7. *We call (V, U) a* weak C^*-pseudo-Kac system *if V is well-behaved and if the equivalent conditions in Lemma 3.6 hold, and a* C^*-pseudo-Kac-system *if $V, \check{V}, \widehat{V}$ are regular and additionally $(\Sigma(1\underset{\mathfrak{b}}{\otimes}U)V)^3 = \mathrm{id}$, where $\Sigma(1\underset{\mathfrak{b}}{\otimes}U)V\colon H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \to H_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H$.*

Remark 3.8. *In leg notation, the equation $\big(\Sigma(1 \underset{\mathfrak{b}}{\otimes} U)V\big)^3 = 1$ can be rewritten as $(\Sigma U_2 V)^3 = 1$. Conjugating by Σ or V, we see that this condition is equivalent to the relation $(U_2 V\Sigma)^3 = 1$ and to the relation $(V\Sigma U_2)^3 = 1$.*

Lemma 3.9. $(\Sigma U_2 V)^3 = 1$ *if and only if* $\widehat{V} V \check{V} = U_1 \Sigma$.

Proof. $U_1 U_2 (\Sigma U_2 V)^3 U_2 \Sigma = \Sigma U_1 V U_1 \Sigma \cdot V \cdot \Sigma U_2 V U_2 \Sigma = \widehat{V} \cdot V \cdot \check{V}$. □

Proposition 3.10. *Every C^*-pseudo-Kac system is a weak C^*-pseudo-Kac system.*

Proof. Let (V, U) be a C^*-pseudo-Kac system. Then $V, \check{V}, \widehat{V}$ are regular and therefore well-behaved. Using diagrams (3.2) and (3.5), we find

$$V_{12}\check{V}_{12}\Sigma_{12}V_{23} = V_{12}\check{V}_{12}V_{13}\Sigma_{12} = V_{12}V_{13}V_{23}\check{V}_{12}\Sigma_{12} = V_{23}V_{12}\check{V}_{12}\Sigma_{12}.$$

By Lemma 3.9, $V_{12}\check{V}_{12}\Sigma_{12} = \widehat{V}_{12}^* U_1$ and hence $\widehat{V}_{12}^* U_1 V_{23} = V_{23}\widehat{V}_{12}^* U_1$. Since $\widehat{V}_{12}$ is unitary and $U_1 V_{23} = V_{23} U_1$, we can conclude $\widehat{V}_{12} V_{23} = V_{23} \widehat{V}_{12}$. A similar argument shows that $\check{V}_{23} V_{12} = V_{12} \check{V}_{23}$. □

The following result is crucial for the duality presented in the next section.

Proposition 3.11. *Let (V, U) be a C^*-pseudo-Kac system. Then $[A\widehat{A}] = [\widehat{\alpha}\widehat{\alpha}^*]$.*

Proof. The relation $[\widehat{\alpha}^* \widehat{A}] = \widehat{\alpha}^*$ (Remark 3.2 (iii)), regularity of V, and the relations $V^* = \Sigma U_2 V \Sigma U_2 V \Sigma U_2$ and $[V|\alpha\rangle_2 \widehat{A}] = [|\beta\rangle_2 \widehat{A}]$ (see Remark 4.7 in [29]) imply

$$\begin{aligned}
[\widehat{\alpha}\widehat{\alpha}^*] = [U\alpha\alpha^* U\widehat{A}] &= [U\langle\alpha|_2 V^* |\alpha\rangle_1 U\widehat{A}] \\
&= [U\langle\alpha|_2 \Sigma U_2 V \Sigma U_2 V \Sigma U_2 |\alpha\rangle_1 U\widehat{A}] \\
&= [\langle\alpha|_1 V \Sigma U_2 V |\alpha\rangle_2 \widehat{A}] \\
&= [\langle\alpha|_1 V \Sigma U_2 |\beta\rangle_2 \widehat{A}] = [\langle\alpha|_1 V |\widehat{\beta}\rangle_1 \widehat{A}] = [A\widehat{A}].
\end{aligned}$$
□

Lemma 3.12. *Let (V, U) be a (weak) C^*-pseudo-Kac system. Then also $(\check{V}, U)$, $(\widehat{V}, U)$, and (V^{op}, U) are (weak) C^*-pseudo-Kac systems.*

Proof. If (V, U) is a weak C^*-pseudo-Kac system, then the tuples above are balanced C^*-pseudo-multiplicative unitaries by Remark 3.2 (i), and the remaining necessary conditions follow easily from Proposition 3.4 and equation (3.3).

If (V, U) is a C^*-pseudo-Kac system, then equation (3.4), the relation $\widecheck{(V^{op})} = U_1 V^* U_1 = (\widehat{V})^{op}$, and the fact that V^{op} is regular, imply that the tuples above satisfy the regularity condition in Definition 3.7. To check that they also satisfy the second condition, we use Remark 3.8 and calculate $(\Sigma U_2 \widehat{V})^3 = (V \Sigma U_2)^3 = 1$, $(\check{V}\Sigma U_2)^3 = (\Sigma U_2 V)^3 = 1$, $(U_2 V^{op} \Sigma)^3 = (U_2 \Sigma V^*)^3 = ((V \Sigma U_2)^3)^* = 1$. □

3.2. **The C^*-pseudo-Kac system of a compact C^*-quantum groupoid.** In [**?**], we introduced compact C^*-quantum groupoids and associated to each such object a regular C^*-pseudo-multiplicative unitary V. We now recall this construction and define a symmetry U such that (V, U) is a C^*-pseudo-Kac system.

A *compact C^*-quantum graph* consists of a unital C^*-algebra B with a faithful KMS-state μ, a unital C^*-algebra A with unital embeddings $r \colon B \to A$ and $s \colon B^{op} \to A$ such that $[r(B), s(B^{op})] = 0$, and faithful conditional expectations $\phi \colon A \to r(B) \cong B$ and $\psi \colon A \to s(B^{op}) \cong B^{op}$ such that the compositions $\nu := \mu \circ \phi$ and $\nu^{-1} := \mu^{op} \circ \psi$ are KMS-states related by some positive invertible element $\delta \in A \cap r(B)' \cap s(B^{op})'$ via the formula $\nu^{-1}(a) = \nu(\delta^{1/2} a \delta^{1/2})$, valid for all $a \in A$. An *involution* for such a compact C^*-quantum graph is a $*$-antiisomorphism $R\colon A \to A$ such that $R \circ R = \mathrm{id}_A$, $R(r(b)) = s(b^{op})$ and $\phi(R(a)) = \psi(a)^{op}$ for all $b \in B$, $a \in A$.

Let $(B, \mu, A, r, s, \phi, \psi)$ be a compact C^*-quantum graph with involution R. We denote by $(H_\mu, \zeta_\mu, J_\mu)$ and $(H_\nu, \zeta_\nu, J_\nu)$ the GNS-spaces, canonical cyclic vectors, and modular conjugations for the KMS-states μ and ν, respectively, and let $\zeta_{\nu^{-1}} = \delta^{1/2}\zeta_\nu$. As usual, we have representations $B^{op} \to \mathcal{L}(H_\mu)$, $b^{op} \mapsto J_\mu b^* J_\mu$, and $A^{op} \to \mathcal{L}(H_\nu)$, $a^{op} \mapsto J_\nu a^* J_\nu$. Using the isometries

$$\zeta_\phi\colon H_\mu \to H_\nu,\ b\zeta_\mu \mapsto r(b)\zeta_\nu, \qquad \zeta_\psi\colon H_{\mu^{op}} \to H_\nu,\ b^{op}\zeta_{\mu^{op}} \mapsto s(b^{op})\zeta_{\nu^{-1}},$$

we define subspaces $\widehat{\alpha}, \widehat{\beta}, \alpha, \beta \subseteq \mathcal{L}(H_\mu, H_\nu)$ by $\widehat{\alpha} := [A\zeta_\phi]$, $\widehat{\beta} := [A\zeta_\psi]$, $\beta := [A^{op}\zeta_\phi]$, $\beta := [A^{op}\zeta_\psi]$. Let $H = H_\nu$ and $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$, where $\mathfrak{K} = H_\mu$, $\mathfrak{B} = B \subseteq \mathcal{L}(H_\mu)$, $\mathfrak{B}^\dagger = B^{op} \subseteq \mathcal{L}(H_\mu)$. Then $(H, \widehat{\alpha}, \widehat{\beta}, \alpha, \beta)$ is a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module and $\mathcal{A} := A_H^{\beta,\alpha}$ a C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-algebra [**?**].

A *compact C^*-quantum groupoid* consists of a compact C^*-quantum graph with involution as above and a morphism $\mathcal{A} \to \mathcal{A} \underset{\mathfrak{b}}{*} \mathcal{A}$ of C^*-$(\mathfrak{b}^\dagger, \mathfrak{b})$-algebras satisfying the following conditions:

(i) $(\Delta * \mathrm{id}) \circ \Delta = (\mathrm{id} * \Delta) \circ \Delta$ as maps from A to $\mathcal{L}(H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H)$;

(ii) $\langle \zeta_\phi|_2 \Delta(a) |\zeta_\phi\rangle_2 = \rho_\beta(\phi(a))$ and $\langle \zeta_\psi|_1 \Delta(a) |\zeta_\psi\rangle_1 = \rho_\alpha(\psi(a))$ for all $a \in A$;

(iii) $[\Delta(A)|\alpha\rangle_1] = [|\alpha\rangle_1 A] = [\Delta(A)|\zeta_\psi\rangle_1 A]$ and similarly $[\Delta(A)|\beta\rangle_2] = [|\beta\rangle_2 A] = [\Delta(A)|\zeta_\phi\rangle_2 A]$;

(iv) $R(\langle \zeta_\psi|_1 \Delta(a)(d^{op} \underset{\mathfrak{b}}{\otimes} 1)|\zeta_\psi\rangle_1) = \langle \zeta_\psi|_1 (a^{op} \underset{\mathfrak{b}}{\otimes} 1)\Delta(d)|\zeta_\psi\rangle_1$ for all $a, d \in A$.

Given a compact C^*-quantum groupoid as above, there exists a regular C^*-pseudo-multiplicative unitary $V\colon H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H$ such that $V|a\zeta_\psi\rangle_1 = \Delta(a)|\zeta_\psi\rangle_1$ for all $a \in A$; see Theorem 5.4 in [**?**]. Denote by $J = J_\nu$ the modular conjugation for ν, by $I\colon H \to H$ the antiunitary given by $Ia\zeta_{\nu^{-1}} = R(a)^*\zeta_\nu$ for all $a \in A$, and let $U = IJ \in \mathcal{L}(H)$.

Proposition 3.13. *(V, U) is a C^*-pseudo-Kac system.*

Proof. First, $U^2 = IJIJ = IJJI = II = \mathrm{id}_H$ because $IJ = JI$, and $U\zeta_\phi = \zeta_\psi$, $U\zeta_\nu = \zeta_{\nu^{-1}}$, $U\widehat{\alpha} = I\beta = \alpha$, $U\widehat{\beta} = I\alpha = \beta$ by Lemma 2.7 and Proposition 3.8 in [**?**]. The relation $(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)V(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I) = V^*$ (see Theorem 5.6 in [**?**]) implies

$$\begin{aligned}\check{V} = \Sigma(1 \underset{\mathfrak{b}}{\otimes} JI)V(1 \underset{\mathfrak{b}^\dagger}{\otimes} JI)\Sigma &= (J_\alpha \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} J)\Sigma(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)V(J_\alpha \underset{J_\mu}{\otimes}{}_\beta I)\Sigma(J_{\widehat{\alpha}} \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} J) \\ &= (J_\alpha \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} J)\Sigma V^* \Sigma(J_{\widehat{\alpha}} \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} J) = (J_\alpha \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} J)V^{op}(J_{\widehat{\alpha}} \underset{J_\mu}{\otimes}{}_{\widehat{\beta}} J).\end{aligned}$$

But V^{op} is a regular C^*-pseudo-multiplicative unitary, so $\check{V}$ is regular as well. In particular, (V, U) is a balanced C^*-pseudo-multiplicative unitary. We shall show that $\widehat{V}V = U_1\Sigma\check{V}^*$, and then the claim follows from Lemma 3.9. Let $a, b \in A$ and $\omega =$

$\widehat{V}V(a\zeta_\psi \ominus Ub\zeta_{\nu^{-1}})$. By Proposition 3.4, $\Delta(a) = \widehat{V}^*(1 \underset{\mathfrak{b}^\dagger}{\otimes} a)\widehat{V}$ and hence

$$\begin{aligned}\omega &= \widehat{V}\Delta(a)(\zeta_\psi \ominus Ub\zeta_{\nu^{-1}})\\ &= (1 \underset{\mathfrak{b}^\dagger}{\otimes} a)\widehat{V}(\zeta_\psi \ominus Ub\zeta_{\nu^{-1}})\\ &= \Sigma(U \underset{\mathfrak{b}}{\otimes} 1)(UaU \underset{\mathfrak{b}}{\otimes} 1)V(b\zeta_{\nu^{-1}} \oslash \zeta_\psi)\\ &= \Sigma(U \underset{\mathfrak{b}}{\otimes} 1)(UaU \underset{\mathfrak{b}}{\otimes} 1)\Delta(b)(\zeta_\psi \ominus \zeta_{\nu^{-1}}).\end{aligned}$$

Since $UaU = JIaIJ = R(a)^{op}$ and $[UaU \underset{\mathfrak{b}}{\otimes} 1, \Delta(b)] \in [A^{op} \underset{\mathfrak{b}}{\otimes} 1, A_\alpha \underset{\mathfrak{b}}{*}{}_\beta A] = 0$,

$$\begin{aligned}\omega &= \Sigma(U \underset{\mathfrak{b}}{\otimes} 1)\Delta(b)(UaU\zeta_\psi \ominus \zeta_{\nu^{-1}})\\ &= (U \underset{\mathfrak{b}^\dagger}{\otimes} 1)\Sigma(U \underset{\mathfrak{b}}{\otimes} U)\Delta(b)(U \underset{\mathfrak{b}}{\otimes} U)(aU\zeta_\psi \ominus \zeta_\nu).\end{aligned}$$

By Proposition 3.4, $\check{V}^*(1 \underset{\mathfrak{b}^\dagger}{\otimes} UbU)\check{V} = (U \underset{\mathfrak{b}}{\otimes} U)\Delta(b)(U \underset{\mathfrak{b}}{\otimes} U)$ and hence

$$\begin{aligned}\omega &= (U \underset{\mathfrak{b}^\dagger}{\otimes} 1)\Sigma\check{V}^*(1 \underset{\mathfrak{b}^\dagger}{\otimes} UbU)\check{V}(aU\zeta_\psi \ominus \zeta_\nu)\\ &= (U \underset{\mathfrak{b}^\dagger}{\otimes} 1)\Sigma\check{V}^*(1 \underset{\mathfrak{b}^\dagger}{\otimes} UbU)\Sigma(1 \underset{\mathfrak{b}}{\otimes} U)V(\zeta_\nu \oslash UaU\zeta_\psi).\end{aligned}$$

Finally, by Proposition 5.5 in [?], $V(\zeta_\nu \oslash UaU\zeta_\psi) = \zeta_\nu \oslash UaU\zeta_\phi$, whence

$$\begin{aligned}\omega &= (U \underset{\mathfrak{b}^\dagger}{\otimes} 1)\Sigma\check{V}^*(1 \underset{\mathfrak{b}^\dagger}{\otimes} UbU)(aU\zeta_\phi \ominus \zeta_\nu)\\ &= (U \underset{\mathfrak{b}^\dagger}{\otimes} 1)\Sigma\check{V}^*(a\zeta_\psi \ominus Ub\zeta_{\nu^{-1}}).\end{aligned}$$

□

4. Reduced crossed products and duality

Let (V, U) be a weak C^*-pseudo-Kac system and let $(\mathcal{A}, \Delta)$, $(\widehat{\mathcal{A}}, \widehat{\Delta})$ be the Hopf C^*-bimodules associated to V as in the preceding section. Generalizing the corresponding constructions and results for coactions of Hopf C^*-algebras [3], we now associate to every coaction of one of these Hopf C^*-bimodules a reduced crossed product that carries a dual coaction of the other Hopf C^*-bimodule, and prove a duality theorem concerning the iteration of this construction.

Reduced crossed products for coactions of $(\mathcal{A}, \Delta)$. Let δ be a coaction of the Hopf C^*-bimodule $(\mathcal{A}, \Delta)$ on a C^*-$\mathfrak{b}$-algebra $\mathcal{C} = C^\gamma_K$ and let[1]

$$C \rtimes_r \widehat{A} := [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})] \subseteq \mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H), \qquad \mathcal{C} \rtimes_r \widehat{\mathcal{A}} := (K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_{\widehat{\beta}}, C \rtimes_r \widehat{A}).$$

Proposition 4.1. (i) $[\delta(C)(\gamma \triangleright \widehat{\beta})] \subseteq \gamma \triangleright \widehat{\beta}$ *with equality if δ is left-full.*

(ii) $C \rtimes_r \widehat{A}$ *is a C^*-algebra and $\mathcal{C} \rtimes_r \widehat{\mathcal{A}}$ is a C^*-$\mathfrak{b}^\dagger$-algebra.*

(iii) *There exist nondegenerate $*$-homomorphisms $C \to M(C \rtimes_r \widehat{A})$ and $\widehat{A} \to M(C \rtimes_r \widehat{A})$, given by $c \mapsto \delta(c)$ and $\widehat{a} \mapsto 1 \underset{\mathfrak{b}}{\otimes} \widehat{a}$, respectively.*

[1]The notation $C \rtimes_r \widehat{A}$ is consistent with [3] but not with [10], where $C \rtimes_r A$ is used instead.

Proof. (i) The relation $\widehat{\beta} = [A\widehat{\beta}]$ (see Proposition 3.2 (ii) in [29]) implies that $[\delta(C)|\gamma\rangle_1\widehat{\beta}] = [\delta(C)|\gamma\rangle_1 A\widehat{\beta}] \subseteq [|\gamma\rangle_1 A\widehat{\beta}] = [|\gamma\rangle_1\widehat{\beta}]$.

(ii) We first show that $[(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})\delta(C)] \subseteq [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})]$. Let $\delta^{(2)} := (\mathrm{id} * \Delta) \circ \delta = (\delta * \mathrm{id}) \circ \delta \colon C \to \mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H)$. By definition of $\widehat{A}$ and Δ,

$$\begin{aligned}[(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})\delta(C)] = [\langle\beta|_3(1 \underset{\mathfrak{b}}{\otimes} V)|\alpha\rangle_3\delta(C)] &= [\langle\beta|_3(1 \underset{\mathfrak{b}}{\otimes} V)(\delta(C) \underset{\mathfrak{b}^\dagger}{\otimes} 1)|\alpha\rangle_3] \\ &= [\langle\beta|_3\delta^{(2)}(C)(1 \underset{\mathfrak{b}}{\otimes} V)|\alpha\rangle_3] \\ &\subseteq [\delta(C)\langle\beta|_3(1 \underset{\mathfrak{b}}{\otimes} V)|\alpha\rangle_3] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})].\end{aligned}$$

Consequently, $C \rtimes_r \widehat{A}$ is a C^*-algebra. By Proposition 3.2 (i) in [29], $[\widehat{A}\rho_{\widehat{\beta}}(\mathfrak{B})] = \widehat{A}$, and hence $[(C \rtimes_r \widehat{A})\rho_{(\gamma \triangleright \widehat{\beta})}(\mathfrak{B})] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A}\rho_{\widehat{\beta}}(\mathfrak{B}))] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})] = C \rtimes_r \widehat{A}$.

(iii) Immediate. □

Theorem 4.2. *There exists a unique coaction $\widehat{\delta}$ of $(\widehat{\mathcal{A}}, \widehat{\Delta})$ on $\mathcal{C} \rtimes_r \widehat{\mathcal{A}}$ such that $\widehat{\delta}(\delta(c)(1 \underset{\mathfrak{b}}{\otimes} \widehat{a})) = (\delta(c) \underset{\mathfrak{b}^\dagger}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\widehat{a}))$ for all $c \in C$, $\widehat{a} \in \widehat{A}$. If $\widehat{\Delta}$ is a fine coaction, then $\widehat{\delta}$ is a very fine coaction. If δ is left-full, then $\widehat{\delta}$ is left-full.*

Proof. Define $\widehat{\delta} \colon C \rtimes_r \widehat{A} \to \mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H)$ by $x \mapsto (1 \underset{\mathfrak{b}}{\otimes} \check{V})(x \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \check{V}^*)$. Then $\widehat{\delta}$ is injective and satisfies $\widehat{\delta}(\delta(c)(1 \underset{\mathfrak{b}}{\otimes} \widehat{a})) = (\delta(c) \underset{\mathfrak{b}^\dagger}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\widehat{a}))$ for all $c \in C$, $\widehat{a} \in \widehat{A}$ because $\check{V}(\widehat{a} \underset{\mathfrak{b}}{\otimes} 1)\check{V}^* = \widehat{\Delta}(\widehat{a})$ by Proposition 3.4 and $(1 \underset{\mathfrak{b}}{\otimes} \check{V})\delta(c)(1 \underset{\mathfrak{b}}{\otimes} \check{V}^*) = \delta(c)$ as a consequence of the relation $\check{V}(a \underset{\mathfrak{b}}{\otimes} 1)\check{V}^* = a \underset{\mathfrak{b}^\dagger}{\otimes} 1$. We show that $\widehat{\delta}$ is a coaction of $(\widehat{\mathcal{A}}, \widehat{\Delta})$. First, $[\widehat{\delta}(C \rtimes_r \widehat{A})|\alpha\rangle_3] \subseteq [|\alpha\rangle_3(C \rtimes_r \widehat{A})]$ because

$$[(\delta(C) \underset{\mathfrak{b}^\dagger}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\widehat{A}))|\alpha\rangle_3] \subseteq [(\delta(C) \underset{\mathfrak{b}^\dagger}{\otimes} 1)|\alpha\rangle_3(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})] = [|\alpha\rangle_3\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A})]. \tag{4.1}$$

Next, $[\widehat{\delta}(C \rtimes_r \widehat{A})|\gamma \triangleright \widehat{\beta}\rangle_1\widehat{A}] \subseteq [|\gamma \triangleright \widehat{\beta}\rangle_1\widehat{A}]$ because by Proposition 4.1 (i),

$$\begin{aligned}[(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\widehat{A}))(\delta(C) \underset{\mathfrak{b}^\dagger}{\otimes} 1)|\gamma \triangleright \widehat{\beta}\rangle_1\widehat{A}] &\subseteq [(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\widehat{A}))|\gamma \triangleright \widehat{\beta}\rangle_1\widehat{A}] \\ &= [|\gamma\rangle_1\widehat{\Delta}(\widehat{A})|\widehat{\beta}\rangle_1\widehat{A}] \subseteq [|\gamma\rangle_1|\widehat{\beta}\rangle_1\widehat{A}]. \end{aligned}\tag{4.2}$$

Furthermore, $\widehat{\delta}(x)(1 \underset{\mathfrak{b}}{\otimes} \check{V})|\xi\rangle_3 = (1 \underset{\mathfrak{b}}{\otimes} \check{V})|\xi\rangle_3 x$ for each $x \in C \rtimes_r \widehat{A}$, $\xi \in \widehat{\beta}$, and by Remark 3.2 (ii), $[(1 \underset{\mathfrak{b}}{\otimes} \check{V})|\widehat{\beta}\rangle_3(\gamma \triangleright \widehat{\beta})] = \gamma \triangleright \widehat{\beta} \triangleright \widehat{\beta}$ and $[\langle\widehat{\beta}|_3(1 \underset{\mathfrak{b}}{\otimes} \check{V})^*(\gamma \triangleright \widehat{\beta} \triangleright \widehat{\beta})] = \gamma \triangleright \widehat{\beta}$. The maps $(\widehat{\delta} * \mathrm{id}) \circ \widehat{\delta}$ and $(\mathrm{id} * \widehat{\Delta}) \circ \widehat{\delta}$ from $C \rtimes_r \widehat{A}$ to $\mathcal{L}\big(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H\big)$ are given by $\delta(c)(1 \underset{\mathfrak{b}}{\otimes} \widehat{a}) \mapsto \big(\delta(c) \underset{\mathfrak{b}^\dagger}{\otimes} 1 \underset{\mathfrak{b}^\dagger}{\otimes} 1\big)\big(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}^{(2)}(\widehat{a})\big)$ for all $c \in C, \widehat{a} \in \widehat{A}$, where $\widehat{\Delta}^{(2)} := (\widehat{\Delta} * \mathrm{id}) \circ \widehat{\Delta} = (\mathrm{id} * \widehat{\Delta}) \circ \widehat{\Delta}$. Thus, $(\mathcal{C} \rtimes_r \widehat{\mathcal{A}}, \widehat{\delta})$ is a coaction of $(\widehat{\mathcal{A}}, \widehat{\Delta})$. If the coactions $\widehat{\Delta}$ is fine, then the inclusion (4.1) is an equality and in any case $[\langle\widehat{\beta}|_3(1 \underset{\mathfrak{b}}{\otimes} \check{V})^*(\gamma \triangleright \widehat{\beta} \triangleright \widehat{\beta})] = \gamma \triangleright \widehat{\beta}$, whence

$\widehat{\delta}$ will be very fine. If δ is left-full, then the inclusion (4.2) is an equality by Proposition 4.1 (i) and hence $\widehat{\delta}$ is left-full. ☐

Definition 4.3. *We call* $\mathcal{C} \rtimes_r \widehat{\mathcal{A}}$ *the* reduced crossed product *and* $(\mathcal{C} \rtimes_r \widehat{\mathcal{A}}, \widehat{\delta})$ *the* reduced dual coaction *of* $(\mathcal{C}, \delta)$.

The construction of reduced dual coactions is functorial in the following sense:

Proposition 4.4. *Let* ρ *be a morphism between coactions* $(\mathcal{C}, \delta_{\mathcal{C}})$ *and* $(\mathcal{D}, \delta_{\mathcal{D}})$ *of* $(\mathcal{A}, \Delta)$. *Then there exists a unique morphism* $\rho \rtimes_r \mathrm{id}$ *from* $(\mathcal{C} \rtimes_r \widehat{\mathcal{A}}, \widehat{\delta_{\mathcal{C}}})$ *to* $(\mathcal{D} \rtimes_r \widehat{\mathcal{A}}, \widehat{\delta_{\mathcal{D}}})$ *such that* $(\rho \rtimes_r \mathrm{id})\big((1 \underset{\mathfrak{b}}{\otimes} \widehat{a})\delta_{\mathcal{C}}(c)\big) \cdot \delta_{\mathcal{D}}(d)(1 \underset{\mathfrak{b}}{\otimes} \widehat{a}') = (1 \underset{\mathfrak{b}}{\otimes} \widehat{a})\delta_{\mathcal{D}}(\rho(c)d)(1 \underset{\mathfrak{b}}{\otimes} \widehat{a}')$ *for all* $c \in C$, $d \in D$, $\widehat{a}, \widehat{a}' \in \widehat{A}$.

Proof. The semi-morphism $\mathrm{Ind}_{|\beta\rangle_2}(\rho)$ of Lemma 2.1 evidently restricts to a semi-morphism $\rho \rtimes_r \mathrm{id}$ from $\mathcal{C} \rtimes_r \widehat{\mathcal{A}}$ to $M(\mathcal{D} \rtimes_r \widehat{\mathcal{A}})$ which satisfies the formula given above, and this formula implies that $\rho \rtimes_r \mathrm{id}$ is a morphism of coactions as claimed. ☐

Corollary 4.5. *There exists a functor* $- \rtimes_r \widehat{\mathcal{A}}$: $\mathbf{Coact}_{(\mathcal{A},\Delta)} \to \mathbf{Coact}_{(\widehat{\mathcal{A}},\widehat{\Delta})}$ *given by* $(\mathcal{C}, \delta) \mapsto (\mathcal{C} \rtimes_r \widehat{\mathcal{A}}, \widehat{\delta})$ *and* $\rho \mapsto \rho \rtimes_r \mathrm{id}$. ☐

4.1. **Reduced crossed products for coactions of** $(\widehat{\mathcal{A}}, \widehat{\Delta})$**.** The construction in the preceding paragraph carries over to coactions of the Hopf C^*-bimodule $(\widehat{\mathcal{A}}, \widehat{\Delta})$ as follows. Let δ be a coaction of $(\widehat{\mathcal{A}}, \widehat{\Delta})$ on a C^*-$\mathfrak{b}^\dagger$-algebra $\mathcal{C} = C_K^\gamma$ and let

$$C \rtimes_r A := [\delta(C)(1 \underset{\mathfrak{b}^\dagger}{\otimes} UAU)] \subseteq \mathcal{L}(K {}_\gamma\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H), \qquad \mathcal{C} \rtimes_r \mathcal{A} = (K {}_\gamma\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H_{\widehat{\alpha}}, C \rtimes_r A).$$

Using straightforward modifications of the preceding proofs, one shows:

Proposition 4.6. (i) $[\delta(C)(\gamma \triangleright \widehat{\alpha})] \subseteq \gamma \triangleright \widehat{\alpha}$ *with equality if* δ *is fine.*
(ii) $C \rtimes_r A$ *is a* C^**-algebra and* $\mathcal{C} \rtimes_r \mathcal{A}$ *is a* C^*-$\mathfrak{b}$*-algebra.*
(iii) *There exist nondegenerate* $*$*-homomorphisms* $C \to M(C \rtimes_r A)$ *and* $A \to M(C \rtimes_r A)$, *given by* $c \mapsto \delta(c)$ *and* $a \mapsto 1 \underset{\mathfrak{b}^\dagger}{\otimes} a$, *respectively.*

Theorem 4.7. *There exists a unique coaction* $(\mathcal{C} \rtimes_r \mathcal{A}, \widehat{\delta})$ *of* $(\mathcal{A}, \Delta)$ *such that* $\widehat{\delta}(\delta(c)(1 \underset{\mathfrak{b}^\dagger}{\otimes} UaU)) = (\delta(c) \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{Ad}_{(U \underset{\mathfrak{b}}{\otimes} 1)} \Delta(a))$ *for all* $c \in C$, $a \in A$. *If* Δ *is a fine coaction, then* $\widehat{\delta}$ *is a very fine coaction. If* δ *is left-full, then* $\widehat{\delta}$ *is left-full.*

Definition 4.8. *Let* $(\mathcal{C}, \delta)$ *be a coaction of* $(\widehat{\mathcal{A}}, \widehat{\Delta})$. *Then we call* $\mathcal{C} \rtimes_r \mathcal{A}$ *the* reduced crossed product *and* $(\mathcal{C} \rtimes_r \mathcal{A}, \widehat{\delta})$ *the* reduced dual coaction *of* $(\mathcal{C}, \delta)$.

Proposition 4.9. *Let* ρ *be a morphism between coactions* $(\mathcal{C}, \delta_{\mathcal{C}})$ *and* $(\mathcal{D}, \delta_{\mathcal{D}})$ *of* $(\widehat{\mathcal{A}}, \widehat{\Delta})$. *Then there exists a unique morphism* $\rho \rtimes_r \mathrm{id}$ *from* $(\mathcal{C} \rtimes_r \mathcal{A}, \widehat{\delta_{\mathcal{C}}})$ *to* $(\mathcal{D} \rtimes_r \mathcal{A}, \widehat{\delta_{\mathcal{D}}})$ *such that* $(\rho \rtimes_r \mathrm{id})\big((1 \underset{\mathfrak{b}^\dagger}{\otimes} UaU)\delta_{\mathcal{C}}(c)\big) \cdot \delta_{\mathcal{D}}(d)(1 \underset{\mathfrak{b}^\dagger}{\otimes} Ua'U) = (1 \underset{\mathfrak{b}^\dagger}{\otimes} UaU)\delta_{\mathcal{D}}(\rho(c)d)(1 \underset{\mathfrak{b}^\dagger}{\otimes} Ua'U)$ *for all* $c \in C$, $d \in D$, $a, a' \in A$. ☐

Corollary 4.10. *There exists a functor* $- \rtimes_r \mathcal{A}$: $\mathbf{Coact}_{(\widehat{\mathcal{A}},\widehat{\Delta})} \to \mathbf{Coact}_{(\mathcal{A},\Delta)}$ *given by* $(\mathcal{C}, \delta) \mapsto (\mathcal{C} \rtimes_r \mathcal{A}, \widehat{\delta})$ *and* $\rho \mapsto \rho \rtimes_r \mathrm{id}$. ☐

4.2. **The duality theorem.** The preceding constructions yield for each coaction $(\mathcal{C}, \delta_{\mathcal{C}})$ of $(\mathcal{A}, \Delta)$ and each coaction $(\mathcal{D}, \delta_{\mathcal{D}})$ of $(\widehat{\mathcal{A}}, \widehat{\Delta})$ a bidual $(\mathcal{C} \rtimes_r \widehat{\mathcal{A}} \rtimes_r \mathcal{A}, \widehat{\widehat{\delta}}_{\mathcal{C}})$ and $(\mathcal{D} \rtimes_r \mathcal{A} \rtimes_r \widehat{\mathcal{A}}, \widehat{\widehat{\delta}}_{\mathcal{D}})$, respectively. The following generalization of the Baaj-Skandalis duality theorem [3] identifies these biduals in the case where (V, U) is a C^*-pseudo-Kac system and the initial coactions are fine. Morally, it says that up to Morita equivalence, the functors $- \rtimes_r \widehat{\mathcal{A}}$ and $- \rtimes_r \mathcal{A}$ implement an equivalence of the categories $\mathbf{Coact}^f_{(\mathcal{A},\Delta)}$ and $\mathbf{Coact}^f_{(\widehat{\mathcal{A}},\widehat{\Delta})}$.

Theorem 4.11. *Assume that (V, U) is a C^*-pseudo-Kac system.*

(i) *Let $(\mathcal{C}, \delta)$ be a (very) fine coaction of $(\mathcal{A}, \Delta)$, where $\mathcal{C} = C^\gamma_K$. Then there exists an isomorphism $\Phi\colon C \rtimes_r \widehat{A} \rtimes_r A \to [|\beta\rangle_2 C\langle\beta|_2] \subseteq \mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H)$ such that Φ^{-1} is an (iso)morphism from $(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_{\widehat{\alpha}}, [|\beta\rangle_2 C\langle\beta|_2])$ to $\mathcal{C} \rtimes_r \widehat{\mathcal{A}} \rtimes_r \mathcal{A}$ and $\widehat{\widehat{\delta}} \circ \Phi^{-1} = (\Phi^{-1} * \mathrm{id}) \circ \mathrm{Ad}_{(1 \underset{\mathfrak{b}}{\otimes} \Sigma\widehat{V})} \circ \mathrm{Ind}_{|\beta\rangle_2}(\delta)$.*

(ii) *Let $(\mathcal{D}, \delta)$ be a (very) fine coaction of $(\widehat{\mathcal{A}}, \widehat{\Delta})$, where $\mathcal{D} = D^\epsilon_L$. Then there exists an isomorphism $\Phi\colon D \rtimes_r A \rtimes_r \widehat{A} \cong [|\alpha\rangle_2 D\langle\alpha|_2] \subseteq \mathcal{L}(L_\epsilon \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H)$ such that Φ^{-1} is an (iso)morphism from $(L_\epsilon \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H_{\widehat{\beta}}, [|\alpha\rangle_2 D\langle\alpha|_2])$ to $\mathcal{D} \rtimes_r \mathcal{A} \rtimes_r \widehat{\mathcal{A}}$ and $\widehat{\widehat{\delta}} \circ \Phi^{-1} = (\Phi^{-1} * \mathrm{id}) \circ \mathrm{Ad}_{(1 \underset{\mathfrak{b}^\dagger}{\otimes} \Sigma V)} \circ \mathrm{Ind}_{|\alpha\rangle_2}(\delta)$.*

Proof. We only prove (i); then (ii) follows after replacing (V, U) by $(\check{V}, U)$. By Proposition 3.4 and Proposition 3.11, applied to the C^*-pseudo-Kac system $(\check{V}, U)$, we have $[\widehat{A}\,\mathrm{Ad}_U(A)] = [A_{\check{V}} \widehat{A}_{\check{V}}] = [\beta\beta^*]$, and since δ is fine,

$$[|\beta\rangle_2 C\langle\beta|_2] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \beta\beta^*)] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} \widehat{A}\,\mathrm{Ad}_U(A))].$$

One easily verifies that the $*$-homomorphism $\mathrm{Ind}_{|\beta\rangle_2}(\delta)$ (see Lemma 2.1) yields an (iso)morphism of C^*-$\mathfrak{b}$-algebras

$$\mathrm{Ind}_{|\beta\rangle_2}(\delta)\colon \big(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_{\widehat{\alpha}}, [|\beta\rangle_2 C\langle\beta|_2]\big) \to \big(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_\alpha \underset{\mathfrak{b}}{\otimes}{}_\beta H_{\widehat{\alpha}}, [|\beta\rangle_2 \delta(C)\langle\beta|_2]\big).$$

Denote by Ψ the composition of this (iso)morphism with $\mathrm{Ad}_{(1 \underset{\mathfrak{b}}{\otimes} V^*)}$ and let $\delta^{(2)} = (\delta * \mathrm{id}) \circ \delta = (\mathrm{id} * \Delta) \circ \delta$. Let $x = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} \widehat{a}UaU) \in [|\beta\rangle_2 C\langle\beta|_2]$, where $c \in C, \widehat{a} \in \widehat{A}, a \in A$. By Lemma 3.6,

$$\Psi(x) = \mathrm{Ad}_{(1 \underset{\mathfrak{b}}{\otimes} V^*)}(\delta^{(2)}(c)(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}}{\otimes} \widehat{a}UaU)) = (\delta(c) \underset{\mathfrak{b}^\dagger}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\widehat{a}))(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}^\dagger}{\otimes} UaU).$$

Consequently, $\Psi([|\beta\rangle_2 C\langle\beta|_2]) = C \rtimes_r \widehat{A} \rtimes_r A$. Next, the relations $\mathcal{C} \rtimes_r \widehat{\mathcal{A}} \rtimes_r \mathcal{A} = (K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_{\widehat{\beta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H_{\widehat{\alpha}}, C \rtimes_r \widehat{A} \rtimes_r A)$ and $(1 \underset{\mathfrak{b}}{\otimes} V^*)(\gamma \triangleright \alpha \triangleright \widehat{\alpha}) = \gamma \triangleright \widehat{\beta} \triangleright \widehat{\alpha}$ imply that Ψ is a morphism of C^*-$\mathfrak{b}$-algebras as claimed. Using the definition of $\widehat{\widehat{\delta}}$, Proposition 3.4, and

Lemma 3.6, we find

$$\begin{aligned}
\hat{\hat{\delta}}(\Psi(x)) &= (\delta(c) \underset{\mathfrak{b}^\dagger}{\otimes} 1 \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \widehat{\Delta}(\hat{a}) \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{Ad}_{(U \underset{\mathfrak{b}}{\otimes} 1)}(\Delta(a))) \\
&= \mathrm{Ad}_{(1 \underset{\mathfrak{b}}{\otimes} V^* \underset{\mathfrak{b}}{\otimes} 1)} \big((\delta^{(2)}(c) \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}}{\otimes} \hat{a} \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}}{\otimes} \mathrm{Ad}_{(U \underset{\mathfrak{b}}{\otimes} 1)}(\Delta(a)))\big) \\
&= (\Psi * \mathrm{id})\big((\delta(c) \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \hat{a} \underset{\mathfrak{b}}{\otimes} 1)(1 \underset{\mathfrak{b}}{\otimes} \mathrm{Ad}_{(U \underset{\mathfrak{b}}{\otimes} 1)}(\Delta(a)))\big) \\
&= (\Psi * \mathrm{id})\big((1 \underset{\mathfrak{b}}{\otimes} \Sigma \widehat{V})\delta^{(2)}(c)(1 \underset{\mathfrak{b}}{\otimes} 1 \underset{\mathfrak{b}}{\otimes} \hat{a} U a U)(1 \underset{\mathfrak{b}}{\otimes} \widehat{V}^* \Sigma)\big) \\
&= (\Psi * \mathrm{id})\big((1 \underset{\mathfrak{b}}{\otimes} \Sigma \widehat{V})(\mathrm{Ind}_{|\beta\rangle_2}(\delta)(x))(1 \underset{\mathfrak{b}}{\otimes} \widehat{V}^* \Sigma)\big). \qquad \square
\end{aligned}$$

5. The C^*-pseudo-Kac system of a groupoid

For the remainder of this article, we fix a locally compact, Hausdorff, second countable groupoid G with a left Haar system λ. In [29], we associated to such a groupoid a regular C^*-pseudo-multiplicative unitary V and identified the underlying C^*-algebras of the Hopf C^*-bimodules $(\widehat{\mathcal{A}}, \widehat{\Delta})$ and $(\mathcal{A}, \Delta)$ of V with the function algebra $C_0(G)$ and the reduced groupoid C^*-algebra $C_r^*(G)$, respectively. We now recall this construction and define a symmetry U such that (V, U) becomes a C^*-pseudo-Kac system. For background on groupoids, see [20], [23].

Denote by λ^{-1} the right Haar system associated to λ and let μ be a measure on the unit space G^0 with full support. We denote the range and the source map of G by r and s, respectively, let $G^u := r^{-1}(u)$ and $G_u := s^{-1}(u)$ for each $u \in G^0$, and define measures ν, ν^{-1} on G such that

$$\int_G f \,\mathrm{d}\nu = \int_{G^0} \int_{G^u} f(x) \,\mathrm{d}\lambda^u(x) \,\mathrm{d}\mu(u), \qquad \int_G f d\nu^{-1} = \int_{G^0} \int_{G_u} f(x) \,\mathrm{d}\lambda_u^{-1}(x) \,\mathrm{d}\mu(u)$$

for all $f \in C_c(G)$. We assume that μ is quasi-invariant in the sense that ν and ν^{-1} are equivalent, and denote by $D := \mathrm{d}\nu / \mathrm{d}\nu^{-1}$ the Radon-Nikodym derivative. One can choose D such that it is a Borel homomorphism (see page 89 in [20]), and we do so.

We identify functions in $C_b(G^0)$ and $C_b(G)$ with multiplication operators on the Hilbert spaces $L^2(G^0, \mu)$ and $L^2(G, \nu)$, respectively, and let $\mathfrak{K} = L^2(G^0, \mu)$, $\mathfrak{B} = \mathfrak{B}^\dagger = C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K})$, $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger) = \mathfrak{b}$, $H = L^2(G, \nu)$.

Pulling functions on G^0 back to G along r or s, we obtain representations $r^*\colon C_0(G^0) \to C_b(G) \hookrightarrow \mathcal{L}(H)$ and $s^*\colon C_0(G^0) \to C_b(G) \hookrightarrow \mathcal{L}(H)$. We define Hilbert C^*-$C_0(G^0)$-modules $L^2(G, \lambda)$ and $L^2(G, \lambda^{-1})$ as the respective completions of the pre-C^*-module $C_c(G)$, the structure maps being given by

$$\langle \xi' | \xi \rangle(u) = \int_{G^u} \overline{\xi'(x)} \xi(x) \,\mathrm{d}\lambda^u(x), \qquad \xi f = r^*(f)\xi \qquad \text{in the case of } L^2(G, \lambda),$$

$$\langle \xi' | \xi \rangle(u) = \int_{G_u} \overline{\xi'(x)} \xi(x) \,\mathrm{d}\lambda_u^{-1}(x), \qquad \xi f = s^*(f)\xi \qquad \text{in the case of } L^2(G, \lambda^{-1})$$

respectively, for all $\xi, \xi' \in C_c(G)$, $u \in G^0$, $f \in C_0(G^0)$. Then there exist isometric embeddings $j\colon L^2(G, \lambda) \to \mathcal{L}(\mathfrak{K}, H)$ and $\hat{j}\colon L^2(G, \lambda^{-1}) \to \mathcal{L}\big(\mathfrak{K}, H\big)$ such that

$$\big(j(\xi)\zeta\big)(x) = \xi(x)\zeta(r(x)), \qquad \big(\hat{j}(\xi)\zeta\big)(x) = \xi(x) D^{-1/2}(x) \zeta(s(x))$$

for all $\xi \in C_c(G)$, $\zeta \in C_c(G^0)$. Let $\alpha = \beta := j(L^2(G,\lambda))$ and $\widehat{\alpha} = \widehat{\beta} := \hat{j}(L^2(G,\lambda^{-1}))$. Then $(H, \widehat{\alpha}, \widehat{\beta}, \alpha, \beta)$ is a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$-module, $\rho_\alpha = \rho_\beta = r^*$ and $\rho_{\widehat{\alpha}} = \rho_{\widehat{\beta}} = s^*$, and $j(\xi)^* j(\xi') = \langle \xi | \xi' \rangle$ and $\hat{j}(\eta)^* \hat{j}(\eta') = \langle \eta | \eta' \rangle$ for all $\xi, \xi' \in L^2(G, \lambda)$, $\eta, \eta' \in L^2(G, \lambda^{-1})$; see §2.3 in [29].

The Hilbert spaces $H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H$ and $H {}_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H$ can be described as follows. Define measures $\nu^2_{s,r}$ on $G_s \times_r G$ and $\nu^2_{r,r}$ on $G_r \times_r G$ such that

$$\begin{aligned} \int_{G_s \times_r G} f \,\mathrm{d}\nu^2_{s,r} &= \int_{G^0} \int_{G^u} \int_{G^{s(x)}} f(x,y) \,\mathrm{d}\lambda^{s(x)}(y) \,\mathrm{d}\lambda^u(x) \,\mathrm{d}\mu(u), \\ \int_{G_r \times_r G} g \,\mathrm{d}\nu^2_{r,r} &= \int_{G^0} \int_{G^u} \int_{G^u} g(x,y) \,\mathrm{d}\lambda^u(y) \,\mathrm{d}\lambda^u(x) \,\mathrm{d}\mu(u) \end{aligned} \tag{5.1}$$

for all $f \in C_c(G_s \times_r G)$, $g \in C_c(G_r \times_r G)$. Then there exist unitaries

$$\Phi \colon H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to L^2(G_s \times_r G, \nu^2_{s,r}) \qquad \text{and} \qquad \Psi \colon H {}_\alpha\underset{\mathfrak{b}}{\otimes}{}_\beta H \to L^2(G_r \times_r G, \nu^2_{r,r})$$

such that for all $\eta, \xi \in C_c(G)$, $\zeta \in C_c(G^0)$,

$$\begin{aligned} \Phi\big(\hat{j}(\eta) \otimes \zeta \otimes j(\xi)\big)(x,y) &= \eta(x) D^{-1/2}(x) \zeta(s(x)) \xi(y), \\ \Psi\big(j(\eta) \otimes \zeta \otimes j(\xi)\big)(x,y) &= \eta(x) \zeta(r(x)) \xi(y). \end{aligned}$$

From now on, we use these isomorphisms without further notice.

Theorem 5.1. *There exists a C^*-pseudo-Kac system (V, U) on $(H, \widehat{\alpha}, \widehat{\beta}, \alpha, \beta)$ such that for all $\omega \in C_c(G_s \times_r G)$, $(x,y) \in G_r \times_r G$, $\xi \in C_c(G)$, $z \in G$,*

$$(V\omega)(x,y) = \omega(x, x^{-1}y) \quad \textit{and} \quad (U\xi)(x) = \xi(x^{-1}) D(x)^{-1/2}. \tag{5.2}$$

Proof. By Theorem 2.5 and Example 4.3 (ii) in [29], there exists a regular C^*-pseudo-multiplicative unitary V as claimed. The second formula in (5.2) defines a unitary $U \in \mathcal{L}(H)$ by definition of the Radon-Nikodym derivative $D = d\nu/d\nu^{-1}$, and $U^2 = \mathrm{id}$ because $(U^2\xi)(x) = (U\xi)(x^{-1}) D(x)^{-1/2} = \xi(x) D(x)^{1/2} D(x)^{-1/2} = \xi(x)$ for all $\xi \in C_c(G)$ and $x \in G$. The unitary $\widehat{V} = \Sigma U_1 V U_1 \Sigma$ is equal to $V^{op} = \Sigma V^* \Sigma$ because

$$\begin{aligned} (U_1 V U_1 \omega)(x,y) &= (V U_1 \omega)(x^{-1}, y) D(x)^{-1/2} \\ &= (U_1 \omega)(x^{-1}, xy) D(x)^{-1/2} \\ &= \omega(x, xy) D(x^{-1})^{-1/2} D(x)^{-1/2} = \omega(x, xy) \end{aligned}$$

for all $\omega \in C_c(G_r \times_r G)$, $(x,y) \in G_s \times_r G$. In particular, $\widehat{V}$ is a regular C^*-pseudo-multiplicative unitary. It remains to show that the map $Z := \Sigma U_2 V \colon H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to H {}_{\widehat{\beta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H$ satisfies $Z^3 = 1$. But for all $\omega \in C_c(G_s \times_r G)$ and $(x,y) \in G_s \times_r G$,

$$\begin{aligned} (Z\omega)(x,y) &= (V\omega)(y, x^{-1}) D(x)^{-1/2} = \omega(y, y^{-1}x^{-1}) D(x)^{-1/2}, \\ (Z^3\omega)(x,y) &= (Z^2\omega)(y, y^{-1}x^{-1}) D(x)^{-1/2} \\ &= (Z\omega)(y^{-1}x^{-1}, xyy^{-1}) \big(D(x)D(y)\big)^{-1/2} \\ &= \omega(x, x^{-1}xy) \big(D(x)D(y)D(y^{-1}x^{-1})\big)^{-1/2} = \omega(x,y). \end{aligned}$$

□

The Hopf C^*-bimodules $(\widehat{\mathcal{A}}, \widehat{\Delta})$ and $(\mathcal{A}, \Delta)$ associated to V can be described as follows; see Theorem 3.16 in [29]. Given $g \in C_c(G)$, define $L(g) \in C_r^*(G) \subseteq \mathcal{L}(H)$ by

$$(L(g)f)(x) = \int_{G^{r(x)}} g(z)f(z^{-1}x)D^{-1/2}(z)\,\mathrm{d}\lambda^{r(x)}(z)$$

for all $x \in G$, $f \in C_c(G) \subseteq L^2(G, \nu) = H$. Then

$$\widehat{A} = C_0(G) \subseteq \mathcal{L}(H), \quad \big(\widehat{\Delta}(f)\omega\big)(x, y) = f(xy)\omega(x, y), \tag{5.3}$$

$$A = C_r^*(G), \quad \big(\Delta(L(g))\omega'\big)(x', y') = \int_{G^{u'}} g(z)D^{-1/2}(z)\omega'(z^{-1}x', z^{-1}y')\,\mathrm{d}\lambda^{u'}(z)$$

for all $f \in C_0(G)$, $\omega \in C_c(G_s{\times_r}G)$, $(x, y) \in G_s{\times_r}G$ and $g \in C_c(G)$, $\omega' \in C_c(G_r{\times_r}G)$, $(x', y') \in G_r{\times_r}G$, where $u' = r(x') = r(y')$. We shall loosely refer to $C_0(G)$ and $C_r^*(G)$ as Hopf C^*-bimodules, having in mind $(\widehat{\mathcal{A}}, \widehat{\Delta})$ and $(\mathcal{A}, \Delta)$, respectively.

6. Actions of G and coactions of $C_0(G)$

Let G be a groupoid and consider $C_0(G)$ as a Hopf C^*-bimodule as in the preceding section. Then coactions of $C_0(G)$ can be related to actions of G as follows. Let us say that a tuple $(\mathbf{F}, \mathbf{G}, \eta, \epsilon)$ is *an embedding of a category* $\mathbf{C}$ *into a category* $\mathbf{D}$ *as a full and coreflective subcategory* if $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ is a full and faithful functor and $\mathbf{G}\colon \mathbf{D} \to \mathbf{C}$ is a faithful right adjoint to $\mathbf{F}$, where $\eta\colon \mathrm{id}_{\mathbf{C}} \to \mathbf{GF}$ is the unit and $\epsilon\colon \mathbf{FG} \to \mathrm{id}_{\mathbf{D}}$ is the counit of the adjunction; see also §IV.3 in [18]. In this section, we construct such an embedding of the category of actions of G on continuous $C_0(G^0)$-algebras into the category of certain admissible coactions of $C_0(G)$. We keep the notation introduced in the preceding section.

6.1. $C_0(G^0)$-algebras and C^*-$\mathfrak{b}$-algebras. We shall embed the category of admissible $C_0(G^0)$-algebras into the category of admissible C^*-$\mathfrak{b}$-algebras as a full and coreflective subcategory.

Recall that a $C_0(X)$*-algebra*, where X is some locally compact Hausdorff space, is a C^*-algebra C with a fixed nondegenerate $*$-homomorphism of $C_0(X)$ into the center of the multiplier algebra $M(C)$ [6], [14]. We denote the fiber of a $C_0(X)$-algebra C at a point $x \in X$ by C_x and write the quotient map $p_x\colon C \to C_x$ as $c \mapsto c_x$. Recall that C is a *continuous* $C_0(X)$*-algebra* if the map $X \to \mathbb{R}$ given by $x \mapsto \|c_x\|$ is continuous for each $c \in C$. A *morphism* of $C_0(X)$-algebras C, D is a nondegenerate $*$-homomorphism $\pi\colon C \to M(D)$ such that $\pi(fc) = f\pi(c)$ for all $f \in C_0(X)$, $c \in C$.

Definition 6.1. *We call a $C_0(G^0)$-algebra C* admissible *if it is continuous and if $C_u \neq 0$ for each $u \in G^0$, and we call a C^*-$\mathfrak{b}$-algebra C_K^γ* admissible *if $[\rho_\gamma(C_0(G^0))C] = C$ and $[C\gamma] = \gamma$. A* morphism *between admissible C^*-$\mathfrak{b}$-algebras C_K^γ, D_L^ϵ is a semi-morphism π from C_K^γ to $M(D)_L^\epsilon$ that is nondegenerate in the sense that $[\pi(C)D] = D$. Denote by $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$ the category of all admissible $C_0(G^0)$-algebras, and by $\mathbf{C^*}$-$\mathfrak{b}$-$\mathbf{alg}^a$ the category of all admissible C^*-$\mathfrak{b}$-algebras.*

Lemma 6.2. (i) *Let C_K^γ be an admissible C^*-$\mathfrak{b}$-algebra. Then C is an admissible $C_0(G^0)$-algebra with respect to ρ_γ.*

(ii) *Let π be a morphism between admissible C^*-$\mathfrak{b}$-algebras C_K^γ and D_L^ϵ. Then π is a morphism of $C_0(G^0)$-algebras from (C, ρ_γ) to (D, ρ_ϵ).*

Proof. (i) First, note that $\rho_\gamma(C_0(G^0)) \subseteq M(C)$ is central because $C \subseteq \mathcal{L}(K_\gamma) \subseteq \rho_\gamma(C_0(G^0))'$. The map $C \hookrightarrow \mathcal{L}(K_\gamma) \cong \mathcal{L}(\gamma)$ is a faithful field of representations in the sense of Theorem 3.3 in [6], and therefore C is a continous $C_0(G^0)$-algebra. We have $C_u \neq 0$ for each $u \in G^0$ because otherwise $C = [CI_u]$, where $I_u = C_0(G^0\backslash\{u\})$, and then $[\gamma^*\gamma] = [\gamma^* C\gamma] = [\gamma^* I_u C\gamma] = [\gamma^*\gamma I_u] = I_u \neq C_0(G^0)$, contradicting the fact that K_γ is a C^*-$\mathfrak{b}$-module.

(ii) This is Lemma 3.4 in [27]. □

We embed $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$ into $\mathbf{C^*}$-$\mathfrak{b}$-$\mathbf{alg}^a$ using a KSGNS-construction for the following kind of weights.

Definition 6.3. *A* $C_0(G^0)$-weight *on a* $C_0(G^0)$*-algebra* C *is a* $C_0(G^0)$*-linear, positive map* $\phi\colon C \to C_0(G^0)$*. We denote the set of all such weights by* $\mathcal{W}(C)$*.*

Let C be an admissible $C_0(G^0)$-algebra. The results in [4] imply:

Lemma 6.4. $\bigcap_{\phi\in\mathcal{W}(C)} \ker\phi = \{0\}$ *and* $[\bigcup_{\phi\in\mathcal{W}(C)} \phi(C)] = C_0(G^0)$*.* □

Let $\phi \in \mathcal{W}(C)$. Then ϕ is completely positive by Theorem 3.9 in [21] and bounded by Lemma 5.1 in [16]. Let $E_\phi = C \otimes_\phi \mathfrak{K}$ (see Section 1) and define $\eta_\phi\colon C \to \mathcal{L}(E_\phi)$ and $l_\phi\colon C \to \mathcal{L}(\mathfrak{K}, E_\phi)$ by $\eta_\phi(c)(d \otimes_\phi \zeta) = cd \otimes_\phi \zeta$ and $l_\phi(c)\zeta = c \otimes_\phi \zeta$ for all $c, d \in C$, $\zeta \in \mathfrak{K}$. One easily verifies that for all $c, d \in C$, $f \in C_0(G^0)$, $\zeta \in \mathfrak{K}$,

$$\begin{gathered} l_\phi(c)^* l_\phi(d) = \phi(c^*d), \quad l_\phi(c)f = l_\phi(cf), \\ \eta_\phi(c)(d \otimes_\phi f\zeta) = cdf \otimes_\phi \zeta = \eta_\phi(cf)(d \otimes_\phi \zeta). \end{gathered} \tag{6.1}$$

The *universal* $C_0(G^0)$*-representation* $\eta_C\colon C \to \mathcal{L}(E_C)$ of C is the direct sum of the representations $\eta_\phi\colon C \to \mathcal{L}(E_\phi)$, where $\phi \in \mathcal{W}(C)$. Denote by $l_C \subseteq \mathcal{L}(\mathfrak{K}, E_C)$ the closed linear span of all maps $l_\phi(c)\colon \mathfrak{K} \to E_\phi \hookrightarrow E_C$, where $c \in C$, $\phi \in \mathcal{W}(C)$.

Lemma 6.5. $\eta_C(C)^{l_C}_{E_C}$ *is an admissible* C^*-$\mathfrak{b}$*-algebra and* η_C *is an isomorphism of* $C_0(G^0)$*-algebras from* C *to* $(\eta_C(C), \rho_{l_C})$*.*

Proof. The definition of l_C, the equations (6.1) and Lemma 6.4 imply that $[l_C\mathfrak{K}] = \bigoplus_\phi E_\phi = E_C$, $[l_C^* l_C] = [\bigcup_\phi \phi(C)] = C_0(G^0)$ and $[l_C C_0(G^0)] = l_C$, whence (E_C, l_C) is a C^*-$\mathfrak{b}$-module, and that $[\eta_C(C)\rho_{l_C}(C_0(G^0))] = [\eta_C(CC_0(G^0))] = \eta_C(C)$ and $[\eta_C(C)l_C] = l_C$, whence $\eta_C(C)^{l_C}_{E_C}$ is an admissible C^*-$\mathfrak{b}$-algebra. Lemma 6.4 implies that η_C is injective and hence an isomorphism of C onto $\eta_C(C)$, and the last equation in (6.1) implies that $\eta_C(c)\rho_\gamma(f) = \eta_C(cf)$ for all $c \in C$, $f \in C_0(G^0)$. □

Theorem 6.6. *There exists an embedding as a full and coreflective subcategory* $(\mathbf{F}, \mathbf{G}, \eta, \epsilon)$ *of* $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$ *into* $\mathbf{C^*}$-$\mathfrak{b}$-$\mathbf{alg}^a$ *such that the following conditions hold:*

(i) $\mathbf{F}$ *is given by* $C \mapsto \eta_C(C)^{l_C}_{E_C}$ *on objects and by* $\mathbf{F}\pi\colon \eta_C(c) \mapsto \eta_D(\pi(c))$ *for each morphism* π *between objects* C*,* D *in* $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$*;*

(ii) $\mathbf{G}$ *is given by* $C^\gamma_K \mapsto (C, \rho_\gamma)$ *on objects and* $\pi \mapsto \pi$ *on morphisms;*

(iii) η_C *is defined as above for each object* C *in* $\mathbf{C_0(G^0)}$-$\mathbf{alg}^a$*;*

(iv) $\epsilon_{\mathcal{C}} = \eta^{-1}_{\mathbf{G}\mathcal{C}}$ *for each object* $\mathcal{C}$ *in* $\mathbf{C^*}$-$\mathfrak{b}$-$\mathbf{alg}^a$*.*

Proof of Theorem 6.6. The functor $\mathbf{G}\colon \mathbf{C^*\text{-}\mathfrak{b}\text{-}alg}^a \to \mathbf{C_0(G^0)\text{-}alg}^a$ is well defined by Lemma 6.2 and evidently faithful.

Let C be an admissible $C_0(G^0)$-algebra, $\mathcal{D} = D_K^\gamma$ an admissible C^*-$\mathfrak{b}$-algebra, and $\pi\colon C \to \mathbf{G}\mathcal{D}$ a morphism in $\mathbf{C_0(G^0)\text{-}alg}^a$. We claim that $\pi \circ \eta_C^{-1}$ is a morphism from $\mathbf{F}C$ to $\mathcal{D}$ in $\mathbf{C^*\text{-}\mathfrak{b}\text{-}alg}^a$. Let $\xi \in \gamma$. Then the map $\phi\colon C \to C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K})$ given by $c \mapsto \xi^*\pi(c)\xi$ is a $C_0(G^0)$-weight, and there exists an isometry $S\colon E_\phi \to K$ such that $S(c \otimes_\phi \zeta) = \pi(c)\xi\zeta$ for all $c \in C$, $\zeta \in \mathfrak{K}$. Denote by $P\colon E_C \to E_\phi$ the natural projection. Then $[SPl_C] = [Sl_\phi(C)] = [\pi(C)\xi]$ lies in γ and contains ξ, and $SP\eta_C(c) = S\eta_\phi(c) = \pi(c)$ for each $c \in C$. Since $\xi \in \gamma$ was arbitrary, the claim follows.

Using Lemma 6.5, we conclude that $\mathbf{F}$ is well defined and that η is a natural isomorphism from id to $\mathbf{GF}$. Indeed, if $\pi\colon C \to D$ is a morphism in $\mathbf{C_0(G^0)\text{-}alg}^a$, then $\mathbf{F}\pi = \eta_D \circ \pi \circ \eta_C^{-1}$ is a morphism from $\mathbf{F}C$ to $\mathbf{F}D$ by the argument above.

Finally, let $\mathcal{D}$ be an admissible C^*-$\mathfrak{b}$-algebra. The argument above, applied to the identity on $\mathbf{G}\mathcal{D}$, yields a morphism $\epsilon_{\mathcal{D}}$ from $\mathbf{FG}\mathcal{D}$ to $\mathcal{D}$ in $\mathbf{C^*\text{-}\mathfrak{b}\text{-}alg}^a$ such that the composition $\mathbf{G}\mathcal{D} \xrightarrow{\eta_{\mathbf{G}\mathcal{D}}} \mathbf{GFG}\mathcal{D} \xrightarrow{\mathbf{G}\epsilon_{\mathcal{D}}} \mathbf{G}\mathcal{D}$ is the identity. Since η is a natural transformation, also $\epsilon\colon \mathbf{FG} \to \mathrm{id}$ is one. For each admissible $C_0(G^0)$-algebra C, the composition $\mathbf{F}C \xrightarrow{\mathbf{F}\eta_C} \mathbf{FGF}C \xrightarrow{\epsilon_{\mathbf{F}C}} \mathbf{F}C$ is the identity by construction. From Theorem 2 of §IV.1 in [18], we can conclude that $\mathbf{F}$ is a left adjoint to $\mathbf{G}$ such that η and ϵ form the unit and counit, respectively, of the adjunction. Since η is a natural isomorphism, $\mathbf{F}$ is full and faithful by Theorem 1 of §IV.3 in [18]. □

6.2. **Actions of G and coactions of $C_0(G)$.** We next embed the category of admissible actions of G as a full and coreflective subcategory into the category of all admissible coactions of $C_0(G)$.

The definition of an action of G requires the following preliminaries. Given $C_0(G^0)$-algebras (C,ρ) and (D,σ), where D is commutative, we denote by $C_\rho \boxtimes_\sigma D$ the $C_0(G^0)$-tensor product [5], and drop the subscript ρ or σ if this map is understood. Given a $C_0(G^0)$-algebra C and a continuous surjection $t\colon G \to G^0$, we consider $C_0(G)$ as a $C_0(G^0)$-algebra via $t^*\colon C_0(G^0) \to M(C_0(G))$ and let $t^*C := C \boxtimes_{t^*} C_0(G)$, which is a $C_0(G)$-algebra in a natural way. Each morphism π of $C_0(G^0)$-algebras C, D induces a morphism of $t^*\pi$ of $C_0(G)$-algebras from t^*C to t^*D via $c \boxtimes f \mapsto \pi(c) \boxtimes f$. An *action* of G on a $C_0(G^0)$-algebra C is an isomorphism $\sigma\colon s^*C \to r^*C$ of $C_0(G)$-algebras such that the restrictions of σ to the fibers satisfy $\sigma_x \circ \sigma_y = \sigma_{xy}$ for all $(x,y) \in G_s {\times_r} G$ [17]. A *morphism* between actions (C,σ^C) and (D,σ^D) of G is a morphism of $C_0(G^0)$-algebras π from C to D satisfying $\sigma^D \circ s^*\pi = r^*\pi \circ \sigma^C$.

Definition 6.7. *We call an action (C,σ) of G* admissible *if the $C_0(G^0)$-algebra C is admissible, and we call a coaction (C_K^γ, δ) of $C_0(G)$* admissible *if C_K^γ is an admissible C^*-$\mathfrak{b}$-algebra and $[\delta(C)(1 \underset{\mathfrak{b}}{\otimes} C_0(G))] = C \underset{\mathfrak{b}}{\otimes} C_0(G)$ in $\mathcal{L}(K {}_\gamma\underset{\mathfrak{b}}{\otimes}{}_\alpha H)$.*

Remark 6.8. *If σ is an action of G on a continuous $C_0(G^0)$-algebra, then the subset $Y := \{u \in G^0 \mid C_u \neq 0\} \subseteq G^0$ is open, C is an admissible $C_0(Y)$-algebra, and σ restricts to an action of the subgroupoid $G|_Y := \{x \in G \mid r(x), s(x) \in Y\} \subseteq G$.*

Lemma 6.9. *Let C_K^γ and D_L^ϵ be admissible C^*-$\mathfrak{b}$-algebras, where D is commutative. Then there exists an isomorphism $C_{\rho_\gamma} \boxtimes_{\rho_\epsilon} D \to C {}_\gamma\underset{\mathfrak{b}}{\otimes}{}_\epsilon D$, $c \boxtimes d \mapsto c \underset{\mathfrak{b}}{\otimes} d$.*

Proof. Use Lemma 2.7 in [5] and apply Proposition 4.1 in [5] to the field of representations $C \hookrightarrow \mathcal{L}(K_\gamma) \cong \mathcal{L}(\gamma)$, noting that $\gamma \circledcirc_{\rho_\epsilon} D \cong [|\gamma\rangle_1 D]$ as a Hilbert C^*-D-module via $\xi \circledcirc d \mapsto |\xi\rangle_1 d$ and that $(C_\gamma \underset{\mathfrak{b}}{\otimes}_\epsilon D)[|\gamma\rangle_1 D] \subseteq [|\gamma\rangle_1 D]$. □

We use the isomorphism above without further notice.

Proposition 6.10. (i) *Let (C^γ_K, δ) be an admissible coaction of $C_0(G)$. There exists a unique action σ_δ of G on (C, ρ_γ) given by $c \boxtimes f \mapsto \delta(c)(1 \underset{\mathfrak{b}}{\otimes} f)$.*

(ii) *Let (C, σ) be an admissible action of G. There exists a unique admissible, injective coaction δ_σ of $C_0(G)$ on $\mathbf{F}C$ given by $\eta_C(c) \mapsto (r^*\eta_C)(\sigma(c \boxtimes 1))$.*

Proof. (i) Since $\delta(C)$ and $1 \underset{\mathfrak{b}}{\otimes} C_0(G)$ commute, there exists a unique $*$-homomorphism $\tilde{\sigma}$ from the algebraic tensor product $C \odot C_0(G)$ to r^*C such that $\tilde{\sigma}(c \odot f) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} f)$ for all $c \in C$, $f \in C_0(G)$. Since δ is a coaction, $\delta(c\rho_\gamma(g)) = \delta(c)\rho_{(\gamma \triangleright \widehat{\beta})}(g) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} s^*(g))$ for all $g \in C_0(G^0)$. From Lemma 2.7 in [5], we can conclude that $\tilde{\sigma}$ factorizes to a $*$-homomorphism $\sigma = \sigma_\delta \colon s^*C \to r^*C$ satisfying the formula in (i). This σ is surjective because $[\delta(C)(1 \underset{\mathfrak{b}}{\otimes} C_0(G))] = C \underset{\mathfrak{b}}{\otimes} C_0(G)$. In particular, σ_x is surjective for each $x \in G$. We claim that $\sigma_x \circ \sigma_y = \sigma_{xy}$ for all $(x, y) \in G_s{\times}_r G$. Define $r_1 \colon G_s{\times}_r G \to G^0$ by $(x, y) \mapsto r(x)$. By Lemma 6.9, we have isomorphisms $C_\gamma \underset{\mathfrak{b}}{\otimes}_\alpha C_0(G)_{\widehat{\beta}} \underset{\mathfrak{b}}{\otimes}_\alpha C_0(G) \cong C \boxtimes_{r^*} C_0(G)_{s^*} \boxtimes_{r^*} C_0(G) \cong C \boxtimes_{r_1^*} C_0(G_s{\times}_r G) \cong r_1^* C$. Using formula (5.3), we find

$$\begin{aligned} \sigma_x \circ \sigma_y \circ p_{s(y)} &= \sigma_x \circ p_y \circ \delta = p_{(x,y)} \circ (\delta * \mathrm{id}) \circ \delta, \\ \sigma_{xy} \circ p_{s(y)} &= p_{xy} \circ \delta = p_{(x,y)} \circ (\mathrm{id} * \widehat{\Delta}) \circ \delta, \end{aligned} \tag{6.2}$$

and the claim follows. Finally, $\sigma_u = \mathrm{id}_{C_u}$ for each $u \in G^0$ because σ_u is surjective and idempotent, and σ_x is injective for each $x \in G$ because $\sigma_{s(x)} = \sigma_{x^{-1}} \circ \sigma_x$ is injective. Therefore, σ is injective.

(ii) Let $D := \eta_C(C)_{l_C} \underset{\mathfrak{b}}{\otimes}_\alpha C_0(G)$. Then $\mathcal{D} := ((E_C)_{l_C} \underset{\mathfrak{b}}{\otimes}_\alpha H_{\widehat{\beta}}, D)$ is an admissible C^*-$\mathfrak{b}$-algebra. Define $\delta \colon C \to D$ by $c \mapsto (r^*\eta_C)(\sigma(c \boxtimes 1))$. Let $c \in C$, $g \in C_0(G^0)$. Then $cg \boxtimes 1 = c \boxtimes s^*(g)$ in $M(C \boxtimes_{s^*} C_0(G))$ and therefore $\delta(cg) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} s^*(g)) = \delta(c)\rho_{(l_C \triangleright \widehat{\beta})}(g)$. Consequently, δ is a morphism of $C_0(G^0)$-algebras from C to $(D, \rho_{l_C \triangleright \widehat{\beta}}) = \mathbf{G}\mathcal{D}$. By definition of $\mathbf{F}$ and ϵ, the morphism $\delta_\sigma := \epsilon_{\mathcal{D}} \circ \mathbf{F}\delta \colon \mathbf{F}C \to \mathbf{F}\mathbf{G}\mathcal{D} \to \mathcal{D}$ satisfies $\delta_\sigma \circ \eta_C = \delta$, and a similar calculation as in (6.2) shows that $(\delta_\sigma * \mathrm{id}) \circ \delta_\sigma = (\mathrm{id} * \widehat{\Delta}) \circ \delta_\sigma$. Consequently, δ_σ is a coaction of $(\widehat{\mathcal{A}}, \widehat{\Delta})$. Since σ is injective, so are δ and δ_σ. Finally, δ_σ is admissible because $[\delta_\sigma(\eta_C(C))(1 \underset{\mathfrak{b}}{\otimes} C_0(G))] = (r^*\eta_C)(\sigma(s^*C)) = r^*\eta_C(C) = [\eta_C(C) \underset{\mathfrak{b}}{\otimes} C_0(G)]$. □

Corollary 6.11. *Every admissible coaction of $C_0(G)$ is injective, left-full, and right-full.*

Proof. If (C^γ_K, δ) is an admissible coaction, then the relations $[C_0(G)\alpha] = \alpha$ and $[C\gamma] = \gamma$ imply $[\delta(C)|\alpha\rangle_2] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} C_0(G))|\alpha\rangle_2] = [(C \underset{\mathfrak{b}}{\otimes} C_0(G))|\alpha\rangle_2] = [|\alpha\rangle_2 C]$ and $[\delta(C)|\gamma\rangle_1 C_0(G)] = [\delta(C)(1 \underset{\mathfrak{b}}{\otimes} C_0(G))|\gamma\rangle_1] = [(C \underset{\mathfrak{b}}{\otimes} C_0(G))|\gamma\rangle_1] = [|\gamma\rangle_1 C_0(G)]$. Finally, δ is injective because σ_δ is injective and $\delta(c) = \sigma_\delta(c \boxtimes 1)$ for all $c \in C$. □

Proposition 6.12. *Let $(\mathcal{C}, \delta^{\mathcal{C}})$, $(\mathcal{D}, \delta^{\mathcal{D}})$ be admissible coactions with associated actions $\sigma^{\mathcal{C}} = \sigma_{\delta^{\mathcal{C}}}$, $\sigma^{\mathcal{D}} = \sigma_{\delta^{\mathcal{D}}}$, and let $\pi \in \mathbf{C^*}\text{-}\mathfrak{b}\text{-}\mathbf{alg}^a(\mathcal{C}, \mathcal{D}) = \mathbf{C_0(G^0)}\text{-}\mathbf{alg}^a(\mathbf{G}\mathcal{C}, \mathbf{G}\mathcal{D})$. Then $(\pi * \mathrm{id}) \circ \delta^{\mathcal{C}} = \delta^{\mathcal{D}} \circ \pi$ if and only if $r^*\pi \circ \sigma^{\mathcal{C}} = \sigma^{\mathcal{D}} \circ s^*\pi$.*

Proof. Write $\mathcal{C} = C^\gamma_K$. The assertion holds because for all $c \in C$ and $f \in C_0(G)$,

$$((\pi * \mathrm{id})(\delta^{\mathcal{C}}(c)))(1 \underset{\mathfrak{b}}{\otimes} f) = (\pi * \mathrm{id})(\delta^{\mathcal{C}}(c)(1 \underset{\mathfrak{b}}{\otimes} f)) = (r^*\pi \circ \sigma^{\mathcal{C}})(c \boxtimes f),$$
$$\delta^{\mathcal{D}}(\pi(c))(1 \underset{\mathfrak{b}}{\otimes} f) = \sigma^{\mathcal{D}}(\pi(c) \boxtimes f) = (\sigma^{\mathcal{D}} \circ s^*\pi)(c \boxtimes f). \qquad \square$$

We denote by $\mathbf{G}\text{-}\mathbf{act}^a$ and $\mathbf{Coact}^a_{C_0(G)}$ the categories of all admissible actions of G and all admissible coactions of $C_0(G)$, respectively.

Theorem 6.13. *There exists an embedding as a full and coreflective subcategory $(\hat{\mathbf{F}}, \hat{\mathbf{G}}, \hat{\eta}, \hat{\epsilon})$ of $\mathbf{G}\text{-}\mathbf{act}^a$ into $\mathbf{Coact}^a_{C_0(G)}$, where*

(i) *$\hat{\mathbf{F}}$ is given by $(C, \sigma) \mapsto (\mathbf{F}C, \delta_\sigma)$ on objects and $\pi \mapsto \mathbf{F}\pi$ on morphisms;*
(ii) *$\hat{\mathbf{G}}$ is given by $(\mathcal{C}, \delta) \mapsto (\mathbf{G}\mathcal{C}, \sigma_\delta)$ on objects and $\pi \mapsto \mathbf{G}\pi = \pi$ on morphisms;*
(iii) *$\hat{\eta}_{(C,\sigma)} = \eta_C$ and $\hat{\epsilon}_{(\mathcal{C},\delta)} = \epsilon_{\mathcal{C}}$ for all objects (C, σ) and $(\mathcal{C}, \delta)$.*

Proof. The assignments $\hat{\mathbf{G}}$ and $\hat{\mathbf{F}}$ are well defined on objects and morphisms by Proposition 6.10 and 6.12. For each admissible action (C, σ), we get $\eta_C \in \mathbf{G}\text{-}\mathbf{act}^a((C, \sigma), \hat{\mathbf{G}}\hat{\mathbf{F}}(C, \sigma))$ because $\sigma_{\delta_\sigma}(\eta_C(c) \boxtimes f) = \delta_\sigma(\eta_C(c))(1 \underset{\mathfrak{b}}{\otimes} f) = r^*\eta_C(\sigma(c \boxtimes f))$ for all $c \in C$, $f \in C_0(G)$, and Proposition 6.12 implies that $\epsilon_{\mathcal{C}} = \eta^{-1}_{\mathbf{G}\mathcal{C}} \in \mathbf{Coact}^a_{C_0(G)}(\hat{\mathbf{F}}\hat{\mathbf{G}}(\mathcal{C}, \delta), (\mathcal{C}, \delta))$ for each admissible coaction $(\mathcal{C}, \delta)$. Now, the assertion follows from Theorem 6.6. $\square$

6.3. **Comparison of the associated reduced crossed products.** The reduced crossed product for an action (C, σ) of G is defined as follows [17]. The subspace $C_c(G; C, \sigma) := C_c(G)r^*C \subseteq r^*C$ carries the structure of a $*$-algebra and the structure of a pre-Hilbert C^*-module over C such that

$$(ab)_x = \int_{G^{r(x)}} a_y \sigma_y(b_{y^{-1}x})\, \mathrm{d}\lambda^{r(x)}(y), \qquad (a^*)_x = \sigma_x(a^*_{x^{-1}}),$$
$$\langle a|b\rangle_u = \int_{G^u} \sigma_y((a_{y^{-1}})^* b_{y^{-1}})\, \mathrm{d}\lambda^u(y) = (a^*b)_u, \qquad (ac)_x = a_x\sigma_x(c_{s(x)})$$

for all $a, b \in C_c(G; C, \sigma)$, $u \in G^0$ and $c \in C$, $x \in G$. Denote the completion of this pre-Hilbert C^*-module by $L^2(G, \lambda^{-1}; C, \sigma)$. Using the relation $\langle a|bd\rangle_u = (abd)_u = \langle b^*a|d\rangle_u$, which holds for all $a, b, d \in C_c(G; C, \sigma)$, $u \in G^0$, and a routine norm estimate, one verifies the existence of a $*$-homomorphism $\pi\colon C_c(G; C, \sigma) \to \mathcal{L}(L^2(G, \lambda^{-1}; C, \sigma))$ such that $\pi(b)d = bd$ for all $b, d \in C_c(G; C, \sigma)$. Then the *reduced crossed product* of (C, σ) is the C^*-algebra $C \rtimes_{\sigma,r} G := [\pi(C_c(G; C, \sigma))] \subseteq \mathcal{L}(L^2(G, \lambda^{-1}; C, \sigma))$.

Proposition 6.14. *Let (C^γ_K, δ) be an admissible coaction of $C_0(G)$, consider C as a $C_0(G^0)$-algebra via ρ_γ, and let $\sigma = \sigma_\delta$. Then there exists an isomorphism $C \rtimes_{\sigma_\delta,r} G \to C \rtimes_r C^*_r(G)$ given by $\pi(c \boxtimes f) \mapsto \delta(c)(\mathrm{id} \underset{\mathfrak{b}}{\otimes} UL(f)U)$ for all $c \in C$, $f \in C_c(G)$.*

Proof. Let $\delta_U := \mathrm{Ad}_{(\mathrm{id} \underset{\mathfrak{b}}{\otimes} U)} \circ \delta \colon C \to \mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_{\hat{\alpha}} H)$. We equip $C_c(G; C, \sigma)$ with the structure of a pre-Hilbert C^*-module over C such that

$$(ac)_x = a_x c_{s(x)} \qquad \text{and} \qquad \langle a|b\rangle_u = \int_{G_u} (a_x)^* b_x \,\mathrm{d}\lambda_u^{-1}(x)$$

for all $a, b \in C_c(G; C, \sigma)$, $c \in C$, $u \in G^0$, and denote by $L^2(G, \lambda^{-1}; C)$ the completion. One easily checks that there exists a unique unitary $\Phi \colon L^2(G, \lambda^{-1}; C) \to [|\hat{\alpha}\rangle_2 C] = [\delta_U(C)|\hat{\alpha}\rangle_2\rangle]$ given by $c \boxtimes f \mapsto |\hat{j}(f)\rangle_2 c$, and that for all $c \in C, f \in C_c(G), y \in G$,

$$\Phi^{-1}(\delta_U(c)|\hat{j}(f)\rangle_2)_y = \sigma_{y^{-1}}(c_{r(y)}) f(y).$$

Hence, there exists a unitary $\Psi \colon L^2(G, \lambda^{-1}; C, \sigma) \to [\delta_U(C)|\hat{\alpha}\rangle_2\rangle]$ given by $c \boxtimes f \mapsto \delta_U(c)|\hat{j}(f)\rangle_2$. Let $c, d \in C$, $f, g \in C_c(G)$ and $\omega = \Phi^{-1}(\Psi(c \boxtimes f))$. Then $\delta_U(d)(\mathrm{id} \underset{\mathfrak{b}}{\otimes} L(g))\Psi = \Psi\pi(d \boxtimes g)$ because for all $x \in G$,

$$\begin{aligned}
\Phi^{-1}(\delta_U(d)(\mathrm{id} \underset{\mathfrak{b}}{\otimes} L(g))\Phi(\omega))_x &= \int_{G_u} \sigma_{x^{-1}}(d_{r(x)}) g(xy^{-1}) \omega_y \,\mathrm{d}\lambda_u^{-1}(y) \\
&= \int_{G_u} \sigma_{x^{-1}}\big(d_{r(xy^{-1})} g(xy^{-1}) \sigma_{xy^{-1}}(c_{r(y)}) f(y)\big) \,\mathrm{d}\lambda_u^{-1}(y) \\
&= \Phi^{-1}(\Psi(\pi(d \boxtimes g)(c \boxtimes f)))_x.
\end{aligned}$$

Since $d \in C$ and $g \in C_c(G)$ were arbitrary, the assertion follows. $\square$

7. Fell bundles on groupoids

We now gather preliminaries on Fell bundles that are needed in Sections 8 and 9. We use the notion of a Banach bundle and standard notation; see [8].

7.1. Fell bundles on groupoids and their C^*-algebras.

We first recall the notion of a Fell bundle on G and the definition of the associated reduced C^*-algebra [15]. Given an upper semicontinuous Banach bundle $p \colon \mathcal{F} \to G$, denote by $\mathcal{F}^0$ the restriction of $\mathcal{F}$ to G^0, by $\mathcal{F}_{sp}\times_{rp}\mathcal{F}$ the restriction of $\mathcal{F} \times \mathcal{F}$ to $G_s\times_r G$, by $\mathcal{F}_x$ for each $x \in G$ the fiber at x, by $\Gamma_c(\mathcal{F})$ the space of continuous sections of $\mathcal{F}$ with compact support, and by $\Gamma_0(\mathcal{F}^0)$ the space of continuous sections of $\mathcal{F}^0$ that vanish at infinity in norm.

Definition 7.1. *A* Fell bundle *on G is an upper semicontinuous Banach bundle $p \colon \mathcal{F} \to G$ with a continuous multiplication $\mathcal{F}_{sp}\times_{rp}\mathcal{F} \to \mathcal{F}$ and a continuous involution $* \colon \mathcal{F} \to \mathcal{F}$ such that for all $e \in \mathcal{F}$, $(e_1, e_2) \in \mathcal{F}_{sp}\times_{rp}\mathcal{F}$, $(x, y) \in G_s\times_r G$,*

(i) *$p(e_1e_2) = p(e_1)p(e_2)$ and $p(e^*) = p(e)^{-1}$;*
(ii) *the map $\mathcal{F}_x \times \mathcal{F}_y \to \mathcal{F}_{xy}, (e'_1, e'_2) \mapsto e'_1e'_2$, is bilinear and the map $\mathcal{F}_x \to \mathcal{F}_{x^{-1}}$, $e' \mapsto e'^*$, is conjugate linear;*
(iii) *$(e_1e_2)e_3 = e_1(e_2e_3)$, $(e_1e_2)^* = e_2^*e_1^*$, and $(e^*)^* = e$;*
(iv) *$\|e_1e_2\| \leqslant \|e_1\|\|e_2\|$, $\|e^*e\| = \|e\|^2$, and $e^*e \geqslant 0$ in the C^*-algebra $\mathcal{F}_{s(p(e))}$.*

We call $\mathcal{F}$ saturated *if $[\mathcal{F}_x\mathcal{F}_y] = \mathcal{F}_{xy}$ for all $(x, y) \in G_s\times_r G$, and* admissible *if $\Gamma_0(\mathcal{F}^0)$ is an admissible $C_0(G^0)$-algebra with respect to the pointwise operations.*

Let $\mathcal{F}$ be a Fell bundle on G. The associated reduced C^*-algebra is defined as follows. The space $\Gamma_c(\mathcal{F})$ is a $*$-algebra with respect to the multiplication and involution given by

$$(cd)(x) = \int_{G^{r(x)}} c(y)d(y^{-1}x)\,\mathrm{d}\lambda^{r(x)}(y) = \int_{G_{s(x)}} c(xz^{-1})d(z)\,\mathrm{d}\lambda^{-1}_{s(x)}(z) \tag{7.1}$$

and $c^*(x) = c(x^{-1})^*$, respectively, and a pre-Hilbert C^*-module over $\Gamma_0(\mathcal{F}^0)$ with respect to the structure maps

$$\langle c|d\rangle(u) = \int_{G_u} c(x)^*d(x)\,\mathrm{d}\lambda_u^{-1}(x) = (c^*d)(u), \qquad (ce)(x) = c(x)e(s(x)),$$

where $c, d \in \Gamma_c(\mathcal{F})$, $e \in \Gamma_0(\mathcal{F}^0)$, $x \in G$. Denote by $\Gamma^2(\mathcal{F}, \lambda^{-1})$ the completion of this pre-Hilbert C^*-module. Then there exists a $*$-homomorphism

$$L_{\mathcal{F}} \colon \Gamma_c(\mathcal{F}) \to \mathcal{L}(\Gamma^2(\mathcal{F}, \lambda^{-1})), \quad L_{\mathcal{F}}(a)b = ab \text{ for all } a, b \in \Gamma_c(\mathcal{F}),$$

and $C_r^*(\mathcal{F}) := [L_{\mathcal{F}}(\Gamma_c(\mathcal{F}))] \subseteq \mathcal{L}(\Gamma^2(\mathcal{F}, \lambda^{-1}))$ is the *reduced C^*-algebra of $\mathcal{F}$*. We identify $\Gamma_c(\mathcal{F})$ with $L_{\mathcal{F}}(\Gamma_c(\mathcal{F})) \subseteq C_r^*(\mathcal{F})$ via $L_{\mathcal{F}}$.

We equip $\Gamma_c(\mathcal{F})$ with the *inductive limit topology*; thus, a net converges if it converges uniformly and if the supports of its members are contained in some compact set. We shall use the following result; see Proposition 2.3 in [8].

Lemma 7.2. *Let $\mathcal{E}$ be an upper semicontinuous Banach bundle on a locally compact, second countable, Hausdorff space X and let $\Gamma' \subseteq \Gamma_c(\mathcal{E})$ be a subspace such that*

(i) *Γ' is closed under pointwise multiplication with elements of $C_c(X)$;*
(ii) *$\{f(x) \mid f \in \Gamma'\} \subseteq \mathcal{E}_x$ is dense for each $x \in X$.*

Then Γ' is dense in $\Gamma_c(\mathcal{E})$.

Given $f \in \Gamma_c(\mathcal{F})$ and $g \in \Gamma_0(\mathcal{F}^0)$, define $fg, gf \in \Gamma_c(\mathcal{F})$ by $(fg)(x) = f(x)g(s(x))$, $(gf)(x) = g(r(x))f(x)$ for all $x \in G$. Using the relation $[\mathcal{F}_x] = [\mathcal{F}_x\mathcal{F}_x^*\mathcal{F}_x]$, where $x \in G$, and Lemma 7.2, we find:

Lemma 7.3. *$\Gamma_c(\mathcal{F})\Gamma_0(\mathcal{F}^0)$ and $\Gamma_0(\mathcal{F}^0)\Gamma_c(\mathcal{F})$ are linearly dense in $\Gamma_c(\mathcal{F})$.* □

7.2. **The multiplier bundle of a Fell bundle.** Given a Fell bundle $\mathcal{F}$ on G, we define a multiplier bundle $\mathcal{M}(\mathcal{F})$ on G, extending the definition in §VIII.2.14 of [12]. Given a subspace $C \subseteq G$, we denote by $\mathcal{F}|_C$ the restriction of $\mathcal{F}$ to C.

Definition 7.4. *Let $x \in G$. A* multiplier of $\mathcal{F}$ of order x *is a map $T \colon \mathcal{F}|_{G^{s(x)}} \to \mathcal{F}|_{G^{r(x)}}$ such that $T\mathcal{F}_y \subseteq \mathcal{F}_{xy}$ for all $y \in G^{s(x)}$ and such that there exists a map $T^* \colon \mathcal{F}|_{G^{r(x)}} \to \mathcal{F}|_{G^{s(x)}}$ such that $e^*Tf = (T^*e)^*f$ for all $e \in \mathcal{F}|_{G^{r(x)}}$, $f \in \mathcal{F}|_{G^{s(x)}}$. We denote by $\mathcal{M}(\mathcal{F})_x$ the set of all multipliers of $\mathcal{F}$ of order x.*

As for adjointable operators of Hilbert C^*-modules, one deduces from the definition the following simple properties. Let $x \in G$. Then for each $T \in \mathcal{M}(\mathcal{F})_x$, the map T^* is uniquely determined, $T^* \in \mathcal{M}(\mathcal{F})_{x^{-1}}$, and $T^{**} = T$. Moreover, each $T \in \mathcal{M}(\mathcal{F}_x)$ is fiberwise linear in the sense that $T(\kappa e + f) = \kappa Te + Tf$ for all $\kappa \in \mathbb{C}$, $e, f \in \mathcal{F}_y$, $y \in G^{s(x)}$. The restrictions $T_{s(x)} \colon \mathcal{F}_{s(x)} \to \mathcal{F}_x$ and $(T^*)_x \colon \mathcal{F}_x \to \mathcal{F}_{s(x)}$ are adjoint operators of Hilbert C^*-modules over $\mathcal{F}_{s(x)}$, and since $\mathcal{F}_y = [\mathcal{F}_{r(y)}\mathcal{F}_y]$ for each $y \in G^{s(x)}$, the map $\mathcal{M}(\mathcal{F})_x \to \mathcal{L}(\mathcal{F}_{s(x)}, \mathcal{F}_x)$, $T \mapsto T_{s(x)}$, is a bijection. Clearly, we have a natural embedding $\mathcal{F}_x \hookrightarrow \mathcal{M}(\mathcal{F})_x$, where each $f \in \mathcal{F}$ acts as a multiplier via left multiplication. For each $y \in G^{s(x)}$, we have $\mathcal{M}(\mathcal{F})_x \mathcal{M}(\mathcal{F})_y \subseteq \mathcal{M}(\mathcal{F})_{xy}$, and for each $f \in \mathcal{F}_z$, $z \in G_{r(x)}$, we let $fT := (T^* f^*)^*$.

Definition 7.5. *For each $x \in G$, consider $\mathcal{M}(\mathcal{F})_x$ as a Banach space via the identification with $\mathcal{L}(\mathcal{F}_{s(x)}, \mathcal{F}_x)$. Let $\mathcal{M}(\mathcal{F}) = \coprod_{x \in G} \mathcal{M}(\mathcal{F})_x$ and denote by $\tilde{p} \colon \mathcal{M}(\mathcal{F}) \to G$ the natural map. The* strict topology *on $\mathcal{M}(\mathcal{F})$ is the weakest topology that makes $\tilde{p}$ and the maps $\mathcal{M}(\mathcal{F}) \to \mathcal{F}$ of the form $c \mapsto c \cdot d(s(\tilde{p}(c)))$ and $c \mapsto d(r(\tilde{p}(c))) \cdot c$ continuous for each $d \in \Gamma_c(\mathcal{F}^0)$. Denote by $\Gamma_c(\mathcal{M}(\mathcal{F}))$ the space of all sections that are strictly continuous, norm-bounded, and compactly supported.*

Remark 7.6. *The bundle $\mathcal{M}(\mathcal{F})$ satisfies all axioms of a Fell bundle except for the fact that it is no Banach bundle with respect to the strict topology unless $\mathcal{M}(\mathcal{F}) = \mathcal{F}$. Indeed, for each $u \in G^0$, the subspace topology on $\mathcal{M}(\mathcal{F})_u \cong \mathcal{L}(\mathcal{F}_u) \cong M(\mathcal{F}_u)$ is the strict topology and coincides with the norm topology only if $M(\mathcal{F}_u) = \mathcal{F}_u$.*

Given $f \in \Gamma_c(\mathcal{M}(\mathcal{F}))$ and $g \in \Gamma_0(\mathcal{F}^0)$, define $fg, gf \in \Gamma_c(\mathcal{F})$ by $(fg)(x) = f(x)g(s(x))$, $(gf)(x) = g(r(x))f(x)$ for all $x \in G$ again.

Lemma 7.7. (i) *Let $c \in \Gamma_c(\mathcal{M}(\mathcal{F}))$ and $d \in \Gamma_c(\mathcal{F})$. Then there exists a section $cd \in \Gamma_c(\mathcal{F})$ such that $(cd)(x) = \int_{G^{r(x)}} c(y)d(y^{-1}x) \,\mathrm{d}\lambda^{r(x)}(y)$ for all $x \in G$.*

(ii) *$\Gamma_c(\mathcal{M}(\mathcal{F}))$ carries a structure of a $*$-algebra such that $c^*(x) = c(x^{-1})^*$ and $(cd)(x)e = \int_{G^{r(x)}} c(y)d(y^{-1}x)e \,\mathrm{d}\lambda^{r(x)}$ for all $c, d \in \Gamma_c(\mathcal{M}(\mathcal{F}))$, $x \in G$, $e \in \mathcal{F}_{s(x)}$.*

(iii) *There exists a $*$-homomorphism $L_{\mathcal{M}(\mathcal{F})} \colon \Gamma_c(\mathcal{M}(\mathcal{F})) \to M(C_r^*(\mathcal{F}))$ such that $L_{\mathcal{M}(\mathcal{F})}(c)L_{\mathcal{F}}(d) = L_{\mathcal{F}}(cd)$ for all $c \in \Gamma_c(\mathcal{M}(\mathcal{F}))$, $d \in \Gamma_c(\mathcal{F})$.*

(iv) *$\Gamma_c(\mathcal{M}(\mathcal{F}))$ is closed under pointwise multiplication with elements of $C_c(G)$.*

Proof. (i) Define $cd \colon G \to \mathcal{F}$ as above, and let $\epsilon > 0$. Using Lemma 7.3, we find a sequence $(g_n)_n$ in the span of $\Gamma_0(\mathcal{F}^0)\Gamma_c(\mathcal{F})$ that converges to d in the inductive limit topology. Since $\Gamma_c(\mathcal{M}(\mathcal{F}))\Gamma_0(\mathcal{F}^0) \subseteq \Gamma_c(\mathcal{F})$, the map $h_n \colon x \mapsto \int_{G^{r(x)}} c(y)g_n(y^{-1}x) \,\mathrm{d}\lambda^{r(x)}(y)$ lies in $\Gamma_c(\mathcal{F})$ for each n. Using the fact that c has compact support and bounded norm, one easily concludes that $(h_n)_n$ converges in the inductive limit topology to cd which therefore is in $\Gamma_c(\mathcal{F})$.

(ii) Note that $(cd)(x)$ is well defined because the map $y \mapsto d(y^{-1}x)e$ is in $\Gamma_c(\mathcal{F})$ and thus i) applies. Now, the assertion follows from standard arguments.

(iii) One easily sees that there exists a representation $L_{\mathcal{M}(\mathcal{F})} \colon \Gamma_c(\mathcal{M}(\mathcal{F})) \to \mathcal{L}(\Gamma^2(\mathcal{F}))$ such that $L_{\mathcal{M}(\mathcal{F})}(c)d = cd$ for all $c \in \Gamma_c(\mathcal{M}(\mathcal{F}))$, $d \in \Gamma_c(\mathcal{F})$, and that $L_{\mathcal{M}(\mathcal{F})}(c)L_{\mathcal{F}}(d)e = cde = L_{\mathcal{F}}(cd)e$ for all $c \in \Gamma_c(\mathcal{M}(\mathcal{F}))$, $d, e \in \Gamma_c(\mathcal{F})$.

(iv) This follows immediately from the fact that $\Gamma_c(\mathcal{F})$ is closed under pointwise multiplication by elements of $C_c(G)$. $\square$

7.3. **Morphisms between Fell bundles.** Let $\mathcal{F}$ and $\mathcal{G}$ be Fell bundles on G.

Definition 7.8. *A* (fibrewise nondegenerate) morphism *from $\mathcal{F}$ to $\mathcal{G}$ is a continuous map $T \colon \mathcal{F} \to \mathcal{M}(\mathcal{G})$ that satisfies the following conditions:*

(i) *for each $x \in G$, the map T restricts to a linear map $T_x \colon \mathcal{F}_x \to \mathcal{M}(\mathcal{G})_x$;*

(ii) *$T(e_1)T(e_2) = T(e_1e_2)$ and $T(e)^* = T(e^*)$ for all $(e_1, e_2) \in \mathcal{F}\,{}_{sp}\!\times_{rp}\mathcal{F}$, $e \in \mathcal{F}$;*

(iii) $\mathcal{G}_x = [T(\mathcal{F}_x)\mathcal{G}_{s(x)}]$ *for each* $x \in G^0$.

Let T be a morphism from $\mathcal{F}$ to $\mathcal{G}$. Then $T_u \colon \mathcal{F}_u \to \mathcal{M}(\mathcal{G})_u$ is a nondegenerate $*$-homomorphism for each $u \in G^0$; in particular, $\|T_u\| \leqslant 1$. One easily concludes that $\|T_x\| \leqslant 1$ for each $x \in G$. Hence, the formula $f \mapsto T \circ f$ defines $*$-homomorphisms $T_* \colon \Gamma_c(\mathcal{F}) \to \Gamma_c(\mathcal{M}(\mathcal{G}))$ and $T^0_* \colon \Gamma_0(\mathcal{F}^0) \to M(\Gamma_0(\mathcal{G}^0))$.

Proposition 7.9. (i) $T^0_* \colon \Gamma_0(\mathcal{F}^0) \to M(\Gamma_0(\mathcal{G}^0))$ *is nondegenerate.*
(ii) $T_*(\Gamma_c(\mathcal{F}))\Gamma_c(\mathcal{G}^0)$ *is dense in* $\Gamma_c(\mathcal{F})$.
(iii) T_* *extends to a nondegenerate* $*$*-homomorphism* $T_* \colon C^*_r(\mathcal{F}) \to M(C^*_r(\mathcal{G}))$.

Proof. Assertions (i) and (ii) follow immediately from Lemma 7.2 and 7.7. Part (ii) and a straightforward calculation show that there exists a unique unitary $\Psi \colon \Gamma^2(\mathcal{F}, \lambda^{-1}) \otimes_{T^0_*} \Gamma_0(\mathcal{G}^0) \to \Gamma^2(\mathcal{G}, \lambda^{-1})$ such that $(\Psi(f \otimes g))(x) = T_*(f)g$ for all $f \in \Gamma_c(\mathcal{F})$, $g \in \Gamma_0(\mathcal{G}^0)$. The map $C^*_r(\mathcal{F}) \to \mathcal{L}(\Gamma^2(\mathcal{G}, \lambda^{-1}))$ given by $f \mapsto \Psi(f \otimes \mathrm{id})\Psi^*$ is the desired extension. Lemma 7.3 and part (ii) imply that $[T_*(\Gamma_c(\mathcal{F}))\Gamma_c(\mathcal{G})] = [T_*(\Gamma_c(\mathcal{F}))\Gamma_0(\mathcal{G}^0)\Gamma_c(\mathcal{G})] = [\Gamma_c(\mathcal{G})\Gamma_c(\mathcal{G})] = C^*_r(\mathcal{G})$. □

8. From Fell bundles on G to coactions of $C^*_r(G)$

Let G be a groupoid, V the associated C^*-pseudo-multiplicative unitary, and $C^*_r(G)$ or, more precisely, $(\mathcal{A}, \Delta)$ the associated Hopf C^*-bimodule as in Section 5. We relate Fell bundles on G to coactions of $C^*_r(G)$ as follows. Let $\mathcal{F}$ be an admissible Fell bundle $\mathcal{F}$ on G. We shall construct a coaction of $C^*_r(G)$ on $C^*_r(\mathcal{F})$ which is unitarily implemented by a representation of V, and identify the reduced crossed product of this coaction with the reduced C^*-algebra of another Fell bundle. Finally, we show that this construction is functorial.

A *representation of the unitary* V is a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-module ${}_\gamma K_{\hat{\delta}}$ together with a unitary $X \colon K_{\hat{\delta}} \underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H \to K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H$ that satisfies $X(\gamma \triangleleft \alpha) = \gamma \triangleright \alpha$, $X(\hat{\delta} \triangleright \beta) = \hat{\delta} \triangleleft \beta$, $X(\hat{\delta} \triangleright \hat{\beta}) = \gamma \triangleright \hat{\beta}$, and $X_{12}X_{13}V_{23} = V_{23}X_{12}$; see §4 in [28]. We construct a coaction out of such a representation as follows.

Lemma 8.1. *Let* $({}_\gamma K_{\hat{\delta}}, X)$ *be a representation of* V, *let* C^γ_K *be a* C^*-$\mathfrak{b}$*-algebra such that* $[C, \rho_{\hat{\beta}}(\mathfrak{B})] = 0$, *define* $\delta \colon C \to \mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H)$ *by* $c \mapsto X(c \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})X^*$, *and assume that* $[\delta(C)|\gamma\rangle_1 A] \subseteq [|\gamma\rangle_1 A]$ *and* $[\delta(C)|\beta\rangle_2] \subseteq [|\beta\rangle_2 C]$. *Then* δ *is injective, a morphism from* (K_γ, C) *to* $(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H_\alpha, C_\gamma \underset{\mathfrak{b}}{*}{}^\beta A)$, *and a coaction of* $(\mathcal{A}, \Delta)$ *on* C^γ_K. *If the inclusions above are equalities, then* δ *is left- or right-full, respectively.*

Proof. Evidently, δ is injective. It is a morphism of C^*-$\mathfrak{b}$-algebras because $X|\xi\rangle_2 c = \delta(c)X|\xi\rangle_2$ for each $\xi \in \alpha$, $c \in C$ and because $[X|\alpha\rangle_2\gamma] = \gamma \triangleright \alpha$ and $[(X|\alpha\rangle_2)^*(\gamma \triangleright \alpha)] = [\langle\alpha|_2(\gamma \triangleleft \alpha)] = \gamma$. Finally, for each $c \in C$,

$$\begin{aligned}(\delta * \mathrm{id})(\delta(c)) &= X_{12}X_{13}c_1X^*_{13}X^*_{12} \\ &= X_{12}X_{13}V_{23}c_1V^*_{23}X^*_{13}X^*_{12} \\ &= V_{23}X_{12}c_1X^*_{12}V^*_{23} = (\mathrm{id} * \Delta)(\delta(c)),\end{aligned}$$

where c_1 denotes c acting on the first factor of an iterated relative tensor product. □

8.1. **The representation of V associated to $\mathcal{F}$.** Denote by $\mathcal{W} = \mathcal{W}(\Gamma_0(\mathcal{F}^0))$ the set of all $C_0(G^0)$-weights on $\Gamma_0(\mathcal{F}^0)$ and let $\phi \in \mathcal{W}$.

Lemma 8.2. *Let $c, d \in \Gamma_c(\mathcal{F})$. Then the map $x \mapsto \phi_{s(x)}(c(x)^* d(x))$ lies in $C_c(G)$.*

Proof. The function $G \to s^*\mathcal{F}^0$ given by $x \mapsto c(x)^* d(x)$ is continuous and has compact support, and the composition $h\colon x \mapsto \phi_{s(x)}(c(x)^* d(x))$ is continuous because the map $\mathcal{F}^0 \to \mathbb{C}$ given by $f \mapsto \phi_{p(f)}(f)$ is continuous. □

Define Hilbert C^*-$C_0(G^0)$-modules $\Gamma^2(\mathcal{F}, \lambda; \phi)$, $\Gamma^2(\mathcal{F}, \lambda^{-1}; \phi)$ and a Hilbert space $K_\phi = \Gamma^2(\mathcal{F}, \nu; \phi)$ as the respective completions of $\Gamma_c(\mathcal{F})$, where for all $c, d \in \Gamma_c(\mathcal{F})$, $f \in C_0(G^0)$, the inner product $\langle c|d\rangle$ and the product cf are given by

$$\begin{aligned} &u \mapsto \int_{G^u} \phi_{s(x)}(c(x)^* d(x))\, \mathrm{d}\lambda^u(x), && y \mapsto c(y) f(r(y)) && \text{in case of } \Gamma^2(\mathcal{F}, \lambda; \phi), \\ &u \mapsto \int_{G_u} \phi_{s(x)}(c(x)^* d(x))\, \mathrm{d}\lambda_u^{-1}(x), && y \mapsto c(y) f(s(y)) && \text{in case of } \Gamma^2(\mathcal{F}, \lambda^{-1}; \phi), \\ &\text{and } \int_{G} \phi_{s(x)}(c(x)^* d(x))\, \mathrm{d}\nu(x) && && \text{in case of } \Gamma^2(\mathcal{F}, \nu; \phi). \end{aligned}$$

Lemma 8.3. $[\langle E|E\rangle] = [\phi(\Gamma_0(\mathcal{F}^0))]$ *for* $E \in \{\Gamma^2(\mathcal{F}, \lambda, \phi), \Gamma^2(\mathcal{F}, \lambda^{-1}; \phi)\}$.

Proof. Assume that $(\phi(c^*c))(u) \neq 0$ for some $c \in \Gamma_c(\mathcal{F}^0)$, $u \in G^0$. Choose $d \in \Gamma_c(\mathcal{F})$ such that $d|_{G^0} = c$. Then the function on G given by $x \mapsto \phi_{s(x)}(d(x)^* d(x))$ is non-negative and nonzero at u, whence $\langle d|d\rangle_E(u) \neq 0$. Now, the assertion follows because $[\langle E|E\rangle]$ and $[\phi(\Gamma_0(\mathcal{F}^0))]$ are closed ideals in $C_0(G^0)$. □

Let $K = \bigoplus_{\phi \in \mathcal{W}} K_\phi$ and identify each K_ϕ with a subspace of K. Given $c \in \Gamma_c(\mathcal{F})$ and $f \in C_0(G^0)$, define $fc, cf, cD^{-1/2} \in \Gamma_c(\mathcal{F})$ by

$$fc\colon x \mapsto f(r(x)) c(x), \qquad cf\colon x \mapsto c(x) f(s(x)), \qquad cD^{-1/2}\colon x \mapsto c(x) D^{-1/2}(x).$$

Let $\phi \in \mathcal{W}$. Straightforward calculations show that there exist maps

$$j_\phi\colon \Gamma^2(\mathcal{F}, \lambda; \phi) \to \mathcal{L}(\mathfrak{K}, K_\phi) \qquad \text{and} \qquad \hat{j}_\phi\colon \Gamma^2(\mathcal{F}, \lambda^{-1}; \phi) \to \mathcal{L}(\mathfrak{K}, K_\phi)$$

such that $j_\phi(c) f = fc$ and $\hat{j}_\phi(c) f = (cD^{-1/2}) f$ for all $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G^0)$, and

$$j_\phi(c)^* j_\phi(d) = \langle c|d\rangle_{\Gamma^2(\mathcal{F},\lambda;\phi)}, \qquad \hat{j}_\phi(c)^* \hat{j}_\phi(d) = \langle c|d\rangle_{\Gamma^2(\mathcal{F},\lambda^{-1};\phi)} \text{for all } c, d \in \Gamma_c(\mathcal{F}).$$

Denote by $\gamma \subseteq \mathcal{L}(\mathfrak{K}, K)$ and $\hat{\delta} \subseteq \mathcal{L}(\mathfrak{K}, K)$ the closed linear span of all subspaces $j_\phi(\Gamma^2(\mathcal{F}, \lambda; \phi))$ and $\hat{j}_\phi(\Gamma^2(\mathcal{F}, \lambda^{-1}; \phi))$, respectively, where $\phi \in \mathcal{W}$. Lemmas 6.4 and 8.3 imply:

Lemma 8.4. *${}_\gamma K_{\hat{\delta}}$ is a C^*-$(\mathfrak{b}, \mathfrak{b}^\dagger)$-module, and for all $f \in C_0(G^0)$ and $(c_\phi)_\phi \in \bigoplus_\phi \Gamma_c(\mathcal{F}) \subseteq K$, we have $\rho_\gamma(f)(c_\phi)_\phi = (fc_\phi)_\phi$ and $\rho_{\hat{\delta}}(f)(c_\phi)_\phi = (c_\phi f)_\phi$.* □

For $t = s, r$, denote by $p_1^{t,r}\colon G_t {\times}_r G \to G$ the projection onto the first component, by $\mathcal{F}^2_{t,r} = (p_1^{t,r})^*\mathcal{F}$ the corresponding pull-back of $\mathcal{F}$, and by $\Gamma^2(\mathcal{F}^2_{t,r}, \nu^2_{t,r}; \phi)$ the Hilbert space that is the completion of $\Gamma_c(\mathcal{F}^2_{t,r})$ with respect to the inner product

$$\langle c|d\rangle = \int_{G_t \times_r G} \phi_{s(x)}(c(x,y)^* d(x,y))\, \mathrm{d}\nu^2_{t,r}(x,y).$$

Straightforward calculations show that there exist unitaries

$$\Phi\colon K_{\hat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_{\alpha}H \to \bigoplus_{\phi\in\mathcal{W}} \Gamma^2(\mathcal{F}^2_{s,r}, \nu^2_{s,r}; \phi), \qquad \Psi\colon K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H \to \bigoplus_{\phi\in\mathcal{W}} \Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi),$$

such that for all $\phi \in \mathcal{W}$, $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G^0)$, $g \in C_c(G)$,

$$\Phi\big(\hat{j}_\phi(c) \otimes f \otimes j(g)\big) \in \Gamma^2(\mathcal{F}^2_{s,r}; \nu^2_{s,r}; \phi) \text{ is given by } (x,y) \mapsto ((cD^{-1/2})f)(x)g(y),$$
$$\Psi\big(j_\phi(c) \otimes f \otimes j(g)\big) \in \Gamma^2(\mathcal{F}^2_{r,r}; \nu^2_{r,r}; \phi) \text{ is given by } (x,y) \mapsto (fc)(x)g(y).$$

We shall use the isomorphisms above without further notice. If $(T_\phi)_\phi$ is a norm-bounded family of operators between Hilbert spaces $(H^1_\phi)_\phi$ and $(H^2_\phi)_\phi$, we denote by $\bigoplus_\phi T_\phi \in \mathcal{L}(\bigoplus_\phi H^1_\phi, \bigoplus_\phi H^2_\phi)$ the operator given by $(\xi_\phi)_\phi \mapsto (T_\phi\xi_\phi)_\phi$. Similar arguments as those used for the construction of V in Theorem 2.5 in [29] show:

Proposition 8.5. *If $\phi \in \mathcal{W}$, then exists a unitary $X_\phi\colon \Gamma^2(\mathcal{F}^2_{s,r}, \nu^2_{s,r}; \phi) \to \Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi)$ such that $(X_\phi f)(x,y) = f(x, x^{-1}y)$ for all $f \in \Gamma_c(\mathcal{F}^2_{s,r})$, $(x,y) \in G_r{\times}_rG$, and the pair $({}_\gamma K_{\hat{\delta}}, \bigoplus_\phi X_\phi)$ is a representation of V.* □

8.2. **The coaction of $C^*_r(G)$ on $C^*_r(\mathcal{F})$.** We apply Lemma 8.1 to the representation $({}_\gamma K_{\hat{\delta}}, X)$ and obtain a coaction of $C^*_r(G)$ on $C^*_r(\mathcal{F})$ as follows.

Lemma 8.6. *Let $\phi \in \mathcal{W}$. There exists a representation $\pi_\phi\colon C^*_r(\mathcal{F}) \to \mathcal{L}(K_\phi)$ such that for all $c, d \in \Gamma_c(\mathcal{F})$, $x \in G$,*

$$(\pi_\phi(c)d)(x) = \int_{G^{r(x)}} c(z)d(z^{-1}x)D^{-1/2}(z)\,\mathrm{d}\lambda^{r(x)}(z)$$

and $\pi_\phi(c)\hat{j}_\phi(d) = \hat{j}_\phi(cd)$ and $\pi_\phi(c)\rho_\gamma(f) = \pi_\phi(cf)$ for all $c, d \in \Gamma_c(\mathcal{F})$, $f \in C_0(G^0)$.

Proof. Identify $\Gamma^2(\mathcal{F}, \lambda^{-1}) \otimes_\phi L^2(G^0, \mu)$ with K_ϕ via $c \otimes f \mapsto \hat{j}_\phi(c)f$ for all $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G^0)$, and define π_ϕ by $c \mapsto c \otimes_\phi \mathrm{id}$. □

Define $\pi\colon C^*_r(\mathcal{F}) \to \mathcal{L}(K)$ by $c \mapsto \bigoplus_\phi \pi_\phi(c)$. Lemmas 6.4 and 8.6 imply:

Lemma 8.7. *The representation π is faithful, $\pi(C^*_r(\mathcal{F}))^\gamma_K$ is a C^*-$\mathfrak{b}$-algebra and one has $[\pi(C^*_r(\mathcal{F}))\widehat{\delta}] = \widehat{\delta}$.* □

Define $\delta\colon \pi(C^*_r(\mathcal{F})) \to \mathcal{L}(K_{\gamma}\underset{\mathfrak{b}}{\otimes}{}_{\beta}H)$ by $\pi(c) \mapsto X(\pi(c) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})X^*$. Let $c \in C^*_r(\mathcal{F})$. Then $\delta(\pi(c)) = \bigoplus_\phi \delta(\pi(c))_\phi$ and each $\delta(\pi(c))_\phi \in \mathcal{L}(\Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi))$ acts as follows.

Lemma 8.8. *For all $c \in \Gamma_c(\mathcal{F})$, $\phi \in \mathcal{W}$, $d \in \Gamma_c(\mathcal{F}^2_{r,r})$, $(x,y) \in G_r{\times}_rG$,*

$$\big(\delta(\pi(c))_\phi d\big)(x,y) = \int_{G^{r(x)}} c(z)d(z^{-1}x, z^{-1}y)D^{-1/2}(z)\,\mathrm{d}\lambda^{r(x)}(z).$$

Proof. The verification is straightforward and similar to the calculation of the comultiplication Δ on $C^*_r(G)$; see §3.4 in [29]. □

Theorem 8.9. *$(\pi(C^*_r(\mathcal{F}))^\gamma_K, \delta)$ is a very fine and left-full coaction of $C^*_r(G)$.*

The proof involves the following two lemmas.

Lemma 8.10. *Let $\phi \in \mathcal{W}$. Then there exist maps*

$$T_\phi \colon \Gamma_c(\mathcal{F}^2_{r,r}) \to \mathcal{L}(K_\phi, \Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi)), S_\phi \quad : \Gamma_c(\mathcal{F}^2_{r,r}) \to \mathcal{L}(H, \Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi))$$

that are continuous with respect to the inductive topology on $\Gamma_c(\mathcal{F}^2_{r,r})$ and the operator norm, respectively, such that for all $c \in \Gamma_c(\mathcal{F}^2_{r,r})$, $d \in \Gamma_c(\mathcal{F})$, $f \in C_c(G)$, $(x,y) \in G_r {\times_r} G$,

$$(T_\phi(c)d)(x,y) = \int_{G^{r(x)}} c(z,y) d(z^{-1}x) D^{-1/2}(z) \,\mathrm{d}\lambda^{r(x)}(z),$$

$$(S_\phi(c)f)(x,y) = \int_{G^{r(y)}} c(x,z) f(z^{-1}y) D^{-1/2}(z) \,\mathrm{d}\lambda^{r(y)}(z).$$

Proof. Let $c, d, T_\phi(c)d$ as above. Then

$$\|T_\phi(c)d\|^2 = \int_G \int_{G^{r(x)}} \int_{G^{r(x)}} \int_{G^{r(x)}} \phi_{s(x)} \big(d(z_1^{-1}x)^* c(z_1,y)^* c(z_2,y) d(z_2^{-1}x)\big) \cdot \\ \cdot D^{-1/2}(z_1) D^{-1/2}(z_2) \,\mathrm{d}\lambda^{r(x)}(y) \,\mathrm{d}\lambda^{r(x)}(z_1) \,\mathrm{d}\lambda^{r(x)}(z_2) \,\mathrm{d}\nu(x).$$

We substitute $x' = z_1^{-1}x$, $z = z_1^{-1}z_2$, use the relations $D(z_2) = D(z_1)D(z)$ and

$$\begin{aligned} D^{-1}(z_1) \,\mathrm{d}\lambda^{r(x)}(z_1) \,\mathrm{d}\nu(x) &= D^{-1}(z_1) \,\mathrm{d}\lambda^{r(z_1)}(x) \,\mathrm{d}\nu(z_1) \\ &= \mathrm{d}\lambda^{s(z_1)}(x') \,\mathrm{d}\nu^{-1}(z_1) = \mathrm{d}\lambda^{-1}_{r(x')}(z_1) \,\mathrm{d}\nu(x'), \end{aligned}$$

and find

$$\begin{aligned} \|T_\phi(c)d = &\int_G \int_{G_{r(x')}} \int_{G^{r(x')}} \int_{G^{s(z_1)}} \phi_{s(x)} (d(x')^* c(z_1,y)^* c(z_1 z, y) d(z^{-1}x')) \cdot \\ &\qquad \cdot D^{-1/2}(z) \,\mathrm{d}\lambda^{s(z_1)}(y) \,\mathrm{d}\lambda^{r(x')}(z) \,\mathrm{d}\lambda^{-1}_{r(x')}(z_1) \,\mathrm{d}\nu(x') \\ = &\int_G \int_{G^{r(x')}} \phi_{s(x')}(d(x') R_c(z) d(z^{-1}x')) \,\mathrm{d}\lambda^{r(x')}(z) \,\mathrm{d}\nu(x') = \langle d | \pi_\phi(R_c) d \rangle_{K_\phi}, \end{aligned}$$

where $R_c \in \Gamma_c(\mathcal{F})$ is given by

$$R_c(z) = \int_{G_{r(z)}} \int_{G^{s(z_1)}} c(z_1,y)^* c(z_1 z, y) \,\mathrm{d}\lambda^{s(z_1)}(y) \,\mathrm{d}\lambda^{-1}_{r(z)}(z_1) \quad \text{for all } z \in G.$$

Hence, $T_\phi(c)$ extends to a bounded linear operator of norm $\|T_\phi(c)\|^2 \leqslant \|\pi_\phi(R_c)\|$. If $(c_n)_n$ is a sequence in $\Gamma_c(\mathcal{F}^2_{r,r})$ converging to c in the inductive limit topology, then the functions $R_{(c-c_n)}$ defined similarly as R_c converge to 0 in the inductive limit topology and hence $\|T_\phi(c - c_n)\|^2 \leqslant \|\pi_\phi(R_{(c-c_n)})\|$ converges to 0.

The proof of the assertion concerning S_ϕ is very similar. □

Given $c, d \in \Gamma_c(\mathcal{F})$ and $f \in C_c(G)$, define $\omega_{c,d,f} \in \Gamma_c(\mathcal{F}^2_{r,r})$ by

$$(x,y) \mapsto \int_{G^{r(x)}} c(z) d(z^{-1}x) f(z^{-1}y) \,\mathrm{d}\lambda^{r(x)}(z).$$

Lemma 8.11. *The linear span of all elements $\omega_{c,d,f}$ as above is dense in $\Gamma_c(\mathcal{F}^2_{r,r})$ with respect to the inductive limit topology.*

Proof. Let $(x, y) \in G_r{\times}_rG$, $e \in \mathcal{F}_x$, let $C \subseteq G_r{\times}_rG$ be a compact neighbourhood of (x, y), and let $\epsilon > 0$. Since $[\mathcal{F}_{r(x)}\mathcal{F}_x] = \mathcal{F}_x$, we can choose $c', d' \in \Gamma_c(\mathcal{F})$ such that $\|c'(z)d'(z^{-1}x) - e\| < \epsilon$ for all z in some neighbourhood of $r(x)$ in $G^{r(x)}$. Next, we can choose $h_c, h_d, f \in C_c(G)$ such that the elements $c, d \in \Gamma_c(\mathcal{F})$ given by $c(z) = c'(z)h_c(z)$ and $d(z) = d'(z)h_d(z)$ for all $z \in G$ satisfy $\|\omega_{c,d,f}(x, y) - e\| < \epsilon$ and $\operatorname{supp} \omega_{c,d,f} \subseteq C$. A standard partition of unity argument concludes the proof. $\square$

Proof of Theorem 8.9. We show that Lemma 8.1 applies. Let $\phi \in \mathcal{W}$, $c, d \in \Gamma_c(\mathcal{F})$, $f, g \in C_c(G)$. Define $e_1, e_2, e_3, e_4 \in \Gamma^2(\mathcal{F}^2_{r,r}, \nu^2_{r,r}; \phi)$ and $\omega_1, \omega_2, \omega_3, \omega_4 \in \Gamma_c(\mathcal{F}^2_{r,r})$ by

$$\begin{aligned}
e_1 &= \delta(\pi(c))_\phi|j(f)\rangle_2 d, & \omega_1(z, y) &= c(z)f(z^{-1}y) \text{ for all } (z, y) \in G_r{\times}_rG,\\
e_2 &= |j(f)\rangle_2 \pi_\phi(c)d, & \omega_2(z, y) &= c(z)f(y) \text{ for all } (z, y) \in G_r{\times}_rG,\\
e_3 &= |j_\phi(c)\rangle_1 L(f)g, & \omega_3(x, z) &= c(x)f(z) \text{ for all } (x, z) \in G_r{\times}_rG,\\
e_4 &= \delta(\pi(c))_\phi|j_\phi(d)\rangle_1 L(f)g, & \omega_4 &= \omega_{c,d,f}.
\end{aligned}$$

Using Lemma 8.8, we find that for all $(x, y) \in G_r{\times}_rG$,

$$\begin{aligned}
e_1(x, y) &= \int_{G^{r(x)}} c(z)D^{-1/2}(z)d(z^{-1}x)f(z^{-1}y)\,\mathrm{d}\lambda^{r(x)}(z) = (T_\phi(\omega_1)d)(x, y),\\
e_2(x, y) &= \int_{G^{r(x)}} c(z)d(z^{-1}x)D^{-1/2}(z)\,\mathrm{d}\lambda^{r(x)}(z)f(y) = (T_\phi(\omega_2)d)(x, y),\\
e_3(x, y) &= c(x)\int_{G^{r(y)}} f(z)D^{-1/2}(z)g(z^{-1}y)\,\mathrm{d}\lambda^{r(y)}(z) = (S_\phi(\omega_3)g)(x, y),\\
e_4(x, y) &= \int_{G^{r(x)}} c(z_1)D^{-1/2}(z_1)d(z_1^{-1}x)(L(f)g)(z_1^{-1}y)\,\mathrm{d}\lambda^{r(x)}(z_1)\\
&= \int_{G^{r(x)}}\int_{G^{s(z_1)}} c(z_1)D^{-1/2}(z_1)d(z_1^{-1}x)f(z_2)\cdot\\
&\qquad\qquad \cdot D^{-1/2}(z_2)g(z_2^{-1}z_1^{-1}y)\,\mathrm{d}\lambda^{s(z_1)}(z_2)\,\mathrm{d}\lambda^{r(x)}(z_1)\\
&= \int_{G^{r(x)}}\int_{G^{r(x)}} c(z_1)d(z_1^{-1}x)f(z_1^{-1}z_2')\cdot\\
&\qquad\qquad \cdot D^{-1/2}(z_2')g(z_2'^{-1}y)\,\mathrm{d}\lambda^{r(x)}(z_2')\,\mathrm{d}\lambda^{r(x)}(z_1)\\
&= (S_\phi(\omega_{c,d,f})g)(x, y).
\end{aligned}$$

By Lemmas 7.2 and 8.11, sections of the form like $\omega_1, \omega_2, \omega_3$ or ω_4, respectively, are linearly dense in $\Gamma_c(\mathcal{F}^2_{r,r})$. Therefore, $[\delta(\pi(C^*_r(\mathcal{F})))_\phi|\alpha\rangle_2] = [T_\phi(\Gamma_c(\mathcal{F}^2_{r,r}))] = [|\alpha\rangle_2\pi_\phi(C^*_r(\mathcal{F}))]$ and similarly $[\delta(\pi(C^*_r(\mathcal{F})))|\gamma\rangle_1 C^*_r(G)] = [\bigcup_{\phi\in\mathcal{W}} S_\phi(\Gamma_c(\mathcal{F}^2_{r,r}))] = [|\gamma\rangle_1 C^*_r(G)]$. $\square$

Given $g, g' \in C_c(G)$, define $h_{g,g'} \in C_c(G)$ by

$$h_{g,g'}(z) = \int_{G^{r(z)}} \overline{g(y)}g'(z^{-1}y)\,\mathrm{d}\lambda^{r(z)}(y) \quad \text{for all } z \in G. \tag{8.1}$$

Lemma 8.12. *Let $c \in \Gamma_c(\mathcal{F})$, $g, g' \in C_c(G)$. Then $\langle j(g)|_2\delta(\pi(c))_\phi|j(g')\rangle_2 = \pi_\phi(c')$, where $c'(x) = c(x)h_{g,g'}(x)$ for all $x \in G$.*

Proof. The operators on both sides map each $d \in \Gamma_c(\mathcal{F})$ to the section

$$x \mapsto \int_{G^{r(x)}} \int_{G^{r(x)}} \overline{g(y)} c(z) d(z^{-1}x) g'(z^{-1}y) D^{-1/2}(z) \, \mathrm{d}\lambda^{r(x)}(z) \, \mathrm{d}\lambda^{r(x)}(y). \qquad \square$$

8.3. **The reduced crossed product of the coaction.** The bundle $\mathcal{F}^2_{s,r}$ carries the structure of a Fell bundle, and the reduced crossed product $\pi(C^*_r(\mathcal{F})) \rtimes_r C_0(G)$ for the coaction δ constructed above can be identified with $C^*_r(\mathcal{F}^2_{s,r})$ as follows.

Denote by $G \ltimes G$ the transformation groupoid for the action of G on itself given by right multiplication. Thus, $G \ltimes G = G_s \times_r G$ as a set, $(G \ltimes G)^0 = \bigcup_{u \in G^0} \{u\} \times G^u$ can be identified with G via $(r(y), y) \equiv y$, the range map $\tilde{r}$, the source map $\tilde{s}$, and the multiplication are given by $(x, y) \overset{\tilde{r}}{\mapsto} xy$, $(x, y) \overset{\tilde{s}}{\mapsto} y$, and $((x, y), (x', y')) \mapsto (xx', y')$, respectively, and the topology on $G \ltimes G$ is the weakest topology that makes $\tilde{r}, \tilde{s}$ and the map $(x, y) \mapsto x$ continuous. We equip $G \ltimes G$ with the right Haar system $\tilde{\lambda}^{-1}$ given by $\tilde{\lambda}^{-1}_y(C \times \{y\}) = \lambda^{-1}_{r(y)}(C)$ for all $C \subseteq G_{r(y)}$, $y \in G$.

The bundle $\mathcal{F}^2_{s,r}$ is a Fell bundle on $G \ltimes G$ with respect to the multiplication and involution given by $((f, y), (f', y')) \mapsto (ff', y')$ and $(f, y) \mapsto (f^*, p(f)y)$. The convolution product in $\Gamma_c(\mathcal{F}^2_{s,r})$ is given by

$$(cd)(x, y) = \int_{G_{r(y)}} c(xz^{-1}, zy) d(z, y) \, \mathrm{d}\lambda^{-1}_{r(y)}(z) \tag{8.2}$$

for all $c, d \in \Gamma_c(\mathcal{F}^2_{s,r})$, $(x, y) \in G_s \times_r G$, because $(G \ltimes G)_{\tilde{s}(x,y)} = G_{r(y)} \times \{y\}$ and $(x, y)(z, y)^{-1} = (xz^{-1}, zy)$ for all $z \in G_{r(y)}$.

Proposition 8.13. *There exists a unique isomorphism $\pi(C^*_r(\mathcal{F})) \rtimes_r C_0(G) \to C^*_r(\mathcal{F}^2_{s,r})$ such that $\delta(\pi(c))(1 \underset{\mathfrak{b}}{\otimes} f) \mapsto L_{\mathcal{F}^2_{s,r}}(d)$ whenever $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G)$, and $d(x, y) = c(x)f(y)$ for all $(x, y) \in G_s \times_r G$.*

Let $\phi \in \mathcal{W}$. Then the map $r^*\phi \colon \Gamma_0((\mathcal{F}^2_{s,r})^0) \to C_0(G)$ given by $(r^*\phi(c))(y) = \phi_{r(y)}(c(r(y), y))$ for all $c \in \Gamma_0((\mathcal{F}^2_{s,r})^0)$ and $y \in G$ is a $C_0(G)$-weight. One easily verifies that there exists a representation $L_{r^*\phi} \colon C^*_r(\mathcal{F}^2_{s,r}) \to \mathcal{L}(\Gamma^2(\mathcal{F}^2_{s,r}, \tilde{\lambda}^{-1}; r^*\phi))$ such that $L_{r^*\phi}(c)d = cd$ for all $c, d \in \Gamma_c(\mathcal{F}^2_{s,r})$.

Lemma 8.14. (i) *There exists a unique unitary $U_\phi \colon \Gamma^2(\mathcal{F}^2_{s,r}, \tilde{\lambda}^{-1}; r^*\phi) \otimes H \to \Gamma^2(\mathcal{F}^2_{s,r}, \nu^2_{s,r}; \phi) \subseteq K {}_{\hat{\delta}}\underset{\mathfrak{b}^\dagger}{\otimes}{}_\alpha H$ such that $(U_\phi(e \otimes g))(x, y) = e(x, y)g(y)D^{-1/2}(x)$ for all $e \in \Gamma_c(\mathcal{F}^2_{s,r})$, $g \in C_c(G)$, $(x, y) \in G_s \times_r G$.*

(ii) *$\delta(\pi(c))(1 \underset{\mathfrak{b}}{\otimes} f) X_\phi U_\phi = X_\phi U_\phi (L_{r^*\phi}(d) \otimes \mathrm{id})$ for all c, d, f as in Proposition 8.13.*

Proof. (i) For all e, g as in above,

$$\|U_\phi(e \otimes g)\|^2 = \int_G \int_{G_{r(y)}} \phi_{s(x)}(e(x, y)^* e(x, y)) |g(y)|^2 \, \mathrm{d}\lambda^{-1}_{r(y)}(x) \, \mathrm{d}\nu(y) = \|e \otimes g\|^2.$$

(ii) Let $c, d, e, f, g, (x, y)$ as above and $\widehat{\Delta}(f)_\phi = X^*_\phi(1 \underset{\mathfrak{b}}{\otimes} f)X_\phi$. A short calculation shows that $\big(\widehat{\Delta}(f)_\phi U_\phi(e \otimes g)\big)(x, y) = f(xy)e(x, y)g(y)D^{-1/2}(x)$. Using (8.2), we find that $\big((\pi_\phi(c) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})\widehat{\Delta}(f)_\phi U_\phi(e \otimes g)\big)(x,y)$ is equal to

$$\begin{aligned}
&\int_{G^{r(x)}} c(z)f(z^{-1}xy)e(z^{-1}x,y)g(y)D^{-1/2}(z)D^{-1/2}(z^{-1}xy)\,\mathrm{d}\lambda^{r(x)}(z)\\
&\qquad = \int_{G_{s(x)}} c(xz^{-1})f(zy)e(z,y)g(y)D^{-1/2}(xy)\,\mathrm{d}\lambda^{-1}_{s(x)}(z)\\
&\qquad = \int_{G_{s(x)}} d(xz^{-1},zy)e(z,y)g(y)D^{-1/2}(xy)\,\mathrm{d}\lambda^{-1}_{s(x)}(z) = (U_\phi(de \otimes g))(x,y).
\end{aligned}$$

So, $\delta(\pi(c))(1 \underset{\mathfrak{b}}{\otimes} f)X_\phi U_\phi = X_\phi(\pi_\phi(c) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})\widehat{\Delta}(f)_\phi X_\phi U_\phi = X_\phi U_\phi(L_{r^*\phi}(d) \otimes \mathrm{id})$. □

Proof of Proposition 8.13. Consider the $*$-homomorphism

$$\Phi\colon C^*_r(\mathcal{F}^2_{s,r}) \to \mathcal{L}(K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H), \quad L_{\mathcal{F}^2_{s,r}}(d) \mapsto \bigoplus_{\phi\in\mathcal{W}} X_\phi U_\phi(L_{r^*\phi}(d) \otimes \mathrm{id})U^*_\phi X^*_\phi.$$

By part (ii) of the lemma above, $\Phi(C^*_r(\mathcal{F}^2_{s,r}))$ contains $[\delta(\pi(C^*_r(\mathcal{F})))(1 \underset{\mathfrak{b}}{\otimes} C_0(G))] = \pi(C^*_r(\mathcal{F})) \rtimes_r C_0(G)$. The same lemma implies that this inclusion is an equality because the map $a \to \bigoplus_\phi a \otimes \mathrm{id}$ is continuous with respect to the inductive limit topology on $\Gamma_c(\mathcal{F}^2_{s,r})$ and sections of the form $(x,y) \mapsto c(x)f(y)$, where $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G)$, are dense in $\Gamma_c(\mathcal{F}^2_{s,r})$ by Lemma 7.2. Finally, Lemma 6.4 implies that $[\bigcap_\phi \ker r^*\phi] = 0$, and therefore Φ is injective. □

Proposition 8.15. *If $\mathcal{F}$ is saturated, then $C^*_r(\mathcal{F}^2_{s,r}) \cong \mathcal{K}(\Gamma^2(\mathcal{F},\lambda^{-1}))$.*

Proof. To simplify notation, let $\Gamma^2 = \Gamma^2(\mathcal{F},\lambda^{-1})$, $\tilde{\Gamma}^2 = \Gamma^2(\mathcal{F}^2_{s,r},\tilde{\lambda}^{-1})$, $\Gamma_0 = \Gamma_0(\mathcal{F}^0)$, $\tilde{\Gamma}_0 = \Gamma_0((\mathcal{F}^2_{s,r})^0)$. There exists a unitary $\Psi\colon \Gamma^2 \otimes_{s^*} C_0(G) \to \tilde{\Gamma}^2$ such that $(\Psi(c \otimes f))(x,y) = c(xy)f(y)$ for all $c \in \Gamma_c(\mathcal{F})$, $f \in C_c(G)$, $(x,y) \in G_s {\times}_r G$, because

$$\begin{aligned}
\langle \Psi(c \otimes f)|\Psi(c' \otimes f')\rangle((r(y),y)) &= \int_{G_{r(y)}} c(xy)^* c'(xy)\,\mathrm{d}\lambda^{-1}_{r(y)}(x)\overline{f(y)}f'(y)\\
&= \overline{f(y)}\langle c|c'\rangle_{\Gamma^2}(s(y))f(y) = \langle c \otimes f|c' \otimes f'\rangle(y)
\end{aligned}$$

for all $c, c' \in \Gamma_c(\mathcal{F}), f, f' \in C_c(G)$, $y \in G$ by right-invariance of λ^{-1}. The $*$-homomorphism $\Phi\colon \mathcal{K}(\Gamma^2) \to \mathcal{L}(\tilde{\Gamma}^2)$ given by $k \mapsto \Psi(k \otimes_{s^*} \mathrm{id})\Psi^*$ is injective because $s^*\colon C_0(G^0) \to \mathcal{L}(C_0(G))$ is injective, and the claim follows once we have shown that $\Phi(\mathcal{K}(\Gamma^2)) = C^*_r(\mathcal{F}^2_{s,r})$. Let $d, d' \in \Gamma_c(\mathcal{F})$ and denote by $|d\rangle\langle d'| \in \mathcal{K}(\Gamma^2)$ the operator given by $e \mapsto d\langle d'|e\rangle$. Then for all $c, f, (x,y)$ as above,

$$\begin{aligned}
\big(\Psi(|d\rangle\langle d'|c \otimes f)\big)(x,y) &= \int_{G_{s(y)}} d(xy)d'(z)^* c(z)f(y)\,\mathrm{d}\lambda^{-1}_{s(y)}(z)\\
&= \int_{G_{s(y)}} d(xy)d'(z)^*\big(\Psi(c \otimes f)\big)(zy^{-1},y)\,\mathrm{d}\lambda^{-1}_{s(y)}(z)\\
&= \int_{G_{r(y)}} d(xy)d'(z'y)^*\big(\Psi(c \otimes f)\big)(z',y)\,\mathrm{d}\lambda^{-1}_{r(y)}(z').
\end{aligned}$$

Comparing with equation (8.2), we find that $\Psi(|d\rangle\langle d'| \otimes \mathrm{id})\Psi^* = L_{\mathcal{F}^2_{s,r}}(e)$, where $e \in \Gamma_c(\mathcal{F}^2_{s,r})$ is given by $e(xz^{-1}, zy) = d(xy)d'(zy)^*$, or equivalently, by $e(x', y') = d(x'y')d'(y')^*$ for all $(x', y') \in G_s{\times}_rG$. Since $\mathcal{F}$ is saturated, Lemma 7.2 implies that sections of this form are dense in $\Gamma_c(\mathcal{F}^2_{s,r})$ with respect to the inductive limit topology, and since the map $e \mapsto L_{\mathcal{F}^2_{s,r}}(e)$ is continuous with respect to this topology, we can conclude that $\Phi(\mathcal{K}(\Gamma^2)) = \Psi(\mathcal{K}(\Gamma^2) \otimes \mathrm{id})\Psi^* = C^*_r(\mathcal{F}^2_{s,r})$. □

Corollary 8.16. *If $\mathcal{F}$ is saturated, then $\pi(C^*_r(\mathcal{F})) \rtimes_r C_0(G)$ and $\Gamma_0(\mathcal{F}^0)$ are Morita equivalent.*

Proof. One easily verifies that $\Gamma^2(\mathcal{F}, \lambda^{-1})$ is full. □

Example 8.17. *Let σ be an action of G on an admissible $C_0(G^0)$-algebra C and let δ_σ be the corresponding coaction of $C_0(G)$ on $\mathbf{F}C$ (Proposition 6.10). Then there exists an admissible Fell bundle $\mathcal{C}$ on G with fibre $\mathcal{C}_x = C_{r(x)}$ for each $x \in G$, continuous sections $\Gamma_0(\mathcal{C}) = r^*C$, and multiplication and involution given by $cd = c\sigma_x(d)$, $c^* = \sigma_{x^{-1}}(c^*)$ for all $c \in \mathcal{C}_x$, $d \in \mathcal{C}_y$, $(x, y) \in G_s{\times}_rG$* [15]*, and the identity on $\Gamma_c(\mathcal{C}) = C_c(G)r^*C$ extends to an isomorphism $C^*_r(\mathcal{C}) \to C \rtimes_r G$. One easily verifies that with respect to the isomorphism $\pi(C^*_r(\mathcal{C})) \cong C^*_r(\mathcal{C}) \cong C \rtimes_r G \cong \mathbf{F}C \rtimes_r C^*_r(G)$ of Proposition 6.14, the coaction of Theorem 8.9 coincides with the dual coaction on $\mathbf{F}C \rtimes_r C^*_r(G)$. Moreover, the Fell bundle $\mathcal{C}$ is saturated and $C^*_r(\mathcal{C}) \rtimes_r C_0(G) \cong \mathbf{F}C \rtimes_r C^*_r(G) \rtimes_r C_0(G)$ is Morita equivalent to $\Gamma_0(\mathcal{C}^0) \cong C$, as we already know by Theorem 4.11.*

Remark 8.18. *The Fell bundle $\mathcal{F}$ can be equipped with the structure of an $\mathcal{F}^2_{s,r}$-$\mathcal{F}^0$-equivalence in the sense of* [19] *in a straightforward way.*

8.4. **Functoriality of the construction.** Let $\mathcal{G}$, $\mathcal{F}$ be admissible Fell bundles on G with associated representations $((K_\mathcal{G}, \gamma_\mathcal{G}, \widehat{\delta}_\mathcal{G}), X_\mathcal{G})$, $((K_\mathcal{F}, \gamma_\mathcal{F}, \widehat{\delta}_\mathcal{F}), X_\mathcal{F})$ and coactions $(\pi_\mathcal{G}(C^*_r(\mathcal{G}))^{\gamma_\mathcal{G}}_{K_\mathcal{G}}, \delta_\mathcal{G})$, $(\pi_\mathcal{F}(C^*_r(\mathcal{F}))^{\gamma_\mathcal{F}}_{K_\mathcal{F}}, \delta_\mathcal{F})$, and let T be a morphism from $\mathcal{G}$ to $\mathcal{F}$.

Proposition 8.19. *There exists a unique morphism $\tilde{T}_*$ from $\big(\pi_\mathcal{G}(C^*_r(\mathcal{G}))^{\gamma_\mathcal{G}}_{K_\mathcal{G}}, \delta_\mathcal{G}\big)$ to $\big(\pi_\mathcal{F}(C^*_r(\mathcal{F}))^{\gamma_\mathcal{F}}_{K_\mathcal{F}}, \delta_\mathcal{F}\big)$ that satisfies $\tilde{T}_*(\pi_\mathcal{G}(a)) = \pi_\mathcal{F}(T_*(a))$ for all $a \in \Gamma_c(\mathcal{G})$.*

The proof involves the following construction.

Lemma 8.20. *Let $\phi \in \mathcal{W}(\Gamma_0(\mathcal{F}^0))$, $f \in \Gamma_0(\mathcal{F}^0)$ and define $\psi \in \mathcal{W}(\Gamma_0(\mathcal{G}^0))$ by $g \mapsto \phi(f^*T^0_*(g)f)$.*

- (i) *There exists an isometry $T^f_\phi \colon K_\psi \to K_\phi$ such that $T^\phi_f g = T_*(g)f$ for all $g \in \Gamma_c(\mathcal{G})$.*
- (ii) *$T^f_\phi j_\psi(g) = j_\phi(T_*(g)f)$, $T^f_\phi \hat{j}_\psi(g) = \hat{j}_\psi(T_*(g)f)$, and $T^f_\phi \pi_\psi(g) = \pi_\phi(T_*(g))T^f_\phi$ for all $g \in \Gamma_c(\mathcal{G})$.*

  Denote also the map $K_\mathcal{G} \to K_\psi \hookrightarrow K_\mathcal{F}$ given by $(\xi_{\psi'})_{\psi'} \mapsto T^f_\phi \xi_\psi$ by T^f_ϕ.
- (iii) *T^f_ϕ is a semi-morphism from $(K_\mathcal{G}, \widehat{\delta}_\mathcal{G}, \gamma_\mathcal{G})$ to $(K_\mathcal{F}, \widehat{\delta}_\mathcal{F}, \gamma_\mathcal{F})$ and $(T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})X_\mathcal{G} = X_\mathcal{F}(T^f_\phi \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})$.*
- (iv) *$\delta_\mathcal{F}(\pi_\mathcal{F}(h))(T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})\delta_\mathcal{G}(\pi_\mathcal{G}(g)) = \delta_\mathcal{F}(\pi_\mathcal{F}(hT_*(g)))(T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})$ for all $h \in \Gamma_c(\mathcal{F})$, $g \in \Gamma_c(\mathcal{G})$.*

Proof. (i) Uniqueness is clear. Existence follows from the fact that for all $g, g' \in \Gamma_c(\mathcal{G})$,

$$\begin{aligned}\langle T_*(g)f|T_*(g')f\rangle_{K_\phi} &= \int_G \phi_{s(x)}\big(f(s(x))^*T(g(x)^*g'(x))f(s(x))\big)\,\mathrm{d}\nu(x)\\ &= \int_G \psi_{s(x)}(g(x)^*g'(x))\,\mathrm{d}\nu(x) = \langle g|g\rangle_{K_\psi}.\end{aligned}$$

(ii) Straightforward.

(iii) By (ii), $T^f_\phi \gamma_{\mathcal{G}} \subseteq \gamma_{\mathcal{F}}$ and $T^f_\phi \widehat{\delta}_{\mathcal{G}} \subseteq \widehat{\delta}_{\mathcal{F}}$. For all $\omega \in \Gamma_c(\mathcal{G}^2_{s,r})$ and $(x,y) \in G_r \times_r G$,

$$((T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})X_{\mathcal{G}}\omega)(x,y) = \omega(x, x^{-1}y)f(s(x)) = (X_{\mathcal{F}}(T^f_\phi \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})\omega)(x,y).$$

(iv) By parts (ii) and (iii),

$$X_{\mathcal{F}}(\pi_{\mathcal{F}}(h) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})X^*_{\mathcal{F}}(T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})X_{\mathcal{G}}(\pi_{\mathcal{G}}(g) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})X^*_{\mathcal{G}} = X_{\mathcal{F}}(\pi_{\mathcal{F}}(hT_*(g)) \underset{\mathfrak{b}^\dagger}{\otimes} \mathrm{id})X^*_{\mathcal{F}}(T^f_\phi \underset{\mathfrak{b}}{\otimes} \mathrm{id})$$

for all $g \in \Gamma_c(\mathcal{G})$ and $h \in \Gamma_c(\mathcal{F})$. ☐

Proof of Proposition 8.19. Denote by $\mathcal{T} \subseteq \mathcal{L}(K_{\mathcal{G}}, K_{\mathcal{F}})$ the closed linear span of all operators T^f_ϕ, where $\phi \in \mathcal{W}(\Gamma_0(\mathcal{F}^0))$ and $f \in \Gamma_0(\mathcal{F}^0)$. Then Lemma 8.20 and Proposition 7.9 imply that $S\pi_{\mathcal{G}}(g) = \pi_{\mathcal{F}}(T_*(g))S$ for all $S \in \mathcal{T}, g \in \Gamma_c(\mathcal{G})$ and that

$$[\mathcal{T}\gamma_{\mathcal{G}}] = \Big[\bigcup_\phi j_\phi\big(T_*(\Gamma_c(\mathcal{G}))\Gamma_0(\mathcal{F}^0)\big)\Big] = \Big[\bigcup_\phi j_\phi(\Gamma_c(\mathcal{F}))\Big] = \gamma_{\mathcal{F}}.$$

By Proposition 7.9, T_* extends to a nondegenerate $*$-homomorphism $C^*_r(\mathcal{G}) \to M(C^*_r(\mathcal{F}))$. Henceforth, there exists a semi-morphism $\tilde{T}_*$ from $\pi_{\mathcal{G}}(C^*_r(\mathcal{G}))^{\gamma_{\mathcal{G}}}_{K_{\mathcal{G}}}$ to $\pi_{\mathcal{F}}(C^*_r(\mathcal{F}))^{\gamma_{\mathcal{F}}}_{K_{\mathcal{F}}}$ such that $\tilde{T}_*(\pi_{\mathcal{G}}(g)) = \pi_{\mathcal{F}}(T_*(g))$ for all $g \in \Gamma_c(\mathcal{G})$. For all $h \in \pi_{\mathcal{F}}(\Gamma_c(\mathcal{F}))$, $g \in \pi_{\mathcal{G}}(\Gamma_c(\mathcal{G}))$, $S \in \mathcal{T}$,

$$\delta_{\mathcal{F}}(h) \cdot (\tilde{T}_* * \mathrm{id})(\delta_{\mathcal{G}}(g)) \cdot (S \underset{\mathfrak{b}}{\otimes} \mathrm{id}) = \delta_{\mathcal{F}}(h)(S \underset{\mathfrak{b}}{\otimes} \mathrm{id})\delta_{\mathcal{G}}(g) = \delta_{\mathcal{F}}(h\tilde{T}_*(g))(S \underset{\mathfrak{b}}{\otimes} \mathrm{id})$$

by Lemma 8.20, and therefore $\delta_{\mathcal{F}}(h) \cdot (\tilde{T}_* * \mathrm{id})(\delta_{\mathcal{G}}(g)) = \delta_{\mathcal{F}}(h\tilde{T}_*(g))$. ☐

Denote by $\mathbf{Fell}^a_G$ the category of all admissible Fell bundles on G, and by $\mathbf{Coact}^a_{C^*_r(G)}$ the category of very fine left-full coactions of $C^*_r(G)$.

Theorem 8.21. *The assignments $\mathcal{F} \mapsto (\pi_{\mathcal{F}}(C^*_r(\mathcal{F}))^{\gamma_{\mathcal{F}}}_{K_{\mathcal{F}}}, \delta_{\mathcal{F}})$ and $T \mapsto \tilde{T}_*$ form a faithful functor $\check{\mathbf{F}}\colon \mathbf{Fell}^a_G \to \mathbf{Coact}^a_{C^*_r(G)}$.*

Proof. Functoriality of the constructions is evident. Assume that $\check{\mathbf{F}}S = \check{\mathbf{F}}T$ for some morphisms S, T from $\mathcal{F}$ to $\mathcal{G}$ in $\mathbf{Fell}^a_G$. Then the maps $S_*, T_*\colon \Gamma_c(\mathcal{F}) \to \Gamma_c(\mathcal{M}(\mathcal{G}))$ coincide because $\pi_{\mathcal{G}}$ is injective. Since $\{a(x) \mid a \in \Gamma_c(\mathcal{F})\} = \mathcal{F}_x$ for each $x \in G$ and $S(a(x)) = (S_*a)(x) = (T_*a)(x) = T(a(x))$ for each $a \in \Gamma_c(\mathcal{F})$, $x \in G$, we can conclude that $S = T$. ☐

9. From coactions of $C_r^*(G)$ to Fell bundles for étale G

We now assume that the groupoid G is étale in the sense that the set $\mathfrak{G}$ of all open subsets $U \subseteq G$ for which the restrictions $r_U = r|_U : U \to r(U)$ and $s_U = s|_U : U \to s(U)$ are homeomorphisms is a cover of G; see [23]. Moreover, we assume that the Haar systems λ and λ^{-1} are the families of counting measures. Then the functor $\check{\mathbf{F}}$ has a right adjoint $\check{\mathbf{G}}$ and embeds the category of admissible Fell bundles into a category of very fine coactions of $C_r^*(G)$ as a full and coreflective subcategory. The construction of the functor $\check{\mathbf{G}}$ uses the correspondence between Banach bundles and convex Banach modules developed in [8].

9.1. The Fell bundle of a coaction of $C_r^*(G)$. Let δ be an injective coaction of $C_r^*(G)$ on a C^*-$\mathfrak{b}$-algebra $\mathcal{C} = C_K^\gamma$. Since G is étale, $\rho_\beta(\mathfrak{B}) \subseteq C_r^*(G)$ and $\delta(C)|\gamma\rangle_1 \subseteq [|\gamma\rangle_1 C_r^*(G)]$. For each $U \in \mathfrak{G}$, we define a closed subspace

$$C_U := \{c \in [C\rho_\gamma(C_0(s(U)))] \mid \delta(c)|\gamma\rangle_1 \subseteq [|\gamma\rangle_1 L(C_0(U))]\} \subseteq C,$$

denote by $s_{U*} : C_0(U) \to C_0(s(U))$ and $r_{U*} : C_0(U) \to C_0(r(U))$ the push-forward of functions along s_U and r_U, respectively, and consider C_U as a right Banach $C_0(U)$-module via the formula $c \cdot f := c\rho_\gamma(s_{U*}(f))$. Denote by $\Gamma_f(\mathcal{F})$ the space of all sections of $\mathcal{F}$ that can be written as finite sums of sections in $\Gamma_0(\mathcal{F}|_U)$, where $U \in \mathfrak{G}$. Then $\Gamma_f(\mathcal{F})$ is a $*$-algebra with respect to the operations defined in (7.1), and one has natural inclusions $\Gamma_c(\mathcal{F}) \subseteq \Gamma_f(\mathcal{F}) \subseteq C_r^*(\mathcal{F})$ of $*$-algebras.

Proposition 9.1. *There exist a continuous Fell bundle $\mathcal{F}$ on G and a $*$-homomorphism $\iota : \Gamma_f(\mathcal{F}) \to C$ such that for each $U \in \mathfrak{G}$, the map ι restricts to an isometric isomorphism $\iota_U : \Gamma_0(\mathcal{F}|_U) \to C_U$ of Banach $C_0(U)$-modules. If $(\mathcal{F}', \iota')$ is another such pair, then there exists an isomorphism $T : \mathcal{F} \to \mathcal{F}'$ such that $\iota' \circ T_* = \iota$.*

The proof requires some preliminaries. First, for all $c \in C$, $f \in C_0(G^0)$,

$$\delta(c\rho_\gamma(f)) = \delta(c)\rho_{(\gamma \triangleright \alpha)}(f) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} \rho_\alpha(f)) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} r^*(f)).$$

Lemma 9.2. *Let $U, V \in \mathfrak{G}$.*

(i) *$c \cdot f = \rho_\gamma(r_{U*}(f))c$ for each $c \in C_U$ and $f \in C_0(U)$.*
(ii) *$C_V C_U \subseteq C_{VU}$, $(C_U)^* = C_{U^{-1}}$, and $C_U = [C_V C_0(U)] \subseteq C_V$ if $U \subseteq V$.*
(iii) *$C_{s(U)}$ is a continuous $C_0(s(U))$-algebra.*
(iv) *C_U is a convex and continuous Banach $C_0(U)$-module.*

Proof. (i) Let c, f as above. Since $L(g)r^*(s_{U*}(f)) = r^*(r_{U*}(f))L(g)$ for all $g \in C_0(U)$, we have $\delta(c \cdot f) = \delta(c)(1 \underset{\mathfrak{b}}{\otimes} r^*(s_{U*}(f))) = (1 \underset{\mathfrak{b}}{\otimes} r^*(r_{U*}(f)))\delta(c) = \delta(\rho_\gamma(r_{U*}(f))c)$ and by injectivity of δ also $c \cdot f = \rho_\gamma(r_{U*}(f))c$.

(ii) Clearly, $\delta(C_V C_U)|\gamma\rangle_1 \subseteq |\gamma\rangle_1 L(C_0(VU))$. Using (i) twice, we find

$$\begin{aligned} C_V C_U &\subseteq [C_V \rho_\gamma(C_0(s(V))C_0(r(U)))C_U] \\ &= [C_V \rho_\gamma(C_0(s(V) \cap r(U))C_U] \subseteq [C\rho_\gamma(C_0(s(VU)))]. \end{aligned}$$

Consequently, $C_V C_U \subseteq C_{VU}$. By (i) again, we have $(C_U)^* = [\rho_\gamma(C_0(r(U)))C_U]^* \subseteq [C\rho_\gamma(C_0(s(U^{-1})))]$, and using the relation $\delta(C_U^*)|\gamma\rangle_1 \subseteq [|\gamma\rangle_1 C_r^*(G)]$, we obtain

$$\begin{aligned}\delta(C_U^*)|\gamma\rangle_1 &\subseteq [|\gamma\rangle_1\langle\gamma|_1\delta(C_U)^*|\gamma\rangle_1] \\ &\subseteq [|\gamma\rangle_1 L(C_0(U))^*\langle\gamma|_1|\gamma\rangle_1] = [|\gamma\rangle_1 L(C_0(U^{-1}))].\end{aligned}$$

If $U \subseteq V$, then $C_U \subseteq [C_V C_0(U)] \subseteq C_V$, and $C_V C_0(U) \subseteq C_U$ because

$$\begin{aligned}\delta(C_V C_0(U))|\gamma\rangle_1 &= \delta(C_V)|\gamma\rangle_1 r^*(C_0(s(U))) \\ &\subseteq [|\gamma\rangle_1 L(C_0(V))r^*(C_0(s(U)))] = [|\gamma\rangle_1 L(C_0(U))].\end{aligned}$$

(iii) By (ii), $C_{s(U)}$ is a C^*-algebra. Consider $|\gamma\rangle_1$ as a Hilbert C^*-module over $r^*(C_0(G^0)) \cong C_0(G^0)$. Since $\delta(C_{G^0})|\gamma\rangle_1 \subseteq |\gamma\rangle_1$ and $\delta(c\cdot f)|\eta\rangle_1 = \delta(c)|\eta\rangle_1 r^*(f)$ for all $c \in C_{G^0}$, $f \in C_0(G^0)$, $\eta \in \gamma$, the formula $c\cdot|\eta\rangle_1 := \delta(c)|\eta\rangle_1$ defines a faithful field of representations $C_{G^0} \to \mathcal{L}(|\gamma\rangle_1)$ in the sense of Theorem 3.3 in [6]. Consequently, C_{G^0} is a continuous $C_0(G^0)$-algebra and $C_{s(U)}$ a continuous $C_0(s(U))$-algebra.

(iv) Let $c, c' \in C_U$ and $f, f' \in C_0(U)$ such that $0 \leqslant f, f'$ and $f + f' \leqslant 1$. Then $\|c\cdot f + c'\cdot f'\|^2 = \|c^*c\cdot g^2 + c^*c'\cdot gg' + c'^*c\cdot g'g + c'^*c'\cdot g'^2\|$, where $g = s_{U*}(f)$, $g' = s_{U*}(f')$. Since $g^2+gg''+g'g+g'^2 \leqslant 1$ and $c^*c, c'^*c', c^*c', c'^*c'$ belong to the continuous $C_0(s(U))$-algebra $C_{U^{-1}U}$, which is a convex Banach $C_0(s(U))$-module, we get $\|cf+c'f'\|^2 \leqslant \max\{\|c\|, \|c'\|\}^2$. Finally, the norm $\|c_u\|^2 = \|(c^*c)_{u^{-1}u}\|$ depends continuously on $u \in U$ because $C_{U^{-1}U}$ is a continuous $C_0(s(U))$-algebra. □

Proof of Proposition 9.1. Using Lemma 9.2 and [8], one easily verifies that there exists a continuous Fell bundle $\mathcal{F}$ on G with an isometric isomorphism $\iota_U\colon \Gamma_0(\mathcal{F}|_U) \to C_U$ of Banach $C_0(U)$-modules for each $U \in \mathfrak{G}$ such that for all $U, V \in \mathfrak{G}$, the following properties hold. First, the map $\Gamma_0(\mathcal{F}|_U) \hookrightarrow \Gamma_0(\mathcal{F}|_V) \xrightarrow{\iota_V} C_V$ is equal to $\Gamma_0(\mathcal{F}|_U) \xrightarrow{\iota_U} C_U \hookrightarrow C_V$ if $U \subseteq V$, and second, $\iota_U(f)^* = \iota_{U^{-1}}(f^*)$, $\iota_{UV}(fg) = \iota_U(f)\iota_V(g)$ for all $f \in \Gamma_0(\mathcal{F}|_U)$, $g \in \Gamma_0(\mathcal{F}|_V)$. Define $\iota\colon \Gamma_f(\mathcal{F}) \to C$ as follows. Given $a = \sum_i a_i \in \Gamma_f(\mathcal{F})$, where $a_i \in \Gamma_0(\mathcal{F}|_{U_i})$ and $U_i \in \mathfrak{G}$, let $\iota(a) = \sum_i \iota_{U_i}(a_i)$. Using the preceding two properties of ι, one easily verifies that ι is well-defined and a $*$-homomorphism. □

Denote by $p_0\colon \Gamma_f(\mathcal{F}) \to \Gamma_0(\mathcal{F}^0)$ the restriction.

Proposition 9.3. *There exists a faithful conditional expectation p from the C^*-algebra $[\iota(\Gamma_f(\mathcal{F}))]$ to C_{G^0} satisfying $p\circ\iota = \iota_{G^0}\circ p_0$.*

In the following lemma, $fh_{\xi,\xi'}$ denotes the pointwise product of functions $f, h_{\xi,\xi'} \in C_c(G)$, where $h_{\xi,\xi'}$ was defined in (8.1).

Lemma 9.4. *Let $\xi, \xi' \in C_c(G)$, $c \in C$, $f \in C_c(G)$, $\eta, \eta' \in \gamma$. Then:*
(i) $\langle\eta|_1\delta(\langle j(\xi)|_2\delta(c)|j(\xi')\rangle_2)|\eta'\rangle_1 = \langle j(\xi)|_2\Delta(\langle\eta|_1\delta(c)|\eta'\rangle_1)|j(\xi')\rangle_2$;
(ii) $\langle j(\xi)|_2\Delta(L(f))|j(\xi')\rangle_2 = L(fh_{\xi,\xi'})$;
(iii) $\langle j(\xi)|_2\delta(c\cdot f)|j(\xi')\rangle_2 = c\cdot(fh_{\xi,\xi'})$ *if $c \in C_U$ and $f \in C_0(U)$ for some $U \in \mathfrak{G}$.*

Proof. (i) If $d = \langle j(\xi)|_2\delta(c)|j(\xi')\rangle_2$, then

$$\delta(d) = \langle j(\xi)|_3(\delta * \mathrm{id})(\delta(c))|j(\xi')\rangle_3 = \langle j(\xi)|_3(\mathrm{id} * \Delta)(\delta(c))|j(\xi')\rangle_3$$

and $\langle\eta|_1\delta(d)|\eta'\rangle_1 = \langle j(\xi)|_2\Delta(\langle\eta|_1\delta(c)|\eta'\rangle_1)|j(\xi')\rangle_2$.

(ii) This is a special case of Lemma 8.12.

(iii) Let $\eta, \eta' \in \gamma$. Since $c \in C_U$, we have $\langle \eta|_1 \delta(c)|\eta'\rangle_1 = L(g)$ for some $g \in C_0(U)$. Let $\xi'' = r^*(s_{U*}(f))\xi$ and denote by $d_l, d_r \in C$ the left and the right hand side of the equation in (iii), respectively. Then $d_l = \langle j(\xi)|_2 \delta(c)|j(\xi'')\rangle_2$, and by (i) and (ii),

$$\langle \eta|_1 \delta(d_l)|\eta'\rangle_1 = \langle j(\xi)|_2 \Delta(\langle \eta|_1 \delta(c)|\eta'\rangle_1)|j(\xi'')\rangle_2 = \langle j(\xi)|_2 L(g)|j(\xi'')\rangle_2 = L(gh_{\xi,\xi''}),$$
$$\langle \eta|_1 \delta(d_r)|\eta'\rangle_1 = \langle \eta|_1 \delta(c)|\eta'\rangle_1 r^*(s_{U*}(fh_{\xi,\xi'})) = L(g)L(s_{U*}(fh_{\xi,\xi'})).$$

We can conclude that $\langle \eta|_1 \delta(d_l)|\eta'\rangle_1 = \langle \eta|_1 \delta(d_r)|\eta'\rangle_1$ because for all $x \in G$,

$$(gh_{\xi,\xi''})(x) = g(x) \int_{G^{r(x)}} \overline{\xi(y)} f(x) \xi'(x^{-1}y) \, \mathrm{d}\lambda^{r(x)}(y) = g(x)(s_{U*}(fh_{\xi,\xi'}))(s(x)).$$

Since $\eta, \eta' \in \gamma$ were arbitrary and δ is injective, we must have $d_l = d_r$. □

Proof of Proposition 9.3. Given a subset $U \subseteq G$, denote by χ_U its characteristic function. Using the same formulas as for elements of $C_c(G)$, we can define a map $j(\xi) \colon \mathfrak{K} \to H$ and the function $h_{\xi,\xi'}$ for the characteristic function $\xi = \xi' = \chi_{G^0}$ of $G^0 \subseteq G$, and then Lemma 9.4 still holds. Define $p \colon C \to C$ by $c \mapsto \langle j(\chi_{G^0})|_2 \delta(c)|j(\chi_{G^0})\rangle_2$. Then $\|p\| \leqslant \|j(\chi_{G^0})\|^2 = 1$, and the relation $h_{\chi_{G^0},\chi_{G^0}} = \chi_{G^0}$ and Lemma 9.4 imply that $p|_{C_{G^0}} = \mathrm{id}$ and $p|_{C_U} = 0$ whenever $U \in \mathfrak{G}$ and $U \cap G^0 = \varnothing$. Using a partition of unity argument and the fact that $G^0 \subseteq G$ is open and closed, we can conclude that $p \circ \iota = \iota_{G^0} \circ p_0$.

It remains to show that p is faithful. Using the right-regular representation of G, one easily verifies that $[C^*_r(G)' j(\chi_{G^0})\mathfrak{K}] = H$. Therefore, the map $q \colon C^*_r(G) \to \mathcal{L}(\mathfrak{K})$, $a \mapsto j(\chi_{G^0})^* a j(\chi_{G^0})$, is faithful in the sense that $q(a^*a) \neq 0$ if $a \neq 0$. If $c \in [\iota(\Gamma_f(\mathcal{F}))]$ and $p(c^*c) = 0$, then $\eta^* p(c^*c)\eta = q(\langle \eta^*|_1 \delta(c^*c)|\eta\rangle_1) = 0$ and hence $\langle \eta^*|_1 \delta(c^*c)|\eta\rangle_1 = 0$ and $\delta(c)|\eta\rangle_1 = 0$ for all $\eta \in \gamma$, whence $\delta(c) = 0$ and $c = 0$ by injectivity of δ. □

Proposition 9.3 and [15, Fact 3.11] imply:

Corollary 9.5. *ι extends to an embedding $C^*_r(\mathcal{F}) \to C$.* □

We denote the extension above by ι again.

Proposition 9.6. *If δ is fine, then $\iota \colon C^*_r(\mathcal{F}) \to C$ is a $*$-isomorphism.*

Proof. We only need to show that C is equal to the linear span of all C_U, where $U \in \mathfrak{G}$. Consider an element $d \in C$ of the form $d = \langle j(\xi)|_2 \delta(c)|j(\xi')\rangle_2$, where $c \in C, \xi \in C_c(V), \xi' \in C_c(V')$ for some $V, V' \in \mathfrak{G}$. Since G is étale and δ is fine, the closed linear span of all elements of the form like d is equal to $[\langle \alpha|_2 \delta(C)|\alpha\rangle_2] = [\langle \alpha|_2|\alpha\rangle_2 C] = C$. We show that $d \in C_U$, where $U = VV'^{-1} \in \mathfrak{G}$, and then the claim follows. Let $\eta, \eta' \in \gamma$. By Lemma 9.4,

$$\langle \eta|_1 \delta(d)|\eta'\rangle_1 \in \langle j(\xi)|_2 \Delta(C^*_r(G))|j(\xi')\rangle_2 \subseteq [L(C_c(G)h_{\xi,\xi'})] \subseteq L(C_0(U)).$$

Using the relation $\delta(d)|\gamma\rangle_1 \subseteq [|\gamma\rangle_1 C^*_r(G)]$, we get

$$\delta(d)|\gamma\rangle_1 \subseteq [|\gamma\rangle_1 \langle \gamma|_1 \delta(d)|\gamma\rangle_1] \subseteq [|\gamma\rangle_1 L(C_0(U))].$$

Moreover, since $h_{\xi,\xi'} \in C_c(U)$, we can choose $g \in C_0(U)$ with $h_{\xi,\xi'} g = h_{\xi,\xi'}$. Then $L(fh_{\xi,\xi'}) r^*(s_{U*}(g)) = L(fh_{\xi,\xi'})$ for each $f \in C_0(U)$, and hence $\langle \eta|_1 \delta(d\rho_\gamma(s_{U*}(g)))|\eta'\rangle_1 = \langle \eta|_1 \delta(d)|\eta'\rangle_1 r^*(s_{U*}(g)) = \langle \eta|_1 \delta(d)|\eta'\rangle_1$. Since δ is injective, we can conclude $d = d\rho_\gamma(s_{U*}(g)) \in C\rho_\gamma(C_0(s(U)))$ and finally $d \in C_U$. □

Proposition 9.7. *If δ is fine, then $\mathcal{F}$ is admissible.*

Proof. The proof is similar to the proof of Lemma 6.2 (i). By 9.2 (iii), $\Gamma_0(\mathcal{F}^0) \cong C_{G^0}$ is a continuous $C_0(G^0)$-algebra. Let $u \in G^0$, denote by $I_u \subset C_0(G^0)$ the ideal of all functions vanishing at u, and assume $\mathcal{F}_u = 0$. Then $\Gamma_0(\mathcal{F}^0) = [\Gamma_0(\mathcal{F}^0)I_u]$ and $[C^*_r(\mathcal{F})] = [C^*_r(\mathcal{F})\Gamma_0(\mathcal{F}^0)] = [C^*_r(\mathcal{F})I_u]$, whence $C = [C\rho_\gamma(I_u)]$. Define $j(\chi_{G^0})$ as in the proof of Proposition 9.3. Then $[\delta(C)|\gamma\rangle_1 C^*_r(G)] = [|\gamma\rangle_1 C^*_r(G)]$ and

$$\begin{aligned}[r^*(C_0(G^0))C^*_r(G)] = [\langle\gamma|_1|\gamma\rangle_1 C^*_r(G)] &= [\langle\gamma|_1\delta(CI_u)|\gamma\rangle_1 C^*_r(G)] \\ &= [\langle\gamma|_1|\gamma\rangle_1 r^*(I_u)C^*_r(G)] = [r^*(I_u)C^*_r(G)],\end{aligned}$$

whence $[j(\chi_{G^0})^* C^*_r(G) j(\chi_{G^0})] = I_u \neq C_0(G^0)$, a contradiction. □

The construction of the Fell bundle is functorial with respect to the following class of morphisms.

Definition 9.8. *A morphism ρ of coactions (C^γ_K, δ_C) and (D^ϵ_L, δ_D) of $C^*_r(G)$ is* strongly nondegenerate *if $[\rho(C)D_{G^0}] = D$.*

Proposition 9.9. *Let π be a strongly nondegenerate morphism of fine coactions (C^γ_K, δ_C), (D^ϵ_L, δ_D) with associated Fell bundles $\mathcal{F}$, $\mathcal{G}$ and $*$-homomorphisms $\iota_{\mathcal{F}}$, $\iota_{\mathcal{G}}$. Then there exists a unique morphism T from $\mathcal{F}$ to $\mathcal{G}$ such that $\iota_{\mathcal{G}} \circ T_* = \pi \circ \iota_{\mathcal{F}}$.*

Proof. Let $U, V \in \mathfrak{G}$. Then $\pi(C_U)D_V \subseteq D_{UV}$ because

$$\begin{aligned}\delta_D(\pi(C_U)D_V)|\epsilon\rangle_1 &= ((\pi * \mathrm{id})(\delta_C(C_U)))\delta_D(D_V)|\epsilon\rangle_1 \\ &\subseteq ((\pi * \mathrm{id})(\delta_C(C_U)))|\epsilon\rangle_1 L(C_0(V)) \\ &\subseteq |\epsilon\rangle_1 L(C_0(U))L(C_0(V)) = |\epsilon\rangle_1 L(C_0(UV)) \\ \text{and } \pi(C_U)D_V &\subseteq [\pi(C\rho_\gamma(C_0(s(U))))D_V] \subseteq [\pi(C)D\rho_\epsilon(C_0(s(UV)))],\end{aligned}$$

where the last inclusion follows similarly as in the proof of Lemma 9.2 (ii). Define a map $S_{U,V} \colon \Gamma_0(\mathcal{F}|_U) \times \Gamma_0(\mathcal{G}|_V) \to \Gamma_0(\mathcal{G}|_{UV})$ by $(f, g) \mapsto \iota_{\mathcal{G}}^{-1}(\pi(\iota_{\mathcal{F}}(f))\iota_{\mathcal{G}}(g))$, let $(x, y) \in (U \times V) \cap G\,{}_s{\times}_r G$, and denote by $I_x \subseteq \Gamma_0(\mathcal{F}|_U)$, $I_y \subseteq \Gamma_0(\mathcal{G}|_V)$, $I_{xy} \subseteq \Gamma_0(\mathcal{G}|_{UV})$ the subspaces of all sections vanishing at x, y, and xy, respectively. Using Lemma 9.2 (i), one easily verifies that $S_{U,V}$ maps $I_x \times \Gamma_0(\mathcal{G}|_V)$ and $\Gamma_0(\mathcal{F}|_U) \times I_y$ into I_{xy}. Hence, there exists a unique map $S_{x,y} \colon \mathcal{F}_x \times \mathcal{G}_y \to \mathcal{G}_{xy}$ such that $S_{x,y}(f(x), g(y)) = (S_{U,V}(f, g))(xy)$ for all $f \in \Gamma_0(\mathcal{F}|_U)$, $g \in \Gamma_0(\mathcal{G}|_V)$, and this map depends on (x, y) but not on (U, V). For each $x \in G$ and $c \in \mathcal{F}_x$, define $T(c) \colon \mathcal{G}|_{G^{s(x)}} \to \mathcal{G}|_{G^{r(x)}}$ by $T(c)d = S_{x,y}(c, d)$ for each $y \in G^{s(x)}$, $d \in \mathcal{G}_y$. One easily checks that then T is a continuous map from $\mathcal{F}$ to $\mathcal{M}(\mathcal{G})$ which satisfies conditions (i) and (ii) of Definition 7.8, and that the representation $\tilde{\pi} := \iota_{\mathcal{G}}^{-1} \circ \pi \circ \iota_{\mathcal{F}} \colon C^*_r(\mathcal{F}) \to M(C^*_r(\mathcal{G}))$ satisfies $\tilde{\pi}(f)g = (T \circ f)g$ for all $f \in \Gamma_c(\mathcal{F})$, $g \in \Gamma_c(\mathcal{G})$. We show that T also satisfies condition (iii) of Definition 7.8. Since π is strongly nondegenerate, $D = [\pi(C)D_{G^0}]$, that is, $C^*_r(\mathcal{G}) = [\tilde{\pi}(C^*_r(\mathcal{F}))\Gamma_0(\mathcal{G}^0)]$ and hence $\Gamma^2(\mathcal{G}, \lambda^{-1}) = [\tilde{\pi}(C^*_r(\mathcal{F}))\Gamma_0(\mathcal{G}^0)]$. In particular, $\mathcal{G}_x = [T(\mathcal{F}_x)\mathcal{G}_{s(x)}]$ for each $x \in G$ because $G_{s(x)}$ is discrete. □

9.2. **The unit and counit of the adjunction.** Denote by $\mathbf{Coact}^{as}_{C^*_r(G)}$ the category of very fine left-full coactions of $C^*_r(G)$ with all strongly nondegenerate morphisms. Then the functor $\check{\mathbf{F}}\colon \mathbf{Fell}^a_G \to \mathbf{Coact}^a_{C^*_r(G)}$ constructed in the preceding section actually takes values in $\mathbf{Coact}^{as}_{C^*_r(G)}$:

Lemma 9.10. *Let T be a morphism of admissible Fell bundles $\mathcal{F}, \mathcal{G}$ on G. Then the morphism $\check{\mathbf{F}}T$ from $\check{\mathbf{F}}\mathcal{F}$ to $\check{\mathbf{F}}\mathcal{G}$ is strongly nondegenerate.*

Proof. Immediate from Proposition 7.9 ii). ☐

The constructions in Proposition 9.1 and Proposition 9.9 yield a functor $\check{\mathbf{G}}$ from $\mathbf{Coact}^{as}_{C^*_r(G)}$ to $\mathbf{Fell}^a_G$. We now obtain an embedding as a full and coreflective subcategory $(\check{\mathbf{F}}, \check{\mathbf{G}}, \check{\eta}, \check{\epsilon})$ of $\mathbf{Fell}^a_G$ into $\mathbf{Coact}^{as}_{C^*_r(G)}$.

Proposition 9.11. *Let $\mathcal{F}$ be an admissible Fell bundle, $(\pi_{\mathcal{F}}(C^*_r(\mathcal{F}))^{\gamma_{\mathcal{F}}}_{K_{\mathcal{F}}}, \delta_{\mathcal{F}}) = \check{\mathbf{F}}\mathcal{F}$ the associated fine coaction, and $\mathcal{G} = \check{\mathbf{G}}\check{\mathbf{F}}\mathcal{F}$ and $\iota_{\mathcal{G}}\colon C^*_r(\mathcal{G}) \to \pi_{\mathcal{F}}(C^*_r(\mathcal{F}))$ the Fell bundle and the $*$-homomorphism associated to this coaction as above. Then there exists a unique isomorphism $\check{\eta}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{G}$ such that $\iota_{\mathcal{G}} \circ (\check{\eta}_{\mathcal{F}})_* = \pi_{\mathcal{F}}$.*

Proof. Let $(C^\gamma_K, \delta) = (\pi_{\mathcal{F}}(C^*_r(\mathcal{F}))^{\gamma_{\mathcal{F}}}_{K_{\mathcal{F}}}, \delta_{\mathcal{F}})$ and $U \in \mathfrak{G}$. We show that $C_U = \pi_{\mathcal{F}}(\Gamma_0(\mathcal{F}|_U))$. Note that $[\{h_{\xi,\xi'} \mid \xi \in C_c(r(U)), \xi' \in C_c(U)\}] = C_0(U)$, where the functions $h_{\xi,\xi'}$ were defined in (8.1). Using Lemma 9.4, we can conclude

$$C_U = [\langle j(C_c(r(U)))|_2 \delta(\pi_{\mathcal{F}}(\Gamma_c(\mathcal{F})))|j(C_c(U))\rangle_2].$$

By Lemma 8.12, we have for all $\xi \in C_c(r(U))$, $f \in \Gamma_c(\mathcal{F})$, $\xi' \in C_c(U)$,

$$\langle j(\xi)|_2 \delta(\pi_{\mathcal{F}}(f))|j(\xi')\rangle_2 = \pi_{\mathcal{F}}(f h_{\xi,\xi'}) \in \pi_{\mathcal{F}}(\Gamma_0(\mathcal{F}|_U)),$$

where $fh_{\xi,\xi'}$ denotes the pointwise product. Consequently, $C_U = \pi_{\mathcal{F}}(\Gamma_0(\mathcal{F}|_U))$. Since $U \in \mathfrak{G}$ was arbitrary, we can conclude that there exists an isomorphism $\check{\eta}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{G}$ of Banach bundles such that $\iota_{\mathcal{G}} \circ (\check{\eta}_{\mathcal{F}})_* = \pi_{\mathcal{F}}\colon \Gamma_c(\mathcal{F}) \to C$. Using the fact that $(\check{\eta}_{\mathcal{F}})_*$ is a $*$-homomorphism and that G is étale, one easily concludes that $\check{\eta}_{\mathcal{F}}$ is an isomorphism of Fell bundles. ☐

Proposition 9.12. *Let $(\mathcal{C}, \delta)$ be a very fine coaction of $C^*_r(G)$, where $\mathcal{C} = C^\gamma_K$, and let $\mathcal{F}, \iota\colon C^*_r(\mathcal{F}) \to C$ be the associated Fell bundle and $*$-isomorphism. Then there exists a unique strongly nondegenerate morphism $\check{\epsilon}_{(\mathcal{C},\delta)}$ from $(\pi_{\mathcal{F}}(C^*_r(\mathcal{F}))^{\gamma_{\mathcal{F}}}_{K_{\mathcal{F}}}, \delta_{\mathcal{F}})$ to $(\mathcal{C}, \delta)$ such that $\check{\epsilon}_{(\mathcal{C},\delta)} \circ \pi_{\mathcal{F}} = \iota$.*

Lemma 9.13. *Let $U \in \mathfrak{G}$, $\xi \in C_c(U)$, $\eta \in \gamma$, and $\omega = |\eta\rangle_1 j(\xi) \in \gamma \triangleright \alpha \subseteq \mathcal{L}(\mathfrak{K}, K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H)$.*

(i) *There exists a $C_0(G^0)$-weight $\phi\colon \Gamma_0(\mathcal{F}^0) \to C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K})$, $f \mapsto \omega^*\delta(\iota(f))\omega$.*

(ii) *There exists a unique isometry $S_\omega\colon K_\phi = \Gamma^2(\mathcal{F}, \nu; \phi) \to K_\gamma \underset{\mathfrak{b}}{\otimes}{}_\beta H$ such that $S_\omega \hat{j}_\phi(f) = \delta(\iota(f))\omega$ for all $f \in \Gamma_c(\mathcal{F})$, and $S_\omega \pi_\phi(f) = \delta(\iota(f)) S_\omega$ for all $f \in \Gamma_c(\mathcal{F})$.*

(iii) *$S_\omega j_\phi(\Gamma_c(\mathcal{F})) \subseteq \gamma \triangleright \alpha$.*

Proof. (i) First, $\omega^*\delta(C_{G^0})\omega \subseteq [\alpha^*\langle\gamma|_1|\gamma\rangle_1 L(C_0(G^0))\alpha] = [\alpha^*\langle\gamma|_1|\gamma\rangle_1\alpha] = C_0(G^0) \subseteq \mathcal{L}(\mathfrak{K})$. Second, observe that for all $c \in C_{G^0}$, $f \in C_0(G^0)$,

$$\begin{aligned}\phi(cf) &= j(\xi)^*\langle\eta|_1\delta(cf)|\eta\rangle_1 j(\xi) \\ &= j(\xi)^*\langle\eta|_1\delta(c)|\eta\rangle_1 r^*(f) j(\xi) = j(\xi)^*\langle\eta|_1\delta(c)|\eta\rangle_1 j(\xi) f = \phi(c)f.\end{aligned}$$

(ii) As before, denote by $p_0\colon \Gamma_f(\mathcal{F}) \to \Gamma_0(\mathcal{F}^0)$ the restriction. Let $U \in \mathfrak{G}$, $f, f' \in \Gamma_c(\mathcal{F})$, and $g = f^*f'$. Using the relation $\operatorname{supp} h_{\xi,\xi} \subseteq G^0$ and Lemma 9.4, we find

$$\begin{aligned}\omega^*\delta(\iota(f))^*\delta(\iota(f'))\omega &= \eta^*\langle j(\xi)|_2\delta(\iota(g))|j(\xi)\rangle_2\eta^* \\ &= \eta^*\iota(g\cdot h_{\xi,\xi})\eta \\ &= \omega^*\delta(\iota(p_0(g)))\omega^* = \phi(p_0(g)) = \langle f|f'\rangle_{\Gamma^2(\mathcal{F},\lambda^{-1},\phi)}.\end{aligned}$$

The existence of S_ω follows. Finally, $S_\omega\pi_\phi(f) = \delta(\iota(f))S_\omega$ since $S_\omega\pi_\phi(f)\hat{j}_\phi(g) = S_\omega\hat{j}_\phi(fg) = \delta(\iota(fg))\omega = \delta(\iota(f))S_\omega\hat{j}_\phi(g)$ for all $f, g \in \Gamma_c(\mathcal{F})$.

(iii) Let $V \in \mathfrak{G}, f \in \Gamma_c(\mathcal{F}|_V), \zeta \in C_c(G^0)$, and define $\zeta' \in L^2(G^0,\mu)$ by $\zeta'(s(x)) = \zeta(r(x))D^{1/2}(x)$ for all $x \in V$ and $\zeta'(y) = 0$ for all $y \in G^0\backslash s(V)$. Then $(j_\phi(f)\zeta)(x) = f(x)\zeta(r(x)) = (\hat{j}_\phi(f)\zeta')(x)$ for all $x \in G$ and therefore

$$S_\omega j_\phi(f)\zeta = S_\omega\hat{j}_\phi(f)\zeta' = \delta(\iota(f))\omega\zeta' = \delta(\iota(f))|\eta\rangle_1 j(\xi)\zeta'.$$

Since $f \in \Gamma_c(\mathcal{F}|_V)$, there exist $f' \in L(C_0(V))$, $\eta' \in \gamma$ such that $\delta(\iota(f))|\eta\rangle_1 = |\eta'\rangle_1 L(f')$. Now,

$$S_\omega j_\phi(f)\zeta = \delta(\iota(f))|\eta\rangle_1 j(\xi)\zeta' = |\eta'\rangle_1 L(f')j(\xi)\zeta' = |\eta'\rangle j(L(f')\xi)\zeta$$

because $(L(f')j(\xi)\zeta')(z) = 0$ for $z \notin VU$ and

$$\begin{aligned}(L(f')j(\xi)\zeta')(xy) &= D^{-1/2}(x)f'(x)\xi(y)\zeta'(r(y)) \\ &= f'(x)\zeta(r(x))\xi(y) = j(L(f')\xi)\zeta(xy)\end{aligned}$$

for all $(x,y) \in (V\times U)\cap G_s\times_r G$. Thus, $S_\omega j_\phi(f)\zeta \in \gamma \triangleright \alpha$. The claim follows. □

Proof of Proposition 9.12. Since $\pi_\mathcal{F}$ is injective, we can define $\check{\epsilon} = \check{\epsilon}_{(\mathcal{C},\delta)} := \iota\circ\pi_\mathcal{F}^{-1}$. We show that $\delta\circ\check{\epsilon}$ is a morphism from $\pi_\mathcal{F}(C_r^*(\mathcal{F}))^{\gamma_\mathcal{F}}_{K_\mathcal{F}}$ to $\delta(C)^{\gamma\triangleright\alpha}_{K_\gamma\underset{\mathfrak{b}}{\otimes}_\beta H}$. For each $C_0(G^0)$-weight ϕ on $\Gamma_0(\mathcal{F}^0)$, denote by $p_\phi\colon K_\mathcal{F} \to K_\phi$ the canonical projection. Let $\mathcal{S} \subseteq \mathcal{L}(K_\mathcal{F}, K_\gamma\underset{\mathfrak{b}}{\otimes}_\beta H)$ be the closed linear span of all operators of the form $S_\omega p_\phi$, where U,ξ,η,ω,ϕ are as in the lemma above. Then $Sa = \delta(\check{\epsilon}(a))$ for each $S \in \mathcal{S}$, $a \in \pi_\mathcal{F}(C_r^*(\mathcal{F}))$, and $[\mathcal{S}\gamma_\mathcal{F}] = [\delta(\iota(\Gamma_c(\mathcal{F})))(\gamma\triangleright\alpha)] = \gamma\triangleright\alpha$. The claim follows. Since δ is an isomorphism from $\mathcal{C}$ to $\delta(C)^{\gamma\triangleright\alpha}_{K_\gamma\underset{\mathfrak{b}}{\otimes}_\beta H}$, we can conclude that $\check{\epsilon}$ is a morphism from $\pi_\mathcal{F}(C_r^*(\mathcal{F}))^{\gamma_\mathcal{F}}_{K_\mathcal{F}}$ to $\mathcal{C}$. The relation $(\check{\epsilon} * \mathrm{id})\circ\delta = \delta\circ\check{\epsilon}$ follows from the fact that

$$\begin{aligned}\langle j(\xi)|_2\delta(\check{\epsilon}(g)\cdot f)|j(\xi')\rangle_2 &= \check{\epsilon}(g)\cdot(fh_{\xi,\xi'}) \\ &= \check{\epsilon}(g\cdot(fh_{\xi,\xi'})) = \check{\epsilon}\big(\langle j(\xi)|_2\delta(\pi_\mathcal{F}(g))|j(\xi')\rangle_2\big)\end{aligned}$$

for all $U \in \mathfrak{G}$, $g \in \Gamma_c(\mathcal{F}|_U)$, $f \in C_c(U)$, $\xi,\xi' \in C_c(G)$ by Lemma 9.4. □

Corollary 9.14. *Every very fine coaction of $C_r^*(G)$ is left-full.*

Proof. Let $(\mathcal{C},\delta)$ be a very fine coaction of $C_r^*(G)$, let $(\pi_\mathcal{F}(C_r^*(\mathcal{F}))^{\gamma_\mathcal{F}}_{K_\mathcal{F}}, \delta_\mathcal{F})$ and $\check{\epsilon}_{(\mathcal{C},\delta)}$ be as above, and let $I := \{T \in \mathcal{L}_s((K_\mathcal{F},\gamma_\mathcal{F}),(K,\gamma)) \mid Tx = \check{\epsilon}_{(\mathcal{C},\delta)}(x)T \text{ for all } x \in \pi_\mathcal{F}(C_r^*(\mathcal{F}))\}$.

Then $\gamma = [I\gamma_{\mathcal{F}}]$ because $\check{\epsilon}_{(\mathcal{C},\delta)}$ is a morphism, and since $\delta_{\mathcal{F}}$ is left-full,

$$\begin{aligned}[\delta(C)|\gamma\rangle_1 C_r^*(G)] &= [(\check{\epsilon}_{(\mathcal{C},\delta)} * \mathrm{id})(\delta_{\mathcal{F}}(\pi_{\mathcal{F}}(C_r^*(\mathcal{F}))))(I \underset{\mathfrak{b}}{\otimes} \mathrm{id})|\gamma_{\mathcal{F}}\rangle_1 C_r^*(G)] \\ &= [(I \underset{\mathfrak{b}}{\otimes} \mathrm{id})\delta_{\mathcal{F}}(\pi_{\mathcal{F}}(C_r^*(\mathcal{F})))|\gamma_{\mathcal{F}}\rangle_1 C_r^*(G)] \\ &= [(I \underset{\mathfrak{b}}{\otimes} \mathrm{id})|\gamma_{\mathcal{F}}\rangle_1 C_r^*(G)] = [|\gamma\rangle_1 C_r^*(G)]. \qquad \square\end{aligned}$$

Theorem 9.15. $(\check{\mathbf{F}}, \check{\mathbf{G}}, \check{\eta}, \check{\epsilon})$ *is an embedding of* $\mathbf{Fell}_G^a$ *into* $\mathbf{Coact}_{C_r^*(G)}^{as}$ *as a full and coreflective subcategory.*

Proof. One easily verifies that $\check{\mathbf{G}}$ is faithful and that the families $(\check{\eta}_{\mathcal{F}})_{\mathcal{F}}$ and $(\check{\epsilon}_{(\mathcal{C},\delta)})_{(\mathcal{C},\delta)}$ are natural transformations as desired. Since $\check{\eta}$ is a natural isomorphism, $\check{\mathbf{F}}$ is full and faithful; see Theorem IV.3.1 in [18]. $\square$

This work was supported by the SFB 478 "Geometrische Strukturen in der Mathematik" and funded by the Deutsche Forschungsgemeinschaft (DFG).

References

[1] S. Baaj, Représentation régulière du groupe quantique des déplacements de Woronowicz. *Astérisque*, **232**(1995), 11–48.

[2] S. Baaj and G. Skandalis, C^*-algèbres de Hopf et théorie de Kasparov équivariante. *K-Theory 2*, **6**(1989), 683–721.

[3] S. Baaj and G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres. *Ann. Sci. École Norm. Sup. (4) 26*, **4**(1993), 425–488.

[4] É. Blanchard, Représentations de champs de C^*-algèbres. *C. R. Acad. Sci. Paris Sér. I Math. 314*, **12**(1992), 911–914.

[5] É. Blanchard, Tensor products of $C(X)$-algebras over $C(X)$. *Astérisque*, **232**(1995), 81–92.

[6] É. Blanchard, Déformations de C^*-algèbres de Hopf. *Bull. Soc. Math. France 124*, **1**(1996), 141–215.

[7] G. Böhm, Galois theory for Hopf algebroids. *Ann. Univ. Ferrara Sez. VII (N.S.) 51* (2005), 233–262.

[8] M.J. Dupré and R.M. Gillette, *Banach bundles, Banach modules and automorphisms of C^*-algebras.* Boston, MA, 1983.

[9] S. Echterhoff, S. Kaliszewski, J. Quigg, and I. Raeburn, A categorical approach to imprimitivity theorems for C^*-dynamical systems. *Mem. Amer. Math. Soc. 180*, **850**(2006), viii+169.

[10] M. Enock, Measured quantum groupoids in action. *Mém. Soc. Math. Fr. (N.S.)*, **114**(2008), ii+150.

[11] M. Enock and J.-M. Vallin, Inclusions of von Neumann algebras and quantum groupoids. II. *J. Funct. Anal. 178*, **1**(2000), 156–225.

[12] J.M.G. Fell and R.S. Doran, *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles. Vol. 2.* Boston, MA, 1988.

[13] L. Kadison, Hopf algebroids and H-separable extensions. *Proc. Amer. Math. Soc. 131*, **10**(2003), 2993–3002.

[14] G.G. Kasparov, Equivariant KK-theory and the Novikov conjecture. *Invent. Math. 91*, **1**(1988), 147–201.

[15] A. Kumjian, Fell bundles over groupoids. *Proc. Amer. Math. Soc. 126*, **4**(1998), 1115–1125.

[16] E.C. Lance, *Hilbert C^*-modules. A toolkit for operator algebraists*, vol. 210 of *London Mathematical Society Lecture Note Series.* Cambridge, 1995.

[17] P.-Y. Le Gall, Théorie de Kasparov équivariante et groupoïdes. I. *K-Theory 16*, **4**(1999), 361-390.

[18] S. Mac Lane, *Categories for the working mathematician*, second ed.. New York, 1998.

[19] P.S. Muhly and D.P. Williams, Equivalence and disintegration theorems for Fell bundles and their C^*-algebras. *Dissertationes Math. (Rozprawy Mat.)*, **456**(2008), 1–57.

[20] A.L.T. Paterson, *Groupoids, inverse semigroups, and their operator algebras.* Boston, MA, 1999.

[21] V. Paulsen, *Completely bounded maps and operator algebras.* Cambridge, 2002.
[22] J.C. Quigg, Discrete C^*-coactions and C^*-algebraic bundles. *J. Austral. Math. Soc. Ser. A 60*, **2**(1996), 204–221.
[23] J. Renault, *A groupoid approach to C^*-algebras*, vol. 793 of *Lecture Notes in Mathematics.* Berlin, 1980.
[24] K. Szlachányi, Weak Hopf algebras. In *Operator algebras and quantum field theory (Rome, 1996).* Cambridge, MA (1997), 621–632.
[25] K. Szlachányi, Galois actions by finite quantum groupoids. In *Locally compact quantum groups and groupoids (Strasbourg, 2002).* Berlin (2003), 105–125.
[26] T. Timmermann, A definition of compact C^*-quantum groupoids, In *Operator Structures and Dynamical Systems (Leiden 2008)*, *Contemp. Math.*, **503** (2009), 267-290.
[27] T. Timmermann, The relative tensor product and a minimal fiber product in the setting of C^*-algebras. *J. Operator Theory*, **68** (2012), no. 2, 101-140.
[28] T. Timmermann, C^*-pseudo-multiplicative unitaries and Hopf C^*-bimodules. preprint, arXiv:0908.1850, 2009.
[29] T. Timmermann, C^*-pseudo-multiplicative unitaries, Hopf C^*-bimodules, and their Fourier algebras. *J. Inst. Math. Jussieu* (2010), doi:10.1017/S1474748010000290.
[30] J.-M. Vallin, Actions and coactions of finite quantum groupoids on von Neumann algebras, extensions of the matched pair procedure. *J. Algebra 314*, **2**(2007), 789–816.

T. TIMMERMANN, University of Muenster,, Einsteinstr. 62, 48149 Muenster, Germany
E-mail address: `timmermt@math.uni-muenster.de`

APPENDIX II.1

FREE DYNAMICAL QUANTUM GROUPS AND THE DYNAMICAL QUANTUM GROUP $\mathrm{SU}_Q^{\mathrm{dyn}}(2)$

THOMAS TIMMERMANN

Banach Center Publications *98:311-341, 2012.*

Abstract. We introduce dynamical analogues of the free orthogonal and free unitary quantum groups, which are no longer Hopf algebras but Hopf algebroids or quantum groupoids. These objects are constructed on the purely algebraic level and on the level of universal C^*-algebras. As an example, we recover the dynamical $\mathrm{SU}_q(2)$ studied by Koelink and Rosengren, and construct a refinement that includes several interesting limit cases.

Contents

1. Introduction

Dynamical quantum groups were introduced by Etingof and Varchenko as an algebraic tool to study the quantum dynamical Yang-Baxter equation appearing in statistical

2000 *Mathematics Subject Classification.* Primary 81R50: Secondary 16T05, 46L99.
Key words and phrases. dynamical quantum group, Hopf algebroid, quantum groupoid.
Supported by the SFB 878 "Groups, geometry and actions".

mechanics [7, 8, 9]. Roughly, one can associate to every dynamical quantum group a monoidal category of dynamical representations, and to every solution R of the dynamical Yang-Baxter equation a dynamical quantum group A_R with a specific dynamical representation π such that R corresponds to a braiding on the monoidal category generated by π.

In this article, we introduce two families of dynamical quantum groups $A_{\mathrm{o}}^{B}(\nabla, F)$ and $A_{\mathrm{u}}^{B}(\nabla, F)$ which are natural generalizations of the free orthogonal and the free unitary quantum groups introduced by Wang [18] and van Daele [17]. Roughly, these dynamical quantum groups are universal with respect to the property that they possess a corepresentation v such that F becomes a morphism of corepresentations from the inverse of the transpose $v^{-\mathsf{T}}$ or from $(v^{-\mathsf{T}})^{-\mathsf{T}}$, respectively, to v.

For a specific choice of B, ∇, F, the free orthogonal dynamical quantum group turns out to coincide with the dynamical analogue of $\mathrm{SU}_q(2)$ that arises from a trigonometric dynamical R-matrix and was studied by Koelink and Rosengren [11]. We refine the definition of this variant of $\mathrm{SU}_q(2)$ so that the resulting global dynamical quantum group includes the classical $\mathrm{SU}(2)$, the non-dynamical $\mathrm{SU}_q(2)$ of Woronowicz [21], the dynamical $\mathrm{SU}_q(2)$ and further interesting limit cases which can be recovered from the global object by suitable base changes.

In the non-dynamical case, free orthogonal and free unitary quantum groups are most conveniently constructed on the level of universal C^*-algebras, where Woronowicz's theory of compact matrix quantum groups applies [20]. We shall, however, start on the purely algebraic level and then pass to the level of universal C^*-algebras, where the main problem is to identify a good definition of a dynamical quantum group.

These new classes of dynamical quantum groups give rise to several interesting questions, for example, whether it is possible to obtain a classification similar as in [19], to determine their categories of representations as in [2] and [3], or to relate their representation theory to special functions as it was done in [11] in the special case of $\mathrm{SU}_q(2)$.

Let us now describe the organization and contents of this article in some more detail.

The first part of this article (§2) is devoted to the purely algebraic setting.

We start with a summary on dynamical quantum groups (§2.1). Roughly, these objects can be regarded as Hopf algebras, that is, as algebras A equipped with a comultiplication Δ, counit ϵ and antipode S, where the field of scalars has been replaced by a commutative algebra B equipped with an action of a group Γ. The comultiplication Δ does not take values in the ordinary tensor product $A \otimes A$, but in a product $A \tilde{\otimes} A$ that takes B and Γ into account, and the counit takes values in the crossed product algebra $B \rtimes \Gamma$ which is the unit for the product $-\tilde{\otimes}-$. If B is trivial, however, these dynamical quantum groups are just Γ-graded Hopf algebras (§2.2). In general, we shall use the term (B, Γ)-Hopf algebroid instead of dynamical quantum group to be more precise.

The free orthogonal and unitary dynamical quantum groups are defined as follows. Let B be a unital, commutative algebra with a left action of a group Γ, let $\nabla = (\gamma_1, \dots, \gamma_n)$ be an n-tuple in Γ and let $F \in \mathrm{GL}_n(B)$ such that $F_{ij} = 0$ whenever $\gamma_i \neq \gamma_j^{-1}$.

Definition. *The* free orthogonal dynamical quantum $A_{\mathrm{o}}^{B}(\nabla, F)$ *is the universal algebra with a homomorphism* $r \times s \colon B \otimes B \to A_{\mathrm{o}}^{B}(\nabla, F)$ *and a* $v \in \mathrm{GL}_n(A_{\mathrm{o}}^{B}(\nabla, F))$ *satisfying*

(a) $v_{ij} r(b) s(b') = r(\gamma_i(b)) s(\gamma_j(b')) v_{ij}$ *for all* $b, b' \in B$ *and* $i, j \in \{1, \dots, n\}$,

(b) $r_n(\hat{F})v^{-\mathsf{T}} = vs_n(F)$, *where* $v^{-\mathsf{T}}$ *denotes the transpose of* v^{-1} *and* $\hat{F} = (\gamma_i(F_{ij}))_{i,j}$.

Theorem. $A_{\mathrm{o}}^B(\nabla, F)$ *can be equipped with the structure of a* (B,Γ)*-Hopf algebroid such that* $\Delta(v_{ij}) = \sum_k v_{ik}\tilde{\otimes}v_{kj}$, $\epsilon(v_{ij}) = \delta_{i,j}\gamma_i$, *and* $S(v_{ij}) = (v^{-1})_{ij}$ *for all* i,j.

Assume now that B is equipped with an involution and let $F \in \mathrm{GL}_n(B)$ such that $F^* = F$ and $F_{ij} = 0$ whenever $\gamma_i \neq \gamma_j$.

Definition. *The* free unitary dynamical quantum $A_{\mathrm{u}}^B(\nabla, F)$ *is the universal* $*$*-algebra with a homomorphism* $r \times s\colon B \otimes B \to A_{\mathrm{u}}^B(\nabla, F)$ *and a unitary* $v \in \mathrm{GL}_n(A_{\mathrm{u}}^B(\nabla, F))$ *satisfying the condition (a) above and (c)* $\bar{v}$ *is invertible and* $r_n(\hat{F})\bar{v}^{-\mathsf{T}} = vs_n(F)$.

Theorem. $A_{\mathrm{u}}^B(\nabla, F)$ *can be equipped with the structure of a* (B,Γ)*-Hopf* $*$*-algebroid such that* $\Delta(v_{ij}) = \sum_k v_{ik}\tilde{\otimes}v_{kj}$, $\epsilon(v_{ij}) = \delta_{i,j}\gamma_i$, $S(v_{ij}) = (v^{-1})_{ij}$ *for all* i,j.

The formulas for $\Delta(v_{ij})$ and $\epsilon(v_{ij})$ above imply that the matrices v above are corepresentations of $A_{\mathrm{o}}^B(\nabla, F)$ and $A_{\mathrm{u}}^B(\nabla, F)$, respectively, and the conditions (b) and (c) assert that F is an intertwiner from $v^{-\mathsf{T}}$ or $\bar{v}^{-\mathsf{T}}$, respectively, to v. Such intertwiner relations admit plenty functorial transformations which are studied systematically in §2.3, and yield short proofs of the results above in §2.4. There, we also consider involutions on certain quotients $A_{\mathrm{o}}^B(\nabla, F, G)$ of $A_{\mathrm{o}}^B(\nabla, F)$ which are parameterized by an additional matrix $G \in \mathrm{GL}_n(B)$.

Interestingly, the square of the antipode on the dynamical quantum groups $A_{\mathrm{o}}^B(\nabla, F)$ and $A_{\mathrm{u}}^B(\nabla, F)$ can be described in terms of a natural family of characters $(\theta^{(k)})_k$ which, like the counit ϵ, take values in $B \rtimes \Gamma$. This family is an analogue of Woronowicz's fundamental family of characters on a compact quantum group.

As a main example of the constructions above, we recover the dynamical quantum group $\mathcal{F}_R(\mathrm{SU}(2))$ of Koelink and Rosengren [11] associated to a deformation parameter $q \neq 1$ as the free orthogonal dynamical quantum group $A_{\mathrm{o}}^B(\nabla, F, G)$, where B is the meromorphic functions on the plane, $\Gamma = \mathbb{Z}$ acting by shifts, $\nabla = (1, -1)$ and $F = \begin{pmatrix} 0 & 1 \\ \tilde{f} & 0 \end{pmatrix}$, where $\tilde{f}$ is the meromorphic function $\lambda \mapsto q^{-1}(q^{2\lambda} - q^{-2})/(q^{2\lambda} - 1)$, and $G = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix}$. In §2.6, we show how this example can be refined such that the resulting dynamical quantum group $A_{\mathrm{o}}^B(\nabla, F, G)$ includes $\mathcal{F}_R(\mathrm{SU}(2))$ and, simultaneously, a number of interesting limit cases which can be recovered from the global object by suitable base changes.

The second part of this article (§3) extends the definition of dynamical quantum groups to the level of universal C^*-algebras. Here, B is assumed to be a unital, commutative C^*-algebra and Γ acts via automorphisms. The main tasks is to find a C^*-algebraic analogue of the product $-\tilde{\otimes}-$ that describes the target of the comultiplication. As in the algebraic setting, we construct this product in two steps, by first forming a cotensor product with respect to the Hopf C^*-algebra $C^*(\Gamma)$ naturally associated to the group Γ (§3.1), and then taking a quotient with respect to B (§3.2). Given the monoidal product, all definitions carry over from the algebraic setting to the setting of universal C^*-algebras easily (§3.3).

2. The purely algebraic level

Throughout this section, we assume all algebras and homomorphisms to be unital over a fixed common ground field, and B to be a commutative algebra equipped with a left action of a group Γ.

2.1. **Preliminaries on dynamical quantum groups.** This subsection summarizes the basics of dynamical quantum groups used in this article. We introduce the monoidal category of (B,Γ)-algebras, then define (B,Γ)-Hopf algebroids, and finally consider base changes and the setting of $*$-algebras. Except for the base change, most of this material is contained in [8] and [11] in slightly different guise. We omit all proofs because they are straightforward.

Let $B^{\mathrm{ev}} = B \otimes B$. A B^{ev}-*algebra* is an algebra with a homomorphism $r \times s \colon B^{\mathrm{ev}} \to A$, or equivalently, with homomorphisms $r_A = r, s_A = s \colon B \to A$ whose images commute. A morphism of B^{ev}-algebras is a B^{ev}-linear homomorphism. Write $\Gamma^{\mathrm{ev}} = \Gamma \times \Gamma$ and let $e \in \Gamma$ be the unit. Given a Γ^{ev}-graded algebra A, we write $\partial_a = (\partial_a^r, \partial_a^s) = (\gamma, \gamma')$ whenever $a \in A_{\gamma,\gamma'}$.

2.1.1. **Definition.** *A* (B,Γ)-algebra *is a* Γ^{ev}*-graded* B^{ev}*-algebra such that* $(r \times s)(B^{\mathrm{ev}}) \subseteq A_{e,e}$ *and* $ar(b) = r(\partial_a^r(b))a$, $as(b) = s(\partial_a^s(b))a$ *for all* $b \in B$, $a \in A$. *A* morphism *of* (B,Γ)*-algebras is a morphism of* Γ^{ev}*-graded* B^{ev}*-algebras. We denote by* $\mathbf{Alg}_{(B,\Gamma)}$ *the category of all* (B,Γ)*-algebras.*

2.1.2. **Example.** Denote by $B \rtimes \Gamma$ the crossed product, that is, the universal algebra containing B and Γ such that $e = 1_B$ and $b\gamma \cdot b'\gamma' = b\gamma(b')\gamma\gamma'$ for all $b, b' \in B$, $\gamma, \gamma' \in \Gamma$. This is a (B,Γ)-algebra, where $\partial_{b\gamma} = (\gamma,\gamma)$ and $r(b) = s(b) = b$ for all $b \in B$, $\gamma \in \Gamma$.

The category of all (B,Γ)-algebras can be equipped with a monoidal structure [12] as follows. Let A and C be (B,Γ)-algebras. Then the subalgebra

$$A \overset{\Gamma}{\otimes} C := \sum_{\gamma,\gamma',\gamma'' \in \Gamma} A_{\gamma,\gamma'} \otimes C_{\gamma',\gamma''} \subset A \otimes C$$

is a (B,Γ)-algebra, where $\partial_{a\otimes c} = (\partial_a^r, \partial_c^s)$ for all $a \in A$, $c \in C$ and $(r \times s)(b \otimes b') = r_A(b) \otimes s_C(b')$ for all $b, b' \in B$. Let $I \subseteq A \overset{\Gamma}{\otimes} C$ be the ideal generated by $\{s_A(b) \otimes 1 - 1 \otimes r_C(b) : b \in B\}$. Then $A \tilde{\otimes} C := A \overset{\Gamma}{\otimes} C/I$ is a (B,Γ)-algebra again, called the *fiber product* of A and C. Write $a \tilde{\otimes} c$ for the image of an element $a \otimes c$ in $A \tilde{\otimes} C$.

The product $(A, C) \mapsto A \tilde{\otimes} C$ is functorial, associative and unital in the following sense.

2.1.3. **Lemma.**
i) *For all morphisms of* (B,Γ)*-algebras* $\pi^1 \colon A^1 \to C^1$, $\pi^2 \colon A^2 \to C^2$, *there exists a morphism* $\pi^1 \tilde{\otimes} \pi^2 \colon A^1 \tilde{\otimes} A^2 \to C^1 \tilde{\otimes} C^2$, $a_1 \tilde{\otimes} a_2 \mapsto \pi^1(a_1) \tilde{\otimes} \pi^2(a_2)$.
ii) *For all* (B,Γ)*-algebras* A, C, D, *there is an isomorphism* $(A \tilde{\otimes} C) \tilde{\otimes} D \to A \tilde{\otimes} (C \tilde{\otimes} D)$, $(a \tilde{\otimes} c) \tilde{\otimes} d \mapsto a \tilde{\otimes} (c \tilde{\otimes} d)$.
iii) *For each* (B,Γ)*-algebra* A, *there exist isomorphisms* $(B \rtimes \Gamma) \tilde{\otimes} A \to A$ *and* $A \tilde{\otimes} (B \rtimes \Gamma) \to A$, *given by* $b\gamma \tilde{\otimes} a \mapsto r(b)a$ *and* $a \tilde{\otimes} b\gamma \mapsto s(b)a$, *respectively.*

Of course, the isomorphisms above are compatible in a natural sense.

2.1.4. **Remark.** The product $-\tilde{\otimes}-$ is related to the left and right Takeuchi products $-{}_B\times-$ and $- \times_B -$ as follows. Given a B^{ev}-algebra A, we write ${}_\bullet A$ or $A_\bullet$ when we regard A as a B-bimodule via $b \cdot a \cdot b' := r(b)s(b')a$ or $b \cdot a \cdot b' := ar(b)s(b')$, respectively. Then the left and right Takeuchi products of B^{ev}-algebras A and C are the B^{ev}-algebras

$$A_B{\times}C := \left\{ \sum_i a_i \underset{B}{\otimes} c_i \in {}_\bullet A \underset{B}{\otimes} {}_\bullet C \,\middle|\, \forall b \in B : \sum_i a_i s_A(b) \underset{B}{\otimes} c_i = \sum_i a_i \underset{B}{\otimes} c_i r_C(b) \right\},$$
$$A \times_B C := \left\{ \sum_i a_i \underset{B}{\otimes} c_i \in A_\bullet \underset{B}{\otimes} C_\bullet \,\middle|\, \forall b \in B : \sum_i s_A(b) a_i \underset{B}{\otimes} c_i = \sum_i a_i \underset{B}{\otimes} r_C(b) c_i \right\},$$

where the multiplication is defined factorwise and the embedding of B^{ev} is given by $b \otimes b' \mapsto r_A(b) \underset{B}{\otimes} s_C(b')$. The assignments $(A, C) \mapsto A_B{\times}C$ and $(A, C) \mapsto A{\times}_B C$ extend to bifunctors on the category of B^{ev}-algebras and turn it into a lax monoidal category [6]. The obvious forgetful functor U from (B, Γ)-algebras to B^{ev}-algebras is compatible with these products in the sense that for every pair of (B, Γ)-algebras A, C, the inclusion $A \overset{\Gamma}{\otimes} C \hookrightarrow A {\otimes} C$ factorizes to inclusions of $A \tilde{\otimes} C$ into $A_B{\times}C$ and $A{\times}_B C$, yielding natural transformations from $U(-\tilde{\otimes}-)$ to $U(-)_B{\times}U(-)$ and $U(-) \times_B U(-)$, respectively.

Briefly, a (B, Γ)-Hopf algebroid is a coalgebra in $\mathbf{Alg}_{(B,\Gamma)}$ equipped with an antipode. To make this definition precise, we need two involutions on $\mathbf{Alg}_{(B,\Gamma)}$. Given an algebra A, we denote by A^{op} its opposite, that is, the same vector space with reversed multiplication.

2.1.5. **Lemma.** *There exist automorphisms $(-)^{\mathsf{op}}$ and $(-)^{\mathsf{co}}$ of $\mathbf{Alg}_{(B,\Gamma)}$ such that for each (B, Γ)-algebra A and each morphism $\phi \colon A \to C$, we have $A^{\mathsf{co}} = A$ as an algebra and*

$$(A^{\mathsf{op}})_{\gamma,\gamma'} = A_{\gamma^{-1},\gamma'^{-1}} \text{ for all } \gamma, \gamma' \in \Gamma, \qquad r_{A^{\mathsf{op}}} = r_A, \qquad s_{A^{\mathsf{op}}} = s_A, \qquad \phi^{\mathsf{op}} = \phi,$$
$$(A^{\mathsf{co}})_{\gamma,\gamma'} = A_{\gamma',\gamma} \text{ for all } \gamma, \gamma' \in \Gamma, \qquad r_{A^{\mathsf{co}}} = s_A, \qquad s_{A^{\mathsf{co}}} = r_A, \qquad \phi^{\mathsf{co}} = \phi.$$

Furthermore, $(-)^{\mathsf{op}} \circ (-)^{\mathsf{op}} = \mathrm{id}$, $(-)^{\mathsf{co}} \circ (-)^{\mathsf{co}} = \mathrm{id}$, $(-)^{\mathsf{op}} \circ (-)^{\mathsf{co}} = (-)^{\mathsf{co}} \circ (-)^{\mathsf{op}}$.

2.1.6. **Remark.** The automorphisms above are compatible with the monoidal structure as follows. Given (B, Γ)-algebras A, C, there exist isomorphisms $(A \tilde{\otimes} C)^{\mathsf{op}} \to (A^{\mathsf{op}} \tilde{\otimes} C)^{\mathsf{op}}$ and $(A \tilde{\otimes} C)^{\mathsf{co}} \to C^{\mathsf{co}} \tilde{\otimes} A^{\mathsf{co}}$ given by $a \tilde{\otimes} c \mapsto a \tilde{\otimes} c$ and $a \tilde{\otimes} c \mapsto c \tilde{\otimes} a$, respectively. Moreover, $(B \rtimes \Gamma)^{\mathsf{co}} = B \rtimes \Gamma$ and there exists an isomorphism $S^{B \rtimes \Gamma} \colon B \rtimes \Gamma \to (B \rtimes \Gamma)^{\mathsf{op}}$, $b\gamma \mapsto \gamma^{-1} b$, and all of these isomorphisms and the isomorphisms in Lemma 2.1.3 are compatible in a natural sense.

2.1.7. **Definition.** *A* (B, Γ)-Hopf algebroid *is a (B, Γ)-algebra A equipped with morphisms $\Delta \colon A \to A \tilde{\otimes} A$, $\epsilon \colon A \to B \rtimes \Gamma$, and $S \colon A \to A^{\mathsf{co},\mathsf{op}}$ such that the diagrams below commute,*

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \tilde{\otimes} A \arrow[d, "\Delta \tilde{\otimes} \mathrm{id}"] \\ A \tilde{\otimes} A \arrow[r, "\mathrm{id} \tilde{\otimes} \Delta"'] & A \tilde{\otimes} A \tilde{\otimes} A, \end{tikzcd} \qquad \begin{tikzcd} A \tilde{\otimes} A \arrow[d, "\epsilon \tilde{\otimes} \mathrm{id}"'] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"] & A \tilde{\otimes} A \arrow[d, "\mathrm{id} \tilde{\otimes} \epsilon"] \\ (B \rtimes \Gamma) \tilde{\otimes} A \arrow[r, "\cong"] & A & A \tilde{\otimes} (B \rtimes \Gamma), \arrow[l, "\cong"'] \end{tikzcd}$$

$$\begin{tikzcd}
A\tilde{\otimes}A \arrow[d, "S\tilde{\otimes}\mathrm{id}"'] & & & A \arrow[lll, "\Delta"'] \arrow[rrr, "\Delta"] \arrow[d, "\epsilon"] & & & A\tilde{\otimes}A \arrow[d, "\mathrm{id}\tilde{\otimes}S"] \\
A^{\mathsf{co,op}}\tilde{\otimes}A \arrow[r, "\check{m}"] & A & & B \rtimes \Gamma \arrow[ll, "\check{s}"'] \arrow[rr, "\hat{r}"] & & A & A\tilde{\otimes}A^{\mathsf{co,op}} \arrow[l, "\hat{m}"']
\end{tikzcd}$$

where the linear maps $\hat{m}, \check{m}, \hat{r}, \check{s}$ are given by

$$\hat{m}(a\tilde{\otimes}a') = aa' = \check{m}(a\tilde{\otimes}a'), \qquad \hat{r}(b\gamma) = r(b), \qquad \check{s}(\gamma b) = s(b)$$

for all $a, a' \in A, b \in B, \gamma \in \Gamma$.

A morphism of (B,Γ)-Hopf algebroids $(A, \Delta_A, \epsilon_A, S_A)$, $(C, \Delta_C, \epsilon_C, S_C)$ is a morphism of (B,Γ)-algebras $\pi\colon A \to C$ such that $\Delta_C \circ \pi = (\pi\tilde{\otimes}\pi) \circ \Delta_A$, $\epsilon_C \circ \pi = \epsilon_A$, $S_C \circ \pi = \pi^{\mathsf{co,op}} \circ S_A$. We denote the category of all (B,Γ)-Hopf algebroids by $\mathbf{Hopf}_{(B,\Gamma)}$.

A (B,Γ)-Hopf algebroid reduces to an $\mathfrak{h}$-Hopf algebroid in the sense of [11] when $\mathfrak{h}$ is a commutative Lie algebra, B is the algebra of meromorphic functions on the dual $\mathfrak{h}^*$, and $\Gamma = \mathfrak{h}^*$ acts by shifting the argument. Let us note that the axioms above can be weakened, see [11, Proposition 2.2], but our examples shall automatically satisfy the apparently stronger conditions above.

2.1.8. **Example.** The (B,Γ)-algebra $B \rtimes \Gamma$ is a (B,Γ)-Hopf algebroid, where $\Delta(b\gamma) = b\gamma\tilde{\otimes}\gamma = \gamma\tilde{\otimes}b\gamma$, $\epsilon(b\gamma) = b\gamma$, and $S(b\gamma) = \gamma^{-1}b$ for all $b \in B, \gamma \in \Gamma$.

Let us comment on some straightforward properties of (B,Γ)-Hopf algebroids:

2.1.9. **Remarks.** Let (A, Δ, ϵ, S) be a (B,Γ)-Hopf algebroid.

i) If $\gamma \neq \gamma'$, then $\epsilon(A_{\gamma,\gamma'}) = 0$ because $(B \rtimes \Gamma)_{\gamma,\gamma'} = 0$.
ii) We have $\Delta(A)(1\tilde{\otimes}A_{e,*}) = A\tilde{\otimes}A = (A_{*,e}\tilde{\otimes}1)\Delta(A)$, where $A_{e,*} = \sum_\gamma A_{e,\gamma}$ and $A_{*,e} = \sum_\gamma A_{\gamma,e}$. Indeed, by [15, Proposition 1.3.7],
$$\sum (xS(y_{(1)})\tilde{\otimes}1)\Delta(y_{(2)}) = x\tilde{\otimes}y = \sum \Delta(x_{(1)})(1\tilde{\otimes}S(x_{(2)})y)$$
for all $x \in A_{\gamma,\gamma'}, y \in A_{\gamma',\gamma''}$, $\gamma, \gamma', \gamma'' \in \Gamma$, where $\sum x_{(1)}\tilde{\otimes}x_{(2)} = \Delta(x)$ and $\sum y_{(1)}\tilde{\otimes}y_{(2)} = \Delta(y)$.

(B,Γ)-Hopf algebroids fit into the general definition of Hopf algebroids [4] as follows.

2.1.10. **Remark.** Let (A, Δ, ϵ, S) be a (B,Γ)-Hopf algebroid. Denote by ${}_\bullet\epsilon$ and $\epsilon_\bullet$ the compositions of $\epsilon\colon A \to B \rtimes \Gamma$ with the linear maps $B \rtimes \Gamma \to B$ given by $b\gamma \mapsto b$ and $\gamma b \mapsto b$, respectively, and denote by ${}_\bullet\Delta$ and $\Delta_\bullet$ the compositions of Δ with the natural inclusions $A\tilde{\otimes}A \to {}_\bullet A \underset{B}{\otimes} {}_\bullet A$ and $A\tilde{\otimes}A \to A_\bullet \underset{B}{\otimes} A_\bullet$, respectively (see Remark 2.1.4).

i) The maps ${}_\bullet\epsilon, \epsilon_\bullet\colon A \to B$ will in general not be homomorphisms, but satisfy
$$\begin{aligned} {}_\bullet\epsilon(ar({}_\bullet\epsilon(a')) &= {}_\bullet\epsilon(aa') = {}_\bullet\epsilon(as({}_\bullet\epsilon(a')), \\ \epsilon_\bullet(r(\epsilon_\bullet(a))a') &= \epsilon_\bullet(aa') = \epsilon_\bullet(s(\epsilon_\bullet(a))a') \end{aligned}$$
for all $a, a' \in A$. Indeed, since $\epsilon(a) = {}_\bullet\epsilon(a)\partial_a$ for all homogeneous $a' \in A$,
$$\begin{aligned} {}_\bullet\epsilon(aa')\partial_{aa'} &= {}_\bullet\epsilon(a)\partial_a \cdot {}_\bullet\epsilon(a')\partial_{a'} \\ &= \partial_a({}_\bullet\epsilon(a')){}_\bullet\epsilon(a)\partial_a\partial_{a'} = {}_\bullet\epsilon(ar({}_\bullet\epsilon(a')))\partial_{aa'} \end{aligned}$$
for all homogeneous $a, a' \in A$, and the remaining equations follow similarly.

ii) One easily verifies that $({}_\bullet A, {}_\bullet\Delta, {}_\bullet\epsilon)$ and $(A_\bullet, \Delta_\bullet, \epsilon_\bullet)$ are B-corings, ${}_\bullet\mathcal{A} := (A, {}_\bullet\Delta, {}_\bullet\epsilon)$ is a left B-bialgebroid, and $\mathcal{A}_\bullet := (A^{\mathsf{co}}, \Delta_\bullet, \epsilon_\bullet)$ is a right B-bialgebroid in the sense of [4]. Using the relations $\check{s} \circ \epsilon = s \circ \epsilon_\bullet$ and $\hat{r} \circ \epsilon = r \circ {}_\bullet\epsilon$, one furthermore finds that $(\mathcal{A}_\bullet, {}_\bullet\mathcal{A}, S)$ is a Hopf algebroid over B. To make the match with Definition 4.1 in [4], one has to take $H, s_L, t_L, \Delta_L, \epsilon_L, s_R, t_R, \Delta_R, \epsilon_R, S$ equal to $A, s, r, {}_\bullet\Delta, {}_\bullet\epsilon, r, s, \Delta_\bullet, \epsilon_\bullet, S$, respectively.

Let B and C be commutative algebras with a left action of Γ and let $\phi\colon B \to C$ be a Γ-equivariant homomorphism. We then obtain base change functors $\phi_*\colon \mathbf{Alg}_{(B,\Gamma)} \to \mathbf{Alg}_{(C,\Gamma)}$ and $\phi_*\colon \mathbf{Hopf}_{(B,\Gamma)} \to \mathbf{Hopf}_{(C,\Gamma)}$ as follows. Let A be a (B,Γ)-algebra. Regard C as a B-module via ϕ, and A as a B-bimodule, where $b \cdot a \cdot b' = r(b)as(b')$ for all $b, b' \in B$, $a \in A$. Then the vector space $\phi_*(A) := C \underset{B}{\otimes} A \underset{B}{\otimes} C$ carries the structure of a (C,Γ)-algebra such that

$$(c \underset{B}{\otimes} a \underset{B}{\otimes} d)(c' \underset{B}{\otimes} a' \underset{B}{\otimes} d') = c\partial^r_a(c') \underset{B}{\otimes} aa' \underset{B}{\otimes} (\partial^s_{a'})^{-1}(d)d',$$
$$\partial_{c \underset{B}{\otimes} a \underset{B}{\otimes} d} = \partial_a, \quad (r \times s)(c \otimes c') = c \underset{B}{\otimes} 1 \underset{B}{\otimes} c' \quad \text{for all } c, c', d, d' \in C, a, a' \in A.$$

Every morphism of (B,Γ)-algebras $\pi\colon A \to A'$ evidently yields a morphism of (C,Γ)-algebras $\phi_*(\pi)\colon \phi_*(A) \to \phi_*(A')$, $c \underset{B}{\otimes} a \underset{B}{\otimes} c' \mapsto c \underset{B}{\otimes} \pi(a) \underset{B}{\otimes} c'$, and the assignments $A \mapsto \phi_*(A)$ and $\phi \mapsto \phi_*(\pi)$ form a functor $\phi_*\colon \mathbf{Alg}_{(B,\Gamma)} \to \mathbf{Alg}_{(C,\Gamma)}$.

2.1.11. **Lemma.** i) *There exists a morphism of (C,Γ)-algebras*

$$\phi^{(0)}\colon \phi_*(B \rtimes \Gamma) \to C \rtimes \Gamma, \ c \underset{B}{\otimes} b\gamma \underset{B}{\otimes} c' \mapsto c\phi(b)\gamma c' = c\phi(b)\gamma(c')\gamma.$$

ii) *For all (B,Γ)-algebras A, D, there exists a unique morphism*

$$\phi^{(2)}_{A,D}\colon \phi_*(A\tilde{\otimes}D) \to \phi_*(A)\tilde{\otimes}\phi_*(D), \ c \underset{B}{\otimes} (a\tilde{\otimes}d) \underset{B}{\otimes} c' \mapsto (c \underset{B}{\otimes} a \underset{B}{\otimes} 1)\tilde{\otimes}(1 \underset{B}{\otimes} d \underset{B}{\otimes} c').$$

2.1.12. **Proposition.** *Let (A, Δ, ϵ, S) be a (B,Γ)-Hopf algebroid. Then $\phi_*(A)$ is (C,Γ)-Hopf algebroid with respect to the morphisms*

i) $\Delta'\colon \phi_*(A) \xrightarrow{\phi_*(\Delta)} \phi_*(A\tilde{\otimes}A) \xrightarrow{\phi^{(2)}_{A,A}} \phi_*(A)\tilde{\otimes}\phi_*(A)$, *given by* $c \underset{B}{\otimes} a \underset{B}{\otimes} c' \mapsto \sum_i (c \underset{B}{\otimes} a'_i \underset{B}{\otimes} 1)\tilde{\otimes}(1 \underset{B}{\otimes} a''_i \underset{B}{\otimes} c')$ *whenever* $\Delta(a) = \sum a'_i\tilde{\otimes}a''_i$;

ii) $\epsilon'\colon \phi_*(A) \xrightarrow{\phi_*(\epsilon)} \phi_*(B \rtimes \Gamma) \xrightarrow{\phi^{(0)}} C \rtimes \Gamma$, *given by* $c \underset{B}{\otimes} a \underset{B}{\otimes} c' \mapsto \sum_i c\phi(b_i)\gamma_i c'$ *whenever* $\epsilon(a) = \sum_i b_i\gamma_i$;

iii) $S'\colon \phi_*(A) \to (\phi_*A)^{\mathsf{co,op}}$ *given by* $c \underset{B}{\otimes} a \underset{B}{\otimes} c' \mapsto c' \underset{B}{\otimes} S(a) \underset{B}{\otimes} c$.

The assignments $(A, \Delta, \epsilon, S) \mapsto (\phi_*(A), \Delta', \epsilon', S')$ as above and $\pi \mapsto \phi_*(\pi)$ evidently form a functor $\phi_*\colon \mathbf{Hopf}_{(B,\Gamma)} \to \mathbf{Hopf}_{(C,\Gamma)}$.

The preceding definitions and results extend to $*$-algebras as follows. Assume that B is a $*$-algebra and that Γ preserves its involution.

A (B,Γ)-$*$-algebra is a (B,Γ)-algebra with an involution that is compatible with the grading and the involution on B, and a morphism of (B,Γ)-$*$-algebras is a morphism of (B,Γ)-algebra that preserves the involution. We denote by $*$-$\mathbf{Alg}_{(B,\Gamma)}$ the category of all

(B,Γ)-$*$-algebras. This subcategory of $\mathbf{Alg}_{(B,\Gamma)}$ is monoidal because the crossed product $B \rtimes \Gamma$ is a (B,Γ)-$*$-algebra with respect to the involution given by $(b\gamma)^* = \gamma^{-1}b^*$, and for all (B,Γ)-$*$-algebras A, C, the fiber product $A\tilde{\otimes}C$ is a (B,Γ)-$*$-algebra with respect to the involution given by $(a\tilde{\otimes}c)^* = a^*\tilde{\otimes}c^*$.

2.1.13. **Definition.** *A (B,Γ)-Hopf $*$-algebroid is a (B,Γ)-Hopf algebroid (A,Δ,ϵ,S) where A is a (B,Γ)-$*$-algebra and Δ and ϵ are morphisms of (B,Γ)-$*$-algebras. A morphism of (B,Γ)-Hopf $*$-algebroids is a morphism of the underlying (B,Γ)-Hopf algebroid and (B,Γ)-$*$-algebras. We denote by $\mathbf{Hopf}^*_{(B,\Gamma)}$ the category of all (B,Γ)-Hopf $*$-algebroids.*

2.1.14. **Remark.** If (A,Δ,ϵ,S) is a (B,Γ)-Hopf $*$-algebroid, then $* \circ S \circ * \circ S = \mathrm{id}$; see [11, Lemma 2.9].

We denote by $\overline{A}$ the conjugate algebra of a complex algebra A; this is the set A with conjugated scalar multiplication and the same addition and multiplication. Thus, the involution of a $*$-algebra A is an automorphism $A \to \overline{A}^{\mathsf{op}}$.

2.1.15. **Lemma.** *The category $*$-$\mathbf{Alg}_{(B,\Gamma)}$ has an automorphism $\overline{(-)}$ such that for every (B,Γ)-$*$-algebra A and every morphism of (B,Γ)-$*$-algebras $\phi\colon A \to C$,*

$$(\overline{A})_{\gamma,\gamma'} = \overline{A_{\gamma,\gamma'}} \text{ for all } \gamma,\gamma' \in \Gamma, \qquad r_{\bar{A}} = r_A \circ *, \qquad s_{\bar{A}} = s_A \circ *, \qquad \overline{\phi} = \phi.$$

Furthermore, $\overline{(-)} \circ \overline{(-)} = \mathrm{id}$, $\overline{(-)} \circ (-)^{\mathsf{op}} = (-)^{\mathsf{op}} \circ \overline{(-)}$, $\overline{(-)} \circ (-)^{\mathsf{co}} = (-)^{\mathsf{co}} \circ \overline{(-)}$.

2.1.16. **Remark.** There exists an isomorphism $B \rtimes \Gamma \to \overline{B \rtimes \Gamma}$, $b\gamma \mapsto b^*\gamma$, and for each pair of (B,Γ)-$*$-algebras A, C, there exists an isomorphism $\overline{A\tilde{\otimes}C} \to \overline{A}\tilde{\otimes}\overline{C}$, $a\tilde{\otimes}c \mapsto a\tilde{\otimes}c$.

Let also C be a commutative $*$-algebra with a left action of Γ and let $\phi\colon B \to C$ be a Γ-equivariant $*$-homomorphism. Then for every (B,Γ)-$*$-algebra A, the (C,Γ)-algebra $\phi_*(A)$ is a (C,Γ)-$*$-algebra with respect to the involution given by $(c \underset{B}{\otimes} a \underset{B}{\otimes} c')^* = (\partial^r_a)^{-1}(c)^* \underset{B}{\otimes} a^* \underset{B}{\otimes} \partial^s_a(c')^*$, and we obtain a functor $\phi_*\colon *\text{-}\mathbf{Alg}_{(B,\Gamma)} \to \mathbf{Alg}^*_{(C,\Gamma)}$. Likewise, we obtain a functor $\phi_*\colon \mathbf{Hopf}^*_{(B,\Gamma)} \to \mathbf{Hopf}^*_{(C,\Gamma)}$.

2.2. **The case of a trivial base algebra.** Assume for this subsection that $B = \mathbb{C}$ equipped with the trivial action of Γ. Then the category of all $(\mathbb{C},\Gamma)$-Hopf algebroids is equivalent to the comma category of all Hopf algebras over $\mathbb{C}\Gamma$ as follows.

Recall that the group algebra $\mathbb{C}\Gamma$ is a Hopf $*$-algebra with involution, comultiplication, counit and antipode given by $\gamma^* = \gamma^{-1}$, $\Delta_{\mathbb{C}\Gamma}(\gamma) = \gamma \otimes \gamma$, $\epsilon_{\mathbb{C}\Gamma}(\gamma) = 1$, $S_{\mathbb{C}\Gamma}(\gamma) = \gamma^{-1}$ for all $\gamma \in \Gamma \subset \mathbb{C}\Gamma$. Objects of the comma category $\mathbf{Hopf}_{\mathbb{C}\Gamma}$ are pairs consisting of a Hopf algebra A and a morphism of Hopf algebras $A \to \mathbb{C}\Gamma$, and morphisms from (A,π_A) to (C,π_C) are all morphisms $A \xrightarrow{\phi} C$ such that $\pi_C \circ \phi = \pi_A$. Likewise, we define the comma category $\mathbf{Hopf}^*_{\mathbb{C}\Gamma}$ of Hopf $*$-algebras over $\mathbb{C}\Gamma$.

Note that a $(\mathbb{C},\Gamma)$-algebra is just a $\Gamma\times\Gamma$-graded algebra and $A\tilde{\otimes}C = A \overset{\Gamma}{\otimes} C \subseteq A \otimes C$ for all $(\mathbb{C},\Gamma)$-algebras A, C. Moreover, $\mathbb{C} \rtimes \Gamma = \mathbb{C}\Gamma$, and for every $(\mathbb{C},\Gamma)$-algebra A, the isomorphisms $(\mathbb{C} \rtimes \Gamma)\tilde{\otimes}A \to A$ and $A\tilde{\otimes}(\mathbb{C} \rtimes \Gamma) \to A$ are equal to $\epsilon_{\mathbb{C}\Gamma} \otimes \mathrm{id}$ and $\mathrm{id} \otimes \epsilon_{\mathbb{C}\Gamma}$.

2.2.1. **Lemma.** *Let (A,Δ,ϵ,S) be a $(\mathbb{C},\Gamma)$-Hopf algebroid and let $\epsilon' := \epsilon_{\mathbb{C}\Gamma} \circ \epsilon\colon A \to \mathbb{C}$. The (A,Δ,ϵ',S) is a Hopf algebra and $\epsilon\colon A \to \mathbb{C}\Gamma$ is morphism of Hopf algebras.*

Proof. The preceding observations easily imply that $(A, \Delta, \epsilon', S)$ is a Hopf algebra. To see that ϵ is a morphism of Hopf algebras, use the fact that Δ, ϵ, S are $\Gamma \times \Gamma$-graded. $\square$

2.2.2. **Lemma.** *Let (A, Δ, ϵ, S) be a Hopf algebra with a morphism $\pi \colon A \to \mathbb{C}\Gamma$. Then A is a $(\mathbb{C}, \Gamma)$-algebra with respect to the grading given by $A_{\gamma,\gamma'} = \{a \in A : (\pi \otimes \mathrm{id} \otimes \pi)(\Delta(a)) = \gamma \otimes a \otimes \gamma'\}$ for all $\gamma, \gamma' \in \Gamma$, and (A, Δ, π, S) is a $(\mathbb{C}, \Gamma)$-Hopf algebroid.*

Proof. The formula above evidently defines a $\Gamma \times \Gamma$-grading on A. Coassociativity of Δ implies that $\Delta(A) \subseteq A \tilde{\otimes} A$. The remark preceding Lemma 2.2.1 and the relation $\epsilon_{\mathbb{C}\Gamma} \circ \pi = \epsilon$ imply $(\pi \tilde{\otimes} \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \tilde{\otimes} \pi) \circ \Delta$. Finally, in the notation of Definition 2.1.7, $\check{m} \circ (S \tilde{\otimes} \mathrm{id}) \circ \Delta = m \circ (S \otimes \mathrm{id}) \circ \Delta = \epsilon = \check{s} \circ \pi$ and similarly $\hat{m} \circ (\mathrm{id} \tilde{\otimes} S) \circ \Delta = \hat{r} \circ \pi$. $\square$

Putting everything together, one easily verifies:

2.2.3. **Proposition.** *There exists an equivalence of categories $\mathbf{Hopf}_{(\mathbb{C},\Gamma)} \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightleftarrows}} \mathbf{Hopf}_{\mathbb{C}\Gamma}$, where $\mathbf{F}(A, \Delta, \epsilon, S) = ((A, \Delta, \epsilon_{\mathbb{C}\Gamma} \circ \epsilon, S), \epsilon)$, $\mathbf{F}\phi = \phi$, $\mathbf{G}((A, \Delta, \epsilon, S), \pi) = (A, \Delta, \pi, S)$ with the grading on A defined as in Lemma 2.2.2, and $\mathbf{G}\phi = \phi$. Likewise, there exists an equivalence $\mathbf{Hopf}^*_{(\mathbb{C},\Gamma)} \rightleftarrows \mathbf{Hopf}^*_{\mathbb{C}\Gamma}$.*

Let us next consider the base change from $\mathbb{C}$ to a commutative algebra C along the unital inclusion $\phi \colon \mathbb{C} \to C$ for a $(\mathbb{C}, \Gamma)$-Hopf algebroid (A, Δ, ϵ, S).

2.2.4. **Remark.** The action of Γ on C and the morphism $\epsilon \colon A \to \mathbb{C}\Gamma$ turn C into a left module algebra over the Hopf algebra $(A, \Delta, \epsilon_{\mathbb{C}\Gamma} \circ \epsilon, S)$, and $\phi_*(A, \Delta, \epsilon, S)$ coincides with the Hopf algebroids considered in [4, §3.4.6] and [10, Theorem 3.1], and is closely related to the quantum transformation groupoid considered in [14, Example 2.6].

Assume that C is an algebra of functions on Γ on which Γ acts by left translations.

2.2.5. **Proposition.** *Define $m \colon A \to \mathrm{End}(A)$ and $m_r, m_s \colon C \to \mathrm{End}(A)$ by $m(a')a = a'a$, $m_r(c)a = c(\partial^r_a)a$, $m_s(c)a = c(\partial^s_a)a$ for all $a, a' \in A$, $c \in C$. Then there exists a homomorphism $\lambda \colon \phi_*(A) \to \mathrm{End}(A)$, $c \otimes a \otimes c' \mapsto m_r(c)m(a)m_s(c')$, and λ is injective if $aA_{\gamma,\gamma'} \neq 0$ for all non-zero $a \in A$ and all $\gamma, \gamma' \in \Gamma$.*

Proof. First, note that

$$m(a')m_r(c)a = a'c(\partial^r_a)a = c((\partial^r_{a'})^{-1}\partial^r_{a'a})a'a = m_r(\partial^r_{a'}(c))m(a)a$$

and likewise $m(a')m_s(c) = m_s(\partial^s_{a'}(c))m(a')$ for all $a, a' \in A$, $c \in C$. The existence of λ follows. Assume that $aA_{\gamma,\gamma'} \neq 0$ for all non-zero $a \in A$ and all $\gamma, \gamma' \in \Gamma$. Let $d := \sum_i c_i \otimes a_i \otimes c'_i \in \phi_*(A)$ be non-zero, where all a_i are homogeneous. Identifying $C \otimes A \otimes C$ with a space of A-valued functions on $\Gamma \times \Gamma$ and using the assumption, we first find $\gamma, \gamma' \in \Gamma$ such that $a := \sum_i c_i(\partial^r_{a_i}\gamma)a_i c'_i(\gamma')$ is non-zero, and then an $a' \in A_{\gamma,\gamma'}$ such that $\lambda(d)a' = aa' \neq 0$. $\square$

2.2.6. **Remark.** Regard elements of C as functionals on $\mathbb{C}\Gamma$ via $c(\sum_i b_i\gamma_i) = \sum_i b_i c(\gamma_i)$. Then $m_r(c)a = (c \circ \epsilon \otimes \mathrm{id})(\Delta(a))$, $m_s(c)a = (\mathrm{id} \otimes c \circ \epsilon)(\Delta(a))$ for all $c \in C$, $a \in A$.

2.2.7. **Example.** Let G be a compact Lie group, $\mathcal{O}(G)$ its Hopf algebra of representative functions [16, §1.2] and $T \subseteq G$ a torus of rank d. We now apply Proposition 2.2.5, where

- $A = \mathcal{O}(G)$, regarded as a Hopf $(\mathbb{C}, \hat{T})$-algebroid as in Lemma 2.2.2 using the homomorphism $\pi\colon \mathcal{O}(G) \to \mathcal{O}(T)$ induced from the inclusion $T \subseteq G$, and the isomorphism $\mathcal{O}(T) \cong \mathbb{C}\hat{T}$,
- $C = U\mathfrak{t}$ is the enveloping algebra of the Lie algebra $\mathfrak{t}$ of T, regarded as a polynomial algebra of functions on $\hat{T}$ such that $X(\chi) = \frac{d}{dt}\big|_{t=0}\chi(e(tX))$, where $e\colon \mathfrak{t} \to T$ denotes the exponential map.

If we regard $U\mathfrak{t}$ as functionals on the algebra $\mathbb{C}\hat{T} \cong \mathcal{O}(T)$ as in Remark 2.2.6, then $X(f) = \frac{d}{dt}\big|_{t=0} f(e(tX))$ and hence $m_r, m_s\colon U\mathfrak{t} \to \operatorname{End}(\mathcal{O}(G))$ are given by

$$(m_r(X)a)(x) = \frac{d}{dt}\Big|_{t=0} a(e(tX)x), \qquad (m_s(X)a)(x) = \frac{d}{dt}\Big|_{t=0} a(xe(tX))$$

for all $X \in \mathfrak{t}$, $a \in \mathcal{O}(G)$, $x \in G$. Thus $\lambda(\mathcal{O}(G)) \subseteq \operatorname{End}(\mathcal{O}(G))$ is the algebra generated by multiplication operators for functions in $\mathcal{O}(G)$ and by left and right differentiation operators along $T \subseteq G$.

If G is connected, then $\mathcal{O}(G)$ has no zero-divisors and hence λ is injective as soon as for all $\chi, \chi' \in \hat{T}$, there exists some non-zero $a \in \mathcal{O}(G)$ such that $a(xyz) = \chi(x)a(y)\chi'(z)$ for all $x, z \in T$ and $y \in G$.

2.3. **Intertwiners for (B, Γ)-algebras.** In this subsection, we study relations of the form used to define the free orthogonal and free unitary dynamical quantum groups $A^B_{\mathrm{o}}(\nabla, F)$ and $A^B_{\mathrm{u}}(\nabla, F)$, and show that such relations admit a number of natural transformations. Conceptually, these relations express that certain matrices are intertwiners or morphisms of corepresentations, and the transformations correspond to certain functors of corepresentation categories. Although elementary, these observations provide short and systematic proofs for the main results in the following subsection.

Regard $M_n(B)$ as a subalgebra of $M_n(B \rtimes \Gamma)$, and let A be a (B, Γ)-algebra. Given a linear map $\phi\colon A \to C$ between algebras, we denote by $\phi_n\colon M_n(A) \to M_n(C)$ its entry-wise extension to $n \times n$-matrices.

2.3.1. **Definition.** *A matrix $u \in M_n(A)$ is* homogeneous *if there exist $\gamma_1, \dots, \gamma_n \in A$ such that $u_{ij} \in A_{\gamma_i,\gamma_j}$ for all i, j. In that case, let $\partial_{u,i} := \gamma_i$ for all i and $\partial_u :=$ $\operatorname{diag}(\gamma_1, \dots, \gamma_n) \in M_n(B \rtimes \Gamma)$. An* intertwiner *for homogeneous matrices $u, v \in M_n(A)$ is an $F \in \mathrm{GL}_n(B)$ satisfying $\partial_v F \partial_u^{-1} \in M_n(B)$ and $r_n(\partial_v F \partial_u^{-1})u = v s_n(F)$. We write such an intertwiner as $u \xrightarrow{F} v$ and let $\hat{F} := \partial_v F \partial_u^{-1}$ if u, v are understood.*

If $u \xrightarrow{F} v$ and $v \xrightarrow{G} w$ are intertwiners, then evidently so are $v \xrightarrow{F^{-1}} u$ and $u \xrightarrow{GF} w$.

2.3.2. **Definition.** *We denote by $\mathcal{R}_n(A)$ the category of all homogeneous matrices in $M_n(A)$ together with their intertwiners as morphisms, and by $\mathcal{R}_n^{\times}(A)$ and $\mathcal{R}_n^{\times\mathsf{T}}(A)$ the full subcategories formed by all homogeneous v in $\mathrm{GL}_n(A)$ or $\mathrm{GL}_n(A)^{\mathsf{T}}$, respectively.*

Evidently, $\mathcal{R}_n(A)$ is a groupoid, and the assignment $A \mapsto \mathcal{R}_n(A)$ extends to a functor from (B, Γ)-algebras to groupoids.

We shall make frequent use of the following straightforward relations.

2.3.3. **Lemma.** *Let $u, v \in M_n(A)$ be homogeneous, $F \in M_n(B)$ and $\hat{F} = \partial_v F \partial_u^{-1}$. Then*

$$\hat{F} \in M_n(B) \;\Leftrightarrow\; (F_{ij} = 0 \textit{ whenever } \partial_{v,i} \neq \partial_{u,j}).$$

Assume that these condition holds. Then $\hat{F} = (\partial_{v,i}(F_{ij}))_{i,j} = (\partial_{u,j}(F_{ij}))_{i,j}$ *and*

$$\hat{F}^{\mathsf{T}} = \partial_u F^{\mathsf{T}} \partial_v^{-1}, \qquad (\partial_v F)^{-\mathsf{T}} = F^{-\mathsf{T}} \partial_u^{-1}, \qquad (F\partial_u^{-1})^{-\mathsf{T}} = \partial_v F^{-\mathsf{T}}.$$

If B is a $*$-algebra and Γ preserves the involution, then $\overline{\partial_v F} = \overline{F}\partial_u^{-1}$ *and* $\overline{F\partial_u} = \partial_v^{-1}\overline{F}$.

Given $u, v \in M_n(A)$ such that $\partial^s_{u_{ik}} = \partial^r_{v_{kj}}$ for all i,k,j, let $u\tilde{\boxtimes} v := (\sum_k u_{ik}\tilde{\otimes} v_{kj})_{i,j} \in M_n(A\tilde{\otimes}A)$.

2.3.4. **Lemma.** *There exist functors*

$$\begin{aligned}
\boldsymbol{\epsilon}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(B \rtimes \Gamma), &\qquad u \mapsto \partial_u, &\qquad (u \xrightarrow{F} v) \mapsto (\partial_u \xrightarrow{F} \partial_v),\\
\boldsymbol{\Delta}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(A\tilde{\otimes}A), &\qquad u \mapsto u\tilde{\boxtimes}u, &\qquad (u \xrightarrow{F} v) \mapsto (u\tilde{\boxtimes}u \xrightarrow{F} v\tilde{\boxtimes}v),\\
(-)^{\mathsf{op}}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(A^{\mathsf{op}}), &\qquad u \mapsto u^{\mathsf{op}} := u, &\qquad (u \xrightarrow{F} v) \mapsto (u^{\mathsf{op}} \xrightarrow{\hat{F}} v^{\mathsf{op}}),
\end{aligned}$$

and $\partial_{u\tilde{\boxtimes}u} = \partial_u$, $\partial_{u^{\mathsf{op}}} = \partial_u^{-1}$ *for all* $u \in \mathcal{R}_n(A)$.

Proof. For each $u \in \mathcal{R}_n(A)$, the matrices $\partial_u, u\tilde{\boxtimes}u, u^{\mathsf{op}}$ evidently are homogeneous, and for every intertwiner $u \xrightarrow{F} v$, Lemma 2.3.3 implies

$$\begin{aligned}
r_n(\hat{F})u\tilde{\boxtimes}u = vs_n(F)\tilde{\boxtimes}u = v\tilde{\boxtimes}r_n(\hat{F})u = v\tilde{\boxtimes}vs_n(F),\\
r_n(\partial_{v^{\mathsf{op}}}\hat{F}\partial_{u^{\mathsf{op}}}^{-1})u^{\mathsf{op}} = r_n(F)^{\mathsf{op}}u^{\mathsf{op}} = (r_n(\hat{F})u)^{\mathsf{op}} = (vs_n(F))^{\mathsf{op}} = v^{\mathsf{op}}s_n(\hat{F})^{\mathsf{op}}.
\end{aligned}$$

Functoriality of the assignments is evident. □

2.3.5. **Lemma.** *There exist contravariant functors*

$$\begin{aligned}
(-)^{\mathsf{T},\mathsf{co}}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(A^{\mathsf{co}}), &\qquad u \mapsto u^{\mathsf{T},\mathsf{co}} := u^{\mathsf{T}}, &\qquad (u \xrightarrow{F} v) \mapsto (v^{\mathsf{T},\mathsf{co}} \xrightarrow{F^{\mathsf{T}}} u^{\mathsf{T},\mathsf{co}}),\\
(-)^{-\mathsf{co}}\colon \mathcal{R}^{\times}_n(A) \to \mathcal{R}_n(A^{\mathsf{co}}), &\qquad u \mapsto u^{-\mathsf{co}} := u^{-1}, &\qquad (u \xrightarrow{F} v) \mapsto (v^{-\mathsf{co}} \xrightarrow{\hat{F}^{-1}} u^{-\mathsf{co}}),
\end{aligned}$$

and $\partial_{u^{\mathsf{T},\mathsf{co}}} = \partial_u$ *and* $\partial_{u^{-\mathsf{co}}} = \partial_u^{-1}$ *for all* u.

Proof. If $u \in \mathcal{R}_n(A)$, then $u^{\mathsf{T},\mathsf{co}}$ evidently is homogeneous as claimed. Assume $u \in \mathcal{R}^{\times}_n(A)$. We claim that $u^{-\mathsf{co}}$ is homogeneous and $\partial_{u^{-\mathsf{co}}} = \partial_u^{-1}$. For each i,j, let w_{ij} be the homogeneous part of $(u^{-1})_{ij}$ of degree $(\partial_{u,j}^{-1}, \partial_{u,i}^{-1})$. Then $\sum_l u_{il}w_{lj}$ is homogeneous of degree $(\partial_{u,i}\partial_{u,j}^{-1}, e)$ and coincides with the homogeneous part of the sum $\sum_l u_{il}(u^{-1})_{lj}$ of the same degree for each i,j. Hence, $uw = uu^{-1}$ and the claim follows.

Let $u \xrightarrow{F} v$ be an intertwiner. Using Lemma 2.3.3, one easily verifies that

$$\begin{aligned}
s_n(\partial_{u^{\mathsf{T},\mathsf{co}}}F^{\mathsf{T}}\partial_{v^{\mathsf{T},\mathsf{co}}}^{-1})v^{\mathsf{T}} = s_n(\hat{F})^{\mathsf{T}}v^{\mathsf{T}} = (vs_n(F))^{\mathsf{T}} = (r_n(\hat{F})u)^{\mathsf{T}} = u^{\mathsf{T}}r_n(F^{\mathsf{T}}),\\
s_n(\partial_{u^{-\mathsf{co}}}\hat{F}^{-1}\partial_{v^{-\mathsf{co}}}^{-1})v^{-1} = s_n(F^{-1})v^{-1} = u^{-1}r_n(\hat{F}^{-1}).
\end{aligned}$$

Finally, functoriality of the assignments is easily checked. □

Forming suitable compositions, we obtain further co- or contravariant functors

$$(-)^{-\mathsf{T}} = (-)^{\mathsf{T},\mathsf{co}} \circ (-)^{-\mathsf{co}} \colon \mathcal{R}_n^{\times}(A) \to \mathcal{R}_n^{\times\mathsf{T}}(A), \quad \begin{cases} u \mapsto u^{-\mathsf{T}} := (u^{-1})^{\mathsf{T}}, \\ (u \xrightarrow{F} v) \mapsto (u^{-\mathsf{T}} \xrightarrow{\hat{F}^{-\mathsf{T}}} v^{-\mathsf{T}}), \end{cases}$$

$$(-)^{-\perp} = (-)^{-\mathsf{co}} \circ (-)^{\mathsf{T},\mathsf{co}} \colon \mathcal{R}_n^{\times\mathsf{T}}(A) \to \mathcal{R}_n^{\times}(A), \quad \begin{cases} u \mapsto u^{-\perp} := (u^{\mathsf{T}})^{-1}, \\ (u \xrightarrow{F} v) \mapsto (u^{-\perp} \xrightarrow{\hat{F}^{-\mathsf{T}}} v^{-\perp}) \end{cases}$$

and

$$(-)^{-\mathsf{co},\mathsf{op}} = (-)^{\mathsf{op}} \circ (-)^{-\mathsf{co}} \colon \mathcal{R}_n^{\times}(A) \to \mathcal{R}_n(A^{\mathsf{co},\mathsf{op}}), \quad \begin{cases} u \mapsto (u^{-\mathsf{co}})^{\mathsf{op}}, \\ (u \xrightarrow{F} v) \mapsto (v^{-\mathsf{co},\mathsf{op}} \xrightarrow{F^{-1}} u^{-\mathsf{co},\mathsf{op}}), \end{cases}$$

where $\partial_{u^{-\mathsf{co},\mathsf{op}}} = \partial_u$ and $\partial_{u^{-\mathsf{T}}} = \partial_{u^{-\perp}} = \partial_u^{-1}$ for all u.

2.3.6. **Lemma.** *The following relations hold:*

i) $(-)^{\mathsf{op}} \circ (-)^{-\mathsf{T}} = (-)^{-\perp} \circ (-)^{\mathsf{op}}$, ii) $(-)^{-\mathsf{T}} \circ (-)^{\mathsf{op}} = (-)^{\mathsf{op}} \circ (-)^{-\perp}$,

iii) $(-)^{-\mathsf{T}} \circ \boldsymbol{\Delta} = \boldsymbol{\Delta} \circ (-)^{-\mathsf{T}}$, iv) $(-)^{-\mathsf{T}} \circ (-)^{-\mathsf{co},\mathsf{op}} = (-)^{-\mathsf{co},\mathsf{op}} \circ (-)^{-\mathsf{T}}$.

Proof. i) We first check that the compositions agree on objects. Let us write v^{op} if we regard $v \in M_n(A)$ as an element of $M_n(A^{\mathsf{op}})$. Then map $M_n(A) \to M_n(A^{\mathsf{op}})$ given by $v \mapsto (v^{\mathsf{T}})^{\mathsf{op}} = (v^{\mathsf{op}})^{\mathsf{T}}$ is an antihomomorphism and hence $(v^{-\mathsf{T}})^{\mathsf{op}} = (v^{\mathsf{T},\mathsf{op}})^{-1} = (v^{\mathsf{op}})^{-\perp}$ for all $v \in \mathrm{GL}_n(A)$. The compositions also agree on morphisms because for every intertwiner $u \xrightarrow{F} v$, we have $\partial_{v^{-\mathsf{T}}}(\partial_v F \partial_u^{-1})^{-\mathsf{T}} \partial_{u^{-\mathsf{T}}}^{-1} = \partial_v^{-\mathsf{T}} \partial_v F^{-\mathsf{T}} \partial_u^{-1} \partial_u = F^{-\mathsf{T}}$.

ii) This equation follows similarly like i).

iii) Let $u \in \mathcal{R}_n^{\times}(A)$. Then $(u \tilde{\boxtimes} u)^{-\mathsf{T}} = u^{-\mathsf{T}} \tilde{\boxtimes} u^{-\mathsf{T}}$ because

$$\sum_k (u \tilde{\boxtimes} u)_{ik} (u^{-\mathsf{T}} \tilde{\boxtimes} u^{-\mathsf{T}})_{jk} = \sum_{k,l,m} u_{il} (u^{-1})_{mj} \tilde{\otimes} u_{lk} (u^{-1})_{km} = \delta_{i,j} 1 \tilde{\otimes} 1.$$

and similarly $\sum_k (u^{-\mathsf{T}} \tilde{\boxtimes} u^{-\mathsf{T}})_{ki} (u \tilde{\boxtimes} u)_{kj} = \delta_{i,j} 1 \tilde{\otimes} 1$. For morphisms, we have nothing to check because $\partial_{u \tilde{\boxtimes} u} = \partial_u$.

iv) This equation follows from the relation $(-)^{-\mathsf{T}} \circ (-)^{\mathsf{op}} \circ (-)^{-\mathsf{co}} = (-)^{\mathsf{op}} \circ (-)^{-\perp} \circ (-)^{-\mathsf{co}} = (-)^{\mathsf{op}} \circ (-)^{-\mathsf{co}} \circ (-)^{\mathsf{T},\mathsf{co}} \circ (-)^{-\mathsf{co}}$. □

Assume for a moment that (A, Δ, ϵ, S) is a Hopf (B, Γ)-algebroid.

2.3.7. **Definition.** *A* matrix corepresentation *of* (A, Δ, ϵ, S) *is a* $v \in \mathcal{R}_n(A)$ *for some* $n \in \mathbb{N}$ *satisfying* $\Delta_n(v) = v \tilde{\boxtimes} v$, $\epsilon_n(v) = \partial_v$, $S_n(v) = v^{-1}$.

2.3.8. **Lemma.** *If* $v \xrightarrow{F} w$ *is a morphism in* $\mathcal{R}_n(A)$ *and* v *is a matrix corepresentation, then so is* w.

Proof. Applying the morphisms Δ, ϵ, S and the functors $\boldsymbol{\Delta}, \boldsymbol{\epsilon}, (-)^{-\mathsf{co},\mathsf{op}}$ to $v \xrightarrow{F} w$ or its inverse, we get intertwiners $w \tilde{\boxtimes} w \xrightarrow{F^{-1}} v \tilde{\boxtimes} v = \Delta_n(v) \xrightarrow{F} \Delta_n(w)$, $\partial_w \xrightarrow{F^{-1}} \partial_v = \epsilon_n(v) \xrightarrow{F} \epsilon_n(w)$ and $w^{-\mathsf{co},\mathsf{op}} \xrightarrow{F^{-1}} v^{-\mathsf{co},\mathsf{op}} = S_n(v) \xrightarrow{F} S_n(w)$. □

Let us now discuss the involutive case.

Given a $*$-algebra C and a matrix $v \in M_n(C)$, we write $\overline{v} := (v_{ij}^*)_{i,j} = (v^*)^{\mathsf{T}}$.

Assume that B is a $*$-algebra, that Γ preserves the involution, and that A is a (B,Γ)-algebra. Then there exists an obvious functor $\mathcal{R}_n(A) \to \mathcal{R}_n(\bar{A})$, given by $u \mapsto u$ and $(u \xrightarrow{F} v) \mapsto (u \xrightarrow{\overline{F}} v)$. Composition with $(-)^{\mathsf{op}}$ gives a functor

$$(-)^{\overline{\mathsf{op}}}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(\overline{A}^{\mathsf{op}}), \qquad u \mapsto u^{\overline{\mathsf{op}}} := u^{\mathsf{op}}, \qquad (u \xrightarrow{F} v) \mapsto (u^{\overline{\mathsf{op}}} \xrightarrow{\overline{\hat{F}}} v^{\overline{\mathsf{op}}}),$$

and $\partial_{u^{\overline{\mathsf{op}}}} = \partial_u^{-1}$ for all u. For later use, we note the following relation.

2.3.9. **Lemma.** *Let $u^{-\mathsf{T}} \xrightarrow{F} v$ be an intertwiner in $\mathcal{R}_n^{\times}(A) \cap \mathcal{R}_n^{\times,\mathsf{T}}(A)$. Then $(v^{\overline{\mathsf{op}}})^{-\mathsf{T}} \xrightarrow{F^*} u^{\overline{\mathsf{op}}}$ is an intertwiner in $\mathcal{R}_n^{\times}(\overline{A}^{\mathsf{op}}) \cap \mathcal{R}_n^{\times,\mathsf{T}}(\overline{A}^{\mathsf{op}})$.*

Proof. Subsequent applications of the functors $(-)^{\overline{\mathsf{op}}}$, $(-)^{-\mathsf{T}}$ yield intertwiners $(v^{-\mathsf{T}})^{\overline{\mathsf{op}}} = (v^{\overline{\mathsf{op}}})^{-\perp} \xrightarrow{\overline{\hat{F}}^{-1}} u^{\overline{\mathsf{op}}}$ and $(u^{\overline{\mathsf{op}}})^{-\mathsf{T}} \xrightarrow{\overline{F}^{-\mathsf{T}}=F^{-*}} v^{\overline{\mathsf{op}}}$. □

Finally, assume that A is a (B,Γ)-$*$-algebra. Then there exists a functor

$$(-)^{*,\mathsf{co}}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(A^{\mathsf{co}}), \qquad u \mapsto u^{*,\mathsf{co}} := u^*, \qquad (u \xrightarrow{F} v) \mapsto (v^{*,\mathsf{co}} \xrightarrow{\hat{F}^*} u^{*,\mathsf{co}}),$$

because $s_n(F^*)v^* = u^* r_n(\hat{F}^*)$ for every intertwiner $u \xrightarrow{F} v$, and $\partial_{u^{*,\mathsf{co}}} = \partial_u^{-1}$. Composing with $(-)^{\mathsf{T},\mathsf{co}}$ for A^{co} and with $(-)^{-\mathsf{T}}$, respectively, we get functors

$$\text{(1)} \qquad \overline{(-)}\colon \mathcal{R}_n(A) \to \mathcal{R}_n(A), \qquad u \mapsto \overline{u} = (u_{ij}^*)_{i,j}, \qquad (u \xrightarrow{F} v) \mapsto (\overline{u} \xrightarrow{\overline{\hat{F}}} \overline{v}),$$

$$\text{(2)} \qquad \mathcal{R}_n(A) \to \mathcal{R}_n(A), \qquad u \mapsto \overline{u} = \overline{u}^{-\mathsf{T}} = \overline{u^{-\perp}}, \qquad (u \xrightarrow{F} v) \mapsto (\overline{u}^{-\mathsf{T}} \xrightarrow{F^{-*}} \overline{v}^{-\mathsf{T}}).$$

2.4. **The free orthogonal and free unitary dynamical quantum groups.** Using the preparations of the last subsection, we now show that the algebras $A_{\mathrm{o}}^B(\nabla, F)$ and $A_{\mathrm{u}}^B(\nabla, F)$ are (B,Γ)-Hopf algebroids as claimed in the introduction.

Let B be a commutative algebra with an action of a group Γ as before, and let $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $\nabla = \mathrm{diag}(\gamma_1, \ldots, \gamma_n) \in M_n(B \rtimes \Gamma)$.

Let $F \in \mathrm{GL}_n(B)$ be ∇-*odd* in the sense that $\nabla F \nabla \in M_n(B)$. The first definition and theorem in the introduction can be reformulated as follows.

2.4.1. **Definition.** *The* free orthogonal dynamical quantum group over B with parameters (∇, F) *is the universal (B,Γ)-algebra $A_{\mathrm{o}}^B(\nabla, F)$ with a $v \in \mathcal{R}_n^{\times}(A_{\mathrm{o}}^B(\nabla, F))$ such that $\partial_v = \nabla$ and $v^{-\mathsf{T}} \xrightarrow{F} v$ is an intertwiner.*

2.4.2. **Theorem.** *The (B,Γ)-algebra $A_{\mathrm{o}}^B(\nabla, F)$ can be equipped with a unique structure of a (B,Γ)-Hopf algebroid such that v becomes a matrix corepresentation.*

Proof. The existence of morphisms $\Delta\colon A \to A \tilde{\otimes} A$, $\epsilon\colon A \to B \rtimes \Gamma$, $S\colon A \to A^{\mathsf{co},\mathsf{op}}$ satisfying $\Delta_n(v) = v \tilde{\boxtimes} v$, $\epsilon_n(v) = \nabla$, $S_n(v) = v^{-1}$ follows from the universal property of A and the relations

$$\boldsymbol{\Delta}(v^{-\mathsf{T}} \xrightarrow{F} v) = ((v \tilde{\boxtimes} v)^{-\mathsf{T}} \xrightarrow{F} v \tilde{\boxtimes} v), \quad \boldsymbol{\epsilon}(v^{-\mathsf{T}} \xrightarrow{F} v) = (\nabla^{-\mathsf{T}} \xrightarrow{F} \nabla),$$

$$(v^{-\mathsf{T}} \xrightarrow{F} v)^{-\mathsf{co},\mathsf{op}} = (v^{-\mathsf{co},\mathsf{op}} \xrightarrow{F^{-1}} (v^{-\mathsf{co},\mathsf{op}})^{-\mathsf{T}});$$

see Lemma 2.3.5 and 2.3.6. Straightforward calculations show that (A, Δ, ϵ, S) is a (B, Γ)-Hopf algebroid. □

2.4.3. **Remarks.** i) In the definition of $A_{\mathrm{o}}^B(\nabla, F)$, we may evidently assume that Γ is generated by the diagonal components $\gamma_1, \ldots, \gamma_n$ of ∇.

ii) Denote by $B_0 \subseteq B$ the smallest Γ-invariant subalgebra containing the entries of F and F^{-1}, and by $\iota\colon B_0 \to B$ the inclusion. Then there exists an obvious isomorphism $A_{\mathrm{o}}^B(\nabla, F) \cong \iota_* A_{\mathrm{o}}^{B_0}(\nabla, F)$.

iii) Let $H \in \mathrm{GL}_n(B)$ be ∇-even and $\hat{H} = \nabla H \nabla^{-1}$. Then there exists an isomorphism $A_{\mathrm{o}}^B(\nabla, HF\hat{H}^{\mathsf{T}}) \to A_{\mathrm{o}}^B(\nabla, F)$ of (B, Γ)-Hopf algebroids whose extension to matrices sends $v \in A_{\mathrm{o}}^B(\nabla, HF\hat{H}^{\mathsf{T}})$ to $w := r_n(\hat{H}) v s_n(H)^{-1} \in A_{\mathrm{o}}^B(\nabla, F)$. Indeed, there exists such a morphism of (B, Γ)-algebras because in $A_{\mathrm{o}}^B(\nabla, F)$, we have intertwiners $v \xrightarrow{H} w$, $v^{-\mathsf{T}} \xrightarrow{\hat{H}^{-\mathsf{T}}} w^{-\mathsf{T}}$ and $v^{-\mathsf{T}} \xrightarrow{F} v$, whence $w^{-\mathsf{T}} \xrightarrow{HF\hat{H}^{\mathsf{T}}} w$, and this morphism is compatible with Δ, ϵ, S because w is a matrix corepresentation by Lemma 2.3.8. A similar argument yields the inverse of this morphism.

Assume that B carries an involution which is preserved by Γ, and let $F \in \mathrm{GL}_n(B)$ be self-adjoint and ∇-*even* in the sense that $\nabla F \nabla^{-1} \in M_n(B)$. The second definition and theorem in the introduction can be reformulated as follows.

2.4.4. **Definition.** *The* free unitary dynamical quantum group over B with parameters (∇, F) *is the universal (B, Γ)-$*$-algebra $A_{\mathrm{u}}^B(\nabla, F)$ with a unitary $u \in \mathcal{R}_n^{\times}(A_{\mathrm{u}}^B(\nabla, F))$ such that $\partial_v = \nabla$ and $(v^{-\mathsf{T}})^{-\mathsf{T}} \xrightarrow{F} v$ is an intertwiner.*

2.4.5. **Theorem.** *The $*$-algebra $A_{\mathrm{u}}^B(\nabla, F)$ can be equipped with a unique structure of a (B, Γ)-Hopf $*$-algebroid such that v becomes a matrix corepresentation.*

To prove this result, we introduce an auxiliary (B, Γ)-algebra which does not involve the involution on B.

2.4.6. **Definition.** *We denote by $A_{\mathrm{u}'}^B(\nabla, F)$ the universal (B, Γ)-algebra with $v, w \in \mathcal{R}_n^{\times}(A)$ such that $\partial_v = \nabla$, $\partial_w = \nabla^{-1}$ and $v^{-\mathsf{T}} \xrightarrow{1} w$, $w^{-\mathsf{T}} \xrightarrow{F} v$ are intertwiners.*

Using the same techniques as in the proof of Theorem 2.4.2, one finds:

2.4.7. **Proposition.** *The (B, Γ)-algebra $A_{\mathrm{u}'}^B(\nabla, F)$ can be equipped with a unique structure of a (B, Γ)-Hopf algebroid such that v and w become matrix corepresentations.*

2.4.8. **Proposition.** *The (B, Γ)-algebra $A_{\mathrm{u}'}^B(\nabla, F)$ can be equipped with an involution such that it becomes a (B, Γ)-Hopf $*$-algebroid and $w = \bar{v}$.*

Proof. Let $A := A_{\mathrm{u}'}^B(\nabla, F)$. By Lemma 2.3.9, we have intertwiners $(\overline{w^{\mathsf{op}}})^{-\mathsf{T}} \xrightarrow{1} \overline{v^{\mathsf{op}}}$ and $(\overline{v^{\mathsf{op}}})^{-\mathsf{T}} \xrightarrow{F^* = F} \overline{w^{\mathsf{op}}}$. The universal property of A yields a homomorphism $j\colon A \to \overline{A}^{\mathsf{op}}$ satisfying $j_n(v) = \overline{w^{\mathsf{op}}}$ and $j_n(w) = \overline{v^{\mathsf{op}}}$. Composition of j with the canonical map $\bar{A}^{\mathsf{op}} \to A$ yields the desired involution, which is easily seen to be compatible with the comultiplication and counit. □

Theorem 2.4.5 now is an immediate corollary to the following result:

2.4.9. **Theorem.** *There exists a unique $*$-isomorphism $A_{\mathrm{u}}^B(\nabla, F) \to A_{\mathrm{u}'}^B(\nabla, F)$ whose extension to matrices sends u to v.*

Proof. One easily verifies that the universal properties of $A := A_{\mathrm{u}}^B(\nabla, F)$ and $A' := A_{\mathrm{u}'}^B(\nabla, F)$ yield homomorphisms $A \to A'$ and $A' \to A$ whose extensions to matrices satisfy $u \mapsto v$ and $v \mapsto u$, $w \mapsto \bar{u}$, respectively. □

The following analogues of Remarks 2.4.3 apply to $A_{\mathrm{u}}^B(\nabla, F)$:

2.4.10. **Remarks.** i) We may assume that Γ is generated by the diagonal components of ∇, and if $\iota\colon B_0 \hookrightarrow B$ denotes the inclusion of the smallest Γ-invariant $*$-subalgebra containing the entries of F and F^{-1}, then $A_{\mathrm{u}}^B(\nabla, F) \cong \iota_* A_{\mathrm{u}}^{B_0}(\nabla, F)$.

ii) Let $H \in \mathrm{GL}_n(B)$ be ∇-even and unitary, and let $\hat{H} = \nabla H \nabla^{-1}$. Then there exists an isomorphism $A_{\mathrm{u}}^B(\nabla, HFH^*) \to A_{\mathrm{u}}^B(\nabla, F)$ of (B, Γ)-Hopf algebroids whose extension to matrices sends the matrix $u \in A_{\mathrm{u}}^B(\nabla, HFH^*)$ to $z := r_n(\hat{H}) u s_n(H)^{-1} \in A_{\mathrm{u}}^B(\nabla, F)$. Indeed, there exists such a morphism of (B, Γ)-algebras because z is a product of unitaries and in $A_{\mathrm{u}}^B(\nabla, F)$, we have intertwiners $u \xrightarrow{H} z$, $\bar{u}^{-\mathsf{T}} \xrightarrow{H^{-*}} \bar{z}^{-\mathsf{T}}$ by (2), and $\bar{u}^{-\mathsf{T}} \xrightarrow{F} u$, whence $\bar{z}^{-\mathsf{T}} \xrightarrow{HFH^*} z$, and this morphism is compatible with Δ, ϵ, S because z is a matrix corepresentation by Lemma 2.3.8. A similar argument yields the inverse of this morphism.

We finally consider involutions on certain quotients of $A_{\mathrm{o}}^B(\nabla, F)$.
Assume that $F, G \in \mathrm{GL}_n(B)$ are ∇-odd and $GF^* = FG^*$. Let $Q := G(\nabla \bar{G} \nabla)$.

2.4.11. **Definition.** *The* free orthogonal dynamical quantum group *over B with parameters (∇, F, G) is the universal (B, Γ)-algebra $A_{\mathrm{o}}^B(\nabla, F, G)$ with a $v \in \mathcal{R}_n^{\times}(A)$ such that $\partial_v = \nabla$ and $v^{-\mathsf{T}} \xrightarrow{F} v$ and $v \xrightarrow{Q} v$ are intertwiners.*

The algebra $A_{\mathrm{o}}^B(\nabla, F, G)$ depends only on Q and not on G, but shall soon be equipped with an involution that does depend on G.
Evidently, there exists a canonical quotient map $A_{\mathrm{o}}^B(\nabla, F) \to A_{\mathrm{o}}^B(\nabla, F, G)$, and

$$A_{\mathrm{o}}^B(\nabla, F, G) \cong A_{\mathrm{o}}^B(\nabla, F)/(r(q) - s(q)) \text{ if } Q = \mathrm{diag}(q, \ldots, q),$$
$$A_{\mathrm{o}}^B(\nabla, F, G) \cong A_{\mathrm{o}}^B(\nabla, F)/(r(q_i) - s(q_j) | i, j = 1, \ldots, n) \text{ if } Q = \mathrm{diag}(q_1, \ldots, q_n),$$

because in the second case $(r_n(\hat{Q})v)_{ij} = r_n(\gamma_i(q_i))v_{ij} = v_{ij} r_n(q_i)$ and $vs_n(Q))_{ij} = v_{ij} s_n(q_j)$ in $A_{\mathrm{o}}^B(\nabla, F)$ for all i, j.

2.4.12. **Theorem.** *The (B, Γ)-algebra $A_{\mathrm{o}}^B(\nabla, F, G)$ can be equipped with a unique structure of a (B, Γ)-Hopf $*$-algebroid such that $\bar{v} \xrightarrow{G} v$ becomes an intertwiner and v a matrix corepresentation.*

Proof. The existence of Δ, ϵ, S follows similarly as in the case of $A_{\mathrm{o}}^B(\nabla, F)$; one only needs to observe that additionally, application of the functors $\boldsymbol{\Delta}, \boldsymbol{\epsilon}$ and $(-)^{\mathsf{co,op}}$ to the intertwiner $(v \xrightarrow{Q} v)$ yield intertwiners $(v \tilde{\boxtimes} v \xrightarrow{Q} v \tilde{\boxtimes} v)$, $(\nabla \xrightarrow{Q} \nabla)$ and $((v^{-1})^{\mathsf{co,op}} \xrightarrow{Q} (v^{-1})^{\mathsf{co,op}})$

Let us prove existence of the involution. Let $w := r_n(\nabla G\nabla)^{-1} v s_n(G)$. Then there exist intertwiners $w \xrightarrow{G} v$ and

$$\begin{aligned}
(v \xrightarrow{G^{-1}} w) \circ (v \xrightarrow{Q} v) &= (v \xrightarrow{\nabla\bar{G}\nabla} w),\\
(v \xrightarrow{\nabla\bar{G}\nabla} w)^{\overline{\mathsf{op}}} \circ (w \xrightarrow{G} v)^{\overline{\mathsf{op}}} &= (v^{\overline{\mathsf{op}}} \xrightarrow{G} w^{\overline{\mathsf{op}}}) \circ (w^{\overline{\mathsf{op}}} \xrightarrow{\nabla\bar{G}\nabla} v^{\overline{\mathsf{op}}})\\
&= (w^{\overline{\mathsf{op}}} \xrightarrow{Q} w^{\overline{\mathsf{op}}}),\\
(v^{-\mathsf{T}} \xrightarrow{F} v) \circ (w \xrightarrow{(\nabla\bar{G}\nabla)^{-1}} v)^{-\mathsf{T}} &= (v^{-\mathsf{T}} \xrightarrow{F} v) \circ (w^{-\mathsf{T}} \xrightarrow{G^*} v^{-\mathsf{T}})\\
&= (w^{-\mathsf{T}} \xrightarrow{FG^*=GF^*} v),\\
((v \xrightarrow{G^{-1}} w) \circ (w^{-\mathsf{T}} \xrightarrow{GF^*} v))^{\overline{\mathsf{op}}} &= (w^{-\mathsf{T}} \xrightarrow{F^*} w)^{\overline{\mathsf{op}}} = ((w^{\overline{\mathsf{op}}})^{-\mathsf{T}} \xrightarrow{F} w^{\overline{\mathsf{op}}});
\end{aligned}$$

where we used Lemma 2.3.5 in the last line. The universal property of $A := A_{\mathrm{o}}^{B}(\nabla, F, G)$ therefore yields a homomorphism $j\colon A \to \overline{A}^{\mathsf{op}}$ such that $j_n(v) = w^{\overline{\mathsf{op}}}$, and this j corresponds to a conjugate-linear antihomomorphism $A \to A$, $a \mapsto a^*$. To see that the map $a \mapsto a^*$ is involutive, we only need to check $\overline{w} = v$. The functor $\overline{(-)}$ of (1) applied to $w \xrightarrow{G} v$ yields $\overline{w} \xrightarrow{\hat{\overline{G}}=\nabla\bar{G}\nabla} \overline{v} = w$, and composition with $w \xrightarrow{G} v$ gives $\overline{w} \xrightarrow{Q} v$. Hence, $\overline{w} = v$.

Finally, the involution is compatible with the comultiplication and counit because w is a matrix corepresentation by Lemma 2.3.8. □

2.4.13. **Remarks.** i) The canonical quotient map from $A_{\mathrm{o}}^{B}(\nabla, F)$ to $A_{\mathrm{o}}^{B}(\nabla, F, G)$ is a morphism of (B,Γ)-Hopf algebroids.

ii) Analogues of Remarks 2.4.3 and 2.4.10 apply to $A_{\mathrm{o}}^{B}(\nabla, F, G)$.

iii) Note that $A_{\mathrm{o}}^{B}(\nabla, F, G)$ is the universal (B,Γ)-$*$-algebra with a $v \in \mathcal{R}_n^{\times}(A)$ such that $\partial_v = \nabla$ and $v^{-\mathsf{T}} \xrightarrow{F} v$ and $\bar{v} \xrightarrow{G} v$ are intertwiners. Indeed, the composition of $\bar{v} \xrightarrow{G} v$ with its image under the functor $\overline{(-)}$ in (1) yields $v \xrightarrow{Q} v$.

iv) If $F = G$, then $\overline{v} = v^{-\mathsf{T}}$ and hence v is unitary. In general, assume that $H \in \mathrm{GL}_n(B)$ satisfies $\nabla H \nabla^{-1} \in M_n(B)$ and $\overline{H}H^{\mathsf{T}} \in \mathbb{C} \cdot G^{-1}F$. Then $u := r_n(H^{-1}) v s_n(\nabla^{-1} H \nabla)$ is a unitary matrix corepresentation whose entries generate $A_{\mathrm{o}}^{B}(\nabla, F, G)$ as a (B,Γ)-algebra. Indeed, $u \xrightarrow{\nabla^{-1}H\nabla} v$ is an intertwiner, and applying $\overline{(-)}$ and $(-)^{-\mathsf{T}}$, respectively, we get $\overline{u} \xrightarrow{\overline{H}} \overline{v} \xrightarrow{G} v \xrightarrow{F^{-1}} v^{-\mathsf{T}} \xrightarrow{H^{\mathsf{T}}} u^{-\mathsf{T}}$ which is scalar by assumption so that $\overline{u} = u^{-\mathsf{T}}$.

We finally consider a simple example; a more complex one is considered in §2.6.

2.4.14. **Example.** Equip $\mathbb{C}[X]$ with an involution such that $X^* = X$ and an action of $\mathbb{Z}$ such that $X \overset{k}{\mapsto} X - k$ for all $k \in \mathbb{Z}$, and let $\gamma_1 = 1$, $\gamma_2 = -1$, $\nabla = \mathrm{diag}(\gamma_1, \gamma_2)$ and $F = G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $A_{\mathrm{o}}^{\mathbb{C}[X]}(\nabla, F, G) \cong \iota_*(A_{\mathrm{o}}^{\mathbb{C}}(\nabla, F, G))$, where $\iota\colon \mathbb{C} \to \mathbb{C}[X]$ is the canonical map.

The algebra $A_{\mathrm{o}}^{\mathbb{C}}(\nabla, F, G)$ equipped with $\Delta, \epsilon_{\mathbb{C}\Gamma} \circ \epsilon, S$ is a Hopf $*$-algebra by Lemma 2.2.1. It is generated by the entries of a unitary matrix v which satisfies $\overline{v} = G^{-1} v G$ and therefore has the form $v = \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$. The relations $vv^* = 1 = v^*v$ then imply that $\alpha, \alpha^*, \gamma, \gamma^*$ commute and $\alpha\alpha^* + \gamma\gamma^* = 1$. Therefore, $A_{\mathrm{o}}^{\mathbb{C}}(\nabla, F, G)$ is isomorphic to the Hopf $*$-algebra $\mathcal{O}(\mathrm{SU}(2))$ of representative functions on $\mathrm{SU}(2)$.

The algebra $A_{\mathrm{o}}^{\mathbb{C}[X]}(\nabla, F, G) \cong \iota_*(A_{\mathrm{o}}^{\mathbb{C}}(\nabla, F, G))$ can be identified with the subalgebra of $\mathrm{End}(\mathcal{O}(\mathrm{SU}(2)))$ generated by multiplication operators associated to elements of $\mathcal{O}(\mathrm{SU}(2))$ and left or right invariant differentiation operators along the diagonal torus in $\mathrm{SU}(2)$; see Example 2.2.7.

2.5. **The square of the antipode and the scaling character groups.** The square of the antipode on the free dynamical quantum groups $A_{\mathrm{o}}^B(\nabla, F)$, $A_{\mathrm{u}}^B(\nabla, F)$, $A_{\mathrm{o}}^B(\nabla, F, G)$ can be described in terms of certain character groups as follows.

Recall the isomorphisms of Lemma 2.1.3 iii) and the anti-automorphism $S^{B\rtimes\Gamma}$ of $B \rtimes \Gamma$ given by $b\gamma \mapsto \gamma^{-1}b$.

2.5.1. **Definition.** *Let (A, Δ, ϵ, S) be a (B, Γ)-Hopf algebroid. A* character group *on A is a family of morphisms $\theta = (\theta^{(k)} \colon A \to B \rtimes \Gamma)_{k\in\mathbb{Z}}$ satisfying $(\theta^{(k)} \tilde{\otimes} \theta^{(l)}) \circ \Delta = \theta^{k+l}$, $\theta^{(0)} = \epsilon$ and $\theta^{(k)} \circ S = S^{B\rtimes\Gamma} \circ \theta^{(-k)}$ for all $k, l \in \mathbb{Z}$. We call a character group θ* scaling *if $S^2 = (\theta^{(1)} \tilde{\otimes} \mathrm{id} \tilde{\otimes} \theta^{(-1)}) \circ \Delta^{(2)}$, where $\Delta^{(2)} = (\Delta \tilde{\otimes} \mathrm{id}) \circ \Delta = (\mathrm{id} \tilde{\otimes} \Delta) \circ \Delta$.*

We construct scaling character groups using intertwiners of the form $u \xrightarrow{H} S_n^2(u)$ for suitable matrix corepresentations u.

2.5.2. **Lemma.** *Let (A, Δ, ϵ, S) be a (B, Γ)-Hopf algebroid, let $\theta = (\theta^{(k)} \colon A \to B \rtimes \Gamma)_{k\in\mathbb{Z}}$ be a family of morphisms satisfying $(\theta^{(k)} \tilde{\otimes} \theta^{(l)}) \circ \Delta = \theta^{k+l}$ for all $k, l \in \mathbb{Z}$, and let $u \in \mathcal{R}_n^\times(A)$ be a matrix corepresentation.*

i) *$S_n^2(u) = (u^{-\mathsf{T}})^{-\mathsf{T}}$.*
ii) *Let $H = \partial_u^{-1}\theta_n^{(1)}(u)$. Then $H \in \mathrm{GL}_n(B)$, $\partial_u H \partial_u^{-1} \in M_n(B)$ and $\theta_n^{(k)}(u) = \partial_u H^k$ for all $k \in \mathbb{Z}$.*
iii) *$\theta_n^{(0)} = \epsilon_n(u)$ and $\theta_n^{(k)}(S_n(u)) = S_n^{B\rtimes\Gamma}(\theta_n^{(-k)}(u))$ for all $k \in \mathbb{Z}$.*
iv) *$S_n^2(u) = ((\theta^{(1)} \tilde{\otimes} \mathrm{id} \tilde{\otimes} \theta^{(-1)}) \circ \Delta^{(2)})_n(u)$ if and only if $u \xrightarrow{H} S_n^2(u)$ is an intertwiner.*

Proof. i) The map $M_n(A) \to M_n(A)$ given by $x \mapsto S_n(x)^{\mathsf{T}}$ is an antihomomorphism and therefore preserves inverses. Hence, $S_n^2(u) = S_n(u^{-\mathsf{T}})^{\mathsf{T}} = (S_n(u)^{\mathsf{T}})^{-\mathsf{T}} = (u^{-\mathsf{T}})^{-\mathsf{T}}$.

ii) Since each $\theta^{(k)}$ preserves the grading, there exists a family $(H_k)_{k\in\mathbb{Z}}$ of elements of $\mathrm{GL}_n(B)$ satisfying $\partial_u H_k \partial_u^{-1} \in M_n(B)$ and $\theta_n^{(k)}(u) = \partial_u H_k$ for all $k \in \mathbb{Z}$. The assumption on θ implies that $H_k H_l = H_{k+l}$ for all $k, l \in \mathbb{Z}$, and consequently, $H_k = H_1^k$ for all $k \in \mathbb{Z}$.

iii) By ii), $\theta_n^{(0)}(u) = \partial_u = \epsilon_n(u)$ and

$$\begin{aligned} \theta_n^{(k)}(S_n(u)) = \theta_n^{(k)}(u^{-1}) = \theta_n^{(k)}(u)^{-1} &= H^{-k}\partial_u^{-1} \\ &= S_n^{B\rtimes\Gamma}(\partial_u H^{-k}) = S_n^{B\rtimes\Gamma}(\theta_n^{(-k)}(u)). \end{aligned}$$

iv) This follows from the relation

$$\begin{aligned} ((\theta^{(1)} \tilde{\otimes} \mathrm{id} \tilde{\otimes} \theta^{(-1)}) \circ \Delta^{(2)})_n(u) &= \theta_n^{(1)}(u) \tilde{\boxtimes} u \tilde{\boxtimes} \theta_n^{(-1)}(u) \\ &= \partial_u H \tilde{\boxtimes} u \tilde{\boxtimes} \partial_u H^{-1} = r_n(\partial_u H \partial_u^{-1}) u s_n(H^{-1}). \qquad \square \end{aligned}$$

We first apply the lemma above to $A_{\mathrm{o}}^B(\nabla, F)$.

2.5.3. **Proposition.** *Let $F \in \mathrm{GL}_n(B)$ be ∇-odd. Then $A_{\mathrm{o}}^B(\nabla, F)$ has an intertwiner $v \xrightarrow{H} S^2(v)$ and a scaling character group θ such that $\theta_n^{(k)}(v) = \nabla H^k$ for all $k \in \mathbb{Z}$, where $H = (\nabla F \nabla)^{\mathsf{T}} F^{-1}$.*

Proof. By Lemma 2.5.2 i), $(v^{-\mathsf{T}} \xrightarrow{F} v) \circ (v^{-\mathsf{T}} \xrightarrow{F} v)^{-\mathsf{T}} = S^2(v) \xrightarrow{H^{-1}} v$. To construct θ, let $k \in \mathbb{Z}$ and $x = \nabla H^k$. By Lemma 2.3.3, $x^{-\mathsf{T}} = (H^{-\mathsf{T}})^k \nabla^{-1}$ and hence

$$(\nabla F \nabla) x^{-\mathsf{T}} = \nabla F \nabla (\nabla^{-1} F^{-1} \nabla^{-1} F^{\mathsf{T}})^k \nabla^{-1} = \nabla (\nabla^{-1} F^{\mathsf{T}} \nabla^{-1} F^{-1})^k F = xF.$$

The universal property of $A_{\mathrm{o}}^B(\nabla, F)$ yields a morphism $\theta^{(k)} \colon A_{\mathrm{o}}^B(\nabla, F) \to B \rtimes \Gamma$ such that $\theta_n^{(k)}(v) = x$. Using Lemma 2.5.2, one easily verifies that the family $(\theta^{(k)})_k$ is a scaling character group. □

Assume that B carries an involution which is preserved by Γ. We call a character group $(\theta^{(k)})_k$ on a (B, Γ)-Hopf $*$-algebroid *imaginary* if $\theta^{(k)} \circ * = * \circ \theta^{(-k)}$ for all $k \in \mathbb{Z}$.

2.5.4. **Proposition.** *Let $F \in \mathrm{GL}_n(B)$ be ∇-even. Then $A_{\mathrm{u}}^B(\nabla, F)$ has intertwiners $u \xrightarrow{F^{-1}} S_n^2(u)$ and $\bar{u} \xrightarrow{(\nabla F \nabla^{-1})^{\mathsf{T}}} S_n^2(\bar{u})$, and an imaginary scaling character group θ such that $\theta_n^{(k)}(u) = \nabla F^{-k}$ and $\theta_n^{(k)}(\bar{u}) = F^{k\mathsf{T}} \nabla^{-1}$ for all $k \in \mathbb{Z}$.*

Proof. By Lemma 2.5.2 i), the first intertwiner is the inverse of $S_n^2(u) = (u^{-\mathsf{T}})^{-\mathsf{T}} = \bar{u}^{-\mathsf{T}} \xrightarrow{F} u$, and the second intertwiner is the inverse of $(\bar{u}^{-\mathsf{T}} \xrightarrow{F} u)^{-\mathsf{T}}$. To construct θ, let $k \in \mathbb{Z}$ and $x = \nabla F^{-k}$, $y = F^{k\mathsf{T}} \nabla^{-1}$. Using Lemma 2.3.3, we find

$$y = x^{-\mathsf{T}}, \qquad y^{-\mathsf{T}} = x, \qquad (\nabla F \nabla^{-1}) y^{-\mathsf{T}} = (\nabla F \nabla^{-1}) x = \nabla F^{1-k} = xF.$$

The universal property of the algebra $A_{\mathrm{u}'}^B(\nabla, F)$ and Theorem 2.4.9 yield a morphism $\theta^{(k)} \colon A_{\mathrm{u}}^B(\nabla, F) \to B \rtimes \Gamma$ such that $\theta_n^{(k)}(u) = x$ and $\theta_n^{(k)}(\bar{u}) = y$. Using Lemma 2.5.2, one easily verifies that the family $(\theta^{(k)})_k$ is a scaling character group. It is imaginary because by Lemma 2.3.3,

$$\overline{\theta_n^{(-k)}(u)} = \overline{\nabla F^k} = \overline{F}^k \nabla^{-1} = F^{\mathsf{T}k} \nabla^{-1} = \theta_n^{(k)}(\bar{u}) \quad \text{for all } k \in \mathbb{Z}.$$ □

The case $A_{\mathrm{o}}^B(\nabla, F, G)$ requires some preparation. Let $F, G \in \mathrm{GL}_n(B)$ be ∇-odd and

$$H = (\nabla F \nabla)^{\mathsf{T}} F^{-1} = \nabla^{-1} F^{\mathsf{T}} \nabla^{-1} F^{-1}, \qquad Q = G \nabla \bar{G} \nabla$$

as before. We say that a diagram with arrows labeled by matrices commutes if for all possible directed paths with the same starting and ending point in the diagram, the products of the labels along the arrows coincide.

2.5.5. **Lemma.** *In the diagram below, (A) commutes if and only if (D) commutes, and (B) commutes if and only (C) commutes:*

$$\begin{tikzcd}
\bullet \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "(A)"] & \bullet \arrow[r, "\nabla F^{-\mathsf{T}}\nabla"] \arrow[d, "\overline{G}^{\mathsf{T}}"] \arrow[dr, phantom, "(C)"] & \bullet \arrow[d, "G"] \\
\bullet \arrow[r, "\overline{F}^{\mathsf{T}}"] \arrow[d, "\nabla\overline{G}\nabla"'] \arrow[dr, phantom, "(B)"] & \bullet \arrow[r, "\nabla^{-1}\bar{F}^{-1}\nabla^{-1}"] \arrow[d, "\nabla^{-1}G^{\mathsf{T}}\nabla^{-1}"] \arrow[dr, phantom, "(D)"] & \bullet \arrow[d, "\nabla\overline{G}\nabla"] \\
\bullet \arrow[r, "F"'] & \bullet \arrow[r, "\nabla F^{-\mathsf{T}}\nabla"'] & \bullet
\end{tikzcd}$$

If all squares commute, then $HQ = QH$, $\overline{G}\nabla H^{-1} = \overline{HG}\nabla$, and $QF = F\nabla Q^{\mathsf{T}}\nabla^{-1}$.

Proof. Applying the transformation $X \mapsto X^{-\mathsf{T}}$ and reversing invertible arrows, one can obtain (D) from (A) and (C) from (B). If all small squares commute, then the three asserted relations follow from the commutativity of the large square, of the lower two squares, and of the left two squares, respectively. □

2.5.6. **Proposition.** *Let $F, G \in \mathrm{GL}_n(B)$ be ∇-odd. Assume that $FG^* = GF^*$ and $F^*(\nabla\bar{G}\nabla)^* = (\nabla\bar{G}\nabla)F$, and let $H = \nabla^{-1}F^{\mathsf{T}}\nabla^{-1}F^{-1}$. Then $A_{\mathrm{o}}^B(\nabla, F, G)$ has an intertwiner $v \xrightarrow{H} S^2(v)$ and an imaginary scaling character group $(\theta^{(k)})_k$ such that $\theta_n^{(k)}(v) = \nabla H^k$.*

Proof. We can re-use the arguments in the proof of Proposition 2.5.3 and only have to show additionally that $\nabla H^k \xrightarrow{Q} \nabla H^k$ is an intertwiner and that $\theta_n^{(-1)}(\bar{v}) = \overline{\theta_n^{(1)}(v)}$. But by the lemma above, $(\nabla Q\nabla^{-1})\nabla H^k = \nabla QH^k = \nabla H^k Q$ and

$$\theta_n^{(-1)}(\bar{v}) = \theta_n^{(1)}(\overline{G}v(\nabla\overline{G}\nabla)^{-1}) = \overline{G}\nabla H^{-1}\nabla^{-1}\overline{G}^{-1}\nabla^{-1} = \overline{H}\nabla^{-1} = \overline{\theta_n^{(1)}(v)}. \qquad \square$$

2.5.7. **Remark.** Applying the functor (2) to $v^{-\mathsf{T}} \xrightarrow{F} v$, $\bar{v} \xrightarrow{G} v$, $v \xrightarrow{\nabla\bar{G}\nabla} \bar{v}$, we obtain intertwiners $\bar{v}^{-\mathsf{T}} \xrightarrow{F^*} \bar{v}$, $\bar{v}^{-\mathsf{T}} \xrightarrow{G^*} v^{-\mathsf{T}}$, $v^{-\mathsf{T}} \xrightarrow{(\nabla\bar{G}\nabla)^*} \bar{v}^{-\mathsf{T}}$, and the conditions $FG^* = GF^*$ and $F^*(\nabla\bar{G}\nabla)^* = (\nabla\bar{G}\nabla)F$ amount to commutativity of the squares

$$\begin{tikzcd}
\bar{v}^{-\mathsf{T}} \arrow[r, "F^*"] \arrow[d, "G^*"'] & \bar{v} \arrow[d, "G"] \\
v^{-\mathsf{T}} \arrow[r, "F"'] & v
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
v^{-\mathsf{T}} \arrow[r, "F"] \arrow[d, "(\nabla\bar{G}\nabla)^*"'] & v \arrow[d, "\nabla\bar{G}\nabla"] \\
\bar{v}^{-\mathsf{T}} \arrow[r, "F^*"'] & \bar{v}.
\end{tikzcd}$$

If $Q = G\nabla\bar{G}\nabla$ is scalar, then both conditions evidently are equivalent.

2.6. **The full dynamical quantum group $\mathrm{SU}_Q^{\mathrm{dyn}}(2)$.** In [11], Koelink and Rosengren studied a dynamical quantum group $\mathcal{F}_R(\mathrm{SU}(2))$ that arises from a dynamical R-matrix via the generalized FRT-construction of Etingof and Varchenko. We first recall its definition, then show that this dynamical quantum group coincides with $A_{\mathrm{o}}^B(\nabla, F, G)$ for specific choice of B, Γ, ∇, F, G, and finally construct a refinement that includes several interesting limit cases.

We shall slightly reformulate the definition of $\mathcal{F}_R(\mathrm{SL}(2))$ and $\mathcal{F}_R(\mathrm{SU}(2))$ given in [11, §2.2] so that it fits better with our approach.

Fix $q \in (0,1)$. Let $\mathfrak{M}(\mathbb{C})$ be the algebra of meromorphic functions on the plane and let $\mathbb{Z}$ act on B such that $b \overset{k}{\mapsto} b_{(k)} := b(\,\cdot - k)$ for all $b \in B$, $k \in \mathbb{Z}$. Define $f \in \mathfrak{M}(\mathbb{C})$ by

$$f(\lambda) = q^{-1}\frac{q^{2(\lambda+1)} - q^{-2}}{q^{2(\lambda+1)} - 1} = \frac{q^{\lambda+2} - q^{-(\lambda+2)}}{q^{\lambda+1} - q^{-(\lambda-1)}} \quad \text{for all } \lambda \in \mathbb{C}. \tag{3}$$

Then the $(\mathfrak{M}(\mathbb{C}),\mathbb{Z})$-Hopf algebroid $\mathcal{F}_R(\mathrm{SL}(2))$ is the universal $(\mathfrak{M}(\mathbb{C}),\mathbb{Z})$-algebra with generators $\alpha, \beta, \gamma, \delta$ satisfying

$$\partial_\alpha = (1,1), \quad \partial_\beta = (1,-1), \quad \partial_\gamma = (-1,1), \quad \partial_\delta = (-1,-1), \tag{4}$$

$$\alpha\beta = s(f_{(1)})\beta\alpha, \quad \alpha\gamma = r(f)\gamma\alpha, \quad \beta\delta = r(f)\delta\beta, \quad \gamma\delta = s(f_{(1)})\delta\gamma, \tag{5}$$

$$\frac{r(f)}{s(f)}\delta\alpha - \frac{1}{s(f)}\beta\gamma = \alpha\delta - r(f)\gamma\beta = \frac{r(f_{(1)})}{s(f_{(1)})}\alpha\delta - r(f_{(1)})\beta\gamma = \delta\alpha - \frac{1}{s(f_{(1)})}\gamma\beta = 1, \tag{6}$$

and with comultiplication, counit and antipode given by

$$\begin{aligned} \Delta(\alpha) &= \alpha\tilde{\otimes}\alpha + \beta\tilde{\otimes}\gamma, & \Delta(\beta) &= \alpha\tilde{\otimes}\beta + \beta\tilde{\otimes}\delta, \\ \Delta(\gamma) &= \gamma\tilde{\otimes}\alpha + \delta\tilde{\otimes}\gamma, & \Delta(\delta) &= \gamma\tilde{\otimes}\beta + \delta\tilde{\otimes}\delta, \end{aligned} \tag{7}$$

$$\epsilon(\alpha) = \partial_\alpha^r = \partial_\alpha^s, \quad \epsilon(\beta) = \epsilon(\gamma) = 0, \quad \epsilon(\delta) = \partial_\delta^r = \partial_\delta^s, \tag{8}$$

$$S(\alpha) = \frac{r(f)}{s(f)}\delta, \quad S(\beta) = -\frac{1}{s(f)}\beta, \quad S(\gamma) = -r(f)\gamma, \quad S(\delta) = \alpha. \tag{9}$$

Equip $\mathfrak{M}(\mathbb{C})$ with the involution given by $b^*(\lambda) = \overline{b(\overline{\lambda})}$ for all $b \in \mathfrak{M}(\mathbb{C})$, $\lambda \in \mathbb{C}$. Then $\mathcal{F}_R(\mathrm{SL}(2))$ can be equipped with an involution such that

$$\alpha^* = \delta, \qquad \beta^* = -q\gamma, \qquad \gamma^* = -q^{-1}\beta, \qquad \delta^* = \alpha, \tag{10}$$

and one obtains a $(\mathfrak{M}(\mathbb{C}),\mathbb{Z})$-Hopf $*$-algebroid which is denoted by $\mathcal{F}_R(\mathrm{SU}(2))$ [11].

2.6.1. **Proposition.** *Let* $\nabla = \mathrm{diag}(1,-1)$*,* $F = \begin{pmatrix} 0 & -1 \\ f_{(1)}^{-1} & 0 \end{pmatrix}$ *and* $G = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix}$*. Then there exist isomorphisms of* $(\mathfrak{M}(\mathbb{C}),\mathbb{Z})$*-Hopf (*-)algebroids* $A_{\mathrm{o}}^{\mathfrak{M}(\mathbb{C})}(\nabla, F) \to \mathcal{F}_R(\mathrm{SL}(2))$ *and* $A_{\mathrm{o}}^{\mathfrak{M}(\mathbb{C})}(\nabla, F, G) \to \mathcal{F}_R(\mathrm{SU}(2))$ *whose extensions to matrices map* v *to* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Proof. First, note that the function $\lambda \mapsto q^\lambda$ and hence also f is self-adjoint, and that

$$\hat{F} := \nabla F \nabla = \begin{pmatrix} 0 & -1 \\ f^{-1} & 0 \end{pmatrix}, \quad \hat{G} := \nabla G \nabla = G, \quad FG^* = \begin{pmatrix} 1 & 0 \\ 0 & (qf_{(1)})^{-1} \end{pmatrix} = GF^*.$$

Therefore, $A := A_{\mathrm{o}}^{\mathfrak{M}(\mathbb{C})}(\nabla, F)$ and $A_{\mathrm{o}}^{\mathfrak{M}(\mathbb{C})}(\nabla, F, G)$ are well-defined. Since $\nabla G \nabla \overline{G} = G^2 = q^{-1} \in M_2(\mathfrak{M}(\mathbb{C}))$, the latter algebra coincides with the former.

Write $v \in M_2(A)$ as $v = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ and write (4)′–(10)′ for the relations (4)–(10) with $\alpha', \beta', \gamma', \delta'$ instead of $\alpha, \beta, \gamma, \delta$. Then the relation $\partial_v = \nabla$ is equivalent to (4)′. The relation $v^{-\mathsf{T}} = r_2(\hat{F}^{-1})vs_2(F)$ is equivalent to

$$v^{-1} = \left(\begin{pmatrix} 0 & r(f) \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ s(f_{(1)}^{-1}) & 0 \end{pmatrix} \right)^{\mathsf{T}} = \begin{pmatrix} \frac{r(f)}{s(f)}\delta' & -\frac{1}{s(f)}\beta' \\ -r(f)\gamma' & \alpha' \end{pmatrix},$$

and multiplying out $v^{-1}v = 1 = vv^{-1}$ and using $(4)'$, we find that this relation is equivalent to $(5)'$ and $(6)'$. Hence, there exists an isomorphism of $(\mathfrak{M}(\mathbb{C}), \mathbb{Z})$-algebras $A \to \mathcal{F}_R(\mathrm{SL}(2))$ sending $\alpha', \beta', \gamma', \delta'$ to $\alpha, \beta, \gamma, \delta$. This isomorphism is compatible with the involution, comultiplication, counit and antipode because $(7)'$–$(10)'$ are equivalent to $\Delta_2(v) = v\tilde{\boxtimes}v$, $\epsilon_2(v) = \partial_v$, $S_2(v) = v^{-1}$ and

$$\bar{v} = r_2(\hat{G}^{-1})vs_2(G) = \begin{pmatrix} 0 & q \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix} = \begin{pmatrix} \delta' & -q\gamma' \\ -q^{-1}\beta' & \alpha' \end{pmatrix}. \qquad \square$$

We now refine the definition above as follows. The first idea is to replace the base $\mathfrak{M}(\mathbb{C})$ by the $\mathbb{Z}$-invariant subalgebra containing f and f^{-1}. This subalgebra can be described in terms of the functions $x(\lambda) = q^{\lambda}$, $y(\lambda) = q^{-\lambda}$ and $z = x - y$ as follows. Since $f = z_{(-2)}/z_{(-1)}$, this subalgebra is generated by all fractions $z_{(k)}/z_{(l)}$, where $k, l \in \mathbb{Z}$, and since $z_{(-1)} - qz = (q^{-1} - q)q^{-\lambda}$, also by all fractions $x/z_{(k)}$ and $y/z_{(k)}$, where $k \in \mathbb{Z}$. The second idea is to drop the relation $xy = 1$ to allow the limit cases $\lambda \to \pm\infty$, and regard x, y as canonical coordinates on $\mathbb{C}P^1$. Finally, we also regard q as a variable.

Let us now turn to the details. Denote by $R \subset \mathbb{C}(Q)$ the localization of $\mathbb{C}[Q]$ with respect to Q and the polynomials

$$S_k = (1 - Q^{2k})/(1 - Q^2) = 1 + Q^2 + \cdots + Q^{2(k-1)}, \quad \text{where } k \in \mathbb{N}.$$

Let $\mathbb{Z}$ act on the algebra $\mathbb{C}(Q, X, Y)$ of rational functions in Q, X, Y by

$$Q_{(k)} = Q, \qquad X_{(k)} = Q^{-k}X, \qquad Y_{(k)} = Q^kY \qquad \text{for all } k \in \mathbb{Z},$$

where the lower index (k) denotes the action of k. Denote by $B \subset \mathbb{C}(Q, X, Y)$ the subalgebra generated by R and all elements

$$Z_{k,l} := (X - Y)_{(k)}/(X - Y)_{(l)}, \quad \text{where } k, l \in \mathbb{Z}.$$

We equip B with the induced action of $\mathbb{Z}$ and the involution given by $Q = Q^*$ and $Z_{k,l}^* = Z_{k,l}$ for all $k, l \in \mathbb{Z}$. Note that this involution is the one inherited from $\mathbb{C}(Q, X, Y)$ when $Q = Q^*$ and either $X^* = X$, $Y^* = Y$ or $X^* = -X$, $Y^* = -Y$. Finally, let

$$\nabla = (1, -1), \qquad F = \begin{pmatrix} 0 & -1 \\ Z_{0,-1} & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & -Q \\ 1 & 0 \end{pmatrix}.$$

Then $FG^* = G^*F$ and $G\nabla\bar{G}\nabla = G^2 = \mathrm{diag}(-Q, -Q)$.

2.6.2. **Definition.** *We let* $\mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2)) := A_{\mathrm{o}}^B(\nabla, F, G)$.

Thus, $\mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2))$ is generated by the entries $\alpha, \beta, \gamma, \delta$ of a 2×2-matrix v which satisfy the relations (4)–(10) with $Z_{-2,-1}$ and Q instead of f and q. This $(B, \mathbb{Z})$-Hopf $*$-algebroid aggregates several other interesting quantum groups and quantum groupoids which can be obtained by suitable base changes as follows.

Denote by $z \in \mathfrak{M}(\mathbb{C})$ the function $\lambda \mapsto q^{\lambda} - q^{-\lambda}$. Equip $\mathbb{C}(\lambda)$ with an involution such that $\lambda^* = \lambda$, and a $\mathbb{Z}$-action such that $\lambda_{(k)} = \lambda - k$. Let $\Omega = (0, 1] \times [-\infty, \infty]$ and let $\mathbb{Z}$ act on $C(\Omega)$ by $g_{(k)}(q, t) = g(q, t - k)$ for all $g \in C(\Omega)$, $(q, t) \in \Omega$, $k \in \mathbb{Z}$.

2.6.3. **Lemma.** *There exist $\mathbb{Z}$-equivariant $*$-homomorphisms*

i) $\pi^q_{\mathfrak{M}(\mathbb{C})} \colon B \to \mathfrak{M}(\mathbb{C})$, $Q \mapsto q$, $\quad Z_{k,l} \mapsto \dfrac{z_{(k)}}{z_{(l)}}$ *for $q \in (0,1) \cup (1,\infty)$,*

ii) $\pi^1_{\mathfrak{M}(\mathbb{C})} \colon B \to \mathbb{C}(\lambda)$, $Q \mapsto 1$, $\quad Z_{k,l} \mapsto \dfrac{\lambda - k}{\lambda - l}$,

iii) $\pi_{\pm\infty} \colon B \to R$, $Q \mapsto Q$, $\quad Z_{k,l} \mapsto \dfrac{Q^{\pm k}}{Q^{\pm l}} = Q^{\pm k \mp l}$,

iv) $\pi^q_{\pm\infty} \colon B \to \mathbb{C}$, $Q \mapsto q$, $\quad Z_{k,l} \mapsto q^{\pm k \mp l}$ *for $q \in (0,\infty)$,*

v) $\pi_\Omega \colon B \to C(\Omega)$, $Q \mapsto ((q,t) \mapsto q)$, $\quad Z_{k,l} \mapsto \left((q,t) \mapsto \begin{cases} \frac{q^{t-k}+q^{k-t}}{q^{t-l}+q^{l-t}}, & t \in \mathbb{R}, \\ q^{\pm k \mp l}, & t = \pm\infty \end{cases} \right)$.

Proof. i) Restrict the homomorphism $\pi \colon \mathbb{C}(Q,X,Y) \to \mathfrak{M}(\mathbb{C})$ given by $Q \mapsto q$, $X \mapsto (\lambda \mapsto q^\lambda)$, $Y \mapsto (\lambda \mapsto q^{-\lambda})$ to B.

ii) Use i) and the fact that for all $k,l \in \mathbb{Z}$ and $\lambda \in \mathbb{C} \setminus \{l\}$,

$$\lim_{q\to 1} \pi^q_{\mathfrak{M}(\mathbb{C})}(Z_{k,l})(\lambda) = \lim_{q \to 1} \frac{q^{\lambda-k} - q^{k-\lambda}}{q^{\lambda-l} - q^{l-\lambda}} = \frac{\lambda-k}{\lambda-l}.$$

iii) Define $\pi \colon \mathbb{C}[Q,X,Y] \to R$ by $Q \mapsto Q$, $X \mapsto 1$, $Y \mapsto 0$. Then π extends to the localization B of $\mathbb{C}[Q,X,Y]$, giving $\pi_{-\infty}$, because $\pi((X-Y)_{(k)}) = Q^{-k}$ is invertible for all $k \in \mathbb{Z}$. This homomorphism $\pi_{-\infty}$ evidently is involutive, and $\mathbb{Z}$-equivariant because $\pi_{-\infty}(Z_{k+j,l+j}) = \pi_{-\infty}(Z_{k,l})$ for all $j \in \mathbb{Z}$. Similarly, one obtains $\pi_{+\infty}$.

iv) Immediate from iii).

v) Define $\pi \colon \mathbb{C}[Q,X,Y] \to C((0,1] \times \mathbb{R})$ by $Q \mapsto ((q,t) \mapsto q)$, $X \mapsto ((q,t) \mapsto iq^t)$, $Y \mapsto ((q,t) \mapsto -iq^t)$. Since $\pi((X-Y)_{(k)}) = i(q^{t-k} + q^{k-t})$ is invertible for all $t \in \mathbb{R}$, $k \in \mathbb{Z}$, this π extends to B. Moreover, each $\pi(Z_{k,l})$ extends to a continuous function on $C(\Omega)$ as desired, giving π_Ω. □

Note that $\pi^1_{+\infty} = \pi^1_{-\infty}$. Using this map, we obtain for each algebra C with an action by $\mathbb{Z}$ an $\mathbb{Z}$-equivariant homomorphism $\pi^1_C \colon B \to C$ sending Q and each $Z_{k,l}$ to 1_C.

2.6.4. **Proposition.** *There exist isomorphisms of Hopf $*$-algebroids as follows:*

i) $(\pi^q_{\mathfrak{M}(\mathbb{C})})_* \mathcal{O}(\mathrm{SU}^{\mathrm{dyn}}_Q(2)) \cong \mathcal{F}_R(\mathrm{SU}(2))$ *for each $q \in (0,1) \cup (1,\infty)$;*

ii) $(\pi^q_{-\infty})_* \mathcal{O}(\mathrm{SU}^{\mathrm{dyn}}_Q(2)) \cong \mathcal{O}(\mathrm{SU}_q(2))$ *for each $q \in (0,\infty)$;*

iii) $(\pi^q_{\infty})_* \mathcal{O}(\mathrm{SU}^{\mathrm{dyn}}_Q(2)) \cong \mathcal{O}(\mathrm{SU}_q(2))^{\mathsf{op}}$ *for each $q \in (0,\infty)$;*

iv) $(\pi^1_{\mathbb{C}[X]})_* \mathcal{O}(\mathrm{SU}^{\mathrm{dyn}}_Q(2))$ *is isomorphic to the $(\mathbb{C}[X],\mathbb{Z})$-Hopf $*$-algebroid in Example 2.4.14.*

Proof. i) This is immediate from the definitions and Proposition 2.6.1.

ii), iii) Let $\pi^\pm = \pi^q_{\pm\infty}$. Then $(\pi^\pm)_* \mathcal{O}(\mathrm{SU}^{\mathrm{dyn}}_Q(2))$ is generated by the entries $\alpha', \beta', \gamma', \delta$ of a matrix v' such that $\beta' = -q\gamma'^*$ and $\delta' = \alpha'^*$. Moreover, $v'^{-\mathsf{T}} = \pi^\pm_2(F)^{-1} v' \pi^\pm_2(F)$ and $\bar{v}' = \pi^\pm_2(G)^{-1} v' \pi^\pm_2(G)$, where

$$\pi^-_2(F) = \begin{pmatrix} 0 & -1 \\ \pi(Z_{0,-1}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix} = \pi^\pm_2(G), \qquad \pi^+_2(F) = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}.$$

In the case of π_-, we find that v' is unitary, and obtain the usual presentation of $\mathcal{O}(\mathrm{SU}_q(2))$. Multiplying out the relation $v'^{-\mathsf{T}} = \pi_2^+(F)^{-1}v'\pi_2^+(F)$, one easily verifies the assertion on π^+.

iv) Immediate from the relations $(\pi^1_{\mathbb{C}[X]})_2(F) = (\pi^1_{\mathbb{C}[X]})_2(G) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. □

We expect most of the results of [11] to carry over from $\mathcal{F}_R(\mathrm{SU}(2))$ to $\mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2))$.

3. The level of universal C^*-algebras

Throughout this section, we shall only work with unital C^*-algebras. We assume all $*$-homomorphisms to be unital, and B to be a commutative, unital C^*-algebra equipped with a left action of a discrete group Γ. Given a subset X of a normed space V, we denote by $\overline{X} \subseteq v$ its closure and by $[X] \subseteq V$ the closed linear span of X.

3.1. **The maximal cotensor product of C^*-algebras with respect to $C^*(\Gamma)$.** This subsection reviews the cotensor product of C^*-algebras with respect to the Hopf C^*-algebra $C^*(\Gamma)$ and develops the main properties that will be needed in §3.2. The material presented here is certainly well known to the experts, but we didn't find a suitable reference.

We first recall a few preliminaries.

Let A be a $*$-algebra. A *representation* of A is a $*$-homomorphism into a C^*-algebra. Such a representation π is *universal* if every other representation of A factorizes uniquely through π. A universal representation exists if and only if for each $a \in A$,

$$|a| := \sup\{\|\pi(a)\| : \pi \text{ is a } *\text{-homomorphism of } A \text{ into some } C^*\text{-algebra}\} < \infty.$$

Indeed, if $|a|$ is finite for all $a \in A$, then the separated completion of A with respect to $|-|$ carries a natural structure of a C^*-algebra, which is denoted by $C^*(A)$ and called the enveloping C^*-algebra of A, and the natural representation $A \to C^*(A)$ is universal.

The maximal tensor product of C^*-algebras A and C is the enveloping C^*-algebra of the algebraic tensor product $A \otimes C$, and will be denoted by $A \hat{\otimes} C$.

The full group C^*-algebra $C^*(\Gamma)$ of Γ is the enveloping C^*-algebra of the group algebra $\mathbb{C}\Gamma$. We denote by $\Delta_\Gamma \colon C^*(\Gamma) \to C^*(\Gamma) \hat{\otimes} C^*(\Gamma)$ the *comultiplication*, given by $\gamma \mapsto \gamma \otimes \gamma$ for all $\gamma \in \Gamma$, and by $\epsilon_\Gamma \colon C^*(\Gamma) \to \mathbb{C}$ the *counit*, given by $\gamma \mapsto 1$ for all $\gamma \in \Gamma$. Clearly, $(\epsilon_\Gamma \hat{\otimes} \mathrm{id})\Delta_\Gamma = \mathrm{id} = (\mathrm{id} \hat{\otimes} \epsilon_\Gamma)\Delta_\Gamma$.

A completely positive (contractive) map, or brielfy c.p.(c.)-map, from a C^*-algebra A to a C^*-algebra C is a linear map $\phi \colon A \to C$ such that $\phi_n \colon M_n(A) \to M_n(C)$ is positive (and $\|\phi_n\| \leq 1$) for all $n \in \mathbb{N}$.

3.1.1. **Definition.** *A* $(\mathbb{C}, \Gamma)$-C^*-algebra *is a unital C^*-algebra A with injective unital $*$-homomorphisms $\delta_A \colon A \to C^*(\Gamma) \hat{\otimes} A$ and $\bar{\delta}_A \colon A \to A \hat{\otimes} C^*(\Gamma)$ such that* $(\mathrm{id} \hat{\otimes} \delta_A) \circ \delta_A = (\Delta_\Gamma \hat{\otimes} \mathrm{id}) \circ \delta_A$, $(\bar{\delta}_A \hat{\otimes} \mathrm{id}) \circ \bar{\delta}_A = (\mathrm{id} \hat{\otimes} \Delta_\Gamma) \circ \bar{\delta}_A$ *and* $(\delta_A \hat{\otimes} \mathrm{id}) \circ \bar{\delta}_A = (\mathrm{id} \hat{\otimes} \bar{\delta}_A) \circ \delta_A$. *A* morphism *of $(\mathbb{C}, \Gamma)$-C^*-algebras A and C is a unital $*$-homomorphism $\pi \colon A \to C$ satisfying $\delta_C \circ \pi = (\mathrm{id} \hat{\otimes} \pi) \circ \delta_A$ and $\bar{\delta}_C \circ \pi = (\pi \hat{\otimes} \mathrm{id}) \circ \bar{\delta}_A$. We denote by $\mathbf{C^*}$-$\mathbf{Alg}_{(\mathbb{C},\Gamma)}$ the category of all $(\mathbb{C}, \Gamma)$-C^*-algebras. Replacing $*$-homomorphisms by c.p.-maps, we define c.p.-maps of $(\mathbb{C}, \Gamma)$-C^*-algebras and the category $\mathbf{C^*}$-$\mathbf{Alg}^{\mathrm{c.p.}}_{(\mathbb{C},\Gamma)}$.*

3.1.2. **Remark.** Let A be a $(\mathbb{C},\Gamma)$-C^*-algebra. Then $(\epsilon_\Gamma\hat{\otimes}\mathrm{id})\circ\delta_A=\mathrm{id}_A$ because

$$\delta_A(\epsilon_\Gamma\hat{\otimes}\mathrm{id})\circ\delta_A=(\epsilon_\Gamma\hat{\otimes}\mathrm{id}\hat{\otimes}\mathrm{id})\circ(\mathrm{id}\hat{\otimes}\delta_A)\circ\delta_A=((\epsilon_\Gamma\hat{\otimes}\mathrm{id})\circ\Delta_\Gamma\hat{\otimes}\mathrm{id})\circ\delta_A=\delta_A,$$

and likewise $(\mathrm{id}\hat{\otimes}\epsilon_\Gamma)\circ\bar{\delta}_A=\mathrm{id}_A$.

Let A and C be $(\mathbb{C},\Gamma)$-C^*-algebras. Then the maximal tensor product $A\hat{\otimes}C$ is a $(\mathbb{C},\Gamma)$-C^*-algebra with respect to $\delta_A\hat{\otimes}\mathrm{id}$ and $\mathrm{id}\hat{\otimes}\bar{\delta}_C$, and the assignments $(A,C)\mapsto A\hat{\otimes}C$ and $(\phi,\psi)\mapsto\phi\hat{\otimes}\psi$ define a product $-\hat{\otimes}-$ on $\mathbf{C}^*\text{-}\mathbf{Alg}^{(\mathrm{c.p.})}_{(\mathbb{C},\Gamma)}$ that is associative in the obvious sense. Unless Γ is trivial, this product can not be unital because it forgets $\bar{\delta}_A$ and δ_C.

With respect to the restrictions of $\delta_A\hat{\otimes}\mathrm{id}$ and $\mathrm{id}\hat{\otimes}\bar{\delta}_C$, the subspace

$$A\overset{\Gamma}{\hat{\otimes}}C:=\{x\in A\hat{\otimes}C:(\bar{\delta}_A\hat{\otimes}\mathrm{id})(x)=(\mathrm{id}\hat{\otimes}\delta_C)(x)\}\subseteq A\hat{\otimes}C$$

evidently is a $(\mathbb{C},\Gamma)$-C^*-algebra again. Moreover, given morphisms of $(\mathbb{C},\Gamma)$-C^*-algebras $\phi\colon A\to C$ and $\psi\colon D\to E$, the product $\phi\hat{\otimes}\psi$ restricts to a morphism $\phi\overset{\Gamma}{\hat{\otimes}}\psi\colon A\overset{\Gamma}{\hat{\otimes}}D\to C\overset{\Gamma}{\hat{\otimes}}E$. We thus obtain a second product $-\overset{\Gamma}{\hat{\otimes}}-$ on $\mathbf{C}^*\text{-}\mathbf{Alg}^{(\mathrm{c.p.})}_{(\mathbb{C},\Gamma)}$ that is associative in the natural sense, and unital in the following sense.

Regard $C^*(\Gamma)$ as a $(\mathbb{C},\Gamma)$-C^*-algebra with respect to Δ_Γ. Then for each $(\mathbb{C},\Gamma)$-C^*-algebra A, the maps δ_A and $\bar{\delta}_A$ are isomorphisms of $(\mathbb{C},\Gamma)$-C^*-algebras

$$\delta_A\colon A\xrightarrow{\cong}C^*(\Gamma)\overset{\Gamma}{\hat{\otimes}}A,\qquad\qquad \bar{\delta}_A\colon A\xrightarrow{\cong}A\overset{\Gamma}{\hat{\otimes}}C^*(\Gamma).$$

Indeed, they evidently are morphisms, and surjective because

$$x=(\epsilon_\Gamma\hat{\otimes}\mathrm{id}\hat{\otimes}\mathrm{id})((\Delta_\Gamma\hat{\otimes}\mathrm{id})(x))=(\epsilon_\Gamma\hat{\otimes}\mathrm{id}\hat{\otimes}\mathrm{id})((\mathrm{id}\hat{\otimes}\delta_A)(x))=\delta_A((\epsilon_\Gamma\hat{\otimes}\mathrm{id})(x))$$

for each $x\in C^*(\Gamma)\overset{\Gamma}{\hat{\otimes}}A$ and likewise $y=\bar{\delta}_A((\mathrm{id}\hat{\otimes}\epsilon_\Gamma)(y))$ for each $y\in A\overset{\Gamma}{\hat{\otimes}}C^*(\Gamma)$.

We next construct a natural transformation $p\colon(-\hat{\otimes}-)\to(-\overset{\Gamma}{\hat{\otimes}}-)$ which will be needed to prove associativity of the product of (B,Γ)-C^*-algebras in §3.2. The construction is based on ideas taken from [1, §7], and carries over from $C^*(\Gamma)$ to any Hopf C^*-algebra H equipped with a Haar mean $H\hat{\otimes}H\to H$; see also [13].

3.1.3. **Lemma.** *There exists a unique state ν on $C^*(\Gamma)\hat{\otimes}C^*(\Gamma)$ such that $\nu(\gamma\otimes\gamma')=\delta_{\gamma,\gamma'}1$ for all $\gamma,\gamma'\in\Gamma$. Moreover, $\nu\circ\Delta_\Gamma=\epsilon_\Gamma$ and $(\mathrm{id}\hat{\otimes}\nu)\circ(\Delta_\Gamma\hat{\otimes}\mathrm{id})=(\nu\hat{\otimes}\mathrm{id})\circ(\mathrm{id}\hat{\otimes}\Delta_\Gamma)$.*

Proof. This follows from [13, Theorem 0.1], but let us include the short direct proof. Uniqueness is clear. To construct ν, denote by $(\epsilon_\gamma)_{\gamma\in\Gamma}$ the canonical orthonormal basis of $l^2(\Gamma)$, by $\lambda,\rho\colon C^*(\Gamma)\to\mathcal{L}(l^2(\Gamma))$ the representations given by $\lambda(\gamma)\epsilon_{\gamma'}=\epsilon_{\gamma\gamma'}$ and $\rho(\gamma)\epsilon_{\gamma'}=\epsilon_{\gamma'\gamma^{-1}}$ for all $\gamma,\gamma'\in\Gamma$, and by $\lambda\times\rho\colon C^*(\Gamma)\hat{\otimes}C^*(\Gamma)\to\mathcal{L}(l^2(\Gamma))$ the representation given by $x\otimes y\mapsto\lambda(x)\rho(y)$. Then $\nu:=\langle\epsilon_e|(\lambda\times\rho)(-)\epsilon_e\rangle$ satisfies $\nu(\gamma\otimes\gamma')=\delta_{\gamma,\gamma'}1$ for all $\gamma,\gamma'\in\Gamma$. The remaining equations follow easily. □

3.1.4. **Lemma.** i) *For every $(\mathbb{C},\Gamma)$-C^*-algebra A, the maps*

$$\bar{p}_A:=(\mathrm{id}\hat{\otimes}\nu)(\bar{\delta}_A\hat{\otimes}\mathrm{id})\colon A\hat{\otimes}C^*(\Gamma)\to A,\qquad p_A:=(\nu\hat{\otimes}\mathrm{id})(\mathrm{id}\hat{\otimes}\delta_A)\colon C^*(\Gamma)\hat{\otimes}A\to A$$

are morphisms in $\mathbf{C}^*\text{-}\mathbf{Alg}^{\mathrm{c.p.}}_{(\mathbb{C},\Gamma)}$ and satisfy $p_A\circ\delta_A$ and $\bar{p}_A\circ\bar{\delta}_A=\mathrm{id}$.

ii) *The families* $(p_A)_A$, $(\bar{p}_A)_A$ *are natural transformations from* $-\hat{\otimes}C^*(\Gamma)$ *and* $C^*(\Gamma)\hat{\otimes}-$, *respectively, to* id, *regarded as functors on* $\mathbf{C^*}\text{-}\mathbf{Alg}^{\text{c.p.}}_{(\mathbb{C},\Gamma)}$.

Proof. i) The map p_A is a morphism in $\mathbf{C^*}\text{-}\mathbf{Alg}^{\text{c.p.}}_{(\mathbb{C},\Gamma)}$ because $\bar{\delta}_A \circ p_A = (p_A \hat{\otimes}\,\mathrm{id}) \circ \bar{\delta}_A$ and

$$\begin{aligned}
\delta_A \circ p_A &= (\nu \hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\delta_A)(\mathrm{id}\,\hat{\otimes}\delta_A)\\
&= (\nu \hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\Delta_\Gamma \hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\delta_A)\\
&= (\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}) \circ (\Delta_\Gamma \hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\delta_A)\\
&= (\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\delta_A) \circ (\Delta_\Gamma \hat{\otimes}\,\mathrm{id}) = (\mathrm{id}\,\hat{\otimes} p_A) \circ (\Delta_\Gamma \hat{\otimes}\,\mathrm{id}).
\end{aligned}$$

Moreover, $p_A \circ \delta_A = (\nu \hat{\otimes}\,\mathrm{id})(\mathrm{id}\,\hat{\otimes}\delta_A)\delta_A = (\nu\Delta_\Gamma \hat{\otimes}\,\mathrm{id})\delta_A = (\epsilon_\Gamma \hat{\otimes}\,\mathrm{id})\delta_A = \mathrm{id}$ and similarly $\bar{p}_A \circ \bar{\delta}_A = \mathrm{id}$.

ii) This follows from the fact that $(\delta_A)_A$ and $(\bar{\delta}_A)_A$ are natural transformations. □

3.1.5. **Proposition.** i) *Let* A, C *be* $(\mathbb{C},\Gamma)$-C^**-algebras. Then the map*

$$p_{A,C} := (\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}) \circ (\bar{\delta}_A \hat{\otimes} \delta_C) \colon A\hat{\otimes}C \to A\hat{\otimes}C$$

is equal to $(\mathrm{id}\,\hat{\otimes} p_C) \circ (\bar{\delta}_A \hat{\otimes}\,\mathrm{id})$ *and* $(\bar{p}_A \hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\delta_C)$, *a morphism in* $\mathbf{C^*}\text{-}\mathbf{Alg}^{\text{c.p.}}_{(\mathbb{C},\Gamma)}$, *and a conditional expectation onto* $A \overset{\Gamma}{\hat{\otimes}} C \subseteq A\hat{\otimes}C$ *in the sense that* $p_{A,C}(xyz) = xp_{A,C}(y)z$ *for all* $x, z \in A \overset{\Gamma}{\hat{\otimes}} C$ *and* $y \in A\hat{\otimes}C$.

ii) *The family* $(p_{A,C})_{A,C}$ *is a natural transformation from* $-\hat{\otimes}-$ *to* $-\overset{\Gamma}{\hat{\otimes}}-$, *regarded as functors on* $\mathbf{C^*}\text{-}\mathbf{Alg}^{\text{c.p.}}_{(\mathbb{C},\Gamma)} \times \mathbf{C^*}\text{-}\mathbf{Alg}^{\text{c.p.}}_{(\mathbb{C},\Gamma)}$.

Proof. i) The equality follows immediately from the definitions and implies that $p_{A,C}$ is a morphism as claimed. Next, $p_{A,C}(A\hat{\otimes}C) \subseteq A \overset{\Gamma}{\hat{\otimes}} C$ because

$$\begin{aligned}
(\bar{\delta}_A \hat{\otimes}\,\mathrm{id}) \circ p_{A,C} &= (\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}) \circ (\bar{\delta}_A \hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\bar{\delta}_A \hat{\otimes}\delta_C)\\
&= (\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\Delta_\Gamma \hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\bar{\delta}_A \hat{\otimes}\delta_C)\\
&= (\mathrm{id}\,\hat{\otimes}\nu\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\Delta_\Gamma \hat{\otimes}\,\mathrm{id}) \circ (\bar{\delta}_A \hat{\otimes}\delta_C) = (\mathrm{id}\,\hat{\otimes}\delta_C) \circ p_{A,C}.
\end{aligned}$$

On the other hand,

$$p_{A,C}(x) = (\bar{p}_A \hat{\otimes}\,\mathrm{id})((\mathrm{id}\,\hat{\otimes}\delta_C)(x)) = (\bar{p}_A \hat{\otimes}\,\mathrm{id})((\delta_A \hat{\otimes}\,\mathrm{id})(x)) = x$$

for all $x \in A \overset{\Gamma}{\hat{\otimes}} C$. Thus, $p_{A,C}$ is a completely positive projection from $A\hat{\otimes}C$ onto $A \overset{\Gamma}{\hat{\otimes}} C$ and hence a conditional expectation (see, e.g., [5, Proposition 1.5.7]).

ii) Straightforward. □

Denote by $*\text{-}\mathbf{Alg}^0_{(\mathbb{C},\Gamma)} \subseteq \mathbf{Alg}^*_{(C,\Gamma)}$ the full subcategory formed by all $(\mathbb{C},\Gamma)$-$*$-algebras that have an enveloping C^*-algebra. We shall need an adjoint pair of functors

$$*\text{-}\mathbf{Alg}^0_{(\mathbb{C},\Gamma)} \underset{(-)_{*,*}}{\overset{C^*(-)}{\rightleftarrows}} \mathbf{C^*}\text{-}\mathbf{Alg}_{(\mathbb{C},\Gamma)}. \tag{11}$$

The functor $C^*(-)$ is defined as follows. Let $A \in *\text{-}\mathbf{Alg}^0_{(B,\Gamma)}$. Using the universal property of $C^*(A)$, we obtain unique $*$-homomorphisms $\delta_{C^*(A)} \colon C^*(A) \to C^*(\Gamma)\hat{\otimes}C^*(A)$

and $\bar{\delta}_{C^*(A)}: C^*(A) \to C^*(A)\hat{\otimes}C^*(\Gamma)$ such that $\delta_{C^*(A)}(a) = \gamma \otimes a$ and $\bar{\delta}_{C^*(A)}(a) = a \otimes \gamma'$ for all $a \in A_{\gamma,\gamma'}, \gamma, \gamma' \in A$, and with respect to these $*$-homomorphisms, $C^*(A)$ becomes a $(\mathbb{C},\Gamma)$-C^*-algebra. Moreover, every morphism $\pi: A \to C$ in $*$-$\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$ extends uniquely to a $*$-homomorphism $C^*(\pi): C^*(A) \to C^*(C)$ which is a morphism in $\mathbf{C}^*$-$\mathbf{Alg}_{(\mathbb{C},\Gamma)}$.

The functor $(-)_{*,*}$ is defined as follows. Let A be a $(\mathbb{C},\Gamma)$-C^*-algebra and let

$$A_{\gamma,\gamma'} := \{a \in A : \delta(a) = \gamma \otimes a, \bar{\delta}(a) = a \otimes \gamma'\} \subseteq A \quad \text{for all } \gamma, \gamma' \in \Gamma.$$

Then the sum $A_{*,*} := \sum_{\gamma,\gamma'} A_{\gamma,\gamma'} \subseteq A$ is a $(\mathbb{C},\Gamma)$-$*$-algebra, and, every morphism $\pi: A \to \mathbb{C}$ of $(\mathbb{C},\Gamma)$-C^*-algebras restricts to a morphism $\pi_{*,*}: A_{*,*} \to C_{*,*}$ of $(\mathbb{C},\Gamma)$-$*$-algebras. We thus obtain a functor $(-)_{*,*}: \mathbf{C}^*\text{-}\mathbf{Alg}_{(\mathbb{C},\Gamma)} \to \mathbf{Alg}^*_{(C,\Gamma)}$.

3.1.6. **Lemma.** *$(-)_{*,*}$ takes values in $*$-$\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$.*

Proof. Let A be a $(\mathbb{C},\Gamma)$-C^*-algebra. Then for every $*$-representation π of $A_{*,*}$, the restriction to the C^*-subalgebra $A_{e,e}$ is contractive and thus $\|\pi(a)\|^2 = \|\pi(a^*a)\| \leq \|a^*a\| = \|a\|^2$ for all $a \in A_{\gamma,\gamma'}$ $\gamma, \gamma' \in \Gamma$. Since such elements a span $A_{*,*}$, we can conclude $|a'| < \infty$ for all $a' \in A_{*,*}$. □

For every $(\mathbb{C},\Gamma)$-C^*-algebra A, the morphisms p_A and $\bar{p}_A$ yield a morphism

$$P_A := \bar{p}_A \circ (p_A \hat{\otimes} \mathrm{id}) = p_A \circ (\mathrm{id} \hat{\otimes} \bar{p}_A): C^*(\Gamma)\hat{\otimes}A\hat{\otimes}C^*(\Gamma) \to A$$

in $\mathbf{C}^*$-$\mathbf{Alg}^{\text{c.p.}}_{(\mathbb{C},\Gamma)}$.

3.1.7. **Lemma.** i) *Let $A \in \mathbf{C}^*$-$\mathbf{Alg}_{(\mathbb{C},\Gamma)}$. Then for all $\gamma, \gamma', \beta, \beta' \in \Gamma$,*

$$P_A(\gamma \otimes A \otimes \gamma') = A_{\gamma,\gamma'}, \qquad P_A(\beta \otimes A_{\gamma,\gamma'} \otimes \beta') = \delta_{\beta,\gamma}\delta_{\beta',\gamma'}A_{\gamma,\gamma'}, \qquad A = \overline{A_{*,*}}.$$

ii) *Let $A \in *$-$\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$. Then $C^*(A)_{\gamma,\gamma'} = \overline{A_{\gamma,\gamma'}}$ for all $\gamma, \gamma' \in \Gamma$.*

Proof. We only prove i); assertion ii) follows similarly. First, $P_A(\gamma \otimes A \otimes \gamma') \subseteq A_{\gamma,\gamma'}$ because P_A is a morphism in $\mathbf{C}^*$-$\mathbf{Alg}_{(\mathbb{C},\Gamma)}$ and $\Delta_\Gamma(\gamma'') = \gamma'' \otimes \gamma''$ for $\gamma'' = \gamma, \gamma'$. This inclusion, the relation $C^*(\Gamma)\hat{\otimes}A\hat{\otimes}C^*(\Gamma) = \overline{\sum_{\gamma,\gamma'} \gamma \otimes A \otimes \gamma'}$ and continuity and surjectivity of P_A imply $\overline{A_{*,*}} = A$. The equation $P_A(\beta \otimes A_{\gamma,\gamma'} \otimes \beta') = \delta_{\beta,\gamma}\delta_{\beta',\gamma'}A_{\gamma,\gamma'}$ follows from the definitions and implies that the inclusion $P_A(\gamma \otimes A \otimes \gamma') \subseteq A_{\gamma,\gamma'}$ is an equality. □

For every A in $*$-$\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$ and C in $\mathbf{C}^*$-$\mathbf{Alg}_{(\mathbb{C},\Gamma)}$, we get canonical morphisms $\eta_A: A \to C^*(A)_{*,*}$ in $*$-$\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$ and $\epsilon_C: C^*(C_{*,*}) \to C$ in $\mathbf{C}^*$-$\mathbf{Alg}_{(\mathbb{C},\Gamma)}$.

3.1.8. **Proposition.** *The functors $C^*(-)$ and $(-)_{*,*}$ are adjoint, where the unit and counit of the adjunction are the families $(\eta_A)_A$ and $(\epsilon_C)_C$, respectively. Furthermore, $(-)_{*,*}$ is faithful.*

Proof. Let $A \in *$-$\mathbf{Alg}^0_{(B,\Gamma)}$ and $C \in \mathbf{C}^*$-$\mathbf{Alg}_{(B,\Gamma)}$. Since the representation $A \to C^*(A)$ has dense image and is universal, the assignment $(C^*(A) \xrightarrow{\pi} C) \mapsto (A \xrightarrow{\eta_A} C^*(A)_{*,*} \xrightarrow{\pi_{*,*}} C_{*,*})$ yields a bijective correspondence between morphisms $C^*(A) \to C$ and morphisms $A \to C_{*,*}$. The functor $(-)_{*,*}$ is faithful because $A_{*,*} \subseteq A$ is dense. □

3.1.9. **Remark.** Similar arguments as in the proof of Lemma 3.1.7 show that for all $A, C \in \mathbf{C^*\text{-}Alg}_{(\mathbb{C},\Gamma)}$, $D, E \in *\text{-}\mathbf{Alg}^0_{(\mathbb{C},\Gamma)}$ and all $\gamma, \gamma'' \in \Gamma$,

$$(A \overset{\Gamma}{\hat{\otimes}} C)_{\gamma,\gamma''} = \overline{\sum_{\gamma'} A_{\gamma,\gamma'} \otimes C_{\gamma',\gamma''}}, \qquad (C^*(D) \overset{\Gamma}{\hat{\otimes}} C^*(E))_{\gamma,\gamma'} = \overline{\sum_{\gamma'} D_{\gamma,\gamma'} \otimes E_{\gamma',\gamma''}}.$$

A *short exact sequence* of $(\mathbb{C},\Gamma)$-C^*-algebra is a sequence of morphisms $J \xrightarrow{\iota} A \xrightarrow{\pi} C$ in $\mathbf{C^*\text{-}Alg}_{(\mathbb{C},\Gamma)}$ such that $\ker \iota = 0$, $\iota(J) = \ker \pi$ and $\pi(A) = C$. A functor on $\mathbf{C^*\text{-}Alg}_{(\mathbb{C},\Gamma)}$ is *exact* if it maps short exact sequences to short exact sequences.

3.1.10. **Proposition.** *For every $(\mathbb{C},\Gamma)$-C^*-algebra D, the functors $-\overset{\Gamma}{\hat{\otimes}} D$ and $D \overset{\Gamma}{\hat{\otimes}} -$ on $\mathbf{C^*\text{-}Alg}_{(\mathbb{C},\Gamma)}$ are exact.*

Proof. If $J \xrightarrow{\iota} A \xrightarrow{\pi} C$ is a short exact sequence in $\mathbf{C^*\text{-}Alg}_{(\mathbb{C},\Gamma)}$, then $J \hat{\otimes} D \xrightarrow{\iota \hat{\otimes} \mathrm{id}} A \hat{\otimes} D \xrightarrow{\pi \hat{\otimes} \mathrm{id}} C \hat{\otimes} D$ is exact (see, e.g., [5, Proposition 3.7]), whence $\ker(\iota \overset{\Gamma}{\hat{\otimes}} \mathrm{id}) = 0$ and

$$\begin{aligned}
\ker(\pi \overset{\Gamma}{\hat{\otimes}} \mathrm{id}) = p_{A,D}(\ker(\pi \hat{\otimes} \mathrm{id})) &= p_{A,D}((\iota \hat{\otimes} \mathrm{id})(J \hat{\otimes} D)) \\
&= (\iota \overset{\Gamma}{\hat{\otimes}} \mathrm{id})(p_{J,D}(J \hat{\otimes} D)) = (\iota \overset{\Gamma}{\hat{\otimes}} \mathrm{id})(J \overset{\Gamma}{\hat{\otimes}} D), \\
(\pi \overset{\Gamma}{\hat{\otimes}} \mathrm{id})(A \overset{\Gamma}{\hat{\otimes}} D) &= (\pi \overset{\Gamma}{\hat{\otimes}} \mathrm{id})(p_{A,D}(A \hat{\otimes} D)) \\
&= p_{C,D}((\pi \hat{\otimes} \mathrm{id})(A \hat{\otimes} D)) = p_{C,D}(C \hat{\otimes} D) = C \overset{\Gamma}{\hat{\otimes}} D. \qquad \square
\end{aligned}$$

3.2. **The monoidal category of (B,Γ)-C^*-algebras.** We now define an analogue of (B,Γ)-$*$-algebras on the level of universal C^*-algebras, and construct a monoidal product which is unital and associative.

3.2.1. **Definition.** *A* (B,Γ)-C^*-algebra *is a $(\mathbb{C},\Gamma)$-C^*-algebra A equipped with unital $*$-homomorphisms $r_A, s_A \colon B \to A_{e,e}$ such that $A_{*,*}$ is a (B,Γ)-$*$-algebra with respect to the map $r_A \times s_A \colon B \otimes B \to A_{e,e}$, $b \otimes b' \mapsto r_A(b) s_A(b')$. A* morphism *of (B,Γ)-C^*-algebras is a $B \otimes B$-linear morphism of $(\mathbb{C},\Gamma)$-C^*-algebras. We denote by $\mathbf{C^*\text{-}Alg}_{(B,\Gamma)}$ the category of all (B,Γ)-C^*-algebras. Replacing $*$-homomorphisms by c.p.-maps, we define c.p.-maps of (B,Γ)-C^*-algebras and the category $\mathbf{C^*\text{-}Alg}^{\mathrm{c.p.}}_{(B,\Gamma)}$.*

Denote by $*\text{-}\mathbf{Alg}^0_{(B,\Gamma)} \subseteq *\text{-}\mathbf{Alg}_{(B,\Gamma)}$ the full subcategory formed by all (B,Γ)-$*$-algebras that have an enveloping C^*-algebra. This category is related to $\mathbf{C^*\text{-}Alg}_{(B,\Gamma)}$ as follows. If $C \in \mathbf{C^*\text{-}Alg}_{(B,\Gamma)}$, then $C_{*,*} \in *\text{-}\mathbf{Alg}^0_{(B,\Gamma)}$ by Lemma 3.1.6. Conversely, if $A \in *\text{-}\mathbf{Alg}^0_{(B,\Gamma)}$, then $C^*(A)$ carries a natural structure of a (B,Γ)-C^*-algebra. The canonical maps $\eta_A \colon A \to C^*(A)_{*,*}$ and $\epsilon_C \colon C^*(C_{*,*}) \to C$ are morphisms in $*\text{-}\mathbf{Alg}^0_{(B,\Gamma)}$ and $\mathbf{C^*\text{-}Alg}_{(B,\Gamma)}$, respectively, and Proposition 3.1.8 therefore implies:

3.2.2. **Corollary.** *The assignments $A \mapsto C^*(A), \pi \mapsto C^*(\pi)$ and $A \mapsto A_{*,*}, \pi \mapsto \pi_{*,*}$ form a pair of adjoint functors*

$$*\text{-}\mathbf{Alg}^0_{(B,\Gamma)} \underset{(-)_{*,*},}{\overset{C^*(-)}{\rightleftarrows}} \mathbf{C^*\text{-}Alg}_{(B,\Gamma)}$$

where the unit and counit of the adjunction are the families $(\eta_A)_A$ *and* $(\epsilon_C)_C$*, respectively. Furthermore,* $(-)_{*,*}$ *is faithful.*

Let A and C be (B,Γ)-C^*-algebras. Then the $(\mathbb{C},\Gamma)$-C^*-algebra $A \overset{\Gamma}{\hat{\otimes}} C$ is a (B,Γ)-C^*-algebra with respect to the $*$-homomorphisms $r\colon b \mapsto r_A(b)\hat{\otimes}1$ and $s\colon b' \mapsto 1\hat{\otimes}s_C(b')$, and the assignments $(A,C) \mapsto A \overset{\Gamma}{\hat{\otimes}} C$ and $(\phi,\psi) \mapsto \phi \overset{\Gamma}{\hat{\otimes}} \psi$ define a product $- \overset{\Gamma}{\hat{\otimes}} -$ on $\mathbf{C}^*\text{-}\mathbf{Alg}^{(\text{c.p.})}_{(B,\Gamma)}$ that is associative in the obvious sense. Using the map

$$t_{A,C}\colon B \to A \overset{\Gamma}{\hat{\otimes}} C, \quad b \mapsto s_A(b)\hat{\otimes}1 - 1\hat{\otimes}r_C(b),$$

we define an ideal $(t_{A,C}(B)) \subseteq A \overset{\Gamma}{\hat{\otimes}} C$. Since $t_{A,C}(B) \subseteq (A \overset{\Gamma}{\hat{\otimes}} C)_{e,e}$, the quotient

$$A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} C := (A \overset{\Gamma}{\hat{\otimes}} C)/(t_{A,C}(B)).$$

inherits the (B,Γ)-C^*-algebra structure of $A \overset{\Gamma}{\hat{\otimes}} C$. For every pair of morphisms $\phi\colon A \to C$ and $\psi\colon D \to E$ in $\mathbf{C}^*\text{-}\mathbf{Alg}^{(\text{c.p.})}_{(B,\Gamma)}$, the morphism $\phi \overset{\Gamma}{\hat{\otimes}} \psi$ maps $t_{A,D}(B)$ to $t_{C,E}(B)$ and thus factorizes to a morphism $\phi \underset{B}{\overset{\Gamma}{\hat{\otimes}}} \psi\colon A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} D \to C \underset{B}{\overset{\Gamma}{\hat{\otimes}}} E$. We thus obtain a product $- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -$ on $\mathbf{C}^*\text{-}\mathbf{Alg}^{(\text{c.p.})}_{(B,\Gamma)}$, and the canonical quotient map $q_{A,C}\colon A \overset{\Gamma}{\hat{\otimes}} C \to A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} C$ yields a natural transformation $q = (q_{A,C})_{A,C}$ from $- \overset{\Gamma}{\hat{\otimes}} -$ to $- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -$.

3.2.3. **Remarks.** i) For all (B,Γ)-C^*-algebras A, C, one has $[t_{A,C}(B)(A \overset{\Gamma}{\hat{\otimes}} C)] = t_{A,C}(B) = [(A \overset{\Gamma}{\hat{\otimes}} C)t_{A,C}(B)]$. Indeed, a short calculation shows that for all $\gamma,\gamma',\gamma'' \in \Gamma$, $a \in A_{\gamma,\gamma'}$, $c \in C_{\gamma',\gamma''}$, $b \in B$, $(a\otimes c)t_{A,C}(b) = t_{A,C}(\gamma'(b))(a\otimes c)$, and now the assertion follows from Remark 3.1.9.

ii) For every (B,Γ)-C^*-algebra D, the functors $- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} D$ and $D \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -$ on $\mathbf{C}^*\text{-}\mathbf{Alg}_{(B,\Gamma)}$ preserve surjections because the functors $- \overset{\Gamma}{\hat{\otimes}} D$ and $D \overset{\Gamma}{\hat{\otimes}} -$ do so by Proposition 3.1.10.

We show that the full crossed product $B \hat{\rtimes} \Gamma := C^*(B \rtimes \Gamma)$ is the unit for the product $- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -$. Denote by $\iota_\Gamma\colon C^*(\Gamma) \to B \hat{\rtimes} \Gamma$ the natural inclusion.

3.2.4. **Proposition.** i) *For each* (B,Γ)-C^**-algebra* A*, the* $*$*-homomorphisms*

$$L_A\colon A \xrightarrow{\delta_A} C^*(\Gamma) \overset{\Gamma}{\hat{\otimes}} A \xrightarrow{\iota_\Gamma \overset{\Gamma}{\hat{\otimes}} \mathrm{id}} (B \hat{\rtimes} \Gamma) \overset{\Gamma}{\hat{\otimes}} A \xrightarrow{q_{B\hat{\rtimes}\Gamma,A}} (B \hat{\rtimes} \Gamma) \underset{B}{\overset{\Gamma}{\hat{\otimes}}} A$$

and

$$R_A\colon A \xrightarrow{\bar{\delta}_A} A \overset{\Gamma}{\hat{\otimes}} C^*(\Gamma) \xrightarrow{\mathrm{id} \overset{\Gamma}{\hat{\otimes}} \iota_\Gamma} A \overset{\Gamma}{\hat{\otimes}} (B \hat{\rtimes} \Gamma) \xrightarrow{q_{A,B\hat{\rtimes}\Gamma}} A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} (B \hat{\rtimes} \Gamma),$$

are isomorphisms of (B,Γ)*-*C^**-algebras.*

ii) *The families* $R=(R_A)_A$ *and* $L=(L_A)_A$ *form natural isomorphism from* id *to* $((B\hat{\rtimes}\Gamma)\underset{B}{\overset{\Gamma}{\hat{\otimes}}}-)$ *and* $(-\underset{B}{\overset{\Gamma}{\hat{\otimes}}}(B\hat{\rtimes}\Gamma))$*, respectively, regarded as functors on* $\mathbf{C}^*\text{-}\mathbf{Alg}^{(\mathrm{c.p.})}_{(B,\Gamma)}$.

Proof. One easily checks that each L_A is a morphism of (B,Γ)-C^*-algebras and that $L=(L_A)_A$ is a natural transformation. We show that L_A is an isomorphism for every (B,Γ)-C^*-algebra A. The assertions concerning $R=(R_A)_A$ then follow similarly.

To prove that L_A is surjective, we only need to show that $(t_{B\hat{\rtimes}\Gamma,A}(B))+C^*(\Gamma)\overset{\Gamma}{\hat{\otimes}}A$ is dense in $(B\hat{\rtimes}\Gamma)\overset{\Gamma}{\hat{\otimes}}A$. But by Remark 3.1.9, elements of the form

$$b\gamma\otimes a=t_{B\hat{\rtimes}\Gamma,A}(b)(\gamma\otimes a)+\gamma\otimes r_A(b)a,\quad \text{where } b\in B, a\in A_{\gamma,\gamma'},\gamma,\gamma'\in\Gamma,$$

are linearly dense in $(B\hat{\rtimes}\Gamma)\overset{\Gamma}{\hat{\otimes}}A$.

To prove that L_A is injective, we only need to show that the intersection

$$J:=(\iota_\Gamma(C^*(\Gamma))\overset{\Gamma}{\hat{\otimes}}A)\cap(t_{B\hat{\rtimes}\Gamma,A}(B))\subseteq(B\hat{\rtimes}\Gamma)\overset{\Gamma}{\hat{\otimes}}A$$

equals 0. Since $J=\overline{J_{*,*}}$ by Lemma 3.1.7, it suffices to show that $J_{\gamma,\gamma'}=0$ for all $\gamma,\gamma'\in\Gamma$. Note that $J_{\gamma,\gamma'}=[\gamma\otimes A_{\gamma,\gamma'}]\cap[(B\gamma\otimes A_{\gamma,\gamma'})t_{B\hat{\rtimes}\Gamma,A}(B)]$. For each $\gamma,\gamma'\in\Gamma$, define a linear map $R_{\gamma,\gamma'}\colon B\gamma\otimes A_{\gamma,\gamma'}\to A_{\gamma,\gamma'}$ by $b\gamma\otimes a\mapsto r(b)a$. Then $R_{e,e}$ extends to a $*$-homomorphism on the C^*-subalgebra $B\hat{\otimes}A_{e,e}\subseteq(B\hat{\rtimes}\Gamma)\overset{\Gamma}{\hat{\otimes}}A$, and each $R_{\gamma,\gamma'}$ extends to a bounded linear map on $[B\gamma\otimes A_{\gamma,\gamma'}]\subseteq(B\hat{\rtimes}\Gamma)\overset{\Gamma}{\hat{\otimes}}A$ because

$$\|R_{\gamma,\gamma'}(z)\|^2=\|R_{\gamma,\gamma'}(z)R_{\gamma,\gamma'}(z)^*\|=\|R_{e,e}(zz^*)\|\le\|zz^*\|=\|z\|^2$$

for all $z\in B\gamma\otimes A_{\gamma,\gamma'}$. Now, $R_{\gamma,\gamma'}(zt_{B\hat{\rtimes}\Gamma,A}(b))=0$ for all $z\in[B\gamma\otimes A_{\gamma,\gamma'}]$ and $b\in B$, and $R_{\gamma,\gamma'}(\gamma\otimes a)=a$ for all $a\in A_{\gamma,\gamma'}$. Consequently, $J_{\gamma,\gamma'}=0$. ☐

We now show that the product $-\underset{B}{\overset{\Gamma}{\hat{\otimes}}}-$ is associative. Let A,C,D be (B,Γ)-C^*-algebras, denote by $a_{A,C,D}\colon(A\overset{\Gamma}{\hat{\otimes}}C)\overset{\Gamma}{\hat{\otimes}}D\to A\overset{\Gamma}{\hat{\otimes}}(C\overset{\Gamma}{\hat{\otimes}}D)$ the canonical isomorphism and let

$$\Phi_{A,C,D}:=q_{\left(A\underset{B}{\overset{\Gamma}{\hat{\otimes}}}C\right),D}\circ(q_{A,C}\overset{\Gamma}{\hat{\otimes}}\mathrm{id})\colon(A\overset{\Gamma}{\hat{\otimes}}C)\overset{\Gamma}{\hat{\otimes}}D\to(A\underset{B}{\overset{\Gamma}{\hat{\otimes}}}C)\underset{B}{\overset{\Gamma}{\hat{\otimes}}}D,$$

$$\Psi_{A,C,D}:=q_{A,\left(C\underset{B}{\overset{\Gamma}{\hat{\otimes}}}D\right)}\circ(\mathrm{id}\overset{\Gamma}{\hat{\otimes}}q_{C,D})\colon A\overset{\Gamma}{\hat{\otimes}}(C\overset{\Gamma}{\hat{\otimes}}D)\to A\underset{B}{\overset{\Gamma}{\hat{\otimes}}}(C\underset{B}{\overset{\Gamma}{\hat{\otimes}}}D).$$

3.2.5. **Lemma.** i) $\ker\Phi_{A,C,D}$ *and* $\ker\Psi_{A,C,D}$ *are generated as ideals by* $t_{A,C}(B)\otimes 1_D+t_{\left(A\overset{\Gamma}{\hat{\otimes}}C,D\right)}(B)$ *and* $1_A\otimes t_{C,D}(B)+t_{A,\left(C\overset{\Gamma}{\hat{\otimes}}D\right)}(B)$*, respectively.*

ii) *There exists a unique isomorphism of* (B,Γ)*-*C^**-algebras* $\tilde{a}_{A,C,D}\colon(A\underset{B}{\overset{\Gamma}{\hat{\otimes}}}C)\underset{B}{\overset{\Gamma}{\hat{\otimes}}}D\to A\underset{B}{\overset{\Gamma}{\hat{\otimes}}}(C\underset{B}{\overset{\Gamma}{\hat{\otimes}}}D)$ *such that* $\tilde{a}_{A,C,D}\circ\Phi_{A,C,D}=\Psi_{A,C,D}\circ a_{A,C,D}$.

Proof. i) By Proposition 3.1.10, $\ker(q_{A,C} \overset{\Gamma}{\hat{\otimes}} \mathrm{id}_D) = (\ker q_{A,C}) \overset{\Gamma}{\hat{\otimes}} D = (t_{A,C}(B)) \overset{\Gamma}{\hat{\otimes}} D$, and $\ker q_{\left(A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} C\right),D}$ is generated as an ideal by $(q_{A,C} \overset{\Gamma}{\hat{\otimes}} \mathrm{id}_D)\big(t_{\left(A \overset{\Gamma}{\hat{\otimes}} C,D\right)}(B)\big)$. The assertion on $\Phi_{A,C,D}$ follows, and the assertion concerning $\Psi_{A,C,D}$ follows similarly.

ii) Using i), one easily verifies that $a_{A,C,D}(\ker \Phi_{A,C,D}) = \ker \Psi_{A,C,D}$. We thus get an isomorphism $\tilde{a}_{A,C,D}$ of C^*-algebras which is easily seen to be an isomorphism of (B, Γ)-C^*-algebras. □

3.2.6. **Proposition.** *The family $(\tilde{a}_{A,C,D})_{A,C,D}$ is a natural isomorphism from $(- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -) \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -$ to $- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} (- \underset{B}{\overset{\Gamma}{\hat{\otimes}}} -)$.*

Proof. By Lemma 3.2.5, we only need to check naturality which is straightforward. □

3.3. **Free dynamical quantum groups on the level of universal C^*-algebras.** Given the monoidal structure on the category of all (B, Γ)-C^*-algebras, the definitions in §2.1–§2.4 carry over as follows:

3.3.1. **Definition.** *A* compact (B, Γ)-Hopf C^*-algebroid *is a (B, Γ)-C^*-algebra A with a morphism $\Delta \colon A \to A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} A$ satisfying*

i) $(\Delta \underset{B}{\overset{\Gamma}{\hat{\otimes}}} \mathrm{id}) \circ \Delta = (\mathrm{id} \underset{B}{\overset{\Gamma}{\hat{\otimes}}} \Delta) \circ \Delta$ *(*coassociativity*),*

ii) $[\Delta(A)(1 \otimes A_{e,*})] = A \underset{B}{\overset{\Gamma}{\hat{\otimes}}} A = [(A_{*,e} \otimes 1)\Delta(A)]$*, where* $A_{e,*} = [\sum_\gamma A_{e,\gamma}] \subseteq A$ *and* $A_{*,e} = [\sum_\gamma A_{\gamma,e}] \subseteq A$ *(*cancellation*).*

A counit *for a compact (B, Γ)-Hopf C^*-algebroid (A, Δ) is a morphism $\epsilon \colon A \to B \hat{\rtimes} \Gamma$ of (B, Γ)-C^*-algebras satisfying $(\epsilon \underset{B}{\overset{\Gamma}{\hat{\otimes}}} \mathrm{id}) \circ \Delta = \mathrm{id}_A = (\mathrm{id} \underset{B}{\overset{\Gamma}{\hat{\otimes}}} \epsilon) \circ \Delta$. A* morphism *of compact (B, Γ)-Hopf C^*-algebroids (A, Δ_A) and (C, Δ_C) is a morphism $\pi \colon A \to C$ satisfying $\Delta_C \circ \pi = (\pi \underset{B}{\overset{\Gamma}{\hat{\otimes}}} \pi) \circ \Delta_A$. We denote the category of all compact (B, Γ)-Hopf C^*-algebroids by $\mathbf{C^*}$-$\mathbf{Hopf}_{(B,\Gamma)}$.*

Denote by $\mathbf{Hopf}^0_{(B,\Gamma)}$ the full subcategory of $\mathbf{Hopf}^*_{(B,\Gamma)}$ formed by all (B, Γ)-Hopf $*$-algebroids (A, Δ, ϵ, S) where $A \in *$-$\mathbf{Alg}^0_{(B,\Gamma)}$.

3.3.2. **Proposition.** *Let $(A, \Delta, \epsilon, S) \in \mathbf{Hopf}^*_{(B,\Gamma)}$. Then Δ extends to a morphism of (B, Γ)-C^*-algebras $\Delta_{C^*(A)} \colon C^*(A) \to C^*(A) \underset{B}{\overset{\Gamma}{\hat{\otimes}}} C^*(A)$ such that $(C^*(A), \Delta_{C^*(A)})$ is a compact (B, Γ)-Hopf C^*-algebroid with the counit $C^*(\epsilon) \colon C^*(A) \to C^*(B \rtimes \Gamma) = B \hat{\rtimes} \Gamma$.*

Proof. The composition of Δ with the canonical map $A \tilde{\otimes} A \to C^*(A) \underset{B}{\overset{\Gamma}{\hat{\otimes}}} C^*(A)$ extends to a morphism $\Delta_{C^*(A)}$ by the universal property of $C^*(A)$. Coassociativity of Δ and density of A in $C^*(A)$ imply coassociativity of $\Delta_{C^*(A)}$, and cancellation follows from Remark 2.1.9 ii). □

The assignments $(A, \Delta, \epsilon, S) \mapsto (C^*(A), \Delta_{C^*(A)})$ and $\pi \mapsto C^*(\pi)$ evidently form a functor $\mathbf{Hopf}^0_{(B,\Gamma)} \to \mathbf{C^*\text{-}Hopf}_{(B,\Gamma)}$.

We now apply this functor to the free unitary and free orthogonal dynamical quantum groups $A^B_{\mathrm{u}}(\nabla, F)$ and $A^B_{\mathrm{o}}(\nabla, F, G)$ introduced in Definition 2.4.4, Theorem 2.4.5 and Definition 2.4.11, Theorem 2.4.12, respectively.

Let $\gamma_1, \dots, \gamma_n \in \Gamma$ and $\nabla = \mathrm{diag}(\gamma_1, \dots, \gamma_n) \in M_n(B \rtimes \Gamma)$.

Assume that $F \in \mathrm{GL}_n(B)$ be ∇-even in the sense that $\nabla F \nabla^{-1} \in M_n(B)$. Then the (B,Γ)-Hopf $*$-algebroid $A^B_{\mathrm{u}}(\nabla, F)$ is generated by a copy of $B \otimes B$ and entries of a unitary matrix $v \in M_n(A^B_{\mathrm{u}}(\nabla, F))$ and therefore has an enveloping C^*-algebra. Applying the functor $C^*(-)$ and unraveling the definitions, we find:

3.3.3. **Corollary.** *$C^*(A^B_{\mathrm{u}}(\nabla, F))$ is the universal C^*-algebra generated by a inclusion $r \times s$ of $B \otimes B$ and by the entries of a unitary $n \times n$-matrix v subject to the relations*

i) *$v_{ij} r(b) = r(\gamma_i(b)) v_{ij}$ and $v_{ij} s(b) = s(\gamma_j(b)) v_{ij}$ for all i, j and $b \in B$,*
ii) *$v^{-\mathsf{T}} = \bar{v}$ is invertible and $r_n(\nabla F \nabla^{-1}) \bar{v}^{-\mathsf{T}} = v s_n(F)$.*

It has the structure of a compact (B,Γ)-Hopf C^*-algebroid with counit, where for all i, j,

$$\delta(v_{ij}) = \gamma_i \otimes v_{ij}, \quad \bar{\delta}(v_{ij}) = v_{ij} \otimes \gamma_j, \quad \Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}, \quad \epsilon(v_{ij}) = \delta_{i,j}. \tag{12}$$

Let $F, G \in \mathrm{GL}_n(B)$ be ∇-odd in the sense that $\nabla F \nabla, \nabla G \nabla \in M_n(B)$, and assume that $GF^* = FG^*$. If $G^{-1}F = \lambda \bar{H} H^{\mathsf{T}}$ for some $\lambda \in \mathbb{C}$ and some ∇-even $H \in \mathrm{GL}_n(B)$, then $A^B_{\mathrm{o}}(\nabla, F, G)$ is generated by a copy of $B \otimes B$ and entries of a unitary matrix $u \in M_n(A^B_{\mathrm{o}}(\nabla, F, G))$ by Remark 2.4.13 iii), and therefore has an enveloping C^*-algebra.

3.3.4. **Corollary.** *$C^*(A^B_{\mathrm{o}}(\nabla, F, G))$ is is the universal C^*-algebra generated by a inclusion $r \times s$ of $B \otimes B$ and by the entries of an invertible $n \times n$-matrix v subject to the relations*

i) *$v_{ij} r(b) = r(\gamma_i(b)) v_{ij}$ and $v_{ij} s(b) = s(\gamma_j(b)) v_{ij}$ for all i, j and $b \in B$,*
ii) *$r_n(\nabla F \nabla) v^{-\mathsf{T}} = v s_n(F)$ and $r_n(\nabla G \nabla) \bar{v} = v s_n(G)$.*

It carries the structure of a compact (B,Γ)-Hopf C^*-algebroid with counit such that (12) *holds.*

Acknowledgments. I thank Erik Koelink for introducing me to dynamical quantum groups and for stimulating discussions, and the referee for helpful suggestions.

References

[1] S. Baaj and G. Skandalis. C^*-algèbres de Hopf et théorie de Kasparov équivariante. *K-Theory*, 2(6):683–721, 1989.

[2] T. Banica. Théorie des représentations du groupe quantique compact libre O(n). *C. R. Acad. Sci. Paris Sér. I Math.*, 322(3):241–244, 1996.

[3] T. Banica. Le groupe quantique compact libre U(n). *Comm. Math. Phys.*, 190(1):143–172, 1997.

[4] G. Böhm. Hopf algebroids. In *Handbook of algebra. Vol. 6*, volume 6 of *Handb. Algebr.*, pages 173–235. Elsevier/North-Holland, Amsterdam, 2009.

[5] N. P. Brown and N. Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[6] B. Day and R. Street. Quantum categories, star autonomy, and quantum groupoids. In *Galois theory, Hopf algebras, and semiabelian categories*, volume 43 of *Fields Inst. Commun.*, pages 187–225. Amer. Math. Soc., Providence, RI, 2004.

[7] P. Etingof and F. Latour. *The dynamical Yang-Baxter equation, representation theory, and quantum integrable systems*, volume 29 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2005.
[8] P. Etingof and A. Varchenko. Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups. *Comm. Math. Phys.*, 196(3):591–640, 1998.
[9] P. Etingof and A. Varchenko. Exchange dynamical quantum groups. *Comm. Math. Phys.*, 205(1):19–52, 1999.
[10] L. Kadison. Pseudo-Galois extensions and Hopf algebroids. In *Modules and comodules*, Trends Math., pages 247–264. Birkhäuser Verlag, Basel, 2008.
[11] E. Koelink and H. Rosengren. Harmonic analysis on the SU(2) dynamical quantum group. *Acta Appl. Math.*, 69(2):163–220, 2001.
[12] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
[13] M. Maghfoul. C^*-algèbre de Woronowicz et moyenne de Haar. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(4):393–396, 1997.
[14] D. Nikshych and L. Vainerman. Finite quantum groupoids and their applications. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 211–262. Cambridge University Press, Cambridge, 2002.
[15] T. Timmermann. Measured quantum groupoids associated to proper dynamical quantum groups. arXiv:1206.6744.
[16] T. Timmermann. *An invitation to quantum groups and duality*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.
[17] A. van Daele and S. Wang. Universal quantum groups. *Internat. J. Math.*, 7(2):255–263, 1996.
[18] S. Wang. *General constructions of compact quantum groups*. PhD thesis, University of California, Berkeley, 1993.
[19] S. Wang. Structure and isomorphism classification of compact quantum groups $A_u(Q)$ and $B_u(Q)$. *J. Operator Theory*, 48(3, suppl.):573–583, 2002.
[20] S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111(4):613–665, 1987.
[21] S. L. Woronowicz. Twisted SU(2) group. An example of a noncommutative differential calculus. *Publ. Res. Inst. Math. Sci.*, 23(1):117–181, 1987.

T. TIMMERMANN, University of Muenster, Einsteinstr. 62, 48149 Muenster, Germany
E-mail address: timmermt@math.uni-muenster.de

APPENDIX II.2

MEASURED QUANTUM GROUPOIDS ASSOCIATED TO PROPER DYNAMICAL QUANTUM GROUPS

THOMAS TIMMERMANN

preliminarily accepted by J. Noncommut. Geom.

Abstract. Dynamical quantum groups were introduced by Etingof and Varchenko in connection with the dynamical quantum Yang-Baxter equation, and measured quantum groupoids were introduced by Enock, Lesieur and Vallin in their study of inclusions of type II_1 factors. In this article, we associate to suitable dynamical quantum groups, which are a purely algebraic objects, Hopf C^*-bimodules and measured quantum groupoids on the level of von Neumann algebras. Assuming invariant integrals on the dynamical quantum group, we first construct a fundamental unitary which yields Hopf bimodules on the level of C^*-algebras and von Neumann algebras. Next, we assume properness of the dynamical quantum group and lift the integrals to the operator algebras. In a subsequent article, this construction shall be applied to the dynamical $\mathrm{SU}_q(2)$ studied by Koelink and Rosengren.

Contents

2010 *Mathematics Subject Classification.* Primary 46L99; Secondary 81R50, 20G42, 16T25.
Key words and phrases. dynamical quantum group, Hopf algebroid, quantum groupoid.
Supported by the SFB 878 "Groups, geometry and actions" funded by the DFG.

Introduction

Dynamical quantum groups were introduced by Etingof and Varchenko as an algebraic framework for the study of the dynamical quantum Yang-Baxter equation [6, 7, 8], a variant of the Yang-Baxter equation arising in statistical mechanics. Every (rigid) solution of this equation has a naturally associated tensor category of representations which turns out to be equivalent to the category of representations of some dynamical quantum group. In the case of the basic rational or basic trigonometric solution, this dynamical quantum group can be regarded as a quantization of the function algebra on some Poisson-Lie-groupoid. In general, it can be regarded as a quantum groupoid and fits into the theory of Hopf algebroids developed by Böhm and others [1].

Measured quantum groupoids were introduced by Enock, Lesieur and Vallin [2, 13] to capture generalized Galois symmetries of certain inclusions of type II_1 factors [3, 4, 15]. Apart from this fundamental example in von Neumann algebra theory, which was also considered in the algebraic setting [9, 19], and from the finite case, only few measured quantum groupoids have been constructed and investigated yet [13, 29].

Up to now, connections between algebraic and operator-algebraic approaches to quantum groupoids have only been explored in the finite case [14, 17, 28] and in the form of a few examples and constructions that exist on both levels. The situation is very different in the area of quantum groups, where Woronowicz's theory of compact quantum groups [35] and van Daele's theory of multiplier Hopf algebras with integrals [12, 31] form a bridge between the algebraic and operator-algebraic approaches, combining the computational convenience of the former with the power and richness of the latter.

Another approach to quantum groupoids which is equivalent to the algebraic and operator algebraic one, at least in finite dimensions, is via fusion categories [5, 19].

In this article, we associate to suitable dynamical quantum groups, which are purely algebraic objects, Hopf C^*-bimodules and measured quantum groupoids on the level of von Neumann algebras. The main example of a dynamical group we have in mind for application is the dynamical $\mathrm{SU}_q(2)$ studied by Koelink and Rosengren [10], and in a subsequent article, we want to study the construction for this example in detail.

On the dynamical quantum groups, we have to impose several assumptions.

First, we need a left- and a right-invariant integral, which correspond to fiber-wise integration on a groupoid, and a weight on the basis that is suitably quasi-invariant, such that the resulting total integrals are faithful, positive, and coincide. In the case of the dynamical $\mathrm{SU}_q(2)$, the left- and right-invariant integrals can be obtained from a Peter-Weyl decomposition due to Koelink and Rosengren [10], while the quasi-invariant weight on the basis can be chosen quite freely.

Second, we assume the dynamical quantum group to be proper, which is the natural analogue of compactness and unitality for quantum groupoids, and to possess a specific approximate unit in the base algebra. The dynamical $\mathrm{SU}_q(2)$ mentioned above even is compact and thus satisfies this second assumption.

In particular, the dynamical quantum group need not be a Hopf algebroid, but only a multiplier Hopf algebroid in the sense of [25]. The latter are closely related to the weak multiplier Hopf algebras that were recently introduced by Van Daele and Wang [33, 34].

Third, we assume that the quasi-invariant weight on the basis admits a bounded GNS-construction. Like the first condition, this one is very natural. In the case of the dynamical $\mathrm{SU}_q(2)$, the base algebra is formed by all meromorphic functions on the plane and does not admit any non-trivial bounded representations. To apply our construction, one therefore has to change the base and check that the Peter-Weyl decomposition persists.

Given these assumptions, the measured quantum groupoid is constructed as follows.

The algebraic GNS-construction, applied to the total integral on the dynamical quantum group, yields a Hilbert space of square-integrable functions on the dynamical quantum group together with a natural representation by densely defined multiplication operators. To obtain a C^*-algebra or von Neumann algebra, one has to show that these multiplication operators are bounded. To prove this and to lift the comultiplication to the resulting C^*-algebra and von Neumann algebra, we proceed as in the case of quantum groups [23] and construct a fundamental unitary which is pseudo-multiplicative on the level of von Neumann algebras and C^*-algebras in the sense of [27] and [24], respectively. The general theory of these unitaries then yields completions of the dynamical quantum group in the form a Hopf C^*-bimodule and a Hopf von-Neumann bimodule, and simultaneously a Pontrjagin dual in the same form. Finally, we extend the invariant integrals to the level of operator algebras, using properness of the dynamical quantum group and standard von Neumann algebra techniques.

This article is organized as follows.

Section 1 provides the algebraic basics on dynamical quantum groups and integration that are needed for the construction in Section 2. We first generalize the definition of a dynamical quantum group or $\mathfrak{h}$-Hopf algebroid, allowing the base to be non-unital, then consider left- and right-invariant integrals on the total algebra and quasi-invariant weights on the basis, and finally construct a $*$-algebra related to the Pontrjagin dual. The main result of this section is the existence of a modular automorphism for the total integral, which follows from a strong invariance property similarly as in the setting of multiplier Hopf algebras [31].

Section 2 presents the construction of the measured quantum groupoid outlined above. It uses Connes spatial theory, in particular the relative tensor product of Hilbert modules, and the C^*-algebraic analogue of that construction [22], and introduces the necessary concepts along the way when they are needed.

We use standard notation and adopt the following conventions. All algebras will be over the ground field $\mathbb{C}$ and we do not assume the existence of a unit element. Given a vector space V with a subset $X \subseteq V$, we denote by $\langle X \rangle \subseteq V$ the linear span and, if V is normed, by $[X] \subseteq V$ the closed linear span of X. Inner products on Hilbert spaces will be linear in the second and anti-linear in the first variable.

1. Dynamical quantum groups with integrals on the algebraic level

This section summarizes and develops the basics on dynamical quantum groups and integration used in this article. Before turning to details, let us outline the main concepts.

A dynamical quantum group is a special quantum groupoid and as such consists of an algebra B called the basis, an algebra A, an embedding $r\colon B \to A$ and an anti-homomorphic embedding $s\colon B \to A$ whose images commute, and a comultiplication, antipode and counit. What makes it special is that the basis B is commutative, that $r(B)$ and $s(B)$ are central in A up to a twist which is controlled by an action of a group Γ on B and a bigrading of A by Γ, and that the target of the comultiplication is a well-behaved monoidal product $A\tilde{\otimes}A$.

Integration on a quantum groupoid involves several ingredients. The analogue of the left- or right-invariance property of Haar measures on groups, Haar systems on groupoids, and Haar weights on quantum groups can be formulated for conditional expectations from A to $r(B)$ or $s(B)$, respectively. To obtain a total integration on A, such a partial integral has to be composed with a suitable functional on B that is quasi-invariant with respect to the action of Γ.

Let us now turn to details. We proceed as follows.

From the beginning, we assume all our algebras to possess an involution but not necessarily a unit. We first recall terminology concerning non-unital algebras (§1.1), then describe the monoidal product $A\tilde{\otimes}A$ (§1.2), and define dynamical quantum groups or, more precisely, multiplier (B,Γ)-Hopf $*$-algebroids (§1.3). Afterwards, we introduce and study integrals (§1.4–§1.6) and prove the existence of a modular automorphism that controls the deviation of the total integral from being a trace. Using integration, we finally construct the dual $*$-algebra of a multiplier (B,Γ)-Hopf $*$-algebroid (§1.7).

1.1. **Preliminaries on non-unital algebras.** To handle non-unital algebras, we use extra non-degeneracy assumptions and multiplier algebras [30, appendix] which are recalled below.

Let R be an algebra, not necessarily unital. Given a left R-module M, we say that R *has local units for* M if for each finite subset $F \subseteq M$, there exists some $r \in R$ such that $rm = m$ for all $m \in F$ [32]. The corresponding notion for right R-modules is defined similarly. We say that R *has local units* if it has local units for R, regarded as a left and as a right R-module.

Let R and S be algebras with local units, let N be an R-S-bimodule and assume that R and S have local units for N. A *multiplier* of N is a pair $T = (T_\rho, T_\lambda)$, where $T_\rho\colon R \to N$ is a left R-module map and $T_\lambda\colon S \to N$ a right S-module map satisfying $T_\rho(r)s = rT_\lambda(s)$ for all $r \in R, s \in S$. Given such a multiplier, we write $rT := T_\rho(r)$ and $Ts := T_\lambda(s)$ for all $r \in R$, $s \in S$. We denote the set of all multipliers of N by $M(N)$. Clearly, N embeds into $M(N)$ and $M(N)$ carries a natural structure of an R-S-bimodule that is compatible with this embedding.

Regarding R as an R-R-bimodule, $M(R)$ becomes an algebra via $TT' = (T'_\rho \circ T_\rho, T_\lambda \circ T'_\lambda)$, and R embeds into $M(R)$ as an essential ideal. If R is a $*$-algebra, then so is $M(R)$, where the adjoint of a multiplier $T = (T_\rho, T_\lambda) \in M(R)$ is the pair $T^* = (T^*_\rho, T^*_\lambda)$ given by $T^*_\rho(r) = (T_\lambda(r^*))^*$ and $T^*_\lambda(r) = (T_\rho(r^*))^*$ for all $r \in R$.

The bimodule N is an $M(R)$-$M(S)$-bimodule via $T(rns)T' := T_\lambda(r)nT'_\rho(s)$ for all $T \in M(R), r \in R, n \in N, s \in S, T' \in M(S)$, and $M(N)$ is an $M(R)$-$M(S)$-bimodule via $TT'T'' := (T''_\rho \circ T'_\rho \circ T_\rho, T_\lambda \circ T'_\lambda \circ T''_\lambda)$ for all $T \in M(R), T' \in M(N), T'' \in M(S)$.

A homomorphism $\pi\colon R \to M(S)$ is *non-degenerate* if $\langle \pi(R)S\rangle = S = \langle S\pi(R)\rangle$; in that case, it extends uniquely to a homomorphism $M(R) \to M(S)$ which is again denoted by π (see [30]).

1.2. **The category of** $(B,\Gamma)^{\mathrm{ev}}$**-algebras.** Let B be a commutative $*$-algebra with local units, let Γ be a group that acts on B on the left, and let $e \in \Gamma$ be the unit.

A (B,Γ)*-module* is a Γ-graded B-bimodule $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ for which B has local units, where each V_γ is a B-bimodule and $vb = \gamma(b)v$ for all $v \in V_\gamma, b \in B, \gamma \in \Gamma$. A *morphism* of (B,Γ)-modules V and W is a morphism of Γ-graded B-bimodules.

A (B,Γ)*-algebra* is a Γ-graded $*$-algebra $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ which has local units in A_e and is equipped with a $*$-homomorphism $B \to M(A)$ that turns A into a (B,Γ)-module. Such a (B,Γ)-algebra is *proper* if B maps into A.

Given a (B,Γ)-algebra A and $\gamma \in \Gamma$, we denote by $M(A)_\gamma \subseteq M(A)$ the space of all multipliers $T \in M(A)$ satisfying $TA_{\gamma'} \subseteq A_{\gamma\gamma'}$ and $A_{\gamma'}T \subseteq A_{\gamma'\gamma}$ for all $\gamma' \in \Gamma$.

A *morphism* of (B,Γ)-algebras A and C is a non-degenerate, B-linear $*$-homomorphism $\pi\colon A \to M(C)$ satisfying $\pi(A_\gamma) \subseteq M(C)_\gamma$ for all $\gamma \in \Gamma$. Such a morphism is *proper* if it maps A into C.

Using the extension of non-degenerate homomorphisms to multipliers, one defines the composition of morphisms and checks that (B,Γ)-algebras form a category.

The tensor product $B \otimes B$ is a $*$-algebra with local units and a natural action of $\Gamma \times \Gamma$. Replacing (B,Γ) by $(B,\Gamma)^{\mathrm{ev}} := (B \otimes B, \Gamma \times \Gamma)$ in the definition above, we obtain the category of all $(B,\Gamma)^{\mathrm{ev}}$*-algebras.*

Let A be a $(B,\Gamma)^{\mathrm{ev}}$-algebra. We call an element $x \in A$ *homogeneous* and write $\partial_x = \gamma$, $\bar\partial_x = \gamma'$ if $x \in A_{\gamma,\gamma'}$ for some $\gamma,\gamma' \in \Gamma$. Thus, $\partial_x\partial_y = \partial_{xy}$, $\bar\partial_x\bar\partial_y = \bar\partial_{xy}$ and $\partial_{x^*} = \partial_x^{-1}$, $\bar\partial_{x^*} = \bar\partial_x^{-1}$ for all homogeneous $x, y \in A$. Define $r = r_A\colon B \to M(A)$ and $s = s_A\colon B \to M(A)$ by $r(b)a = (b \otimes 1)a$ and $s(b)a = (1 \otimes b)a$ for all $a \in A$, $b \in B$. We write ${}_rA, A_r, {}_sA, A_s$ if we consider A as a B-module via left or right multiplication via r or s, respectively.

Clearly, B is a (B,Γ)-algebra and $B \otimes B$ is a $(B,\Gamma)^{\mathrm{ev}}$-algebra with respect to the trivial gradings. Every (B,Γ)-algebra A can be regarded as a $(B,\Gamma)^{\mathrm{ev}}$-algebra, where $A_{(\gamma,\gamma)} = A_\gamma$ and $A_{(\gamma,\gamma')} = 0$ whenever $\gamma \neq \gamma'$, and $(b \otimes b')a = bb'a$ for all $b, b' \in B$, $a \in A$. Conversely, every $(B,\Gamma)^{\mathrm{ev}}$-algebra A can be considered as a (B,Γ)-algebra via $r\colon B \to M(A)$ and the grading given by $A_\gamma := \bigoplus_{\gamma'} A_{\gamma,\gamma'}$, or via $s\colon B \to M(A)$ and the grading given by $A_{\gamma'} := \bigoplus_\gamma A_{\gamma,\gamma'}$. We write (A, r) and (A, s), respectively, to denote the resulting (B,Γ)-algebras.

Denote by $B \rtimes \Gamma$ the crossed product for the action of Γ on B, that is, the universal algebra containing B and Γ such that $e = 1_B$ and $b\gamma \cdot b'\gamma' = b\gamma(b')\gamma\gamma'$ for all $b, b' \in B$, $\gamma,\gamma' \in \Gamma$. This is a (B,Γ)-algebra with respect to the natural inclusion $B \to B \rtimes \Gamma$ and the involution and grading given by $(b\gamma)^* = \gamma^{-1}b^*$ and $(B \rtimes \Gamma)_\gamma = B\gamma$ for all $b \in B$, $\gamma \in \Gamma$.

The *fiber product* of $(B,\Gamma)^{\mathrm{ev}}$-algebras A and C is defined as follows. The subalgebra

$$A \overset{\Gamma}{\otimes} C := \sum_{\gamma,\gamma',\gamma''\in\Gamma} A_{\gamma,\gamma'} \otimes C_{\gamma',\gamma''} \subset A \otimes C$$

is a $(B,\Gamma)^{\mathrm{ev}}$-algebra, where $(A \overset{\Gamma}{\otimes} C)_{\gamma,\gamma''} = \sum_{\gamma'} A_{\gamma,\gamma'} \otimes C_{\gamma',\gamma''}$ for all $\gamma,\gamma'' \in \Gamma$ and $(r\times s)(b\otimes b') = r_A(b)\otimes s_C(b')$ for all $b,b' \in B$. Let $I \subseteq M(A \overset{\Gamma}{\otimes} C)$ be the ideal generated by $\{s_A(b)\otimes 1 - 1\otimes r_C(b) : b \in B\}$. Then the quotient

$$A\tilde{\otimes}C := A \overset{\Gamma}{\otimes} C/(I(A \overset{\Gamma}{\otimes} C))$$

is a $(B,\Gamma)^{\mathrm{ev}}$-algebra again, called the fiber product of A and C. Write $a\tilde{\otimes}c$ for the image of an element $a\otimes c$ in $A\tilde{\otimes}C$.

The assignment $(A,C) \mapsto A\tilde{\otimes}C$ is functorial, associative and unital. Indeed, for all morphisms of $(B,\Gamma)^{\mathrm{ev}}$-algebras $\pi^1\colon A^1 \to C^1$, $\pi^2\colon A^2 \to C^2$, there exists a morphism

$$\pi^1\tilde{\otimes}\pi^2\colon A^1\tilde{\otimes}A^2 \to C^1\tilde{\otimes}C^2, \quad a_1\tilde{\otimes}a_2 \mapsto \pi^1(a_1)\tilde{\otimes}\pi^2(a_2); \tag{1}$$

for all $(B,\Gamma)^{\mathrm{ev}}$-algebras A,C,D, there exists an isomorphism

$$(A\tilde{\otimes}C)\tilde{\otimes}D \to A\tilde{\otimes}(C\tilde{\otimes}D), \quad (a\tilde{\otimes}c)\tilde{\otimes}d \mapsto a\tilde{\otimes}(c\tilde{\otimes}d), \tag{2}$$

and for each $(B,\Gamma)^{\mathrm{ev}}$-algebra A, there exist isomorphisms

$$(B\rtimes\Gamma)\tilde{\otimes}A \to A,\ b\gamma\tilde{\otimes}a \mapsto r(b)a, \qquad A\tilde{\otimes}(B\rtimes\Gamma) \to A,\ a\tilde{\otimes}b\gamma \mapsto s(b)a. \tag{3}$$

These isomorphisms are compatible in a natural sense and endow the category of $(B,\Gamma)^{\mathrm{ev}}$-algebras with a monoidal structure. From now on, we shall use them without further notice.

The category of $(B,\Gamma)^{\mathrm{ev}}$-algebras carries automorphisms $(-)^{\mathrm{op}}$ and $(-)^{\mathrm{co}}$ such that for each (B,Γ)-algebra A and each morphism $\phi\colon A \to C$, we have $A^{\mathrm{co}} = A$ as an algebra, A^{op} is the opposite $*$-algebra of A, that is, the same vector space with the same involution and reversed multiplication, and

$$(A^{\mathrm{op}})_{\gamma,\gamma'} = A_{\gamma^{-1},\gamma'^{-1}} \text{ for all } \gamma,\gamma' \in \Gamma, \quad r_{A^{\mathrm{op}}} = r_A, \quad s_{A^{\mathrm{op}}} = s_A, \quad \phi^{\mathrm{op}} = \phi, \tag{4}$$

$$(A^{\mathrm{co}})_{\gamma,\gamma'} = A_{\gamma',\gamma} \text{ for all } \gamma,\gamma' \in \Gamma, \quad r_{A^{\mathrm{co}}} = s_A, \quad s_{A^{\mathrm{co}}} = r_A, \quad \phi^{\mathrm{co}} = \phi. \tag{5}$$

These automorphisms are involutive and commute, that is,

$$(-)^{\mathrm{op}} \circ (-)^{\mathrm{op}} = \mathrm{id}, \qquad (-)^{\mathrm{co}} \circ (-)^{\mathrm{co}} = \mathrm{id}, \qquad (-)^{\mathrm{op}} \circ (-)^{\mathrm{co}} = (-)^{\mathrm{co}} \circ (-)^{\mathrm{op}}.$$

Furthermore, they are compatible with the monoidal structure as follows. Given (B,Γ)-algebras A,C, there exist isomorphisms $(A\tilde{\otimes}C)^{\mathrm{op}} \to A^{\mathrm{op}}\tilde{\otimes}C^{\mathrm{op}}$ and $(A\tilde{\otimes}C)^{\mathrm{co}} \to C^{\mathrm{co}}\tilde{\otimes}A^{\mathrm{co}}$ given by $a\tilde{\otimes}c \mapsto a\tilde{\otimes}c$ and $a\tilde{\otimes}c \mapsto c\tilde{\otimes}a$, respectively. Moreover, $(B\rtimes\Gamma)^{\mathrm{co}} = B\rtimes\Gamma$, there exists an isomorphism $S^{B\rtimes\Gamma}\colon B\rtimes\Gamma \to (B\rtimes\Gamma)^{\mathrm{op}}$, $b\gamma \mapsto \gamma^{-1}b$, and all of these isomorphisms and the isomorphisms in (2) and (3) are compatible in a natural sense.

1.3. **Multiplier (B,Γ)-Hopf $*$-algebroids.** We shall work with variants of the $\mathfrak{h}$-Hopf algebroids and (B,Γ)-Hopf $*$-algebroids considered in [7, 10] and [21], respectively, where the basis need no longer be unital. These variants consist of a $(B,\Gamma)^{\mathrm{ev}}$-algebra and a comultiplication, counit and antipode, which will be introduced one after the other. To quickly proceed to the main part of this article, we postulate all the usual properties of these maps as axioms and leave a study of the axiomatics for later.

Given a $(B,\Gamma)^{\mathrm{ev}}$-algebra A, we denote by $\tilde{M}(A\tilde{\otimes}A) \subseteq M(A\tilde{\otimes}A)$ the set of all $T \in M(A\tilde{\otimes}A)$ for which all products of the form

$$T(x\tilde{\otimes}1_{M(A)}), \qquad (x\tilde{\otimes}1_{M(A)})T, \qquad T(1_{M(A)}\tilde{\otimes}y), \qquad (1_{M(A)}\tilde{\otimes}y)T$$

where $x \in A_{\gamma,e}, y \in A_{e,\gamma}, \gamma \in \Gamma$, lie in $A\tilde{\otimes}A$. Evidently, $\tilde{M}(A\tilde{\otimes}A)$ is a $*$-subalgebra of $M(A\tilde{\otimes}A)$.

1.3.1. **Definition.** A *comultiplication* on a $(B,\Gamma)^{\mathrm{ev}}$-algebra A is a morphism Δ from A to $A\tilde{\otimes}A$ satisfying $\Delta(A) \subseteq \tilde{M}(A\tilde{\otimes}A)$ and $(\Delta\tilde{\otimes}\mathrm{id})\circ\Delta = (\mathrm{id}\tilde{\otimes}\Delta)\circ\Delta$. A *(proper) multiplier* (B,Γ)*-$*$-bialgebroid* is a (proper) $(B,\Gamma)^{\mathrm{ev}}$-algebra with a comultiplication. A *morphism* of multiplier (B,Γ)-$*$-bialgebroids (A,Δ_A), (B,Δ_B) is a morphism ϕ from A to B satisfying $\Delta_B\circ\phi = (\phi\tilde{\otimes}\phi)\circ\Delta_A$.

Let (A,Δ) be a multiplier (B,Γ)-$*$-bialgebroid.

We shall need to form products of the form $\Delta(x)(1\otimes y)$ or $(y\otimes 1)\Delta(x)$ when $\partial_y \neq e$ or $\bar{\partial}_y \neq e$, respectively, which are defined as follows. Let $x,y \in A$. The multiplication on $A\otimes A$ induces a canonical $A\tilde{\otimes}A$-$A\otimes A$-bimodule structure on ${}_sA \underset{B}{\otimes} {}_rA$ and a canonical $A\otimes A$-$A\tilde{\otimes}A$-bimodule structure on $A_s \underset{B}{\otimes} A_r$. Using the natural maps ${}_sM(A)\underset{B}{\otimes}{}_rM(A) \to M({}_sA \underset{B}{\otimes} {}_rA)$ and $M(A)_s \underset{B}{\otimes} M(A)_r \to M(A_s \underset{B}{\otimes} A_r)$, we define multipliers $1 \underset{B}{\otimes} y, x \underset{B}{\otimes} 1 \in M({}_sA \underset{B}{\otimes} {}_rA)$ and $x \underset{B}{\otimes} 1, 1 \underset{B}{\otimes} y \in M(A_s \underset{B}{\otimes} A_r)$. Regarding $M({}_sA \underset{B}{\otimes} {}_rA)$ as an $M(A\tilde{\otimes}A)$-$M(A\otimes A)$-bimodule and $M(A_s \underset{B}{\otimes} A_r)$ as an $M(A\otimes A)$-$M(A\tilde{\otimes}A)$-bimodule (see §1.1), we can then multiply these multipliers with $\Delta(x)$ or $\Delta(y)$, respectively.

1.3.2. **Lemma.** *The following linear maps are well-defined:*

$$\begin{aligned}
T_1 &\colon A_s \underset{B}{\otimes} {}_sA \to {}_sA \underset{B}{\otimes} {}_rA, \ x \underset{B}{\otimes} y \mapsto \Delta(x)(1 \underset{B}{\otimes} y),\\
T_2 &\colon A_r \underset{B}{\otimes} {}_rA \to A_s \underset{B}{\otimes} A_r, \ x \underset{B}{\otimes} y \mapsto (x \underset{B}{\otimes} 1)\Delta(y),\\
T_3 &\colon {}_sA \underset{B}{\otimes} A_s \to A_s \underset{B}{\otimes} A_r, \ x \underset{B}{\otimes} y \mapsto (1 \underset{B}{\otimes} y)\Delta(x),\\
T_4 &\colon {}_rA \underset{B}{\otimes} A_r \to {}_sA \underset{B}{\otimes} {}_rA, \ x \underset{B}{\otimes} y \mapsto \Delta(y)(x \underset{B}{\otimes} 1).
\end{aligned}$$

Proof. We only prove the assertion concerning T_1, the cases of $T_2,\dots,T_4$ being similar. Using the explanations above, we obtain a linear map $A\otimes A \to M({}_sA \underset{B}{\otimes} {}_rA)$, $x\otimes y \mapsto \Delta(x)(1 \underset{B}{\otimes} y)$. This map factorizes through the quotient map $A\otimes A \to A_s \underset{B}{\otimes} {}_sA$ because $\Delta(xs(b)) = \Delta(x)(1\tilde{\otimes}s(b))$ for all $x \in A, b \in B$, and takes values in ${}_sA\underset{B}{\otimes}{}_rA$ because $\Delta(A)$ is contained in $\tilde{M}(A\tilde{\otimes}A)$. □

We adopt the Sweedler notation and write $\Delta(x) = \sum x_{(1)} \tilde{\otimes} x_{(2)}$ for $x \in A$. This notation requires extra care because $\Delta(x)$ need not lie in $A \tilde{\otimes} A$ but only in $\tilde{M}(A \tilde{\otimes} A)$, so that $x_{(1)}$ and $x_{(2)}$ do not simply represent elements of A. In this notation, the maps introduced above take the form

$$T_1 \colon x \underset{B}{\otimes} y \mapsto \sum x_{(1)} \underset{B}{\otimes} x_{(2)} y, \qquad T_2 \colon x \underset{B}{\otimes} y \mapsto \sum x y_{(1)} \underset{B}{\otimes} y_{(2)},$$
$$T_3 \colon x \underset{B}{\otimes} y \mapsto \sum x_{(1)} \underset{B}{\otimes} y x_{(2)}, \qquad T_4 \colon x \underset{B}{\otimes} y \mapsto \sum y_{(1)} x \underset{B}{\otimes} y_{(2)}.$$

We shall almost exclusively use the Sweedler notation for products as above. A detailed explanation of this notation in the context of multiplier Hopf algebras is given in [30, 32]. Apart from the fact that we use tensor products of B-modules instead of tensor products of vector spaces, this explanation carries over easily. As in the theory of (multiplier) Hopf algebras, we extend the Sweedler notation to iterated applications of Δ, writing

$$(\Delta \tilde{\otimes} \mathrm{id})(\Delta(x)) = \sum x_{(1)} \tilde{\otimes} x_{(2)} \tilde{\otimes} x_{(3)} = (\mathrm{id} \tilde{\otimes} \Delta)(\Delta(x))$$

for $x \in A$, and to iterated applications of the maps $T_1, \dots, T_4$, writing, for example,

$$(T_2 \underset{B}{\otimes} \mathrm{id})((\mathrm{id} \underset{B}{\otimes} T_1)(x \underset{B}{\otimes} y \underset{B}{\otimes} z)) = \sum x y_{(1)} \underset{B}{\otimes} y_{(2)} \underset{B}{\otimes} y_{(3)} z = (\mathrm{id} \underset{B}{\otimes} T_1)((T_2 \underset{B}{\otimes} \mathrm{id})(x \underset{B}{\otimes} y \underset{B}{\otimes} z))$$

for all $x, y, z \in A$.

1.3.3. **Definition.** A *counit* for a multiplier (B, Γ)-$*$-bialgebroid (A, Δ) is a proper morphism of $(B, \Gamma)^{\mathrm{ev}}$-algebras $\epsilon \colon A \to B \rtimes \Gamma$ satisfying $(\epsilon \tilde{\otimes} \mathrm{id}) \circ \Delta = \mathrm{id}_A = (\mathrm{id} \tilde{\otimes} \epsilon) \circ \Delta$.

Let (A, Δ) be a multiplier (B, Γ)-$*$-bialgebroid with counit ϵ. Using the linear maps

$$\sharp \colon B \rtimes \Gamma \to B, \ \sum_\gamma b_\gamma \gamma \mapsto \sum_\gamma b_\gamma, \qquad \flat \colon B \rtimes \Gamma \to B, \ \sum_\gamma \gamma b_\gamma \mapsto \sum_\gamma b_\gamma,$$

we define $\epsilon^\sharp, \epsilon^\flat \colon A \to B$ by $\epsilon^\sharp := \sharp \circ \epsilon$ and $\epsilon^\flat := \flat \circ \epsilon$. Define $m_r \colon A_r \underset{B}{\otimes} {}_rA \to A$ and $m_s \colon A_s \underset{B}{\otimes} {}_sA \to A$ by $\sum_i x_i \underset{B}{\otimes} y_i \mapsto \sum_i x_i y_i$.

1.3.4. **Remarks.**
i) Clearly, $\epsilon(A_{\gamma, \gamma'}) \subseteq (B \rtimes \Gamma)_{\gamma, \gamma'} = 0$ whenever $\gamma, \gamma' \in \Gamma$ and $\gamma \neq \gamma'$.
ii) If ϵ' is a counit as well, then $\epsilon = \epsilon \circ (\mathrm{id} \tilde{\otimes} \epsilon') \circ \Delta = \epsilon' \circ (\epsilon \tilde{\otimes} \mathrm{id}) \circ \Delta = \epsilon'$.
iii) The condition $(\epsilon \tilde{\otimes} \mathrm{id}) \circ \Delta = \mathrm{id}_A = (\mathrm{id} \tilde{\otimes} \epsilon) \circ \Delta$ is equivalent to the relations

$$\sum r(\epsilon^\sharp(x_{(1)})) x_{(2)} y = xy = \sum x y_{(1)} s(\epsilon^\flat(y_{(2)})) \quad \text{for all } x, y \in A,$$

and hence to commutativity of the diagrams

$$\begin{tikzcd} & {}_sA \underset{B}{\otimes} {}_rA \arrow[dr, "\epsilon^\sharp \underset{B}{\otimes} \mathrm{id}"] & \\ A_s \underset{B}{\otimes} {}_sA \arrow[ur, "T_1"] \arrow[rr, "m_s"'] & & A, \end{tikzcd} \qquad \begin{tikzcd} & A_s \underset{B}{\otimes} A_r \arrow[dr, "\mathrm{id} \underset{B}{\otimes} \epsilon^\flat"] & \\ A_r \underset{B}{\otimes} {}_rA \arrow[ur, "T_2"] \arrow[rr, "m_r"'] & & A. \end{tikzcd}$$

Furthermore, this condition is equivalent to the relations

$$\sum x y_{(2)} r(\epsilon^\flat(y_{(1)})) = xy = \sum s(\epsilon^\sharp(x_{(2)})) x_{(1)} y \quad \text{for all } x, y \in A.$$

The definition of the antipode involves the isomorphism

$$\sigma_{A,A} \colon (A\tilde{\otimes}A)^{\mathrm{co,op}} \to A^{\mathrm{co,op}}\tilde{\otimes}A^{\mathrm{co,op}}, \quad x\tilde{\otimes}y \mapsto y\tilde{\otimes}x.$$

1.3.5. **Definition.** An *antipode* for a multiplier (B,Γ)-$*$-bialgebroid (A,Δ) with counit ϵ is an isomorphism $S\colon A \to A^{\mathrm{co,op}}$ of $(B,\Gamma)^{\mathrm{ev}}$-algebras that makes the following diagrams commute:

$$\begin{tikzcd} A_s \underset{B}{\otimes} {}_sA \arrow[r, "T_1"] \arrow[d, "\epsilon^\flat \underset{B}{\otimes} \mathrm{id}"'] & {}_sA \underset{B}{\otimes} {}_rA \arrow[d, "S \underset{B}{\otimes} \mathrm{id}"] \\ A & A_r \underset{B}{\otimes} {}_rA \arrow[l, "m_r"] \end{tikzcd}, \qquad \begin{tikzcd} A_r \underset{B}{\otimes} {}_rA \arrow[r, "T_2"] \arrow[d, "\mathrm{id} \underset{B}{\otimes} \epsilon^\sharp"'] & A_s \underset{B}{\otimes} A_r \arrow[d, "\mathrm{id} \underset{B}{\otimes} S"] \\ A & A_s \underset{B}{\otimes} {}_sA \arrow[l, "m_s"] \end{tikzcd},$$

$$\begin{tikzcd} A \arrow[rr, "S"] \arrow[d, "\Delta"'] & & A^{\mathrm{co,op}} \arrow[d, "\Delta^{\mathrm{co,op}}"] \\ A\tilde{\otimes}A \arrow[r, "S\tilde{\otimes}S"] & A^{\mathrm{co,op}}\tilde{\otimes}A^{\mathrm{co,op}} & (A\tilde{\otimes}A)^{\mathrm{co,op}}. \arrow[l, "\sigma_{A,A}"'] \end{tikzcd}$$

A *multiplier* (B,Γ)*-Hopf* $*$*-algebroid* is a multiplier (B,Γ)-$*$-bialgebroid with counit and antipode.

1.3.6. **Examples.** i) The tensor product $B\otimes B$ is a multiplier (B,Γ)-Hopf $*$-algebroid, where $\Delta(b\otimes b') = (b\otimes 1)\tilde{\otimes}(1\otimes b')$, $\epsilon(b\otimes b') = bb'$, $S(b\otimes b') = b'\otimes b$ for all $b, b' \in B$.

ii) The crossed product $B \rtimes \Gamma$ is a multiplier (B,Γ)-Hopf $*$-algebroid, where $\Delta(b\gamma) = b\gamma\tilde{\otimes}\gamma = \gamma\tilde{\otimes}b\gamma$, $\epsilon = \mathrm{id}$ and $S(\gamma b) = b\gamma^{-1}$ for all $b \in B, \gamma \in \Gamma$.

Given an antipode S on a multiplier (B,Γ)-$*$-bialgebroid (A,Δ) and an element $a \in A$, we shall henceforth always regard $S(a)$ as an element of A and not of $A^{\mathrm{co,op}}$.

1.3.7. **Remarks.** Let (A,Δ,ϵ,S) be a multiplier (B,Γ)-Hopf $*$-algebroid.

i) In Sweedler notation, commutativity of the diagrams in Definition 1.3.5 amount to

$$\sum S(x_{(1)})x_{(2)}y = s(\epsilon^\flat(x))y, \quad \sum xy_{(1)}S(y_{(2)}) = xr(\epsilon^\sharp(y)) \quad \text{for all } x, y \in A, \tag{6}$$

$$\sum S(x_{(1)})\tilde{\otimes}S(x_{(2)}) = \sum S(x)_{(2)}\tilde{\otimes}S(x)_{(1)} \quad \text{for all } x \in A. \tag{7}$$

ii) If S' is an antipode as well, then $S' = S$ because for all $x, y, z \in A$,

$$\begin{aligned} xS(y)z = S(yS^{-1}(x))z &= \sum S(s(\epsilon^\sharp(y_{(2)}))y_{(1)}S^{-1}(x))z \\ &= \sum S(y_{(2)}S^{-1}(x))r(\epsilon^\sharp(y_{(2)}))z \\ &= \sum S(y_{(1)}S^{-1}(x))y_{(2)}S'(S'^{-1}(z)y_{(3)}) = xS'(y)z. \end{aligned}$$

For every multiplier (B,Γ)-Hopf $*$-algebroid, the maps $T_1, \ldots, T_4$ defined above are bijections.

1.3.8. **Proposition.** *Let (A,Δ) be a multiplier (B,Γ)-$*$-bialgebroid. If (A,Δ) has a counit ϵ and an antipode S, then the maps $T_1, \ldots, T_4$ are bijective and for all $x, y \in A$,*

$$T_1^{-1}(x \underset{B}{\otimes} y) = \sum x_{(1)} \underset{B}{\otimes} S(S^{-1}(y)x_{(2)}), \qquad T_2^{-1}(x \underset{B}{\otimes} y) = \sum S(y_{(1)}S^{-1}(x)) \underset{B}{\otimes} y_{(2)},$$

$$T_3^{-1}(x \underset{B}{\otimes} y) = \sum x_{(1)} \underset{B}{\otimes} S^{-1}(x_{(2)}S(y)), \qquad T_4^{-1}(x \underset{B}{\otimes} y) = \sum S^{-1}(S(x)y_{(1)}) \underset{B}{\otimes} y_{(2)}.$$

Proof. We only prove the assertion concerning T_1. One first checks that the formula given for T_1^{-1} yields a well-defined map $T_1'\colon {}_sA \underset{B}{\otimes} {}_rA \to A_s \underset{B}{\otimes} {}_sA$, and then that for all $x, y \in A$ and $u, v \in A_{e,e}$,

$$\begin{aligned}
(u \otimes v) \cdot (T_1 \circ T_1')(x \underset{B}{\otimes} y) &= \sum ux_{(1)} \underset{B}{\otimes} vx_{(2)}S(S^{-1}(y)x_{(3)}) \\
&= \sum ux_{(1)} \underset{B}{\otimes} vx_{(2)}S(x_{(3)})y \\
&= \sum ux_{(1)} \underset{B}{\otimes} vr(\epsilon^{\sharp}(x_{(2)}))y \\
&= \sum us(\epsilon^{\sharp}(x_{(2)}))x_{(1)} \underset{B}{\otimes} vy = ux \underset{B}{\otimes} vy, \\
(u \otimes v) \cdot (T_1' \circ T_1)(x \underset{B}{\otimes} y) &= \sum ux_{(1)} \underset{B}{\otimes} vS(S^{-1}(x_{(3)}y)x_{(2)}) \\
&= \sum ux_{(1)} \underset{B}{\otimes} vS(x_{(2)})x_{(3)}y \\
&= \sum ux_{(1)} \underset{B}{\otimes} vs(\epsilon^{\flat}(x_{(2)}))y \\
&= \sum ux_{(1)}s(\epsilon^{\flat}(x_{(2)})) \underset{B}{\otimes} vy = ux \underset{B}{\otimes} vy. \qquad \square
\end{aligned}$$

1.4. **Bi-measured multiplier (B,Γ)-$*$-bialgebroids.** We now introduce the main objects of this article — multiplier (B,Γ)-Hopf $*$-algebroids equipped with certain integrals. In §2, we shall construct completions of such objects in the form of measured quantum groupoids.

As on a groupoid, integration on a multiplier (B,Γ)-$*$-bialgebroid (A,Δ) proceeds in stages. First, one needs partial integrals $\phi,\psi\colon A \to B$ with suitable left or right invariance properties, and second a suitable weight $\mu\colon B \to \mathbb{C}$ that is compatible with the action of Γ. The results in [10] suggest that dynamical quantum groups that are compact in a suitable sense even possess a bi-invariant integral $h\colon A \to B \otimes B$ that can be obtained from a Peter-Weyl decomposition of A.

We first focus on the weight μ and the bi-integral h, and discuss left and right integrals in the next subsection.

Let us briefly recall some terminology. Let C be a $*$-algebra with local units. A linear map $\mu\colon C \to \mathbb{C}$ is *faithful* if $\mu(Cc) = 0$ implies $c = 0$, and *positive* if $\mu(c^*c) \geqslant 0$ for all $c \in C$. Assume that μ is positive. Then μ is $*$-linear, because positivity of $\phi((b+c)^*(b+c))$ and $\phi((b+ic)^*(b+ic))$ implies $\mu(b^*c) = \overline{\phi(c^*b)}$ for all $b,c \in C$, and faithful as soon as $\mu(c^*c) \neq 0$ whenever $c \neq 0$.

1.4.1. **Definition.** A *weight* for (B,Γ) is a faithful, positive linear map $\mu\colon B \to \mathbb{C}$ that is *quasi-invariant with respect to* Γ in the sense that for each $\gamma \in \Gamma$, there exists some $D_\gamma \in M(B)$ such that $\mu(\gamma(bD_\gamma)) = \mu(b)$ for all $b \in B$.

1.4.2. **Remark.** Let μ be a weight for (B,Γ). Then

i) each D_γ is uniquely determined and self-adjoint,
ii) $D_{\gamma\gamma'} = \gamma'^{-1}(D_\gamma)D_{\gamma'}$ and $1 = \gamma^{-1}(D_{\gamma^{-1}})D_\gamma$ for all $\gamma,\gamma' \in \Gamma$,
iii) $\mu(\gamma^{-1}(b)c) = \mu(b\gamma(c)D_{\gamma^{-1}}^{-1}))) = \mu(b\gamma(cD_\gamma))$ for all $b,c \in B, \gamma \in \Gamma$.

Indeed, i) and ii) follow easily from the fact that μ is faithful and the relations $\mu(\gamma(bD_\gamma^*)) = \overline{\mu(\gamma(D_\gamma b^*))} = \overline{\mu(b^*)} = \mu(b)$ and $\mu(\gamma(\gamma'(bD_{\gamma\gamma'}))) = \mu(b) = \mu(\gamma'(bD_{\gamma'})) = \mu(\gamma(\gamma'(bD_{\gamma'})D_\gamma))$.

We henceforth call the family $(D_\gamma)_{\gamma\in\Gamma}$ the *Radon-Nikodym cocycle* of μ.

The following definition is inspired by the notion of a Haar functional introduced in [10].

1.4.3. **Definition.** A *bi-integral* on (A,Δ) is a morphism of $(B,\Gamma)^{\mathrm{ev}}$-modules $h\colon A \to B\otimes B$ satisfying $\Delta(\ker h)(1\tilde{\otimes}A_{e,e}) \subseteq \ker h\tilde{\otimes}A$ and $\Delta(\ker h)(A_{e,e}\tilde{\otimes}1) \subseteq A\tilde{\otimes}\ker h$. If (A,Δ) is proper and $h(r(b)s(b')) = b\otimes b'$ for all $b,b'\in B$, we call such a bi-integral *normalized.*

1.4.4. **Lemma.** *Let (A,Δ) be proper and let h be a normalized bi-integral on (A,Δ).*

i) $(\mathrm{id}\tilde{\otimes}m_B\circ h)\circ\Delta = h = (m_B\circ h\tilde{\otimes}\mathrm{id})\circ\Delta$, *where $m_B\colon B\otimes B\to B$ denotes the multiplication.*

ii) *If h' is a normalized bi-integral on (A,Δ), then $h'=h$.*

iii) *If (A,Δ,ϵ,S) is a proper multiplier (B,Γ)-Hopf $*$-algebroid, then $h\circ S = \sigma_B\circ h$, where $\sigma_B\colon B\otimes B\to B\otimes B$ denotes the flip $b\otimes c\mapsto c\otimes b$.*

Proof. i) We only prove the first equation. Let $\omega\colon (A,r)\to B$ be a morphism of (B,Γ)-modules sending $I := \ker h$ to 0. Then

$$(\mathrm{id}\tilde{\otimes}\omega)(\Delta(I))A_{e,e} = (\mathrm{id}\underset{B}{\otimes}\omega)(\Delta(I)(A_{e,e}\tilde{\otimes}1)) \subseteq (\mathrm{id}\underset{B}{\otimes}\omega)(A\underset{B}{\otimes}I) = 0$$

and hence $(\mathrm{id}\tilde{\otimes}\omega)(\Delta(I)) = 0$. Moreover, if $b,b',b''\in B$ and $u\in A_{e,e}$, then

$$(\mathrm{id}\tilde{\otimes}\omega)(\Delta(r(b)s(b')))s(b'')u = (\mathrm{id}\underset{B}{\otimes}\omega)(r(b)s(b'')u\underset{B}{\otimes}s(b')) = r(b)s(\omega(s(b')r(b'')))u.$$

For $\omega = m_B\circ h$, these calculations imply for all $a\in I$ and $b,b'\in B$

$$(\mathrm{id}\tilde{\otimes}m_B\circ h)(\Delta(a)) = 0 = h(a), \quad (\mathrm{id}\tilde{\otimes}m_B\circ h)(\Delta(r(b)s(b'))) = r(b)s(b') = h(r(b)s(b')).$$

Since $A = I + r(B)s(B)$, we can conclude $(\mathrm{id}\tilde{\otimes}m_B\circ h)\circ\Delta = h$.

ii) Let $x\in\ker h$ and choose $u,u'\in B\otimes B$ such that $u(1\otimes m_B(u'))h'(x) = h'(x)$. Then

$$h'(x) = h(uh'(x)s(m_B(u'))) = \sum h(ux_{(1)}s(m_B(h'(x_{(2)}u')))) = 0$$

because $\sum ux_{(1)}\otimes x_{(2)}u' \in u(\ker h)\underset{B}{\otimes}A$. Thus, $\ker h\subseteq\ker h'$. Since h and h' are normalized and $\ker h + B\otimes B = A$, we can conclude $h=h'$.

iii) One easily verifies that $\sigma_B\circ h\circ S$ is a normalized bi-integral. By ii), it equals h. □

1.4.5. **Definition.** A proper multiplier (B,Γ)-$*$-bialgebroid (A,Δ) is *bi-measured* if it is equipped with a normalized bi-integral $h\colon A\to B\otimes B$ and a weight μ for (B,Γ) such that $\nu := (\mu\otimes\mu)\circ h$ is faithful and positive.

1.4.6. **Remark.** Given a bi-measured proper multiplier (B,Γ)-Hopf $*$-algebroid as above, h is evidently faithful, and also $*$-linear. To see this, note that $(\mu\otimes\mu)(h(a^*)(b\otimes c)) = \nu(a^*r(b)s(c)) = \overline{\nu(s(c^*)r(b^*)a)} = \overline{(\mu\otimes\mu)((b\otimes c)^*h(a))} = (\mu\otimes\mu)(h(a)^*(b\otimes c))$ for all $a\in A, b,c\in B$.

1.5. **Left and right integrals.** For large parts of this article, the multiplier (B,Γ)-Hopf $*$-algebroids under consideration need not be equipped with a bi-integral, but only with left and right integrals ϕ,ψ. The definition of these integrals involves slice maps of the following form.

Let (A,Δ) be a multiplier (B,Γ)-$*$-bialgebroid and let $\phi\colon (A,r)\to B$ be a morphism of (B,Γ)-modules. Then there exists a unique linear map $\mathrm{id}\tilde{\otimes}\phi\colon \tilde{M}(A\tilde{\otimes}A)\to M(A)$ such that

$$((\mathrm{id}\tilde{\otimes}\phi)(T))a=(\mathrm{id}\underset{B}{\otimes}\phi)(T(a\otimes 1)),\qquad a((\mathrm{id}\tilde{\otimes}\phi)(T))=(\mathrm{id}\underset{B}{\otimes}\phi)((a\otimes 1)T)$$

for all $T\in\tilde{M}(A\tilde{\otimes}A)$ and $a\in A$, where we regard $T(a\otimes 1)$ and $(a\otimes 1)T$ as elements of ${}_sA\underset{B}{\otimes}{}_rA$ and $A_s\underset{B}{\otimes}A_r$, respectively. In the case $T=\Delta(x)$ for some $x\in A$,

$$(\mathrm{id}\tilde{\otimes}\phi)(\Delta(x))a=\sum s(\phi(x_{(2)}))x_{(1)}a,\qquad a(\mathrm{id}\tilde{\otimes}\phi)(\Delta(x))=\sum ax_{(1)}s(\phi(x_{(2)})).\tag{8}$$

Likewise, every morphism $\psi\colon (A,s)\to B$ yields a slice map $\psi\tilde{\otimes}\,\mathrm{id}\colon \tilde{M}(A\tilde{\otimes}A)\to M(A)$.

1.5.1. **Definition.** A *left integral* on (A,Δ) is a morphism $\phi\colon (A,r)\to B$ satisfying $(\mathrm{id}\tilde{\otimes}\phi)\circ\Delta=r\circ\phi$. A *right integral* on (A,Δ) is a morphism $\psi\colon (A,s)\to B$ satisfying $(\psi\tilde{\otimes}\,\mathrm{id})\circ\Delta=s\circ\psi$.

1.5.2. **Remarks.** i) In Sweedler notation, the invariance conditions on ϕ and ψ become

$$\sum s(\phi(x_{(2)}))x_{(1)}a=r(\phi(x))a,\qquad \sum ax_{(2)}r(\psi(x_{(1)}))=as(\psi(x))\quad\text{for all } a,x\in A.$$

ii) If (A,Δ,ϵ,S) is a (B,Γ)-Hopf $*$-algebroid, then the map $\phi\mapsto\phi\circ S$ gives a bijection between left and right integrals on (A,Δ). This follows easily from (7).

iii) If ϕ is a left integral, then also $\phi(-s(b))$ is left integral for each $b\in B$. Likewise, if ψ is a right integral, then also $\psi(-r(b))$ is a right integral for each $b\in B$.

We shall frequently use the following strong invariance relations:

1.5.3. **Proposition.** *Assume that (A,Δ,ϵ,S) is a (B,Γ)-Hopf $*$-algebroid. Then*

i) $(\mathrm{id}\underset{B}{\otimes}\phi)((1\tilde{\otimes}z)\Delta(x))=S((\mathrm{id}\underset{B}{\otimes}\phi)(\Delta(z)(1\tilde{\otimes}x)))$ *for every left integral ϕ and all $x,z\in A$;*

ii) $(\psi\underset{B}{\otimes}\mathrm{id})(\Delta(x)(z\tilde{\otimes}1))=S((\psi\underset{B}{\otimes}\mathrm{id})((x\tilde{\otimes}1)\Delta(z))$ *for every right integral ψ and all $x,z\in A$.*

Proof. Using Sweedler notation, we calculate

$$\begin{aligned}\sum x_{(1)}s(\phi(zx_{(2)}))&=\sum x_{(1)}s(\phi(z_{(2)}r(\epsilon^\flat(z_{(1)}))x_{(2)}))\\&=\sum s(\epsilon^\flat(z_{(1)}))x_{(1)}s(\phi(z_{(2)}x_{(2)}))\\&=\sum S(z_{(1)})z_{(2)}x_{(1)}s(\phi(z_{(3)}x_{(2)}))=\sum S(z_{(1)})r(\phi(z_{(2)}x))\end{aligned}$$

and

$$\begin{aligned}\sum r(\psi(x_{(1)}z)x_{(2)} &= \sum r(\psi(x_{(1)}s(\epsilon^{\sharp}(z_{(2)}))z_{(1)}))x_{(2)} \\ &= \sum r(\psi(x_{(1)}z_{(1)}))x_{(2)}r(\epsilon^{\sharp}(z_{(2)})) \\ &= \sum r(\psi(x_{(1)}z_{(1)})x_{(2)}z_{(2)}S(z_{(3)}) = s(\psi(xz_{(1)}))S(z_{(2)}). \qquad \square\end{aligned}$$

Normalized bi-integrals yield left and right integrals as follows:

1.5.4. **Lemma.** *Assume that (A,Δ) is proper, h is a normalized bi-integral on (A,Δ), and $\mu\colon B\to\mathbb{C}$ is linear. Then $\phi := (\mathrm{id}\otimes\mu)\circ h$ and $\psi := (\mu\otimes\mathrm{id})\circ h$ are a left and a right integral, respectively, and $\phi\circ S^{\pm 1} = \psi$.*

Proof. Repeating the proof of Lemma 1.4.4 i) with $\omega := \phi = (\mathrm{id}\otimes\mu)\circ h$, we find

$$(\mathrm{id}\,\tilde{\otimes}\phi)(\Delta(a)) = 0 = r(\phi(a)), \qquad (\mathrm{id}\,\tilde{\otimes}\phi)(\Delta(r(b)s(b'))) = r(b\mu(b')) = \phi(r(b)s(b'))$$

for all $a\in\ker h$ and $b,b'\in B$. Since $A = (\ker h)+r(B)s(B)$, we can conclude $(\mathrm{id}\,\tilde{\otimes}\phi)\circ\Delta = r\circ\phi$. The assertion on ψ follows similarly, and the last equation follows from Lemma 1.4.4 iii). $\square$

1.6. **Measured multiplier (B,Γ)-$*$-bialgebroids.** Much of the ensuing material applies not only to bi-measured proper multiplier (B,Γ)-Hopf $*$-algebroids but also to the following class of objects.

1.6.1. **Definition.** A a multiplier (B,Γ)-$*$-bialgebroid (A,Δ) is *measured* if it is equipped with a left integral ϕ, a right integral ψ, and a weight μ for (B,Γ) such that $\nu := \mu\circ\phi$ and $\nu^{-1} := \mu\circ\psi$ are faithful, positive, and coincide, and $\psi(A) = B = \phi(A)$.

1.6.2. **Remarks.**
i) Given a measured multiplier (B,Γ)-Hopf $*$-algebroid as above, the maps ϕ and ψ are $*$-linear. This can be seen from a similar argument as in Remark 1.4.6.
ii) If $(A,\Delta,\epsilon,S,h,\mu)$ is a bi-measured proper multiplier (B,Γ)-Hopf $*$-algebroid and $\phi = (\mu\otimes\mathrm{id})\circ h$ and $\psi = (\mathrm{id}\otimes\mu)\circ h$, then $(A,\Delta,\epsilon,S,\phi,\psi,\mu)$ is a measured multiplier (B,Γ)-Hopf $*$-algebroid by Lemma 1.5.4. In that case, $\phi\circ S^{\pm 1} = \psi$ and $\nu\circ S = \nu$ by Lemma 1.4.4 iii).

Till the end of this subsection, let $(A,\Delta,\epsilon,S,\phi,\psi,\mu)$ be a measured multiplier (B,Γ)-Hopf $*$-algebroid and let $(D_\gamma)_\gamma$ be the Radon-Nikodym cocycle for μ. Define $D,\bar{D}\colon A\to A$ by

$$D(a) = r(D_{\partial_a^{-1}})a = ar(D^{-1}_{\partial_a}), \qquad \bar{D}(a) = s(D_{\bar{\partial}_a^{-1}})a = as(D^{-1}_{\bar{\partial}_a}) \tag{9}$$

for all homogeneous $a\in A$.

1.6.3. **Lemma.** *D and $\bar{D}$ both are algebra and $(B,\Gamma)^{\mathrm{ev}}$-module automorphisms of A, and satisfy*

$$(D\tilde{\otimes}\,\mathrm{id})\circ\Delta = \Delta\circ D, \quad (\mathrm{id}\,\tilde{\otimes}\bar{D})\circ\Delta = \Delta\circ\bar{D}, \quad (\bar{D}\tilde{\otimes}\,\mathrm{id})\circ\Delta = (\mathrm{id}\,\tilde{\otimes}D)\circ\Delta,$$

$$D\circ\bar{D} = \bar{D}\circ D, \quad S\circ D = \bar{D}^{-1}\circ S, \quad S\circ\bar{D} = D^{-1}\circ S,$$

$$*\circ D = D^{-1}\circ *, \quad *\circ\bar{D} = \bar{D}^{-1}\circ *.$$

Proof. The maps D and $\bar{D}$ are bijective because D_γ is invertible for each $\gamma \in \Gamma$. The remaining assertions follow from straightforward calculations, for example,

$$D(xy) = r(D_{\partial_{xy}^{-1}})xy = r(D_{\partial_x^{-1}}\partial_x(D_{\partial_y^{-1}}))xy = r(D_{\partial_x^{-1}})xr(D_{\partial_y^{-1}})y = D(x)D(y),$$
$$S(D(x)) = S(r(D_{\partial_x}^{-1})x) = S(x)s(D_{\bar{\partial}_{S(x)}}) = \bar{D}^{-1}(S(x)),$$
$$D(x)^* = x^*r(D^*_{\partial_x^{-1}}) = x^*r(D_{\partial_{x^*}}) = D^{-1}(x^*) \quad \text{for all } x, y \in A. \qquad \square$$

1.6.4. **Lemma.** *Let $\omega \in \{\phi, \psi, \nu\}$.*

i) $\omega(A_{\gamma,\gamma'}) = 0$ *whenever* $(\gamma, \gamma') \neq (e, e)$.
ii) $\omega(r(b)s(b')a) = \omega(ar(b)s(b'))$ *for all* $a \in A$, $b, b' \in B$.
iii) $\omega(D(a)a') = \omega(aD^{-1}(a'))$ *and* $\omega(\bar{D}(a)a') = \omega(a\bar{D}^{-1}(a'))$ *for all* $a, a' \in A$.

Proof. i) For $\omega = \nu$, the assertion follows from the relation $\ker\phi + \ker\psi \subseteq \ker\nu$. To obtain the assertion for $\omega = \phi, \psi$, use the fact that μ is faithful.

ii) Let $a \in A$ and $b, b' \in B$. Then $\nu(r(b)a) = \mu(b\phi(a)) = \mu(\phi(a)b) = \nu(ar(b))$ and similarly $\nu(s(b')a) = \nu(as(b'))$. To obtain the assertion for $\omega = \phi, \psi$, use the fact that μ is faithful again.

iii) This follows immediately equation (9) and i). $\square$

We shall now show that $\nu = \mu \circ \phi$ has a modular automorphism and thus satisfies an algebraic variant of the KMS-condition. Let us briefly recall this concept.

Let C be a $*$-algebra with local units and a faithful, positive, linear map $\omega\colon C \to \mathbb{C}$. A *modular automorphism* for ω is a bijection $\theta_\omega\colon C \to C$ satisfying $\omega(cc') = \omega(c'\theta_\omega(c))$ for all $c, c' \in C$. If it exists, a modular automorphism θ_ω for ω is uniquely determined, an algebra automorphism, and satisfies $\omega \circ \theta_\omega = \omega$ and $\theta_\omega \circ * \circ \theta_\omega \circ * = \mathrm{id}_C$. This follows easily from the relations

$$\omega(z\theta_\omega(xy)) = \omega(xyz) = \omega(yz\theta_\omega(x)) = \omega(z\theta_\omega(x)\theta_\omega(y)),$$
$$\omega(yx) = \overline{\omega(x^*y^*)} = \overline{\omega(y^*\theta_\omega(x^*))} = \omega(\theta_\omega(x^*)^*y) = \omega(y\theta_\omega(\theta_\omega(x^*)^*)),$$

where $x, y, z \in C$.

As before, let $(A, \Delta, \epsilon, S, \phi, \psi, \mu)$ be a measured multiplier (B, Γ)-Hopf $*$-algebroid.

1.6.5. **Theorem.** *Let $(A, \Delta, \epsilon, S, \phi, \psi, \mu)$ be a measured multiplier (B, Γ)-Hopf $*$-algebroid and let $\nu = \mu \circ \phi = \mu \circ \psi$.*

i) *There exists a modular automorphism θ for ν.*
ii) *θ is a $(B, \Gamma)^{\mathrm{ev}}$-module automorphism of A.*
iii) *If $\nu \circ S = \nu$, then $\theta \circ S = S \circ \theta^{-1}$.*

Proof. i) The proof repeatedly uses strong invariance of ϕ and ψ, and closely follows [31], where the corresponding result was obtained for multiplier Hopf algebras. We proceed in three steps.

Step 1. Repeatedly using Remark 1.4.2 iii), we find that for all homogeneous $x, x', y, y' \in A$,

$$\begin{aligned} \bar\partial_{x'} = \bar\partial_{y'}^{-1} \quad \Rightarrow \quad \nu^{-1}(ys(\psi(xx'))y') &= \mu(\psi(yy')\bar\partial_{y'}^{-1}(\psi(xx'))) \\ &= \mu(\psi(xx')\bar\partial_{y'}(\psi(yy')D_{\bar\partial_{y'}})) = \nu^{-1}(xs(\psi(yy'))\bar D(x')), \quad (10) \\ \bar\partial_x = \partial_{y'} \quad \Rightarrow \quad \nu(yr(\psi(xx'))y') &= \mu(\phi(yy')\partial_{y'}^{-1}(\psi(xx'))) \\ &= \mu(\partial_{y'}(\phi(yy')D_{\partial y'})\psi(xx')) = \nu^{-1}(xs(\phi(D(y)y'))x'), \quad (11) \\ \partial_x = \partial_y^{-1} \quad \Rightarrow \quad \nu(yr(\phi(xx'))y') &= \mu(\partial_y(\phi(xx'))\phi(yy')) \\ &= \mu(\partial_y^{-1}(\phi(yy')D_{\partial_y^{-1}})\phi(xx')) = \nu(xr(\phi(D(y)y'))x'). \quad (12) \end{aligned}$$

Step 2. Let $c, d \in A$ be homogeneous and let

$$a = \sum \bar D(s(\psi(dS(c_{(2)})))c_{(1)}) \in A, \qquad a' = \sum d_{(2)}r(\phi(D(S(d_{(1)}))\bar D(c))) \in A. \tag{13}$$

Then the equations above and Proposition 1.5.3 imply

$$\begin{aligned} \nu(za) &= \sum \nu(z\bar D(s(\psi(dS(c_{(2)})))c_{(1)})) \\ &= \sum \nu(ds(\psi(zc_{(1)}))S(c_{(2)})) && \text{(Equation (10))} \\ &= \sum \nu(dr(\psi(z_{(1)}c))z_{(2)}) && \text{(Proposition 1.5.3)} \\ &= \sum \nu(z_{(1)}s(\phi(D(d)z_{(2)}))c) && \text{(Equation (11))} \\ &= \sum \nu(S(D(d_{(1)}))r(\phi(d_{(2)}z))c) && \text{(Proposition 1.5.3)} \\ &= \sum \nu(S(d_{(1)})r(\phi(d_{(2)}z))\bar D(c)) && \text{(use } S\circ D = \bar D^{-1}\circ S \text{ and 1.6.4 iii))} \\ &= \sum \nu(d_{(2)}r(\phi(D(S(d_{(1)}))\bar D(c)))z) = \nu(a'z). && \text{(Equation 12)} \end{aligned}$$

Step 3. Using bijectivity of the maps $\bar D, S, T_1$ and the relation $\langle s(\psi(A))A\rangle = A$, one finds that all elements of the form like a in (13) span A. A similar argument shows that the same is true for elements of the form like a'. Hence, there exists a bijection $\theta\colon A \to A$ such that $\nu(az) = \nu(z\theta(a))$ for all $a \in A$, and uniqueness of such a bijection follows from faithfulness of ν.

ii) We first show that θ respects the grading. Let $c, d \in A$ be homogeneous. Then the element a in (13) is homogeneous as well, with grading given by $\partial_a = \partial_c$ and $\bar\partial_a = \bar\partial_d$ because $\psi(dS(c_{(2)})) = 0$ unless $\bar\partial_d = \partial_{c_{(2)}} = \bar\partial_{c_{(1)}}$, and similarly a' in (13) is homogeneous with the same degree like a. To see that θ is $B\otimes B$-linear, use the relation $\nu(y\theta(r(b)s(b')x)) = \nu(r(b)s(b')xy) = \nu(xyr(b)s(b')) = \nu(yr(b)s(b')\theta(x))$, where $x, y \in A$ and $b, b' \in B$, and faithfulness of ν.

iii) If $\nu\circ S = \nu$, then $\nu(y\theta(S(x))) = \nu(S(x)y) = \nu(S^{-1}(y)x) = \nu(\theta^{-1}(x)S^{-1}(y)) = \nu(yS(\theta^{-1}(x)))$ for all $x, y \in A$. □

Define $\theta_D, \theta_{\bar D}, \theta_{D,\bar D}\colon A \to A$ by

$$\theta_D := \theta\circ D^{-1} = D^{-1}\circ\theta, \quad \theta_{\bar D} := \theta\circ\bar D^{-1} = \bar D^{-1}\circ\theta, \quad \theta_{D,\bar D} := \theta\circ D^{-1}\circ\bar D^{-1}.$$

1.6.6. **Proposition.** *Let $x, y \in A$ be homogeneous. Then*

i) $\phi\circ\theta = \phi$ *and* $\phi(xy) = \partial_x(\phi(y\theta_D(x)))$*;*

ii) $\psi \circ \theta = \psi$ *and* $\psi(xy) = \bar{\partial}_x(\psi(y\theta_{\bar{D}}(x)))$;
iii) $h \circ \theta = h$ *and* $h(xy) = (\partial_x \otimes \bar{\partial}_x)(h(y\theta_{D,\bar{D}}(x)))$ *if* h *is a bi-invariant integral and* $\nu = (\mu \otimes \mu) \circ h$.

Proof. Assertion i) follows from the fact that μ is faithful and that for all homogeneous $x, y \in A$ and all $b \in B$,

$$\begin{aligned}
\mu(b\phi(\theta(x))) &= \nu(r(b)\theta(x)) = \nu(\theta(r(b)x)) = \nu(r(b)x) = \mu(b\phi(x)),\\
\mu(b\phi(y\theta(x))) &= \nu(r(b)y\theta(x)) = \nu(xr(b)y)\\
&= \nu(r(\partial_x(bD_{\partial_x}))xr(D_{\partial_x}^{-1})y)\\
&= \mu(\partial_x(bD_{\partial_x})\phi(D(x)y)) = \mu(b\partial_x^{-1}(\phi(D(x)y))).
\end{aligned}$$

Assertions ii) and iii) follow similarly. □

Recall that a B-module N is called *flat* if the functor $N \underset{B}{\otimes} -$ on the category of B-modules is exact or, equivalently, preserves injectivity of morphisms.

1.6.7. **Proposition.** *Assume that* A_s *is a flat* B*-module. Then* $\Delta \circ \theta_D = (S^2 \tilde{\otimes} \theta_D) \circ \Delta$.

Proof. Let $x, y \in A$ be homogeneous. Using Sweedler notation, we calculate

$$\begin{aligned}
\sum \theta_D(x)_{(1)} s(\phi(y\theta_D(x)_{(2)})) &= \sum S(s(\phi(y_{(2)}\theta_D(x)))y_{(1)}) && \text{(Proposition 1.5.3))}\\
&= \sum S(s(\partial_x^{-1}(\phi(xy_{(2)})))y_{(1)}) && \text{(Proposition 1.6.6)}\\
&= \sum S(y_{(1)}s(\phi(xy_{(2)})))\\
&= \sum S^2(s(\phi(x_{(2)}y))x_{(1)}) && \text{(Proposition 1.5.3)}\\
&= \sum S^2(s(\partial_{x_{(2)}}(\phi(y\theta_D(x_{(2)}))))x_{(1)}) && \text{(Proposition (1.6.6))}\\
&= \sum S^2(x_{(1)})s(\phi(y\theta_D(x_{(2)}))).
\end{aligned}$$

Since A_s is a flat B-module and maps of the form $a \mapsto \phi(ya)$, where $y \in A$ is homogeneous, separate the points of A, we can conclude $\sum \theta_D(x)_{(1)} \tilde{\otimes} \theta_D(x)_{(2)} = \sum S^2(x_{(1)}) \tilde{\otimes} \theta_D(x_{(2)})$. □

1.7. **The dual $*$-algebra.** Let $(A, \Delta, \epsilon, S, \phi, \psi, \mu)$ be a measured multiplier (B, Γ)-Hopf $*$-algebroid. Denote by $M(A)'$ the dual vector space of $M(A)$ and let

$$\hat{A} := \{\nu(x-) : x \in A\} \subseteq M(A)'$$

Then $\hat{A} = \{\nu(-x) : x \in A\}$ by Theorem 1.6.5 and for each $\omega \in \hat{A}$, there exist unique B-module maps ${}_r\omega\colon {}_rM(A) \to B$, $\omega_r\colon M(A)_r \to B$, ${}_s\omega\colon {}_sM(A) \to B$, $\omega_s\colon M(A)_s \to B$ whose compositions with μ are equal to ω, because $\nu = \mu \circ \phi = \mu \circ \psi$ and μ is faithful. Using either of these B-module maps, one can equip $\hat{A}$ with the structure of a $*$-algebra. We shall choose an approach that fits well with the duality on the operator-algebraic level in the next section.

First, we define an abstract Fourier transform

$$A \to \hat{A}, \quad x \mapsto \hat{x} := \nu(S(x)-),$$

which is a linear bijection because ν is faithful. Evidently, $\hat{x}_s = \psi(S(x)-)$ and $\hat{x}_r = \phi(S(x)-)$, and by Proposition 1.6.6, ${}_s\hat{x} = \psi(-\theta(S(x)))$ and ${}_r\hat{x} = \phi(-\theta(S(x)))$. For all $x, a \in A$, we define a right convolution

$$a * \hat{x} := \sum a_{(2)} r(\hat{x}_s(a_{(1)})) = \sum a_{(2)} r(\psi(S(x)a_{(1)})) \in A. \tag{14}$$

1.7.1. **Remark.** One could also work with the transform $A \to \hat{A}$, $x \mapsto \check{x} := \nu(-S(x))$, and the left convolution defined by

$$\check{x} * a := \sum s({}_r\check{x}(a_{(2)}))a_{(1)} = \sum s(\phi(a_{(2)}S(x)))a_{(1)} \in A \quad \text{for all } x, a \in A. \tag{15}$$

If $\phi \circ S = \psi$, for example, if we are in the bi-measured case (see Remark 1.6.2 ii)), then

$$\begin{aligned}\widetilde{S(x)} * S(a) &= \sum s(\phi(S(a)_{(2)}S^2(x)))S(a)_{(1)} \\ &= S(a_{(2)}r(\psi(S(x)a_{(1)}))) = S(a * \hat{x}) \quad \text{for all } a, x \in A.\end{aligned}$$

We collect a few useful formulas. First, for all $a, x \in A$,

$$a * \hat{x} = \sum r(\psi(a_{(1)}\theta_{\bar{D}}(x)))a_{(2)}, \qquad \text{(Prop. 1.6.6)} \tag{16}$$

$$a * \hat{x} = \sum S^{-1}(r(\psi(S(x)_{(1)}a))S(x)_{(2)}) = \sum x_{(1)}s(\psi(S(x_{(2)})a)) \qquad \text{(Prop. 1.5.3)} \tag{17}$$

Next, for all $a, x, y \in A$, $b \in B$, $\gamma, \gamma', \delta, \delta' \in \Gamma$,

$$\begin{aligned} r(b)a * \hat{x} &= a * \widehat{s(b)x}, & ar(b) * \hat{x} &= a * \widehat{xs(b)}, \\ s(b)a * \hat{x} &= s(b)(a * \hat{x}), & as(b) * \hat{x} &= (a * \hat{x})s(b), \end{aligned} \tag{18}$$

$$\begin{aligned}(a * \hat{x}) * \hat{y} &= \sum a_{(3)}r(\psi(S(y)a_{(2)}r(\psi(S(x)a_{(1)})))) \\ &= \sum a_{(2)}r(\psi(S(y)x_{(1)}s(\psi(S(x_{(2)})a_{(1)})))) \\ &= \sum a_{(2)}r(\psi(S(x_{(2)}r(\psi(S(y)x_{(1)})))a)) = a * \widehat{(x * \hat{y})},\end{aligned} \tag{19}$$

$$A_{\gamma,\gamma'} * \widehat{A_{\delta,\delta'}} \subseteq \sum_{\gamma''} s(\psi(A_{\delta'^{-1},\delta^{-1}}A_{\gamma,\gamma''}))A_{\gamma'',\gamma'} \subseteq \delta_{\gamma,\delta'}A_{\delta,\gamma'}, \tag{20}$$

where we used Lemma 1.6.4 in the last line. Finally, note that surjectivity of T_2 (Proposition 1.3.8 and of ψ imply

$$\langle A * \hat{A} \rangle = (\psi \underset{B}{\otimes} \mathrm{id})(T_2(A_r \underset{B}{\otimes} {}_rA)) = \langle Ar(\psi(A)) \rangle = \langle Ar(B) \rangle = A. \tag{21}$$

The $(B, \Gamma)^{\mathrm{ev}}$-algebra structure on A induces the following structure on $\hat{A}$:

1.7.2. **Proposition.** i) *$\hat{A}$ carries the structure of a non-degenerate $*$-algebra, where $\hat{y}\hat{x} = \widehat{x * \hat{y}}$ and $\hat{x}^* = \widehat{S(x)^*}$ for all $x, y \in A$.*

ii) *There exist non-degenerate $*$-homomorphisms $\hat{r}, \hat{s} \colon B \to M(\hat{A})$ such that*

$$\hat{r}(b)\hat{x} = \widehat{xr(b)}, \qquad \hat{x}\hat{r}(b) = \widehat{xs(b)}, \qquad \hat{s}(b)\hat{x} = \widehat{r(b)x}, \qquad \hat{x}\hat{s}(b) = \widehat{s(b)x}$$

for all $x \in A, b \in B$. The images of $\hat{r}$ and $\hat{s}$ commute.

iii) *Let $\hat{A}^{\gamma,\gamma'} = \widehat{(A_{\gamma,\gamma'})}$ for all $\gamma, \gamma' \in \Gamma$. Then $\hat{A} = \bigoplus_{\gamma,\gamma' \in \Gamma} \hat{A}^{\gamma,\gamma'}$ as a vector space and*

$$\hat{A}^{\gamma,\gamma'}\hat{A}^{\delta,\delta'} \subseteq \delta_{\gamma',\delta}\hat{A}^{\gamma,\delta'}, \quad (\hat{A}^{\gamma,\gamma'})^* = \hat{A}^{\gamma',\gamma}, \quad \hat{r}(B)\hat{s}(B)\hat{A}^{\gamma,\gamma'} \subseteq \hat{A}^{\gamma,\gamma'} \quad \textit{for all } \gamma, \gamma', \delta, \delta' \in \Gamma.$$

Furthermore, for all $\gamma, \gamma', \delta, \delta' \in \Gamma$, $\hat{a} \in \hat{A}^{\gamma,\gamma'}$, $b, b' \in B$,

$$\hat{r}(b)\hat{s}(b')\hat{a} = \hat{r}(\gamma^{-1}(b'))\hat{s}(\gamma(b))\hat{a} \qquad \text{and} \qquad \hat{a}\hat{r}(b)\hat{s}(b') = \hat{a}\hat{r}(\gamma'^{-1}(b'))\hat{s}(\gamma'(b)).$$

Proof. i) The multiplication is associative and turns $\hat{A}$ into a non-degenerate algebra by (19) and (21). The $*$-operation is involutive because $* \circ S$ is involutive, and anti-multiplicative because

$$\begin{aligned} S(y * \hat{x})^* &= \sum S(y_{(2)}r(\psi(S(x)y_{(1)}))^* \\ &= \sum S(y_{(2)})^* s(\psi(y^*_{(1)}S(x)^*)) \\ &= \sum S(y)^*_{(1)} s(\psi(S(S(y)^*_{(2)})S(x)^*)) = S(x)^* * \widehat{S(y)^*}. \end{aligned}$$

ii) For each $b \in B$, the formulas above define multipliers $\hat{r}(b), \hat{s}(b) \in M(\hat{A})$ because

$$\hat{y}(\hat{r}(b)\hat{x}) = (xr(b) * \hat{y})\hat{} = (x * \widehat{ys(b)})\hat{} = (\hat{y}\hat{r}(b))\hat{x}$$

and similarly $\hat{y}(\hat{s}(b)\hat{x}) = (\hat{y}\hat{s}(b))\hat{x}$ for all $x, y \in A$ by (18). The maps $\hat{r}, \hat{s} \colon B \to M(\hat{A})$ are non-degenerate homomorphisms because $r, s \colon B \to M(A)$ have the same properties, their images evidently commute, and they are involutive because

$$(\hat{x}\hat{r}(b))^* = (\widehat{xs(b)})^* = (S(xs(b))^*)\hat{} = (S(x)^*r(b^*))\hat{} = \hat{r}(b^*)\hat{x}^*$$

and similarly $(\hat{x}\hat{s}(b))^* = \hat{s}(b^*)\hat{x}^*$ for all $x \in A$, $b \in B$.

iii) All of these assertions follow easily from the definitions and relation (20), for example, $\hat{r}(b)\hat{x} = \widehat{xr(b)} = (r(\gamma(b))x)\hat{} = \hat{s}(\gamma(b))\hat{x}$ for all $\gamma, \gamma' \in \Gamma, x \in A_{\gamma,\gamma'}, b \in B$. ☐

2. Construction of associated measured quantum groupoids

In this section, we fix a measured multiplier (B, Γ)-Hopf $*$-algebroid $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$ and construct operator-algebraic completions of this algebraic object in the form of a Hopf C^*-bimodule, Hopf-von Neumann bimodule and a measured quantum groupoid. Along the way, we shall impose further assumptions on B, Γ, μ, A which were mentioned already in the introduction, most notably properness.

The basic idea is to use the GNS-representations for the weight μ on the basis B and the functional ν on the total algebra A, respectively. Naturally, some restrictions have to be made on B, Γ, μ. To show that ν admits a bounded GNS-representation and to lift the comultiplication to the level of operator algebras, we use a fundamental unitary. To take full advantage of this unitary, we describe its domain and range as relative tensor products, and show that it is a pseudo-multiplicative unitary in the sense of [24] and [27]. The necessary modules are introduced in §2.2, and the unitary itself is constructed in §2.3. This part uses Connes' spatial theory [20], and the relative tensor product of Hilbert spaces over C^*-algebras which was introduced in [22]. The fundamental unitary then gives rise to completions of A and $\hat{A}$ in the form of Hopf C^*-bimodules and two Hopf-von Neumann bimodules; see §2.4–§2.6.To obtain the full structure of a measured quantum groupoid, we finally extend the integrals ϕ, ψ to the level of von Neumann algebras and show that these extensions are left or right invariant again in §2.7.

Before we turn to details, let us briefly sketch the construction of the fundamental unitary, which we denote by W. Its domain and range can be described as separated completions of the relative tensor products ${}_sA \underset{B}{\otimes} {}_rA$ and ${}_rA \underset{B}{\otimes} A_r$ with respect to the sesquilinear forms given by

$$\begin{aligned} \langle x \underset{B}{\otimes} y | x' \underset{B}{\otimes} y' \rangle_{({}_sA \underset{B}{\otimes} {}_rA)} &= \nu(x^* s(\partial_y(\phi(y^*y')))x'), \\ \langle x \underset{B}{\otimes} y) | x' \underset{B}{\otimes} y' \rangle_{({}_rA \underset{B}{\otimes} A_r)} &= \nu(x^* r(\phi(y^*y'))x') \end{aligned} \tag{22}$$

for all $x, y \in A$, where y is assumed to be homogeneous in the upper line. Note that positivity of these forms is not evident because ϕ is not assumed to be completely positive in any sense. Given that positivity, the map

$$T_4 \colon {}_rA \underset{B}{\otimes} A_r \to {}_sA \underset{B}{\otimes} {}_rA, \quad x \underset{B}{\otimes} y \mapsto \Delta(y)(x \underset{B}{\otimes} 1) = \sum y_{(1)}x \underset{B}{\otimes} y_{(2)},$$

extends to a unitary on the respective completions because it is surjective by Proposition 1.3.8 and isometric as shown by the calculation

(23)

$$\begin{aligned} \sum \langle y_{(1)}x \underset{B}{\otimes} y_{(2)} | y'_{(1)}x' \underset{B}{\otimes} y'_{(2)} \rangle_{({}_sA \underset{B}{\otimes} {}_rA)} &= \sum \nu(x^* y^*_{(1)} s(\partial_{y_{(2)}}(\phi(y^*_{(2)}y'_{(2)})))y'_{(1)}x') \\ &= \sum \nu(x^* s(\phi(y^*_{(2)}y'_{(2)}))y^*_{(1)}y'_{(1)}x') \\ &= \nu(x^* r(\phi(y^*y'))x') = \langle x \underset{B}{\otimes} y | x' \underset{B}{\otimes} y' \rangle_{({}_rA \underset{B}{\otimes} A_r)}, \end{aligned}$$

where $y_{(2)}$ is assumed to be homogeneous without loss of generality. The adjoint of this extension is the fundamental unitary W.

Similarly, one can construct and employ another unitary V which is an extension of the map $T_1 \colon A_s \underset{B}{\otimes} {}_sA \to {}_sA \underset{B}{\otimes} {}_rA$, $x \underset{B}{\otimes} y \mapsto \Delta(x)(1 \underset{B}{\otimes} y)$. We shall focus on W because this unitary is given preference in the theory of locally compact quantum groups and measured quantum groupoids.

2.1. **Preparations concerning the base.** Denote by $\Lambda_\mu \colon B \to \mathcal{L}(B)$ the GNS-map for μ as before. From now on, we assume:

(A1) The weight μ *admits a GNS-representation via bounded operators* in the sense that the following equivalent conditions hold:
  i) for each $b \in B$, there is a $\lambda > 0$ such that $\mu(c^*b^*bc) \leqslant \lambda\mu(c^*c)$ for all $c \in B$;
  ii) the formula $\pi_\mu(b)\Lambda_\mu(c) = \Lambda_\mu(bc)$ defines a $*$-homomorphism $\pi_\mu \colon B \to \mathcal{L}(K)$.

2.1.1. **Remarks.** i) Assumption (A1) holds if B is a pre-C^*-algebra since then $\mu(c^*b^*bc) \leqslant \mu(c^*\|b^*b\|c) = \|b^*b\|\mu(c^*c)$ for all $b, c \in B$. Conversely, if (A1) holds, then B can be regarded as a pre-C^*-algebra with respect to the norm given by $b \mapsto \|\pi_\mu(b)\|$.

ii) To apply the constructions below, it may be useful to first perform a base change, similarly as described in [21, §2], to replace B by a pre-C^*-algebra of the form $C_c(\Omega)$, where Ω is a locally compact space with an action of Γ. For example, one can take Ω to be the set of all $*$-homomorphisms $\chi \colon B \to \mathbb{C}$, equipped with the weakest topology that makes the function $\Omega \to \mathbb{C}$, $\chi \mapsto \chi(b)$, continuous for each $b \in B$, and perform a base change along the canonical map $B \to M(C_c(\Omega))$. Note, however, that such a base change can not simply be applied to left and right integrals, but only to bi-integrals.

Recall that a *Hilbert algebra* is a $*$-algebra with an inner product such that left multiplication by each element is bounded, the resulting $*$-representation is non-degenerate, and the involution is pre-closed with respect to the norm induced by the inner product. Since B is commutative, the map $\Lambda_\mu(B) \to \Lambda_\mu(B)$ given by $\Lambda_\mu(b) \mapsto \Lambda_\mu(b^*)$ extends to an anti-unitary operator J_μ on K, and hence $\Lambda_\mu(B) \subseteq K$ together with the $*$-algebra structure inherited from B is a Hilbert algebra. We thus obtain

- a von Neumann algebra $N := \pi_\mu(B)'' \subseteq \mathcal{L}(K)$,
- a n.s.f. weight $\tilde{\mu}$ on N such that $\tilde{\mu}(\pi_\mu(b^*b)) = \langle \Lambda_\mu(b)|\Lambda_\mu(b)\rangle = \mu(b^*b)$ for all $b \in B$,
- a left ideal $\mathfrak{N}_{\tilde{\mu}} := \{x \in N : \tilde{\mu}(x^*x) < \infty\} \subseteq N$ of square-integrable elements,
- a closed map $\Lambda_{\tilde{\mu}} \colon \mathfrak{N}_{\tilde{\mu}} \to K$ such that $(K, \Lambda_{\tilde{\mu}}, \mathrm{id}_N)$ is a GNS-representation for $\tilde{\mu}$; this is the closure of the map $\pi_\mu(B) \to K$ given by $\pi_\mu(b) \to \Lambda_\mu(b)$.

2.2. **Various module structures.** We define an inner product on A by $\langle a|a'\rangle := \nu(a^*a')$ for all $a, a' \in A$ and denote by H the Hilbert space obtained by completion. We call the canonical inclusion of A into H the *GNS-map* for ν and denote it by Λ_ν.

2.2.1. **Lemma.** *There exist maps $\Lambda_\phi, \Lambda_\psi, \Lambda^\dagger_\phi, \Lambda^\dagger_\psi \colon A \to \mathcal{L}(K, H)$ such that for all $x, y \in A$, $b \in B$,*

$$\begin{aligned}
&\Lambda_\phi(x)\Lambda_\mu(b) = \Lambda_\nu(xr(b)), && \Lambda_\phi(x)^*\Lambda_\nu(y) = \Lambda_\mu(\phi(x^*y)), && \Lambda_\phi(x)^*\Lambda_\phi(y) = \pi_\mu(\phi(x^*y)),\\
&\Lambda_\psi(x)\Lambda_\mu(b) = \Lambda_\nu(xs(b)), && \Lambda_\psi(x)^*\Lambda_\nu(y) = \Lambda_\mu(\psi(x^*y)), && \Lambda_\psi(x)^*\Lambda_\psi(y) = \pi_\mu(\psi(x^*y)),\\
&\Lambda^\dagger_\phi(x)\Lambda_\mu(b) = \Lambda_\nu(r(b)x), && \Lambda^\dagger_\phi(x)^*\Lambda_\nu(y) = \Lambda_\mu(\phi(y\theta(x^*))), && \Lambda^\dagger_\phi(x)^*\Lambda^\dagger_\phi(y) = \pi_\mu(\phi(y\theta(x^*))),\\
&\Lambda^\dagger_\psi(x)\Lambda_\mu(b) = \Lambda_\nu(s(b)x), && \Lambda^\dagger_\psi(x)^*\Lambda_\nu(y) = \Lambda_\mu(\psi(y\theta(x^*))), && \Lambda^\dagger_\psi(x)^*\Lambda^\dagger_\psi(y) = \pi_\mu(\psi(y\theta(x^*))).
\end{aligned}$$

Proof. We only prove the assertions concerning Λ_ϕ and $\Lambda^\dagger_\phi$. They follow from the relations

$$\begin{aligned}
\|\Lambda_\nu(xr(b))\|^2 &= \nu(r(b)^*x^*xr(b)) = \mu(b^*\phi(x^*x)b) \leqslant \|\pi_\mu(\phi(x^*x))\|\|\Lambda_\mu(b)\|^2,\\
\langle \Lambda_\nu(y)|\Lambda_\nu(xr(b))\rangle &= \nu(y^*xr(b)) = \mu(\phi(y^*x)b) = \langle \Lambda_\mu(\phi(x^*y))|\Lambda_\mu(b)\rangle
\end{aligned}$$

and

$$\begin{aligned}
\|\Lambda_\nu(r(b)x)\|^2 = \nu(x^*r(b^*b)x) &= \nu(\theta^{-1}(x)x^*r(b^*b))\\
&= \mu(\phi(\theta^{-1}(x)x^*)b^*b) \leqslant \|\Lambda_\mu(b)\|^2\|\pi_\mu(\theta^{-1}(x)x^*)\|,\\
\langle \Lambda_\nu(y)|\Lambda_\nu(r(b)x)\rangle = \nu(y^*r(b)x) &= \nu(\theta^{-1}(x)y^*r(b))\\
&= \mu(\phi(\theta^{-1}(x)y^*)b) = \langle \Lambda_\mu(\phi(y\theta(x^*)))|\Lambda_\mu(b)\rangle,
\end{aligned}$$

which hold for all $x, y \in A$ and $b \in B$. $\square$

The maps introduced above yield various module structures on H as follows. Let

$$E_\phi := [\Lambda_\phi(A)], \qquad E_\psi := [\Lambda_\psi(A)], \qquad E^\dagger_\phi := [\Lambda^\dagger_\phi(A)], \qquad E^\dagger_\psi := [\Lambda^\dagger_\psi(A)]. \tag{24}$$

We shall use the following concepts introduced in [22, 24]. A *C^*-$\mathfrak{b}$-module*, where $\mathfrak{b} = (K, [\pi_\mu(B)], [\pi_\mu(B)])$, consists of a Hilbert space L and a closed subset $E \subseteq \mathcal{L}(K, L)$ such that $[EK] = L$, $[E\pi_\mu(B)] = E$, $[E^*E] = [\pi_\mu(B)]$. Each such C^*-$\mathfrak{b}$-module gives rise to a normal, faithful, non-degenerate representation $\rho_E \colon N = \pi_\mu(B)'' \to \mathcal{L}(L)$ such that $\rho_E(x)\xi = \xi x$ for all $x \in N$, $\xi \in E$. A C^*-$(\mathfrak{b},\mathfrak{b})$-*module* is a triple (L, E, F) such that (L, E) and (L, F) are C^*-$\mathfrak{b}$-modules and $[\rho_E(\pi_\mu(B))F] = F$ and $[\rho_F(\pi_\mu(B))E] = [E]$.

2.2.2. **Lemma.** *The Hilbert space H is a C^*-$(\mathfrak{b},\mathfrak{b})$-module with respect to either two of the spaces $E_\phi, E_\psi, E^\dagger_\phi, E^\dagger_\psi$. The representations $\alpha := \rho_{E^\dagger_\phi}$, $\beta := \rho_{E^\dagger_\psi}$, $\widehat{\alpha} := \rho_{E_\psi}$, $\widehat{\beta} := \rho_{E_\phi}$ of N on H are given by*

$$\begin{aligned} &\alpha(\pi_\mu(b))\Lambda_\nu(a) = \Lambda_\nu(r(b)a), \quad \beta(\pi_\mu(b))\Lambda_\nu(a) = \Lambda_\nu(s(b)a), \\ &\widehat{\beta}(\pi_\mu(b))\Lambda_\nu(a) = \Lambda_\nu(ar(b)), \quad \widehat{\alpha}(\pi_\mu(b))\Lambda_\nu(a) = \Lambda_\nu(as(b)) \end{aligned} \tag{25}$$

for all $b \in B, a \in A$.

Proof. Let E, F be any two of the spaces listed above. Then $[EH] = H$ and $[E\pi_\mu(B)] = E$ because $\langle r(B)s(B)Ar(B)s(B)\rangle = A$, and $[E^*E] = [\pi_\mu(B)]$ because $\phi(A) = B = \psi(A)$. Thus, (H, E) is a C^*-$\mathfrak{b}$-module. The formulas for the associated representations are easily verified. Using these formulas and the relation $\langle r(B)s(B)Ar(B)s(B)\rangle = A$, one easily checks that $[\rho_E(\pi_\mu(B))F] = F$ and $[\rho_F(\pi_\mu(B))E] = E$. □

Recall that a vector ζ in a Hilbert space L is *bounded* with respect to a normal, non-degenerate representation $\rho\colon N \to L$ and the weight $\tilde{\mu}$ if the following equivalent conditions hold:

i) there exists a $K \geqslant 0$ such that $\|\rho(x)\zeta\| \leqslant K\tilde{\mu}(x^*x)$ for all $x \in \mathfrak{N}_{\tilde{\mu}}$;
ii) there exists an operator $R^{\rho,\tilde{\mu}}_\zeta \in \mathcal{L}(K, L)$ such that $R^{\rho,\tilde{\mu}}_\zeta\Lambda_\mu(x) = \rho(x)\zeta$ for all $x \in \mathfrak{N}_{\tilde{\mu}}$.

The set of all such bounded vectors is denoted by $D(L_\rho, \tilde{\mu})$. This spaces carries an N-valued inner product $\langle -|-\rangle_{\rho,\tilde{\mu}}$, given by $\langle\zeta|\zeta'\rangle_{\rho,\tilde{\mu}} = (R^{\rho,\tilde{\mu}}_\zeta)^*R^{\rho,\tilde{\mu}}_{\zeta'}$ for all $\zeta, \zeta' \in D(L_\rho, \tilde{\mu})$, and $\rho(N)'D(L_\rho, \tilde{\mu}) = D(L_\rho, \tilde{\mu})$ and

$$\Lambda_{\tilde{\mu}}(\langle\zeta|\zeta'\rangle_{\rho,\tilde{\mu}}) = (R^{\rho,\tilde{\mu}}_\zeta)^*\zeta', \quad R^{\rho,\tilde{\mu}}_{T\zeta} = TR^{\rho,\tilde{\mu}}_\zeta \quad \text{for all } T \in \rho(N)', \zeta, \zeta' \in D(L_\rho, \tilde{\mu}). \tag{26}$$

2.2.3. **Lemma.** $\Lambda_\nu(A) \subseteq D(H_\alpha, \tilde{\mu}) \cap D(H_\beta, \tilde{\mu}) \cap D(H_{\widehat{\alpha}}, \tilde{\mu}) \cap D(H_{\widehat{\beta}}, \tilde{\mu})$ *and for all $x, y \in A$,*

$$R^{\alpha,\tilde{\mu}}_{\Lambda_\nu(x)} = \Lambda^\dagger_\phi(x), \qquad R^{\beta,\tilde{\mu}}_{\Lambda_\nu(x)} = \Lambda^\dagger_\psi(x), \qquad R^{\widehat{\alpha},\tilde{\mu}}_{\Lambda_\nu(x)} = \Lambda_\psi(x), \qquad R^{\widehat{\beta},\tilde{\mu}}_{\Lambda_\nu(x)} = \Lambda_\phi(x).$$

Proof. We shall only prove the assertion concerning α. Let $a \in A$. Then $\Lambda^\dagger_\phi(a)\Lambda_{\tilde{\mu}}(\pi_\mu(b)) = \Lambda_\nu(r(b)a) = \alpha(\pi_\mu(b))\Lambda_\nu(a)$ for all $b \in B$, and since $\pi_\mu(B)$ is a core for $\Lambda_{\tilde{\mu}}$, we can conclude $\Lambda^\dagger_\phi(a)\Lambda_{\tilde{\mu}}(x) = \alpha(x)\Lambda_\nu(a)$ for all $x \in \mathfrak{N}_{\tilde{\mu}}$. □

The preceding result and Lemma 2.2.1 imply that for all $x, y \in A$,

$$\begin{aligned} &\langle\Lambda_\nu(x)|\Lambda_\nu(y)\rangle_{\alpha,\tilde{\mu}} = \pi_\mu(\phi(y\theta(x^*))), \quad \langle\Lambda_\nu(x)|\Lambda_\nu(y)\rangle_{\beta,\tilde{\mu}} = \pi_\mu(\psi(y\theta(x^*))), \\ &\langle\Lambda_\nu(x)|\Lambda_\nu(y)\rangle_{\widehat{\alpha},\tilde{\mu}} = \pi_\mu(\psi(x^*y)), \qquad \langle\Lambda_\nu(x)|\Lambda_\nu(y)\rangle_{\widehat{\beta},\tilde{\mu}} = \pi_\mu(\phi(x^*y)). \end{aligned} \tag{27}$$

2.3. **The fundamental unitary.** To define the domain and the range of the fundamental unitary, we use Connes' relative tensor product of Hilbert modules and the module structures introduced above. Connes' original manuscript on the construction remained unpublished; we therefore refer to [20] and [23] for details.

The *relative tensor product* $H_{\beta}\underset{\tilde{\mu}}{\otimes}{}_{\alpha}H$ is the separated completion of the algebraic tensor product $D(H_\beta,\tilde{\mu})\otimes K\otimes D(H_\alpha,\tilde{\mu})$ with respect to the sesquilinear form given by

$$\langle \xi\otimes\zeta\otimes\eta|\xi'\otimes\zeta'\otimes\eta'\rangle = \langle\zeta|\langle\xi|\xi'\rangle_{\beta,\tilde{\mu}}\langle\eta|\eta'\rangle_{\alpha,\tilde{\mu}}\zeta'\rangle. \tag{28}$$

This Hilbert space can naturally be identified with the separated completions of the algebraic tensor products $D(H_\beta,\tilde{\mu})\otimes H$ and $H\otimes D(H_\alpha,\tilde{\mu})$ with respect to the sesquilinear forms given by

$$\langle\xi\otimes\eta|\xi'\otimes\eta'\rangle = \langle\eta|\alpha(\langle\xi|\xi'\rangle_{\beta,\tilde{\mu}})\eta'\rangle \quad\text{and}\quad \langle\xi\otimes\eta|\xi'\otimes\eta'\rangle = \langle\xi|\beta(\langle\eta|\eta'\rangle_{\alpha,\tilde{\mu}})\xi'\rangle, \tag{29}$$

respectively, via

$$\xi\otimes R^{\alpha,\tilde{\mu}}_{\xi}\zeta \equiv \xi\otimes\zeta\otimes\eta \equiv R^{\beta,\tilde{\mu}}_{\xi}\zeta\otimes\eta, \tag{30}$$

and we shall use these identifications without further notice. Replacing the representations β,α by $\alpha,\widehat{\beta}$ or $\widehat{\alpha},\beta$, respectively, one obtains the relative tensor products $H_{\alpha}\underset{\tilde{\mu}}{\otimes}{}_{\widehat{\beta}}H$ and $H_{\widehat{\alpha}}\underset{\tilde{\mu}}{\otimes}{}_{\beta}H$.

To proceed, we impose the following simplifying assumption:

(A2) The Radon-Nikodym cocycle $(D_\gamma)_\gamma$ of μ *has a positive square root in* $M(B)$ in the sense that there exists a family $(D_\gamma^{\frac{1}{2}})_{\gamma\in\Gamma}$ in $M(B)$ such that for all $\gamma,\gamma'\in\Gamma$, $c\in B$,

$$D_e^{\frac{1}{2}}=1,\quad (D_\gamma^{\frac{1}{2}})^*=D_\gamma^{\frac{1}{2}},\quad (D_\gamma^{\frac{1}{2}})^2=D_\gamma,\quad D^{\frac{1}{2}}_{\gamma\gamma'}=\gamma'^{-1}(D_\gamma^{\frac{1}{2}})D^{\frac{1}{2}}_{\gamma'},\quad \mu(c^*D_\gamma^{\frac{1}{2}}c)\geqslant 0.$$

Clearly, this condition implies the existence of a unitary representation $U\colon\Gamma\to\mathcal{L}(K)$ such that

$$U_\gamma\Lambda_\mu(c)=\Lambda_\mu(\gamma(cD_\gamma^{\frac{1}{2}})),\qquad U_\gamma\pi_\mu(b)U_\gamma^*=\pi_\mu(\gamma(b))\qquad\text{for all } b,c\in B,\gamma\in\Gamma. \tag{31}$$

Similarly as in (9), we define linear maps $D^{\frac{1}{2}},\bar{D}^{\frac{1}{2}}\colon A\to A$ by

$$D^{\frac{1}{2}}(a)=r(D^{\frac{1}{2}}_{\partial_a^{-1}})a=ar(D^{-\frac{1}{2}}_{\partial_a}),\qquad\qquad \bar{D}^{\frac{1}{2}}(a)=s(D^{\frac{1}{2}}_{\bar{\partial}_a^{-1}})a=as(D^{-\frac{1}{2}}_{\bar{\partial}_a})$$

for all homogeneous $a\in A$. These maps share all the properties of the maps $D,\bar{D}$ listed in Lemma 1.6.3. Short calculations show that for all homogeneous $x,y\in A$,

$$\Lambda_\phi(x)U_{\partial_x^{-1}}=\Lambda_\phi^\dagger(D^{\frac{1}{2}}(x)),\qquad \langle\Lambda_\nu(D^{\frac{1}{2}}(x))|\Lambda_\nu(D^{\frac{1}{2}}(y))\rangle_{\alpha,\tilde{\mu}}=\pi_\mu(\partial_x(\phi(x^*y))), \tag{32}$$

$$\Lambda_\psi(x)U_{\bar{\partial}_x^{-1}}=\Lambda_\psi^\dagger(\bar{D}^{\frac{1}{2}}(x)),\qquad \langle\Lambda_\nu(\bar{D}^{\frac{1}{2}}(x))|\Lambda_\nu(\bar{D}^{\frac{1}{2}}(y))\rangle_{\beta,\tilde{\mu}}=\pi_\mu(\bar{\partial}_x(\psi(x^*y))). \tag{33}$$

Indeed, for all homogeneous $x,y\in A$ and $b\in B$,

$$\begin{aligned}
\Lambda_\phi(x)U_{\partial_x^{-1}}\Lambda_\mu(b) &= \Lambda_\nu(xr(\partial_x^{-1}(bD^{\frac{1}{2}}_{\partial_x^{-1}}))=\Lambda_\nu(r(bD^{\frac{1}{2}}_{\partial_x^{-1}}x))=\Lambda_\psi^\dagger(D^{\frac{1}{2}}(x))\Lambda_\mu(b),\\
\Lambda_\phi^\dagger(D^{\frac{1}{2}}(x))^*\Lambda_\phi^\dagger(D^{\frac{1}{2}}(y)) &= U^*_{\partial_x^{-1}}\Lambda_\phi(x)^*\Lambda_\phi(y)U_{\partial_y^{-1}}\\
&= U_{\partial_x}\pi_\mu(\phi(x^*y))U_{\partial_y^{-1}}=\pi_\mu(\partial_x(\phi(x^*y))).
\end{aligned}$$

2.3.1. **Lemma.** *The sesquilinear forms on* ${}_sA \underset{B}{\otimes} {}_rA$ *and* ${}_rA \underset{B}{\otimes} A_r$ *defined in* (22) *are positive. Denote by* $\overline{{}_sA \underset{B}{\otimes} {}_rA}$ *and* $\overline{{}_rA \underset{B}{\otimes} A_r}$ *the respective separated completions. Then there exist isomorphisms*

$$\Lambda\colon \overline{{}_rA \underset{B}{\otimes} A_r} \to H_{\alpha}\underset{\tilde{\mu}}{\otimes}{}_{\widehat{\beta}}H, \quad x \underset{B}{\otimes} y \mapsto \Lambda_\nu(x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y),$$
$$\Lambda'\colon \overline{{}_sA \underset{B}{\otimes} {}_rA} \to H_{\beta}\underset{\tilde{\mu}}{\otimes}{}_{\alpha}H, \quad x \underset{B}{\otimes} y \mapsto \Lambda_\nu(x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(D^{\frac{1}{2}}(y)).$$

Proof. The maps Λ, Λ' are surjective because $\Lambda_\nu(A) \subseteq H$ is dense, and they are well-defined and isometric because (29), (27) and (32) imply for all homogeneous $x, y \in A$

$$\langle \Lambda(x \otimes y) | \Lambda(x' \otimes y')\rangle = \nu(x^* s(\phi(y^* y'))x'),$$
$$\langle \Lambda'(x \otimes y) | \Lambda'(x' \otimes y')\rangle = \nu(x^* r(\partial_y(\phi(y^* y')))x'). \qquad \square$$

2.3.2. **Proposition.** *There exists a unitary* $W\colon H_{\beta}\underset{\tilde{\mu}}{\otimes}{}_{\alpha}H \to H_{\alpha}\underset{\tilde{\mu}}{\otimes}{}_{\widehat{\beta}}H$ *such that* $W^* \circ \Lambda = \Lambda' \circ T_4$ *as maps from* ${}_rA \underset{B}{\otimes} A_r$ *to* $H_{\beta}\underset{\tilde{\mu}}{\otimes}{}_{\alpha}H$*, that is, for all homogeneous* $x, y \in A$*,*

$$W^*(\Lambda_\nu(x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y)) = \sum \Lambda_\nu(\bar{D}^{\frac{1}{2}}(y_{(1)})x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y_{(2)}) = \sum \Lambda_\nu(y_{(1)}x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(D^{\frac{1}{2}}(y_{(2)})),$$
$$W(\Lambda_\nu(x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y)) = \sum \Lambda_\nu(S^{-1}(D^{-\frac{1}{2}}(y_{(1)}))x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y_{(2)})$$
$$= \sum (\bar{D}^{\frac{1}{2}}(S^{-1}(y_{(1)}))x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y_{(2)}).$$

Proof. The calculation (23) and Lemma 2.3.1 imply that the map $\Lambda_\nu(x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y) \mapsto \sum \Lambda_\nu(y_{(1)}x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(D^{\frac{1}{2}}(y_{(2)}))$ extends to an isometry $H_{\alpha}\underset{\tilde{\mu}}{\otimes}{}_{\widehat{\beta}}H \to H_{\beta}\underset{\tilde{\mu}}{\otimes}{}_{\alpha}H$. Bijectivity of this isometry and the formula for W follow from Proposition 1.3.8 and Lemma 1.6.3. $\square$

Similarly, the map T_1 yields a second fundamental unitary:

2.3.3. **Proposition.** *There exists a unitary* $V\colon H_{\widehat{\alpha}}\underset{\tilde{\mu}}{\otimes}{}_{\beta}H \to H_{\beta}\underset{\tilde{\mu}}{\otimes}{}_{\alpha}H$ *such that for all homogeneous* $x, y \in A$*,*

$$V(\Lambda_\nu(x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y)) = \sum \Lambda_\nu(\bar{D}^{\frac{1}{2}}(x_{(1)})) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(x_{(2)}y) = \sum \Lambda_\nu(x_{(1)}) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(D^{\frac{1}{2}}(x_{(2)})y),$$
$$V^*(\Lambda_\nu(x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y)) = \sum \Lambda_\nu(x_{(1)}) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(S(\bar{D}^{-\frac{1}{2}}(x_{(2)}))y) \sum \Lambda_\nu(x_{(1)}) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(D^{\frac{1}{2}}(S(x_{(2)}))y).$$

Proof. The formula above defines an isometry V. Indeed, (29), (27) and (33) imply

$$\sum \langle \Lambda_\nu(\bar{D}^{\frac{1}{2}}(x_{(1)})) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(x_{(2)}y) | \Lambda_\nu(\bar{D}^{\frac{1}{2}}(x'_{(1)})) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(x'_{(2)}y')\rangle_{(H_{\beta}\underset{\tilde{\mu}}{\otimes}{}_{\alpha}H)}$$
$$= \sum \nu(y^* x^*_{(2)} r(\bar{\partial}_{x_{(1)}}(\psi(x^*_{(1)}x'_{(1)}))x'_{(2)}y')$$
$$= \sum \nu(y^* x^*_{(2)} x'_{(2)} r(\psi(x^*_{(1)}x'_{(1)}))y'),$$
$$\langle \Lambda_\nu(x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y) | \Lambda_\nu(x') \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y')\rangle_{(H_{\widehat{\alpha}}\underset{\tilde{\mu}}{\otimes}{}_{\beta}H)} = \nu(y^* s(\psi(x^* x'))y')$$

for all homogeneous $x, x', y, y' \in A$, where $x_{(1)}$ is assumed to be homogeneous without loss of generality, and by right-invariance of ψ (see Remark 1.5.2 i)), the expressions above coincide. Bijectivity of V and the inversion formula follow from Proposition 1.3.8. $\square$

2.4. **Boundedness of the canonical representations.** The first application of the fundamental unitary W is to show that left multiplication on A and right convolution by $\hat{A}$ extend to representations on the Hilbert space H.

The proof of Theorem 2.4.2 involves operators and slice maps of the following form. For each $\xi \in D(H_\beta, \tilde{\mu})$ and $\eta \in D(H_\alpha, \tilde{\mu})$, there exist bounded linear operators

$$\lambda_\xi^{\beta,\alpha} \colon H \to H_\beta \underset{\tilde{\mu}}{\otimes}_\alpha H,\ \eta' \mapsto \xi \underset{\tilde{\mu}}{\otimes} \eta', \qquad \rho_\eta^{\beta,\alpha} \colon H \to H_\beta \underset{\tilde{\mu}}{\otimes}_\alpha H,\ \xi' \mapsto \xi' \underset{\tilde{\mu}}{\otimes} \eta, \tag{34}$$

whose adjoints are given by

$$(\lambda_\xi^{\beta,\alpha})^*(\xi' \otimes \eta') = \alpha(\langle \xi | \xi' \rangle_{\beta,\tilde{\mu}})\eta', \qquad (\rho_\eta^{\beta,\alpha})^*(\xi' \otimes \eta') = \beta(\langle \eta | \eta' \rangle_{\alpha,\tilde{\mu}})\xi'; \tag{35}$$

see also [4]. Likewise, there exist operators $\lambda_\xi^{\alpha,\hat{\beta}}, \rho_\eta^{\alpha,\hat{\beta}} \colon H \to H_\alpha \underset{\tilde{\mu}}{\otimes}_{\hat{\beta}} H$ for all $\xi \in D(H_\alpha, \tilde{\mu})$ and $\eta \in D(H_{\hat{\beta}}, \tilde{\mu})$ which are defined similarly. Using these operators, one defines slice maps

$$\omega_{\xi,\xi'} * \mathrm{id} \colon \mathcal{L}(H_\alpha \underset{\tilde{\mu}}{\otimes}_{\hat{\beta}} H, H_\beta \underset{\tilde{\mu}}{\otimes}_\alpha H) \to \mathcal{L}(H), \quad T \mapsto (\lambda_\xi^{\beta,\alpha})^* T \lambda_{\xi'}^{\alpha,\hat{\beta}},$$
$$\mathrm{id} * \omega_{\eta,\eta'} \colon \mathcal{L}(H_\alpha \underset{\tilde{\mu}}{\otimes}_{\hat{\beta}} H, H_\beta \underset{\tilde{\mu}}{\otimes}_\alpha H) \to \mathcal{L}(H), \quad T \mapsto (\rho_\eta^{\beta,\alpha})^* T \rho_{\eta'}^{\alpha,\hat{\beta}}$$

for all $\xi \in D(H_\beta, \tilde{\mu})$, $\xi' \in D(H_\alpha, \tilde{\mu})$, $\eta \in D(H_\alpha, \tilde{\mu})$, $\eta' \in D(H_{\hat{\beta}}, \tilde{\mu})$; see [3].

2.4.1. **Lemma.** *Let $x, x', y, y' \in A$. Then*

$$(\mathrm{id} * \omega_{\Lambda_\nu(y),\Lambda_\nu(y')})(W^*)\Lambda_\nu(x) = \Lambda_\nu(ax), \qquad \text{where } a = \sum \bar{D}^{-\frac{1}{2}}(y'_{(1)} s(\phi(y^* y'_{(2)}))),$$
$$(\omega_{\Lambda_\nu(x),\Lambda_\nu(x')} * \mathrm{id})(W^*)\Lambda_\nu(y) = \Lambda_\nu(y * \hat{c}), \qquad \text{where } c = S^{-1}(\bar{D}^{\frac{1}{2}}(\theta^{-1}(x')x^*)).$$

Proof. Without loss of generality, we may assume y to be homogeneous. Then

$$\begin{aligned}
(\rho_{\Lambda_\nu(y)}^{\beta,\alpha})^* W^* \rho_{\Lambda_\nu(y')}^{\alpha,\hat{\beta}} \Lambda_\nu(x) &= \sum (\rho_{\Lambda_\nu(y)}^{\beta,\alpha})^* (\Lambda_\nu(y'_{(1)}x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(D^{\frac{1}{2}}(y'_{(2)}))) \\
&= \sum \beta(\langle \Lambda_\nu(y) | \Lambda_\nu(D^{\frac{1}{2}}(y'_{(2)})) \rangle_{\alpha,\tilde{\mu}} \Lambda_\nu(y'_{(1)}x) && \text{(Equation (35))} \\
&= \sum \Lambda_\nu(s(\partial_y(\phi(D^{-\frac{1}{2}}(y)^* y'_{(2)})))y'_{(1)}x) && \text{(Equation (32))} \\
&= \sum \Lambda_\nu(y'_{(1)} s(\phi(y^* D^{-\frac{1}{2}}(y'_{(2)})))x) && \text{(Lemma 1.6.3)} \\
&= \sum \Lambda_\nu(\bar{D}^{-\frac{1}{2}}(y'_{(1)} s(\phi(y^* y'_{(2)})))x), && \text{(Lemma 1.6.3),}
\end{aligned}$$

$$\begin{aligned}
(\lambda^{\beta,\alpha}_{\Lambda_\nu(x)})^* W^* \lambda^{\alpha,\hat{\beta}}_{\Lambda_\nu(x')} \Lambda_\nu(y) &= \sum (\lambda^{\beta,\alpha}_{\Lambda_\nu(x)})^* (\Lambda_\nu(\bar{D}^{\frac{1}{2}}(y_{(1)})x') \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y_{(2)})) \\
&= \sum \alpha(\langle \Lambda_\nu(x) | \Lambda_\nu(\bar{D}^{\frac{1}{2}}(y_{(1)})x')\rangle_{\beta,\tilde{\mu}}) \Lambda_\nu(y_{(2)}) && \text{(Equation (35))} \\
&= \sum \Lambda_\nu(r(\psi(\bar{D}^{\frac{1}{2}}(y_{(1)})x'\theta(x^*)))y_{(2)}) && \text{(Equation (27))} \\
&= \sum \Lambda_\nu(r(\psi(y_{(1)}\bar{D}^{-\frac{1}{2}}(x'\theta(x^*))))y_{(2)}) && \text{(Lemma 1.6.3)} \\
&= \sum \Lambda_\nu(y_{(2)} r(\psi(\bar{D}^{\frac{1}{2}}(\theta^{-1}(x')x^*)y_{(1)}))). && \text{(Equation (16))}
\end{aligned}$$

□

2.4.2. **Theorem.** *Let $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$ be a measured multiplier (B, Γ)-Hopf $*$-algebroid such that μ admits a GNS-representation via bounded operators $(K, \Lambda_\mu, \pi_\mu)$ and its Radon-Nikodym cocycle has a positive square root in $M(B)$. Denote by $\Lambda_\nu \colon A \to \mathcal{L}(H)$ the GNS-map for $\nu = \mu \circ \phi = \mu \circ \psi$. Then there exist non-degenerate $*$-homomorphisms $\pi_\nu \colon A \to \mathcal{L}(H)$ and $\rho \colon \hat{A} \to \mathcal{L}(H)$ such that*

$$\pi_\nu(x)\Lambda_\nu(y) = \Lambda_\nu(xy) \quad \textit{and} \quad \rho(\omega)\Lambda_\nu(y) = \Lambda_\nu(y * \omega) \quad \textit{for all } x, y \in A, \omega \in \hat{A}. \tag{36}$$

Proof of Theorem 2.4.2. For elements a and c of the form in Lemma 2.4.1, the maps $\Lambda_\nu(y) \mapsto \Lambda_\nu(ay)$ and $\Lambda_\nu(x) \mapsto \Lambda_\nu(x * \hat{c})$ coincide with compositions of bounded operators and therefore are bounded. Since elements of this form span A, we obtain maps $\pi_\nu \colon A \to \mathcal{L}(H)$ and $\rho \colon \hat{A} \to \mathcal{L}(H)$ as in (36). Evidently, π_ν is a $*$-homomorphism. The map ρ is multiplicative by (19) and Proposition 1.7.2, and it is involutive because by (17) and Proposition 1.7.2,

$$\begin{aligned}
\langle \rho(\hat{x})^* \Lambda_\nu(z) | \Lambda_\nu(y) \rangle &= \langle \rho(\widehat{S(x)^*}) \Lambda_\nu(z) | \Lambda_\nu(y) \rangle \\
&= \sum \langle \Lambda_\nu(S(x)^*_{(1)} s(\psi(S(S(x)^*_{(2)})z))) | \Lambda_\nu(y) \rangle \\
&= \nu(s(\psi(z^* x_{(1)})S(x_{(2)})y) \\
&= \nu(z^* x_{(1)} s(\psi(S(x_{(2)})y))) = \langle \Lambda_\nu(z) | \rho(\hat{x}) \Lambda_\nu(y) \rangle.
\end{aligned}$$

Finally, π_ν and ρ are non-degenerate because $\langle AA \rangle = A$ and $\langle A * \hat{A} \rangle = A$ (see (21)). □

2.4.3. **Remark.** Lemma 2.4.1, Theorem 2.4.2 and self-adjointness of $\pi_\nu(A)$ and $\rho(\hat{A})$ imply

$$\begin{aligned}
\pi_\nu(A) &= \operatorname{span}\{(\operatorname{id} * \omega_{\Lambda_\nu(y), \Lambda_\nu(y')})(W^*) | y, y' \in A\} = \operatorname{span}\{(\operatorname{id} * \omega_{\Lambda_\nu(y), \Lambda_\nu(y')})(W) | y, y' \in A\}, \\
\rho(\hat{A}) &= \operatorname{span}\{(\omega_{\Lambda_\nu(x), \Lambda_\nu(x')} * \operatorname{id})(W^*) | x, x' \in A\} = \operatorname{span}\{(\omega_{\Lambda_\nu(x), \Lambda_\nu(x')} * \operatorname{id})(W) | x, x' \in A\}.
\end{aligned}$$

For later use, we calculate the slices of V, which are defined similarly as those of W^*.

2.4.4. **Lemma.** *Let $x, x', y, y' \in A$. Then*

$$\begin{aligned}
(\omega_{\Lambda_\nu(x), \Lambda_\nu(x')} * \operatorname{id})(V) \Lambda_\nu(y) &= \Lambda_\nu(ay), && \textit{where } a = \sum D^{-\frac{1}{2}}(x'_{(2)} r(\psi(x^* x'_{(1)}))), \\
(\operatorname{id} * \omega_{\Lambda_\nu(y), \Lambda_\nu(y')})(V) \Lambda_\nu(x) &= \Lambda_\nu(\check{c} * x), && \textit{where } c = S^{-1}(D^{-\frac{1}{2}}(y'\theta(y^*)).
\end{aligned}$$

Proof. Without loss of generality, we assume x to be homogeneous. Proceeding as in the proof of that Lemma 2.4.1, we then find

$$\begin{aligned}
(\lambda^{\beta,\alpha}_{\Lambda_\nu(x)})^* V \lambda^{\widehat{\alpha},\beta}_{\Lambda_\nu(x')}\Lambda_\nu(y) &= \sum (\lambda^{\beta,\alpha}_{\Lambda_\nu(x)})^*(\Lambda_\nu(\bar{D}^{\frac{1}{2}}(x'_{(1)})) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(x'_{(2)}y)) && \text{(Definition of } V\text{)} \\
&= \sum \alpha(\langle \Lambda_\nu(x) | \Lambda_\nu(\bar{D}^{\frac{1}{2}}(x'_{(1)}))\rangle_{\beta,\tilde{\mu}})\Lambda_\nu(x'_{(2)}y) \\
&= \sum \Lambda_\nu(r(\bar{\partial}_x(\psi(\bar{D}^{-\frac{1}{2}}(x)^* x'_{(1)})))x'_{(2)}y) && \text{(Equation (33))} \\
&= \sum \Lambda_\nu(D^{-\frac{1}{2}}(x'_{(2)} r(\psi(x^* x'_{(1)})))y),
\end{aligned}$$

$$\begin{aligned}
(\rho^{\beta,\alpha}_{\Lambda_\nu(y)})^* V \rho^{\widehat{\alpha},\beta}_{\Lambda_\nu(y')}\Lambda_\nu(x) &= \sum (\rho^{\beta,\alpha}_{\Lambda_\nu(y)})^*(\Lambda_\nu(x_{(1)}) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(D^{\frac{1}{2}}(x_{(2)})y')) && \text{(Definition of } V\text{)} \\
&= \sum \beta(\langle \Lambda_\nu(y) | \Lambda_\nu(D^{\frac{1}{2}}(x_{(2)})y')\rangle_{\alpha,\tilde{\mu}})\Lambda_\nu(x_{(1)}) \\
&= \sum \Lambda_\nu(s(\phi(D^{\frac{1}{2}}(x_{(2)})y'\theta(y^*))x_{(1)}) && \text{(Equation (27))} \\
&= \sum \Lambda_\nu(s(\phi(x_{(2)}D^{-\frac{1}{2}}(y'\theta(y^*)))x_{(1)}).
\end{aligned}$$

$\square$

2.5. **The Hopf-von Neumann bimodules.** We next show that the fundamental unitary W is pseudo-multiplicative in the sense of [27] and therefore yields two Hopf-von Neumann bimodules, which are completions of A and $\hat{A}$, respectively. First, we need further preliminaries.

The relative tensor product is functorial so that there exist bounded linear operators $S \underset{\tilde{\mu}}{\otimes} T \in \mathcal{L}(H_\beta \underset{\tilde{\mu}}{\otimes}_\alpha H)$ for all $S \in \beta(N)', T \in \alpha(N)'$, as well as $S \underset{\tilde{\mu}}{\otimes} T \in \mathcal{L}(H_\alpha \underset{\tilde{\mu}}{\otimes}_{\widehat{\beta}} H)$ for all $S \in \alpha(N)', T \in \widehat{\beta}(N)'$, both times given by $\xi \underset{\tilde{\mu}}{\otimes} \eta \mapsto S\xi \underset{\tilde{\mu}}{\otimes} T\eta$.

In particular, the commuting representations $\alpha, \beta, \widehat{\alpha}, \widehat{\beta}$ yield six representations $\alpha \underset{\tilde{\mu}}{\otimes} \mathrm{id}$, $\widehat{\alpha} \underset{\tilde{\mu}}{\otimes} \mathrm{id}$, $\widehat{\beta} \underset{\tilde{\mu}}{\otimes} \mathrm{id}$, $\mathrm{id} \underset{\tilde{\mu}}{\otimes} \beta$, $\mathrm{id} \underset{\tilde{\mu}}{\otimes} \widehat{\alpha}$, $\mathrm{id} \underset{\tilde{\mu}}{\otimes} \widehat{\beta}$ of N on $H_\beta \underset{\tilde{\mu}}{\otimes}_\alpha H$, and further six representations of N on $H_\alpha \underset{\tilde{\mu}}{\otimes}_{\widehat{\beta}} H$.

2.5.1. **Lemma.** *The following relations hold for all $x \in N$:*

$$\begin{aligned}
W(\mathrm{id} \underset{\tilde{\mu}}{\otimes} \widehat{\beta}(x)) &= (\beta(x) \underset{\tilde{\mu}}{\otimes} \mathrm{id})W, & W(\alpha(x) \underset{\tilde{\mu}}{\otimes} \mathrm{id}) &= (\mathrm{id} \underset{\tilde{\mu}}{\otimes} \alpha(x))W, \\
W(\widehat{\beta}(x) \underset{\tilde{\mu}}{\otimes} \mathrm{id}) &= (\widehat{\beta}(x) \underset{\tilde{\mu}}{\otimes} \mathrm{id})W, & W(\mathrm{id} \underset{\tilde{\mu}}{\otimes} \beta(x)) &= (\mathrm{id} \underset{\tilde{\mu}}{\otimes} \beta(x))W, \\
W(\widehat{\alpha}(x) \underset{\tilde{\mu}}{\otimes} \mathrm{id}) &= (\widehat{\alpha}(x) \underset{\tilde{\mu}}{\otimes} \mathrm{id})W, & W(\mathrm{id} \underset{\tilde{\mu}}{\otimes} \widehat{\alpha}(x)) &= (\mathrm{id} \underset{\tilde{\mu}}{\otimes} \widehat{\alpha}(x))W.
\end{aligned}$$

Proof. This follows immediately from the fact that $\pi_\mu(B) \subseteq N$ is weakly dense, the definition of W, and the formulas for $\alpha, \beta, \widehat{\alpha}, \widehat{\beta}$ given in Lemma 2.2.2. $\square$

The relative tensor product is associative in a natural sense. The intertwining relations for W obtained above imply that all operators in the diagram below are well-defined,

$$
\begin{tikzcd}
H_\beta\underset{\tilde\mu}{\otimes}{}_\alpha H_\beta\underset{\tilde\mu}{\otimes}{}_\alpha H \arrow[r, "W_{12}"] \arrow[d, "W_{23}"] & H_\alpha\underset{\tilde\mu}{\otimes}{}_{\hat\beta} H_\beta\underset{\tilde\mu}{\otimes}{}_\alpha H \arrow[r, "W_{23}"] & H_\alpha\underset{\tilde\mu}{\otimes}{}_{\hat\beta} H_\alpha\underset{\tilde\mu}{\otimes}{}_{\hat\beta} H, \\
H_\beta\underset{\tilde\mu}{\otimes}{}_{(\mathrm{id}\underset{\tilde\mu}{\otimes}\alpha)}(H_\alpha\underset{\tilde\mu}{\otimes}{}_{\hat\beta}H) \arrow[rr, "W_{13}"] & & (H_\beta\underset{\tilde\mu}{\otimes}{}_\alpha H)_{(\alpha\underset{\tilde\mu}{\otimes}\mathrm{id})}\underset{\tilde\mu}{\otimes}{}_{\hat\beta}H \arrow[u, "W_{12}"']
\end{tikzcd}
\tag{37}
$$

where $W_{12} = W\underset{\tilde\mu}{\otimes}\mathrm{id}$, $W_{23} = \mathrm{id}\underset{\tilde\mu}{\otimes}W$, and W_{13} acts on the first and third tensor factor; see [27] for details.

2.5.2. **Lemma.** *Diagram* (37) *commutes, that is,* $W_{23}W_{12} = W_{12}W_{13}W_{23}$.

Proof. A short calculation shows that the adjoints of both compositions are given by

$$\Lambda_\nu(x)\underset{\tilde\mu}{\otimes}\Lambda_\nu(y)\underset{\tilde\mu}{\otimes}\Lambda_\nu(z) \mapsto \sum \Lambda_\nu(z_{(1)}y_{(1)}x)\underset{\tilde\mu}{\otimes}\Lambda_\nu(D^{\frac{1}{2}}(z_{(2)}y_{(2)}))\underset{\tilde\mu}{\otimes}\Lambda_\nu(D^{\frac{1}{2}}(z_{(3)})). \qquad \square$$

2.5.3. **Theorem.** *Let* $(A,\Delta,\epsilon,S,\mu,\phi,\psi)$ *be a measured multiplier* (B,Γ)*-Hopf* $*$*-algebroid such that* μ *admits a GNS-representation via bounded operators* (K,Λ_μ,π_μ) *and its Radon-Nikodym cocycle has a positive square root in* $M(B)$*. Let* $\tilde\mu$ *be the weight on* $N = \pi_\mu(B)''$ *associated to the Hilbert algebra* $\Lambda_\mu(B)$*, let* $\Lambda_\nu\colon A \to \mathcal{L}(H)$ *be the GNS-map for* $\nu = \mu\circ\phi = \mu\circ\psi$*, and define* $\alpha,\beta,\hat\beta\colon N \to \mathcal{L}(H)$ *as in* (25)*. Then the unitaries* $W\colon H_\beta\underset{\tilde\mu}{\otimes}{}_\alpha H \to H_\alpha\underset{\tilde\mu}{\otimes}{}_{\hat\beta}H$ *and* $V\colon H_{\hat\alpha}\underset{\tilde\mu}{\otimes}{}_\beta H \to H_\beta\underset{\tilde\mu}{\otimes}{}_\alpha H$ *defined in Proposition 2.3.2 and 2.3.3 are pseudo-multiplicative in the sense of* [27].

Proof. The assertion on W is just Lemma 2.5.1 and Lemma 2.5.2. For V, the proof is similar. $\square$

2.5.4. **Definition.** Let $(A,\Delta,\epsilon,S,\mu,\phi,\psi)$, W and V as in Theorem 2.5.3. Then we call W and V the *left* and the *right pseudo-multiplicative unitary of* $(A,\Delta,\epsilon,S,\mu,\phi,\psi)$, respectively.

Recall from [26] that a *Hopf-von Neumann bimodule* over $(N,\tilde\mu)$ is a von Neumann algebra M acting on a Hilbert space L together with faithful, non-degenerate, commuting normal representations $\gamma,\delta\colon N \to M$ and a non-degenerate, normal $*$-homomorphism $\Delta_M\colon M \to M_\delta\underset{\tilde\mu}{*}{}_\gamma M$ such that $\Delta_M\circ\gamma = \gamma\underset{\tilde\mu}{\otimes}\mathrm{id}$, $\Delta_M\circ\delta = \mathrm{id}\underset{\tilde\mu}{\otimes}\delta$ and $(\Delta_M\underset{\tilde\mu}{*}\mathrm{id})\circ\Delta_M = (\mathrm{id}\underset{\tilde\mu}{*}\Delta_M)$, where $M_\delta\underset{\tilde\mu}{*}{}_\gamma M = (M'\underset{\tilde\mu}{\otimes}M')' \subseteq \mathcal{L}(L_\delta\underset{\tilde\mu}{\otimes}{}_\gamma L)$, and $\Delta_M\underset{\tilde\mu}{*}\mathrm{id}$ and $\mathrm{id}\underset{\tilde\mu}{*}\Delta_M$ are suitably defined [16].

2.5.5. **Lemma.** *The following relations hold:*

$$
\begin{aligned}
&\alpha(N)\cup\beta(N) \subseteq \pi_\nu(A)'' \subseteq \hat\beta(N)'\cap\hat\alpha(N)',\\
&\hat\beta(N)\cup\alpha(N) \subseteq \rho(\hat A)'' \subseteq \beta(N)'\cap\hat\alpha(N)'
\end{aligned}
\tag{38}
$$

and

$$\pi_\nu(A)' = \{S \in \beta(N)' \cap \alpha(N)' \mid (S {}_\alpha\underset{\tilde\mu}{\otimes}{}_{\hat\beta} 1)W = W(S {}_\beta\underset{\tilde\mu}{\otimes}{}_\alpha 1)\},$$
(39)
$$\rho(\hat A)' = \{T \in \alpha(N)' \cap \hat\beta(N)' \mid (1 {}_\alpha\underset{\tilde\mu}{\otimes}{}_{\hat\beta} T)W = W(1 {}_\beta\underset{\tilde\mu}{\otimes}{}_\alpha T)\}.$$

Proof. The inclusions in (38) follow from Lemma 2.2.2, non-degeneracy of $\pi_\nu(A)$ and $\rho(\hat A)$ and equation (18). The equations (39) follow from (38) and Remark 2.4.3. □

Using (38) and (39) and slightly abusing notation, we define faithful, normal, non-degenerate $*$-homomorphisms

$$\Delta\colon \pi_\nu(A)'' \to \mathcal{L}(H {}_\beta\underset{\tilde\mu}{\otimes}{}_\alpha H), \quad x \mapsto W^*(\mathrm{id} \underset{\tilde\mu}{\otimes} x)W,$$
(40)
$$\hat\Delta\colon \rho(\hat A)'' \to \mathcal{L}(H {}_\alpha\underset{\tilde\mu}{\otimes}{}_{\hat\beta} H), \quad y \mapsto \Sigma W(y \underset{\tilde\mu}{\otimes} \mathrm{id})W^*\Sigma.$$

2.5.6. **Theorem.** *Let $\mathcal{A} = (A, \Delta, \epsilon, S, \mu, \phi, \psi)$ be a measured multiplier (B, Γ)-Hopf $*$-algebroid such that μ admits a GNS-representation via bounded operators $(K, \Lambda_\mu, \pi_\mu)$ and the Radon-Nikodym cocycle of μ has a positive square root in $M(B)$. Let $\tilde\mu$ be the n.s.f. weight on $N = \pi_\mu(B)''$ associated to the Hilbert algebra $\Lambda_\mu(B)$ and let $\Lambda_\nu\colon A \to \mathcal{L}(H)$ be the GNS-map for $\nu = \mu \circ \phi = \mu \circ \psi$. Define $\pi_\nu\colon A \to \mathcal{L}(H)$, $\rho\colon \hat A \to \mathcal{L}(H)$ as in (36), $\alpha, \beta, \hat\beta\colon N \to \mathcal{L}(H)$ as in (25) and $\Delta, \hat\Delta$ as in (40), where $W\colon H {}_\beta\underset{\tilde\mu}{\otimes}{}_\alpha H \to H {}_\alpha\underset{\tilde\mu}{\otimes}{}_{\hat\beta} H$ is the left pseudo-multiplicative unitary of $\mathcal{A}$. Then $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ and $(\rho(\hat A)'', \widehat\beta, \alpha, \hat\Delta)$ are Hopf-von Neumann bimodules over $(N, \tilde\mu)$.*

Proof. The tuples $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ and $(\rho(\hat A)'', \widehat\beta, \alpha, \hat\Delta)$ are the Hopf-von Neumann bimodules associated to the pseudo-multiplicative unitary W. More precisely, the assertion follows from Proposition 10.3.10 and Theorem 10.3.11 in [23] and equation (39). □

2.5.7. **Definition.** Let $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$, $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ and $(\rho(\hat A)'', \widehat\beta, \alpha, \hat\Delta)$ be as in Theorem 2.5.6. Then we call $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ the *Hopf-von Neumann bimodule of* $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$ and $(\rho(\hat A)'', \widehat\beta, \alpha, \hat\Delta)$ the *dual Hopf-von Neumann bimodule of* $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$.

Theorem 2.5.6 above can also be deduced from the following explicit formulas for Δ and $\hat\Delta$:

2.5.8. **Lemma.** *For all $a, c, x, y \in A$,*

$$\Delta(\pi_\nu(a))(\Lambda_\nu(x) \underset{\tilde\mu}{\otimes} \Lambda_\nu(y)) = \sum \Lambda_\nu(a_{(1)}x) \underset{\tilde\mu}{\otimes} \Lambda_\nu(D^{\frac12}(a_{(2)})y),$$
$$\hat\Delta(\rho(\hat c))(\Lambda_\nu(x) \underset{\tilde\mu}{\otimes} \Lambda_\nu(y)) = \sum \Lambda_\nu(x_{(2)} r(\psi(S(c)x_{(1)}y_{(1)}))) \underset{\tilde\mu}{\otimes} \Lambda_\nu(y_{(2)}),$$

Proof. We calculate

$$\begin{aligned}\Delta(\pi_\nu(a))\sum \Lambda_\nu(y_{(1)}x) \underset{\tilde\mu}{\otimes} \Lambda_\nu(D^{\frac12}(y_{(2)})) &= W^*(\mathrm{id}\underset{\tilde\mu}{\otimes}\pi_\nu(a))WW^*(\Lambda_\nu(x)\underset{\tilde\mu}{\otimes}\Lambda_\nu(y)) \\ &= W^*(\Lambda_\nu(x)\underset{\tilde\mu}{\otimes}\Lambda_\nu(ay)) \\ &= \sum \Lambda_\nu(a_{(1)}y_{(1)}x)\underset{\tilde\mu}{\otimes}\Lambda_\nu(D^{\frac12}(a_{(2)}y_{(2)})),\end{aligned}$$

$$\begin{aligned}W^*\hat\Delta(\rho(\hat c))(\Lambda_\nu(x)\underset{\tilde\mu}{\otimes}\Lambda_\nu(y)) &= (\rho(\hat c)\underset{\tilde\mu}{\otimes}\mathrm{id})W^*(\Lambda_\nu(x)\underset{\tilde\mu}{\otimes}\Lambda_\nu(y)) \\ &= \sum \rho(\hat c)\Lambda_\nu(y_{(1)}x)\underset{\tilde\mu}{\otimes}\Lambda_\nu(D^{\frac12}(y_{(2)})) \\ &= \sum \Lambda_\nu(y_{(2)}x_{(2)}r(\psi(S(c)y_{(1)}x_{(1)})))\underset{\tilde\mu}{\otimes}\Lambda_\nu(D^{\frac12}(y_{(3)})) \\ &= W^*\sum \Lambda_\nu(x_{(2)}r(\psi(S(c)y_{(1)}x_{(1)})))\underset{\tilde\mu}{\otimes}\Lambda_\nu(y_{(2)}). \qquad \square\end{aligned}$$

2.5.9. **Remark.** Under the identification (30), for all homogeneous $a, x, y \in A$ and $\zeta \in K$

$$\Delta(\pi_\nu(a))(\Lambda_\nu(x)\otimes\zeta\otimes\Lambda_\nu(y)) = \sum \Lambda_\nu(a_{(1)}x)\otimes U_{\bar\partial_{a_{(1)}}}\zeta\otimes\Lambda_\nu(a_{(2)}y),$$

where $a_{(1)}$ is assumed to be homogeneous without loss of generality.

2.6. **The Hopf C^*-bimodules.** The fundamental unitary W is regular C^*-pseudo-multiplicative unitaries in the sense of [24], and therefore yields Hopf C^*-bimodules which are completions of A and $\hat A$. To prove this, we again need some preliminaries concerning the relative tensor product in the setting of C^*-algebras; for details, see [22] and [24]. The construction is parallel to the von Neumann-algebraic setting and differs mainly in notation.

As before, let $\mathfrak{b} = (K, [\pi_\mu(B)], [\pi_\mu(B)])$. The relative tensor product $H_{E_\psi^\dagger}\underset{\mathfrak b}{\otimes}{}_{E_\phi^\dagger}H$ of the C^*-$\mathfrak b$-modules $(H, E_\psi^\dagger)$ and $(H, E_\phi^\dagger)$ is the separated completion of the algebraic tensor product $E_\psi^\dagger\otimes K\otimes E_\phi^\dagger$ with respect to the sesquilinear form given by

$$\langle\xi\otimes\zeta\otimes\eta|\xi'\otimes\zeta'\otimes\eta'\rangle = \langle\zeta|(\xi^*\xi')(\eta^*\eta')\zeta'\rangle. \tag{41}$$

It can be regarded as a twofold internal tensor product of Hilbert C^*-modules and identified with certain separated completions $E_\psi^\dagger \circleddash_\alpha H$ and $H_\beta \oslash E_\phi^\dagger$ of the algebraic tensor products $E_\psi^\dagger\otimes H$ and $H\otimes E_\phi^\dagger$, respectively, such that

$$E_\psi^\dagger \circleddash{}_\alpha H \cong H_{E_\psi^\dagger}\underset{\mathfrak b}{\otimes}{}_{E_\phi^\dagger}H \cong H_\beta \oslash E_\phi^\dagger, \quad \xi\circleddash\eta\zeta \equiv \xi\otimes\zeta\otimes\eta \equiv \xi\zeta\oslash\eta. \tag{42}$$

Comparing the sesquilinear forms (28) with (41) and using (27), one finds that there exists an isomorphism

$$H_\beta\underset{\tilde\mu}{\otimes}{}_\alpha H \cong H_{E_\psi^\dagger}\underset{\mathfrak b}{\otimes}{}_{E_\phi^\dagger}H, \quad \Lambda_\nu(x)\otimes\zeta\otimes\Lambda_\nu(y) \equiv \Lambda_\psi^\dagger(x)\otimes\zeta\otimes\Lambda_\phi^\dagger(y). \tag{43}$$

For each $\xi\in E_\psi^\dagger$ and $\eta\in E_\phi^\dagger$, there exist bounded linear operators

$$|\xi\rangle_1\colon H\to H_{E_\psi^\dagger}\underset{\mathfrak b}{\otimes}{}_{E_\phi^\dagger}H,\ \eta'\mapsto\xi\circleddash\eta', \qquad |\eta\rangle_2\colon H\to H_{E_\psi^\dagger}\underset{\mathfrak b}{\otimes}{}_{E_\phi^\dagger}H,\ \xi'\mapsto\xi'\oslash\eta.$$

We denote their adjoints by $\langle\xi|_1$ and $\langle\eta|_2$, respectively, and write $|E^\dagger_\psi\rangle_1 = \{|\xi\rangle_1 : \xi \in E^\dagger_\psi\}$, $|E^\dagger_\phi\rangle_2 = \{|\eta\rangle_2 : \eta \in E^\dagger_\phi\}$ et cetera. Comparing with (34), we see that under the identification (43), $\lambda^{\widehat{\alpha},\beta}_{\Lambda_\nu(x)} \equiv |\Lambda^\dagger_\psi(x)\rangle_1$ and $\rho^{\widehat{\alpha},\beta}_{\Lambda_\nu(y)} \equiv |\Lambda^\dagger_\phi(y)\rangle_2$ for all $x, y \in A$.

Replacing $E^\dagger_\psi$ and $E^\dagger_\phi$ by $E^\dagger_\phi$ and E_ϕ, respectively, one similarly defines the relative tensor product $H_{E^\dagger_\phi} \underset{\mathfrak{b}}{\otimes}_{E_\phi} H$ with a canonical isomorphism $H_{E^\dagger_\phi} \underset{\mathfrak{b}}{\otimes}_{E_\phi} H \cong H_\alpha \underset{\tilde{\mu}}{\otimes}_{\widehat{\beta}} H$, and operators $|\xi\rangle_1, |\eta\rangle_2 \colon H \to H_{E^\dagger_\phi} \underset{\mathfrak{b}}{\otimes}_{E_\phi} H$ for all $\xi \in E^\dagger_\phi$ and $\eta \in E_\phi$.

Thus, W can be regarded as a unitary $H_{E^\dagger_\psi} \underset{\mathfrak{b}}{\otimes}_{E^\dagger_\phi} H \to H_{E^\dagger_\phi} \underset{\mathfrak{b}}{\otimes}_{E_\phi} H$.

2.6.1. **Lemma.** *For all $x, x', y, y' \in A$ and $\gamma \in \{\alpha, \beta, \widehat{\beta}\}$, $\gamma' \in \{\alpha, \widehat{\alpha}, \beta\}$,*

$$\Lambda_\nu(x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y) \in D((H_\beta \underset{\tilde{\mu}}{\otimes}_\alpha H)_{\mathrm{id} \underset{\tilde{\mu}}{\otimes} \gamma}, \tilde{\mu}),$$

$$R^{\mathrm{id} \underset{\tilde{\mu}}{\otimes} \gamma, \tilde{\mu}}_{\Lambda_\nu(x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y)} = \lambda^{\beta,\alpha}_{\Lambda_\nu(x)} R^{\gamma,\tilde{\mu}}_{\Lambda_\nu(y)} = |\Lambda^\dagger_\psi(x)\rangle_1 R^{\gamma,\tilde{\mu}}_{\Lambda_\nu(y)},$$

$$\Lambda_\nu(x') \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y') \in D((H_\alpha \underset{\tilde{\mu}}{\otimes}_{\widehat{\beta}} H)_{\gamma' \underset{\tilde{\mu}}{\otimes} \mathrm{id}}, \tilde{\mu}),$$

$$R^{\gamma' \underset{\tilde{\mu}}{\otimes} \mathrm{id}, \tilde{\mu}}_{\Lambda_\nu(x') \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y')} = \rho^{\alpha,\widehat{\beta}}_{\Lambda_\nu(y')} R^{\gamma',\tilde{\mu}}_{\Lambda_\nu(x')} = |\Lambda_\phi(y')\rangle_2 R^{\gamma',\tilde{\mu}}_{\Lambda_\nu(x')}.$$

2.6.2. **Proposition.** *The following equations for subspaces of $\mathcal{L}(H, H_{E^\dagger_\psi} \underset{\mathfrak{b}}{\otimes}_{E^\dagger_\phi} H)$ hold:*

$$W[|E^\dagger_\psi\rangle_1 E_\phi] = [|E_\phi\rangle_2 E^\dagger_\psi], \quad W[|E^\dagger_\phi\rangle_2 E_\psi] = [|E_\phi\rangle_2 E_\psi], \quad W[|E^\dagger_\phi\rangle_2 E_\phi] = [|E_\phi\rangle_2 E_\phi],$$
$$W[|E^\dagger_\phi\rangle_2 E^\dagger_\phi] = [|E^\dagger_\phi\rangle_1 E^\dagger_\phi], \quad W[|E^\dagger_\psi\rangle_1 E^\dagger_\psi] = [|E^\dagger_\phi\rangle_1 E^\dagger_\psi], \quad W[|E^\dagger_\psi\rangle_1 E_\psi] = [|E^\dagger_\phi\rangle_1 E_\psi].$$

Proof. We only prove the first equation; the others follow similarly:

$$\begin{aligned} W[|E^\dagger_\psi\rangle_1 E_\phi] &= [\{W R^{\mathrm{id} \underset{\tilde{\mu}}{\otimes} \widehat{\beta}, \tilde{\mu}}_\omega : \omega \in \Lambda_\nu(A) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(A)\}] && \text{(Lemma 2.6.1 and (2.2.3))} \\ &= [\{R^{\beta \underset{\tilde{\mu}}{\otimes} \mathrm{id}, \tilde{\mu}}_{W\omega} : \omega \in \Lambda_\nu(A) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(A)\}] && \text{(Lemma 2.5.1)} \\ &= [\{R^{\beta \underset{\tilde{\mu}}{\otimes} \mathrm{id}, \tilde{\mu}}_{\omega'} : \omega' \in \Lambda_\nu(A) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(A)\}] && \text{(Definition of } W\text{)} \\ &= [|E_\phi\rangle_2 E^\dagger_\psi]. && \text{(Lemma 2.6.1 and 2.2.3)} \quad \square \end{aligned}$$

2.6.3. **Theorem.** *Let $\mathcal{A} = (A, \Delta, \epsilon, S, \mu, \phi, \psi)$ be a measured multiplier (B, Γ)-Hopf $*$-algebroid such that μ admits a GNS-representation via bounded operators $(K, \Lambda_\mu, \pi_\mu)$ and the Radon-Nikodym cocycle of μ has a positive square root in $M(B)$. Let $\mathfrak{b} = (K, [\pi_\mu(B)], [\pi_\mu(B)])$, let $\Lambda_\nu \colon A \to \mathcal{L}(H)$ be the GNS-map of $\nu = \mu \circ \phi = \mu \circ \psi$ and define $E_\phi, E^\dagger_\phi, E_\psi, E^\dagger_\psi \subseteq \mathcal{L}(K, H)$ as in* (24). *Then the left and the right pseudo-multiplicative unitary W and V of $\mathcal{A}$, regarded as operators $H_{E^\dagger_\psi} \underset{\mathfrak{b}}{\otimes}_{E^\dagger_\phi} H \to H_{E^\dagger_\phi} \underset{\mathfrak{b}}{\otimes}_{E_\phi} H$ and $H_{E_\psi} \underset{\mathfrak{b}}{\otimes}_{E^\dagger_\psi} H \to H_{E^\dagger_\psi} \underset{\mathfrak{b}}{\otimes}_{E^\dagger_\phi} H$ as above, are C^*-pseudo-multiplicative unitaries in the sense of* [24].

Proof. The assertion on W is Proposition 2.6.2 and Lemma 2.5.2. For V, the proof is similar. $\square$

2.6.4. **Proposition.** *W and V are regular in the sense that $[\langle E^{\dagger}_{\phi}|_1 W|E^{\dagger}_{\phi}\rangle_2] = [E^{\dagger}_{\phi}(E^{\dagger}_{\phi})^*] \subseteq \mathcal{L}(H)$ and $[\langle E^{\dagger}_{\psi}|_1 V|E^{\dagger}_{\psi}\rangle_2] = [E^{\dagger}_{\psi}(E^{\dagger}_{\psi})^*] \subseteq \mathcal{L}(H)$.*

Proof. Let $x, x', y \in A$. Then $\Lambda^{\dagger}_{\phi}(y)\Lambda^{\dagger}_{\phi}(x)^*\Lambda_\nu(y') = \Lambda_\nu(r(\phi(y'\theta(x^*)))y)$ by Lemma 2.2.1 and

$$\begin{aligned}
\langle \Lambda^{\dagger}_{\phi}(y)|_2 W^*|\Lambda^{\dagger}_{\phi}(x)\rangle_1 \Lambda_\nu(y') &= (\rho^{\beta,\alpha}_{\Lambda_\nu(y)})^* W^*(\Lambda_\nu(x) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(y')) \\
&= \sum \beta(\langle \Lambda_\nu(y)|\Lambda_\nu(D^{\frac{1}{2}}(y'_{(2)}))\rangle_{\alpha,\tilde{\mu}})\Lambda_\nu(y'_{(1)}x) \\
&= \sum \Lambda_\nu(s(\phi(D^{\frac{1}{2}}(y'_{(2)})\theta(y^*)))y'_{(1)}x) && \text{(Eq. (27))} \\
&= \sum \Lambda_\nu(s(\phi(y'_{(2)}z))y'_{(1)}x) && (z := D^{-\frac{1}{2}}(\theta(y^*))) \\
&= \sum \Lambda_\nu(r(\phi(y'z_{(2)}))S^{-1}(z_{(1)})x). && \text{(Prop. 1.5.3)}
\end{aligned}$$

Since the maps $\theta, D^{-\frac{1}{2}}, S$ and T_3 are bijections, we can conclude

$$[\{\Lambda^{\dagger}_{\phi}(y)\Lambda^{\dagger}_{\phi}(x)^* : x, y \in A\}] = [\{\langle \Lambda^{\dagger}_{\phi}(x)|_2 W^*|\Lambda^{\dagger}_{\phi}(y)\rangle_1 : x, y \in A\}].$$

The assertion on V follows from a similar calculation. $\square$

Recall from [24] that a *Hopf C^*-bimodule over $\mathfrak{b}$* consists of a C^*-$(\mathfrak{b},\mathfrak{b})$-module (L, E, F), a non-degenerate C^*-algebra $C \subseteq \mathcal{L}(L)$ satisfying $\rho_E(\pi_\mu(B)) \subseteq M(C)$ and $\rho_F(\pi_\mu(B)) \subseteq M(C)$, and a non-degenerate $*$-homomorphism $\Delta_C \colon C \to C_F \underset{\mathfrak{b}}{*}{}_E C$ that is co-associative and compatible with E and F in a suitable sense, where

$$\begin{aligned}
C_F \underset{\mathfrak{b}}{*}{}_E C = \{T \in \mathcal{L}(L_F \underset{\mathfrak{b}}{\otimes}{}_E L) : T|F\rangle_1 + T^*|F\rangle_1 \subseteq [|F\rangle_1 C] \\
\text{and } T|E\rangle_2 + T^*|E\rangle_2 \subseteq [|E\rangle_2 C]\}
\end{aligned}$$

is the *fiber product* of C with itself relative to F and E.

2.6.5. **Theorem.** *Let $\mathcal{A} = (A, \Delta, \epsilon, S, \mu, \phi, \psi)$ be a measured multiplier (B, Γ)-Hopf $*$-algebroid such that μ admits a GNS-representation via bounded operators $(K, \Lambda_\mu, \pi_\mu)$ and its Radon-Nikodym cocycle has a positive square root in $M(B)$. Let $\mathfrak{b} = (K, [\pi_\mu(B)], [\pi_\mu(B)])$, let $\Lambda_\nu \colon A \to \mathcal{L}(H)$ be the GNS-map for $\nu = \mu \circ \phi = \mu \circ \psi$, let $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ be the Hopf-von Neumann bimodule of $\mathcal{A}$, let $(\rho(\hat{A})'', \hat{\beta}, \alpha, \hat{\Delta})$ be the dual Hopf-von Neumann bimodule of $\mathcal{A}$, define $E_\phi, E^{\dagger}_{\phi}, E_\psi, E^{\dagger}_{\psi} \subseteq \mathcal{L}(K, H)$ as in (24) and regard Δ and $\hat{\Delta}$ as maps $\pi_\nu(A)'' \to \mathcal{L}(H_{E^{\dagger}_{\psi}} \underset{\mathfrak{b}}{\otimes}{}_{E^{\dagger}_{\phi}} H)$ and $\rho(\hat{A})'' \to \mathcal{L}(H_{E^{\dagger}_{\phi}} \underset{\mathfrak{b}}{\otimes}{}_{E_\phi} H)$ as above. Then $((H, E^{\dagger}_{\phi}, E^{\dagger}_{\psi}), [\pi_\nu(A)], \Delta|_{[\pi_\nu(A)]})$ and $((H, E_\psi, E^{\dagger}_{\phi}), [\rho(\hat{A})], \hat{\Delta}|_{[\rho(\hat{A})]})$ are Hopf C^*-bimodules over $\mathfrak{b}$.*

Proof. By Theorem 2.6.3 and Proposition 2.6.4, the left pseudo-multiplicative unitary W of $\mathcal{A}$, regarded as a unitary $H_{E^{\dagger}_{\psi}} \underset{\mathfrak{b}}{\otimes}{}_{E^{\dagger}_{\phi}} H \to H_{E^{\dagger}_{\phi}} \underset{\mathfrak{b}}{\otimes}{}_{E_\phi} H$, is a regular C^*-pseudo-multiplicative unitary. By [24], the latter gives rise to two Hopf C^*-bimodules

$$((H, E^{\dagger}_{\phi}, E^{\dagger}_{\psi}), [\langle E_\phi|_2 W|E^{\dagger}_{\phi}\rangle_2], \Delta) \qquad \text{and} \qquad ((H, E_\psi, E^{\dagger}_{\psi}), [\langle E^{\dagger}_{\phi}|_1 W|E^{\dagger}_{\psi}\rangle_1], \hat{\Delta}).$$

Finally, by Lemma 2.4.1, $[\langle E_\phi^\dagger|_1 W|E_\psi^\dagger\rangle_1] = [\rho(\hat{A})]$ and $[\langle E_\phi|_2 W|E_\phi^\dagger\rangle_2] = [\lambda(\hat{A})]$. □

2.7. **The measured quantum groupoid.** To obtain a measured quantum groupoid, we finally extend ν, ϕ, ψ to normal, semi-finite, faithful weights on the level of von Neumann algebras. We impose the following simplifying assumptions:

- **(A3)** the bimodule ${}_rA_s$ is *proper* in the sense that $r(B)s(B) \subseteq A$.
- **(A4)** There exists a net $(u_i)_i$ in B that is *truncating for* μ in the sense that $(\pi_\mu(u_i))_i$ is a net of positive elements in the unit ball of $\pi_\mu(B)$ that converges in $M([\pi_\mu(B)])$ strictly to 1 and such that $(\pi_\mu(u_i^2))_i$ is increasing.

Note that a net $(u_i)_i$ as in (A4) exists always if we drop the condition that $(\pi_\mu(u_i^2))_i$ should be increasing.

Let us also note that in the bi-measured case where ϕ, ψ and ν arise from a bi-integral h on (A, Δ), the extensions of ϕ, ψ, ν and the invariance of these extensions can be proved quite easily, see Remark 2.7.5 and 2.7.10.

For the extension of ν, we do not need the assumptions (A3) and (A4), but use the modular automorphism θ for ν obtained in Theorem 1.6.5, the theory of Hilbert algebras [20], and results of Kustermans and van Daele [11].

2.7.1. **Lemma.** *$\Lambda_\nu(A) \subseteq H$ is a Hilbert algebra with respect to the $*$-algebra structure inherited from A.*

Proof. The multiplication $\Lambda_\nu(y) \mapsto \Lambda_\nu(xy)$ is bounded for each $x \in A$ by Theorem 2.4.2, and the involution $\Lambda_\nu(x) \mapsto \Lambda_\nu(x^*)$ is pre-closed because

$$\langle \Lambda_\nu(x)|\Lambda_\nu(y^*)\rangle = \nu(x^*y^*) = \nu(y^*\theta(x^*)) = \langle \Lambda_\nu(y)|\Lambda_\nu(\theta(x^*))\rangle \quad \text{for all } x, y \in A. \qquad \square$$

The general theory of Hilbert algebras [20] now yields

- $M = \pi_\nu(A)'' \subseteq \mathcal{L}(H)$ as the associated von Neumann algebra,
- a n.s.f. weight $\tilde{\nu}$ on M such that $\tilde{\nu}(\pi_\nu(a^*a)) = \langle \Lambda_\nu(a)|\Lambda_\nu(a)\rangle = \nu(a^*a)$ for all $a \in A$,
- a left ideal $\mathfrak{N}_{\tilde{\nu}} := \{x \in M : \tilde{\nu}(x^*x) < \infty\} \subseteq M$ of square-integrable elements,
- a closed map $\Lambda_{\tilde{\nu}} \colon \mathfrak{N}_{\tilde{\nu}} \to H$ such that $(H, \Lambda_{\tilde{\nu}}, \mathrm{id}_M)$ is a GNS-representation for $\tilde{\nu}$; this is the closure of the map $\pi_\nu(A) \to H$ given by $\pi_\nu(a) \to \Lambda_\nu(a)$;
- the usual objects $J_{\tilde{\nu}}, \Delta_{\tilde{\nu}}, \sigma^{\tilde{\nu}}, \mathcal{T}_{\tilde{\nu}}, \ldots$ of Tomita-Takesaki theory.

The modular automorphism θ is related to the modular automorphism group $\sigma^{\tilde{\nu}}$ as follows:

2.7.2. **Proposition.** *$\pi_\nu(A) \subseteq \mathcal{T}_{\tilde{\nu}}$ and $\sigma^{\tilde{\nu}}_{ni}(\pi_\nu(a)) = \pi_\nu(\theta^{-n}(a))$ for all $a \in A$, $n \in \mathbb{Z}$.*

Proof. Use the arguments in [12, §3], in particular from Lemma 3.16 till Proposition 3.22. □

Let $A^\theta := \{a \in A : \theta(a) = a\} \subseteq A$. Note that this space is a $*$-subalgebra and, by (A3), contains $r(B)s(B)$.

2.7.3. **Lemma.** i) *$\sigma^{\tilde{\nu}}$ acts trivially on $\pi_\nu(A^\theta)''$, in particular on $\alpha(N)$ and $\beta(N)$.*
ii) *$J_{\tilde{\nu}}\alpha(x)^*J_{\tilde{\nu}} = \widehat{\beta}(x)$ and $J_{\tilde{\nu}}\beta(x)^*J_{\tilde{\nu}} = \widehat{\alpha}(x)$ for all $x \in N$.*

Proof. i) The first assertion follows from the fact that $\sigma_t^{\tilde{\nu}}(x) = \Delta_{\tilde{\nu}}^{it} x \Delta_{\tilde{\nu}}^{-it}$ and $\Delta_{\tilde{\nu}}^{-1} x \Delta_{\tilde{\nu}} = x$ for each $x \in \pi_{\tilde{\nu}}(A^\theta)$ by Proposition 2.7.2, and the second assertion follows from the fact that $\sigma_t^{\tilde{\nu}}$ is normal for all $t \in \mathbb{R}$ and acts trivially on $\pi_\nu(r(B)s(B))$.

ii) Combine i) and Lemma 2.2.2. □

2.7.4. **Proposition.** *There exist unique n.s.f. weights T_L from M to $\alpha(N)$ and T_R from M to $\beta(N)$ such that $\tilde{\mu} \circ \alpha^{-1} \circ T_L = \tilde{\nu} = \tilde{\mu} \circ \beta^{-1} \circ T_R$.*

Proof. This follows from Lemma 2.7.3 i) and [18, 10.1] or [20, IX Theorem 4.18]. □

We thus obtain extensions $\tilde{\phi} := \alpha^{-1} \circ T_L$ and $\tilde{\psi} := \beta^{-1} \circ T_R$ of ϕ and ψ.

2.7.5. **Remark.** Assume that $\phi = (\mathrm{id} \otimes \mu) \circ h$ and $\psi = (\mu \otimes \mathrm{id}) \circ h$ for a normalized bi-integral h on (A, Δ). Then the map $\Lambda_\mu(B) \otimes \Lambda_\mu(B) \to \Lambda_\nu(A)$ given by $\Lambda_\mu(b) \otimes \Lambda_\mu(b') \mapsto \Lambda_\nu(r(b)s(b'))$ extends to an isometry $\iota\colon K \otimes K \to H$, and a short calculation shows that $\iota^* \pi_\nu(a) \iota = (\pi_\mu \otimes \pi_\mu)(h(a))$ for all $a \in A$. We therefore get a positive, normal, linear extension $\tilde{h}\colon M \to N$, $x \mapsto \iota^* x \iota$, of h, and thereby the desired extensions $\tilde{\phi} = (\mathrm{id} \bar{\otimes} \tilde{\mu}) \circ \tilde{h}$, $\tilde{\psi} = (\tilde{\mu} \bar{\otimes} \tilde{\mu}) \circ \tilde{h}$ and $\tilde{\nu} = (\tilde{\mu} \bar{\otimes} \tilde{\mu}) \circ \tilde{h}$.

Recall that an element $\xi \in H$ is *right-bounded* with respect to the Hilbert algebra $\Lambda_\nu(A)$ if there exists an operator $R_\xi \in \mathcal{L}(H)$ such that $\pi_\nu(a)\xi = R_\xi \Lambda_\nu(a)$ for all $a \in A$. Note that then $R_\xi \in M'$.

2.7.6. **Lemma.** i) *If $x \in A^\theta$, then $\Lambda_\nu(x) \in H$ is right-bounded, $R_{\Lambda_\nu(x)} = J_\nu \pi_\nu(x)^* J_\nu$ and $\|R_{\Lambda_\nu(x)}\| = \|\pi_\nu(x)\|$.*

ii) *If $x \in A^\theta \cap r(B)'$, then $\pi_\nu(a)\Lambda_\phi(x) = R_{\Lambda_\nu(x)} \Lambda_\phi(a)$ for all $a \in A$.*

iii) *If $a \in A$ and $\xi \in K$ is right-bounded with respect to $\Lambda_\mu(B)$, then $\Lambda_\phi(a)\xi = \widehat{\beta}(R_\xi)\Lambda_\nu(a)$.*

Proof. i) For all $x \in A^\theta, a \in A$, we have $\pi_\nu(a)\Lambda_\nu(x) = \Lambda_\nu(ax) = J_\nu \pi_\nu(x)^* J_\nu \Lambda_\nu(a)$.

ii) For all $x \in A^\theta \cap r(B)', a \in A, b \in B$,

$$\begin{aligned}\pi_\nu(a)\Lambda_\phi(x)\Lambda_\mu(b) &= \Lambda_\nu(axr(b)) = \Lambda_\nu(ar(b)x) \\ &= \pi_\nu(ar(b))\Lambda_\nu(x) = R_{\Lambda_\nu(x)}\Lambda_\nu(ar(b)) = R_{\Lambda_\nu(x)}\Lambda_\phi(a)\Lambda_\mu(b).\end{aligned}$$

iii) If $a \in A$ and $\xi = \Lambda_\mu(b)$ for some $b \in B$, then $R_\xi = \pi_\mu(b)$ and $\Lambda_\phi(a)\xi = \Lambda_\nu(ar(b)) = \widehat{\beta}(\pi_\mu(b))\Lambda_\nu(a)$. Now, the assertion follows for all right-bounded ξ because $\Lambda_\mu(B)$ is a core for $\Lambda_{\tilde{\mu}}$ and the right-bounded elements coincide with $\Lambda_{\tilde{\mu}}(\mathfrak{N}_{\tilde{\mu}})$. □

To prove Theorem 2.7.9, we construct increasing approximations of the weights $\tilde{\mu}, \tilde{\nu}, \tilde{\phi}, \tilde{\psi}$ by bounded positive maps, using an approximate unit $(u_i)_i$ in B with the properties assumed in (A4). Let $u_{i,j} := r(u_i)s(u_j) \in A$, and define for all i, j bounded, normal, positive, linear maps

$$\mu_i\colon N \to \mathbb{C},\ x \mapsto \langle \Lambda_\mu(u_i) | x \Lambda_\mu(u_i) \rangle, \qquad \nu_{i,j}\colon M \to \mathbb{C},\ x \mapsto \langle \Lambda_\nu(u_{i,j}) | x \Lambda_\nu(u_{i,j}) \rangle,$$
$$\phi_{i,j}\colon M \to N,\ x \mapsto \Lambda_\phi(u_{i,j})^* x \Lambda_\phi(u_{i,j}), \qquad \psi_{i,j}\colon M \to N,\ x \mapsto \Lambda_\psi(u_{i,j})^* x \Lambda_\psi(u_{i,j}).$$

Given a net $(\lambda_\kappa)_\kappa$ of real numbers, we write $(\lambda_\kappa)_\kappa \nearrow \lambda$ if it is increasing and converges to λ. Likewise, given a von Neumann algebra C with a net $(\omega_\kappa)_\kappa$ in C_*^+ and a n.s.f. weight ω, we write $(\omega_\kappa)_\kappa \nearrow \omega$ if $\omega_\kappa(x^*x) \nearrow \omega(x^*x)$ for all $x \in C$.

2.7.7. **Proposition.** i) $(\mu_i)_i \nearrow \tilde{\mu}$ *and* $(\nu_{i,j})_{i,j} \nearrow \tilde{\nu}$;
ii) $(\upsilon \circ \phi_{i,j})_{i,j} \nearrow \upsilon \circ \tilde{\phi}$ *and* $(\upsilon \circ \psi_{i,j})_{i,j} \nearrow \upsilon \circ \tilde{\psi}$ *for all* $\upsilon \in N_*^+$.

Proof. i) We only prove the assertion concerning $\tilde{\nu}$.

Let $\xi_{i,j} := \Lambda_\nu(u_{i,j})$ and $R_{i,j} := R_{\xi_{i,j}} = J_\nu \pi_\nu(u_{i,j}) J_\nu$ for all i, j.

The net $(\nu_{i,j})_{i,j}$ in M_*^+ is increasing because $(R_{i,j}^* R_{i,j})_{i,j}$ is increasing by assumption on $(u_i)_i$, $\nu_{i,j}(\pi_\nu(a^*a)) = \|R_{i,j}\Lambda_\nu(a)\|^2$ for all $a \in A$ and $\pi_\nu(A) \subseteq M$ is weakly dense.

Call $\xi \in H$ *right-contractive* if ξ is right-bounded and $\|R_\xi\| \leqslant 1$. Let $x \in M$. Then

$$\tilde{\nu}(x^*x) = \sup\left\{\|x\xi\|^2 \,\middle|\, \xi \in H \text{ is right-contractive}\right\}.$$

Each $\xi_{i,j}$ is right-contractive by Lemma 2.7.6 and hence $\nu_{i,j}(x^*x) = \|x\Lambda_\nu(u_{i,j})\|^2 \leqslant \tilde{\nu}(x^*x)$ for all i, j. Conversely, for each right-contractive $\xi \in H$,

$$\|x\xi\|^2 = \lim_{i,j} \|x\pi_\nu(u_{i,j})\xi\|^2 = \lim_{i,j} \|xR_\xi\Lambda_\nu(u_{i,j})\|^2 \leqslant \lim_{i,j} \|x\Lambda_\nu(u_{i,j})\|^2 = \lim_{i,j} \nu_{i,j}(x^*x)$$

because $R_\xi \in M'$ and $R_\xi^* R_\xi \leqslant 1$. Therefore, $\tilde{\nu}(x^*x) \leqslant \lim_{i,j} \nu_{i,j}(x^*x)$.

ii) We only prove the assertion concerning $\tilde{\phi}$. A similar argument as above and Lemma 2.7.6 ii) show that for each $\upsilon \in N_*^+$, the net $(\upsilon \circ \phi_{i,j})_{i,j}$ is increasing. Taking pointwise limits, we obtain a normal semi-finite weight ω from M to N such that for each $y \in M$, the element $\omega(y^*y)$ in the extended positive part $\hat{N}_+$ is defined by $\upsilon(\omega(y)) = \sup_{i,j} \upsilon(\phi_{i,j}(y^*y))$ for all $\upsilon \in N_*^+$. Then for all $y \in M$,

$$\begin{aligned}\tilde{\mu}(\omega(y^*y)) \underset{i,j,k}{\nwarrow} \|y\Lambda_\phi(u_{i,j})\Lambda_\mu(u_k)\|^2 &= \|y\hat{\beta}(\pi_\mu(u_k))\xi_{i,j}\|^2 \\ &\xrightarrow{k\to\infty} \|y\xi_{i,j}\|^2 = \nu_{i,j}(y^*y) \underset{i,j}{\nearrow} \tilde{\nu}(y^*y)\end{aligned}$$

and hence $\tilde{\mu} \circ \omega = \tilde{\nu}$. By [20, Theorem 4.18], $\omega = \tilde{\phi}$. □

2.7.8. **Lemma.** $W^* \rho^{\alpha,\hat{\beta}}_{\Lambda_\nu(r(b)s(b'))}\beta(\pi_\mu(b'')) = \rho^{\beta,\alpha}_{\Lambda_\nu(r(b'')s(b'))}\alpha(\pi_\mu(b))$ *for all* $b, b', b'' \in B$.

Proof. Applying both sides to $\Lambda_\nu(a)$, where $a \in A$ is arbitrary, we obtain $W^*(\Lambda_\nu(s(b'')a) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(r(b)s(b')))$ and $\Lambda_\nu(r(b)a) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(r(b'')s(b'))$, respectively, which coincide. □

As usual, let $\mathfrak{N}_{T_L} := \{x \in M : T_L(x^*x) \in N\}$ and similarly define $\mathfrak{N}_{T_R}$.

2.7.9. **Theorem.** *Let* $\mathcal{A} = (A, \Delta, \epsilon, S, \mu, \phi, \psi)$ *be a measured multiplier* (B, Γ)*-Hopf* $*$*-algebroid satisfying the following conditions:*

(A1) μ *admits a GNS-representation via bounded operators* $(K, \Lambda_\mu, \pi_\mu)$,
(A2) *the Radon-Nikodym cocycle of* μ *has a positive square root in* $M(B)$,
(A3) *the bimodule* ${}_rA_s$ *is proper,*
(A4) *there exists a truncating net for* μ.

Let $\tilde{\mu}$ *be the weight on* $N = \pi_\mu(B)''$ *associated to the Hilbert algebra* $\Lambda_\mu(B)$*, let* $\Lambda_\nu \colon A \to \mathcal{L}(H)$ *be the GNS-map for* $\nu = \mu \circ \phi = \mu \circ \psi$*, let* $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ *be the Hopf-von Neumann bimodule of* $\mathcal{A}$ *(Definition 2.5.7), let* $\tilde{\nu}$ *be the weight on* $M = \pi_\nu(A)''$ *associated to the Hilbert algebra* $\Lambda_\nu(A)$*, and let* T_L *and* T_R *be the n.s.f. weights from* M *to* $\alpha(N)$ *and* $\beta(N)$ *given by* $\tilde{\mu} \circ \alpha^{-1} \circ T_L = \tilde{\nu} = \tilde{\mu} \circ \beta^{-1} \circ T_R$ *(see Proposition 2.7.4).*

Then $(N, \tilde{\mu}, M, \alpha, \beta, \Delta, T_L, T_R, \tilde{\nu})$ *is a measured quantum groupoid in the sense of* [2]. *In particular,* T_L *and* T_R *are left- and right-invariant with respect to* Δ *in the sense that*

$$T_L((\lambda_\xi^{\beta,\alpha})^* \Delta(x^*x)\lambda_\xi^{\beta,\alpha}) = \alpha((R_\xi^{\beta,\tilde{\mu}})^* T_L(x^*x) R_\xi^{\beta,\tilde{\mu}}) \quad \textit{for all } x \in \mathfrak{N}_{T_L}, \xi \in D(H_\beta, \tilde{\mu}),$$
$$T_R((\rho_\eta^{\beta,\alpha})^* \Delta(x^*x)\rho_\eta^{\beta,\alpha}) = \beta((R_\eta^{\alpha,\tilde{\mu}})^* T_R(x^*x) R_\eta^{\alpha,\tilde{\mu}}) \quad \textit{for all } x \in \mathfrak{N}_{T_R}, \eta \in D(H_\alpha, \tilde{\mu}).$$

Proof. We use the same notation as before. To prove the assertion concerning $\tilde{\phi}$ and T_L, we show that

$$\langle \zeta | \tilde{\phi}((\lambda_\xi^{\beta,\alpha})^* \Delta(x^*x)\lambda_\xi^{\beta,\alpha})\zeta\rangle = \|\alpha(\tilde{\phi}(x^*x))^{\frac{1}{2}} R_\xi^{\beta,\tilde{\mu}}\zeta\|^2 \tag{44}$$

for all $x \in \mathfrak{N}_{T_L}$, $\xi \in D(H_\beta, \tilde{\mu})$ and $\zeta \in K$. Given such x, ξ, ζ, let $\xi_k := \alpha(\pi_\mu(u_k))\xi$ and

$$c_{i,j,k} := \langle \zeta | \phi_{i,j}((\lambda_{\xi_k}^{\beta,\alpha})^* \Delta(x^*x)\lambda_{\xi_k}^{\beta,\alpha})\zeta\rangle \quad \text{for all } i, j, k.$$

Then $R_{\xi_k}^{\beta,\tilde{\mu}} = \alpha(\pi_\mu(u_k))R_\xi^{\beta,\tilde{\mu}}$, $\lambda_{\xi_k}^{\beta,\alpha} = (\alpha(\pi_\mu(u_k)) \underset{\tilde{\mu}}{\otimes} \mathrm{id})\lambda_\xi^{\beta,\alpha}$, and by Proposition 2.7.7,

$$c_{i,j,k} \xrightarrow{k\to\infty} \langle \zeta | \phi_{i,j}((\lambda_\xi^{\beta,\alpha})^* \Delta(x^*x)\lambda_\xi^{\beta,\alpha})\zeta\rangle \underset{i,j}{\nearrow} \langle \zeta | \tilde{\phi}((\lambda_\xi^{\beta,\alpha})^* \Delta(x^*x)\lambda_\xi^{\beta,\alpha})\zeta\rangle.$$

On the other hand, using the relation $\Lambda_\phi(u_{i,j}) = \Lambda_\phi^\dagger(u_{i,j})$, we find

$$\begin{aligned}
c_{i,j,k} &= \|(1 \underset{\mu}{\otimes} x)W\lambda_{\xi_k}^{\beta,\alpha}\Lambda_\phi(u_{i,j})\zeta\|^2 && \text{(Definition of } \Delta_W \text{ and } \phi_{i,j})\\
&= \|(1 \underset{\mu}{\otimes} x)W\rho_{\Lambda_\nu(u_{i,j})}^{\beta,\alpha}\alpha(\pi_\mu(u_k))R_\xi^{\beta,\tilde{\mu}}\zeta\|^2 && \text{(Definition of } H_\beta \underset{\tilde{\mu}}{\otimes}{}_\alpha H)\\
&= \|(1 \underset{\mu}{\otimes} x)\rho_{\Lambda_\nu(u_{k,j})}^{\hat{\alpha},\beta}\beta(\pi_\mu(u_i))R_\eta^{\beta,\tilde{\mu}}\zeta\|^2 && \text{(Lemma 2.7.8)}\\
&= \|\hat{\alpha}(\phi_{k,j}(x^*x))^{\frac{1}{2}}\beta(\pi_\mu(u_i))R_\xi^{\beta,\tilde{\mu}}\zeta\|^2 &&\\
&\underset{i,j,k}{\nearrow} \|\hat{\alpha}(\tilde{\phi}(x^*x))^{\frac{1}{2}}R_\xi^{\beta,\tilde{\mu}}\zeta\|^2. && \text{(Proposition 2.7.7)}
\end{aligned}$$

Thus, (44) follows. The assertion concerning $\tilde{\psi}$ and T_R can be proven similarly, where W has to be replaced by the unitary V. □

2.7.10. **Remark.** Assume that $\phi = (\mathrm{id} \otimes \mu) \circ h$ for a normalized bi-integral h on (A, Δ). Then for each $b \in B$, the map $\Lambda_\mu(B) \to \Lambda_\nu(A)$ given by $\Lambda_\mu(c) \mapsto \Lambda_\nu(s(b)r(c))$ is bounded with norm less than or equal to $\mu(b^*b)^{\frac{1}{2}}$, and therefore extends to an operator $\Lambda_\phi(s(b)) \in \mathcal{L}(K, H)$. One can then approximate $\tilde{\phi}$ monotonously by the maps $\phi_i \colon M \to N$, $x \mapsto \Lambda_\phi(s(u_i))^* x \Lambda_\phi(s(u_i))$, and a similar calculation as in Lemma 2.7.8 shows that each ϕ_i is right-invariant.

Associated to the measured quantum groupoid $(N, \tilde{\mu}, M, \alpha, \beta, \Delta, T_L, T_R, \tilde{\nu})$ are two fundamental unitaries $U'_H \colon H_{\hat{\alpha}} \underset{\tilde{\mu}}{\otimes}{}_\beta H \to H_\beta \underset{\tilde{\mu}}{\otimes}{}_\alpha H$ and $U_H \colon H_\alpha \underset{\tilde{\mu}}{\otimes}{}_{\hat{\beta}} H \to H_\beta \underset{\tilde{\mu}}{\otimes}{}_\alpha H$, characterized by

$$(\lambda_w^{\beta,\alpha})^* U_H(v \underset{\tilde{\mu}}{\otimes} \Lambda_{\tilde{\nu}}(a)) = \Lambda_{\tilde{\nu}}((\omega_{w,v} * \mathrm{id})(\Delta(a))) \quad \text{for } v, w \in D(H_\beta, \tilde{\mu}), a \in \mathfrak{N}_{\tilde{\nu}} \cap \mathfrak{N}_{T_L},$$
$$(\rho_{w'}^{\beta,\alpha})^* U'_H(\Lambda_{\tilde{\nu}}(a') \underset{\tilde{\mu}}{\otimes} v') = \Lambda_{\tilde{\nu}}((\mathrm{id} * \omega_{w',v'})(\Delta(a'))) \quad \text{for } v', w' \in D(H_\alpha, \tilde{\mu}), a' \in \mathfrak{N}_{\tilde{\nu}} \cap \mathfrak{N}_{T_R};$$

see [13, Proposition 3.17].

2.7.11. **Proposition.** *$W^* = U_H$ and $V = U'_H$.*

Proof. Let $x, y, y', z \in A$ and choose $v_i, w_i \in A$ such that $\sum \bar{D}^{\frac{1}{2}}(y_{(1)})x' \underset{B}{\otimes} y_{(2)} = \sum v_i \underset{B}{\otimes} w_i$ in ${}_sA \underset{B}{\otimes} {}_rA$. Then

$$\begin{aligned}
(\omega_{\Lambda_\nu(x),\Lambda_\nu(x')} * \mathrm{id})(W^*)\Lambda_\nu(y) &= \sum_i (\lambda^{\beta,\alpha}_{\Lambda_\nu(x)})^*(\Lambda_\nu(v_i) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(w_i)) \\
&= \sum_i \Lambda_\nu(r(\psi(v_i\theta(x^*)))w_i), \\
(\omega_{\Lambda_\nu(x),\Lambda_\nu(x')} * \mathrm{id})(\Delta(y))\Lambda_\nu(z) &= \sum_i (\lambda^{\beta,\alpha}_{\Lambda_\nu(x)})^*(\Lambda_\nu(v_i) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(w_i z)) \\
&= \sum_i \pi_\nu(r(\psi(v_i\theta(x^*))))\Lambda_\nu(w_i z),
\end{aligned}$$

and hence $(\omega_{\Lambda_\nu(x),\Lambda_\nu(x')} * \mathrm{id})(W^*)\Lambda_\nu(y) = \Lambda_{\tilde{\nu}}((\omega_{\Lambda_\nu(x),\Lambda_\nu(x')} * \mathrm{id})(\Delta(y)))$. Likewise, with $v'_i, w'_i \in A$ such that $\sum \bar{D}^{\frac{1}{2}}(x_{(1)}) \underset{B}{\otimes} x_{(2)}y' = \sum v'_i \underset{B}{\otimes} w'_i \in {}_sA \underset{B}{\otimes} {}_rA$, we find

$$\begin{aligned}
(\mathrm{id} * \omega_{\Lambda_\nu(y),\Lambda_\nu(y')})(V)\Lambda_\nu(x) &= \sum_i (\rho^{\beta,\alpha}_{\Lambda_\nu(y)})^*(\Lambda_\nu(v'_i) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(w'_i)) \\
&= \sum_i \Lambda_\nu(s(\phi(w'_i\theta(y^*)))v'_i), \\
(\mathrm{id} * \omega_{\Lambda_\nu(y'),\Lambda_\nu(y)})(\Delta(\pi_\nu(x)))\Lambda_\nu(z) &= \sum_i (\rho^{\beta,\alpha}_{\Lambda_\nu(y)})^*(\Lambda_\nu(v'_i z) \underset{\tilde{\mu}}{\otimes} \Lambda_\nu(w'_i)) \\
&= \sum_i \pi_\nu(s(w'_i\theta(y^*)))\Lambda_\nu(v'_i z)
\end{aligned}$$

and hence $(\mathrm{id} * \omega_{\Lambda_\nu(y),\Lambda_\nu(y')})(V)\Lambda_\nu(x) = \Lambda_{\tilde{\nu}}\big((\mathrm{id} * \omega_{\Lambda_\nu(y'),\Lambda_\nu(y)})(\Delta(\pi_\nu(x)))\big)$. □

The adapted measured quantum groupoid $(N, \tilde{\mu}, M, \alpha, \beta, \Delta, T_L, T_R, \tilde{\nu})$ has an antipode $\tilde{S}$ which is characterized by the following properties:

i) $\mathrm{span}\{(\mathrm{id} * \omega_{v,w} * \mathrm{id})(V) : w, v \in \mathcal{T}_{\tilde{\nu},T_R}\}$ is a core for $\tilde{S}$,
ii) $\tilde{S}((\omega_{w,v} * \mathrm{id})(V)) = (\omega_{w,v} * \mathrm{id})(V^*)$ for all $w, v \in \mathcal{T}_{\tilde{\nu},T_R}$,

where $\mathcal{T}_{\tilde{\nu},T_R}$ is the set of all $x \in M$ that are analytic with respect to $\sigma^{\tilde{\nu}}$ and satisfy $\sigma^{\tilde{\nu}}_z \in \mathfrak{N}_{\tilde{\nu}} \cap \mathfrak{N}^*_{\tilde{\nu}} \cap \mathfrak{N}_{T_R} \cap \mathfrak{N}^*_{T_R}$ for all $z \in \mathbb{C}$. Likewise, one defines $\mathcal{T}_{\tilde{\nu},T_L}$.

2.7.12. **Lemma.** *$\pi_\nu(A) \subseteq \mathcal{T}_{\tilde{\nu},T_R} \cap \mathcal{T}_{\tilde{\nu},T_L}$.*

Proof. Recall that $\pi_\nu(A) \subseteq \mathcal{T}_{\tilde{\nu}}$ by Proposition 2.7.2. Using Lemma 2.7.3 i), we find

$$\sigma^{\tilde{\nu}}_z(\pi_\nu(A)) = \sigma^{\tilde{\nu}}_z(\pi_\nu(As(B))) = \sigma^{\tilde{\nu}}_z(\pi_\nu(A))\beta(\pi_\mu(B)) \subseteq \mathfrak{N}_{\tilde{\nu}}\beta(\mathfrak{N}_{\tilde{\mu}}) \subseteq \mathfrak{N}_{T_R}$$

for all $z \in \mathbb{C}$. Consequently, $\pi_\nu(A) \subseteq \mathcal{T}_{\tilde{\nu},T_R}$. A similar argument shows that $\pi_\nu(A) \subseteq \mathcal{T}_{\tilde{\nu},T_L}$. □

2.7.13. **Proposition.** *$\pi_\nu(A) \subseteq \mathrm{Dom}(\tilde{S})$ and $\tilde{S}(\pi_\nu(a)) = \pi_\nu(D^{\frac{1}{2}}SD^{\frac{1}{2}}(a))$ for all $a \in A$.*

Proof. Let $x, x' \in A$ and $a = \sum D^{-\frac{1}{2}}(x'_{(2)}r(\psi(x^*x'_{(1)})))$. Then

$$\begin{aligned}
(\omega_{\Lambda_\nu(x),\Lambda_\nu(x')} * \mathrm{id})(V) &= \pi_\nu(a), && \text{(Lemma 2.4.4)} \\
(\omega_{\Lambda_\nu(x),\Lambda_\nu(x')} * \mathrm{id})(V^*) &= \Big((\lambda^{\beta,\alpha}_{\Lambda_\nu(x')})^* V \lambda^{\widehat{\alpha},\beta}_{\Lambda_\nu(x)}\Big)^* \\
&= \sum \pi_\nu(D^{-\frac{1}{2}}(x_{(2)}r(\psi(x'^* x_{(1)}))))^* && \text{(Lemma 2.4.4)} \\
&= \sum \pi_\nu(D^{\frac{1}{2}}(r(\psi(x^*_{(1)}x'))x^*_{(2)})) \\
&= \sum \pi_\nu(D^{\frac{1}{2}}(S(x'_{(2)}r(\psi(x^*x'_{(1)}))))) && \text{(Proposition 1.5.3)} \\
&= \pi_\nu(D^{\frac{1}{2}}SD^{\frac{1}{2}}(a)). && \square
\end{aligned}$$

Acknowledgments. I thank Erik Koelink for introducing me to dynamical quantum groups and for stimulating discussion, and the referee for very helpful remarks that improved the presentation.

References

[1] G. Böhm. Hopf algebroids. In *Handbook of algebra. Vol. 6*, volume 6 of *Handb. Algebr.*, pages 173–235. Elsevier/North-Holland, Amsterdam, 2009.
[2] M. Enock. Measured quantum groupoids in action. *Mém. Soc. Math. Fr. (N.S.)*, (114):ii+150 pp. (2009), 2008.
[3] M. Enock. Inclusions of von Neumann algebras and quantum groupoids III. *J. Funct. Anal.*, 223(2):311-364, 2005.
[4] M. Enock and R. Nest. Irreducible inclusions of factors, multiplicative unitaries, and Kac algebras. *J. Funct. Anal.*, 137(2):466–543, 1996.
[5] P. Etingof and D. Calaque. Lectures on tensor categories. In *Quantum groups*, volume 12 of *IRMA Lect. Math. Theor. Phys.*, pages 1–38. Eur. Math. Soc., Zürich, 2008.
[6] P. Etingof and F. Latour. *The dynamical Yang-Baxter equation, representation theory, and quantum integrable systems*, volume 29 of *Oxford Lecture Series in Mathematics and its Applications.* Oxford University Press, Oxford, 2005.
[7] P. Etingof and A. Varchenko. Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups. *Comm. Math. Phys.*, 196(3):591–640, 1998.
[8] P. Etingof and A. Varchenko. Exchange dynamical quantum groups. *Comm. Math. Phys.*, 205(1):19–52, 1999.
[9] L. Kadison and K. Szlachányi. Bialgebroid actions on depth two extensions and duality. *Adv. Math.*, 179(1):75–121, 2003.
[10] E. Koelink and H. Rosengren. Harmonic analysis on the SU(2) dynamical quantum group. *Acta Appl. Math.*, 69(2):163–220, 2001.
[11] J. Kustermans and A. van Daele. C^*-algebraic quantum groups arising from algebraic quantum groups. *Internat. J. Math.*, 8(8):1067–1139, 1997.
[12] J. Kustermans and A. van Daele. C^*-algebraic quantum groups arising from algebraic quantum groups. *Internat. J. Math.*, 8(8):1067–1139, 1997.
[13] F. Lesieur. Measured quantum groupoids. *Mém. Soc. Math. Fr. (N.S.)*, (109):iv+158 pp. (2008), 2007.
[14] D. Nikshych and L. Vainerman. Algebraic versions of a finite-dimensional quantum groupoid. In *Hopf algebras and quantum groups (Brussels, 1998)*, volume 209 of *Lecture Notes in Pure and Appl. Math.*, pages 189–220. Dekker, New York, 2000.
[15] D. Nikshych and L. Vainerman. A characterization of depth 2 subfactors of II_1 factors. *J. Funct. Anal.*, 171(2):278–307, 2000.

[16] J.-L. Sauvageot. Produits tensoriels de Z-modules et applications. In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, volume 1132 of *Lecture Notes in Mathematics*, pages 468–485. Springer, Berlin, 1985.

[17] P. Schauenburg. Weak Hopf algebras and quantum groupoids. In *Noncommutative geometry and quantum groups (Warsaw, 2001)*, volume 61 of *Banach Center Publ.*, pages 171–188. Polish Acad. Sci., Warsaw, 2003.

[18] Ş. Strătilă. *Modular theory in operator algebras.* Editura Academiei Republicii Socialiste România, Bucharest, 1981. Translated from the Romanian by the author.

[19] K. Szlachányi. Finite quantum groupoids and inclusions of finite type. In *Mathematical physics in mathematics and physics (Siena, 2000)*, volume 30 of *Fields Inst. Commun.*, pages 393–407. Amer. Math. Soc., Providence, RI, 2001.

[20] M. Takesaki. *Theory of operator algebras. II*, volume 125 of *Encyclopaedia of Mathematical Sciences.* Springer-Verlag, Berlin, 2003.

[21] T. Timmermann. Free dynamical quantum groups and the dynamical quantum group $\mathrm{SU}_q^{\mathrm{d}}(2)$. In *Operator Algebras and Quantum Groups (Warsaw, 2011)*, volume 98 of *Banach Center Publ.*, pages 311–341. Polish Acad. Sci., Warsaw, 2012.

[22] T. Timmermann. The relative tensor product and a minimal fiber product in the setting of C^*-algebras. *J. Operator Theory*, 68(2):101-140, 2012.

[23] T. Timmermann. *An invitation to quantum groups and duality.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[24] T. Timmermann. C^*-pseudo-multiplicative unitaries, Hopf C^*-bimodules and their Fourier algebras. *J. Inst. Math. Jussieu*, 11:189–229, 2011.

[25] T. Timmermann and A. Van Daele. Regular multiplier Hopf algebroids. Basic theory and examples. in preparation.

[26] J.-M. Vallin. Bimodules de Hopf et poids opératoriels de Haar. *J. Operator Theory*, 35(1):39–65, 1996.

[27] J.-M. Vallin. Unitaire pseudo-multiplicatif associé à un groupoïde. Applications à la moyennabilité. *J. Operator Theory*, 44(2):347–368, 2000.

[28] J.-M. Vallin. Groupoïdes quantiques finis. *J. Algebra*, 239(1):215–261, 2001.

[29] J.-M. Vallin. Measured quantum groupoids associated with matched pairs of locally compact groupoids. arXiv:0906.5247.

[30] A. van Daele. Multiplier Hopf algebras. *Trans. Amer. Math. Soc.*, 342(2):917–932, 1994.

[31] A. van Daele. An algebraic framework for group duality. *Adv. Math.*, 140(2):323–366, 1998.

[32] A. Van Daele. Tools for working with multiplier Hopf algebras. *Arab. J. Sci. Eng. Sect. C Theme Issues*, 33(2):505–527, 2008.

[33] A. Van Daele and S. Wang. Weak multiplier Hopf algebras. Preliminaries, motivation and basic examples. In *Operator Algebras and Quantum Groups (Warsaw, 2011)*, volume 98 of *Banach Center Publ.*, pages 367–415. Polish Acad. Sci., Warsaw, 2012.

[34] A. Van Daele and S. Wang. Weak multiplier Hopf algebras. The main theory. arXiv:1210.4395.

[35] S. L. Woronowicz. Compact quantum groups. In *Symétries quantiques (Les Houches, 1995)*, pages 845–884. North-Holland, Amsterdam, 1998.

T. TIMMERMANN, University of Muenster, Einsteinstr. 62, 48149 Muenster, Germany
E-mail address: timmermt@math.uni-muenster.de