C^*-pseudo-multiplicative unitaries

and

compact Hopf C^*-bimodules

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What is a Hopf bimodule?

compact groupoid:

$$\begin{tikzcd} G^0 & G \arrow[l, shift right, "r"'] \arrow[l, shift left, "s"] & G\,{}_s\!\times_r G \arrow[l, "m"'] \end{tikzcd}$$

unital Hopf bimodule:

$$\begin{tikzcd} B \arrow[r, shift left, "\rho"] \arrow[r, shift right, "\sigma"'] & A \arrow[r, "\Delta"] \arrow[d, phantom, "\circlearrowleft"] & A_\sigma *_\rho A \arrow[d, phantom, "\circlearrowleft"] \\ & H & H_\sigma \otimes_\rho H \end{tikzcd}$$

fiber product

relative tensor product

$$\begin{tikzcd} C(G^0) \arrow[r, shift left, "r^*"] \arrow[r, shift right, "s^*"'] & C(G) \arrow[r, "m^*"] & C(G)_{s^*} \otimes_{r^*} C(G) \end{tikzcd}$$

$$\begin{tikzcd} C(G^0) \arrow[r, shift left] \arrow[r, shift right] & M(C^*_{(r)}(G)) \arrow[r] & ??? \end{tikzcd}$$

Quantum groupoids in the setting of operator algebras

Enock, Lesieur, Vallin: Hopf-von Neumann-bimodules,

examples arising from inclusions of factors,

quantum groupoids in the setting of von Neumann algebras

Blanchard; Enock: continuous bundles of locally compact

quantum groups in the setting of C^*-algebras

Böhm, Szlachányi, Nikshych: finite quantum groupoids/weak

Hopf C^*-algebras

O'uchi; T.: pseudo-multiplicative unitaries on C^*-modules

The fiber product of von Neumann algebras

$$H_1 \circlearrowleft M_1 \xleftarrow{\sigma} N^{op} \circlearrowright K \circlearrowleft N \xrightarrow{\rho} M_2 \circlearrowright H_2$$

Fact: $x, y \in \mathcal{L}_{N^{op}}(K, H_1) \Rightarrow x^* y \in \mathcal{L}_{N^{op}}(K) = N$

$\Rightarrow \mathcal{L}_{N^{op}}(K, H_1)$ Hilbert C^*-module over N,

$\mathcal{L}_N(K, H_2)$ Hilbert C^*-module over N^{op}

Defn:

- $H_1 {}_\sigma\!\otimes_\rho H_2 := \mathcal{L}_{N^{op}}(K, H_1) \otimes_N K_{N^{op}} \otimes \mathcal{L}_N(K, H_2)$

  $\big(\langle x \otimes \zeta \otimes y | x' \otimes \zeta' \otimes y' \rangle = \langle \zeta | x^* x' \cdot y^* y' \cdot \zeta' \rangle\big)$
- $M_1 {}_\sigma\!*_\rho M_2 := (M_1' \otimes_N \mathrm{id}_{N^{op}} \otimes M_2')' \circlearrowright H_1 {}_\sigma\!\otimes_\rho H_2$

What is a Hilbert module over a KMS-weight?

Given:

- C^*-algebra B with faithful KMS-weight μ

  $\rightsquigarrow$ GNS-space $H_\mu \circlearrowleft B$

Defn:

- *Hilbert C^*-module over* $\mu := (H, \alpha)$ or H_α, where

  H Hilbert space, $\alpha \subseteq \mathcal{L}(H_\mu, H)$ closed subspace,

  $[\alpha H_\mu] = H, \ [\alpha^* \alpha] = B, \ [\alpha B] = \alpha$
- $\mathcal{L}(H_\alpha, K_\beta) = \{T \in \mathcal{L}(H, K) \mid T\alpha \subseteq \beta, T^*\beta \subseteq \alpha\}$

Facts:

- $\alpha \otimes_B H_\mu \ni \xi \otimes_\beta \eta \mapsto \xi\eta \in H$ is an isomorphism
- $\mathcal{L}(H_\mu) \supseteq B'$ acts on $H \cong \alpha \otimes_B H_\mu$ via $\rho_\alpha : x \mapsto \mathrm{id} \otimes_B x$

Examples of Hilbert C^*-modules over KMS-weights

1. B finite-dimensional $\Rightarrow \alpha = \mathcal{L}_{B'}(H_\mu, H)$

   $\{\text{Hilbert } C^*\text{-modules over } \mu\} \leftrightarrows \{\text{nd. reps of } B' \subseteq \mathcal{L}(H_\mu)\}$

2. B commutative $\Rightarrow \alpha \equiv$ cont. field of Hilbert spaces $\mathcal{H}$ on $\widehat{B}$,

   H = direct integral of $\mathcal{H}$ over $(\widehat{B}, \mu)$

3. $B \subseteq A$, cond. expectation $\phi: A \to B$ s.t. $\mu \circ \phi$ KMS-weight

   $\rightsquigarrow$ GNS-map $\Lambda_\phi: A \to \mathcal{L}(H_\mu, H_{\mu\circ\phi})$, $\Lambda_\phi(a)\Lambda_\mu(b) = \Lambda_{\mu\circ\phi}(ab)$

   $\rightsquigarrow \left(H_{\mu\circ\phi}, [\Lambda_\phi(A)]\right)$ Hilbert C^*-module over μ

What is a Hilbert bimodule over KMS-weights?

Given: C^*-algebras B, C with KMS-weights μ, ν

$\rightsquigarrow B^{op}$ with $\mu^{op}: b^{op} \mapsto \mu(b)$ $\rightsquigarrow$ $B^{op} \circlearrowright H_{\mu^{op}} \cong H_\mu \circlearrowleft B$

Defn: *Hilbert C^*-bimodule over* $(\mu^{op}, \nu) := (H, \alpha, \beta)$ or ${}_\alpha H_\beta$

- H_α, H_β Hilbert C^*-modules over μ^{op}, ν
- $$\begin{tikzcd} H_\mu \arrow[r, dashed, "\alpha"] \arrow[rr, dashed, bend right, "\alpha"'] & H \arrow[r, dashed, "\rho_\beta(C^{op})"] & H \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H_\nu \arrow[r, dashed, "\beta"] \arrow[rr, dashed, bend right, "\beta"'] & H \arrow[r, dashed, "\rho_\alpha(B)"] & H \end{tikzcd}$$

Ex: $B \subseteq A$, cond. expectation $\phi: A \to B$ s.t. $\mu \circ \phi$ KMS-weight

$\rightsquigarrow \big(H_{\mu\circ\phi}, [\Lambda_{\phi^{op}}(A^{op})], [\Lambda_\phi(A)]\big)$ Hilbert C^*-bim./(μ^{op}, μ)

The relative tensor product of Hilbert C^*-bimodules

Given: Hilbert C^*-bimodules ${}_\alpha H_\beta$, ${}_\gamma K_\delta$ over (τ^{op}, μ), (μ^{op}, ν)

$$
\rightsquigarrow \quad
\begin{tikzcd}
H_{\tau^{op}} \arrow[dr, dashed, "\alpha"] \arrow[dddrr, dashed, bend right] & & H_\mu = H_{\mu^{op}} \arrow[dl, dashed, "\beta"'] \arrow[dr, dashed, "\gamma"] & & H_\nu \arrow[dl, dashed, "\delta"'] \arrow[dddll, dashed, bend left] \\
& H \arrow[d, equal] & & K \arrow[d, equal] & \\
& \beta \otimes_B H_\mu \arrow[dr, dashed, "r(\gamma)"] & & H_\mu {}_{B^{op}}\otimes \gamma \arrow[dl, dashed, "l(\beta)"'] & \\
& & \beta \otimes_B H_\mu {}_{B^{op}}\otimes \gamma & &
\end{tikzcd}
$$

$H_\beta \otimes_\gamma K$

Hilbert C^*-bimodule over (τ^{op}, ν)

Thm:

- obtain bicategory of KMS-weights, Hilbert C^*-bimod.s over KMS-weights, and their operators
- ${}_\alpha H_\beta \mapsto (H, \rho_\alpha, \rho_\beta)$ yields embedding into bicategory of Hilbert modules over von Neumann algebras

What is a C^*-algebra over KMS-weights?

Defn:

- *C^*-algebra over μ* := (H_α, A) or A^α_H, where
  H_α Hilbert C^*-module over μ,
  $A \circlearrowright H$ nd. C^*-algebra s.t. $\rho_\alpha(B^{op}) \subseteq M(A)$
- $\mathrm{Mor}(A^\alpha_H, B^\beta_K) := \{$ morphisms $A \to B$ with sufficiently many intertwiners in $\mathcal{L}(H_\alpha, K_\beta)\}$
- similarly: category of C^*-algebras over (μ^{op}, ν)

Ex: $B \subseteq A$, cond. expectation $\phi: A \to B$ s.t. $\mu \circ \phi$ KMS-weight

$\rightsquigarrow (H_{\mu\circ\phi}, [\Lambda_{\phi^{op}}(A^{op})])$ and A form a C^*-algebra over μ^{op}

The fiber product of C^*-algebras over KMS-weights

Given: $A_H^{\alpha,\beta}$, $B_K^{\gamma,\delta}$ C^*-algebras over (μ^{op},τ), (τ^{op},ν)

$$\rightsquigarrow \quad \begin{tikzcd} H \arrow[r, dashed, "r(\gamma)"] \arrow[d, dashed, "A"'] & H {}_\beta\otimes_\gamma K \arrow[d, "A {}_\beta *_\gamma B"] & K \arrow[l, dashed, "l(\beta)"'] \arrow[d, dashed, "B"] \\ H \arrow[r, dashed, "r(\gamma)"] & H {}_\beta\otimes_\gamma K & K \arrow[l, dashed, "l(\beta)"'] \end{tikzcd}$$

Defn: $A {}_\beta *_\gamma B := \{T \,:\, T^{(*)} l(\beta) \subseteq [l(\beta)B],\ T^{(*)} r(\gamma) \subseteq [r(\gamma)A]\}$

Facts:

- $A {}_\beta *_\gamma B$ C^*-algebra over (μ^{op},ν) if it is nd.
- $\big(A \cap \mathcal{L}(H_\alpha)\big) {}_\beta\otimes_\gamma \big(B \cap \mathcal{L}(K_\beta)\big) \subseteq A {}_\beta *_\gamma B \subseteq A'' {}_{\rho_\beta} *_{\rho_\gamma} B''$
- fiber product is functorial but not associative

What is a Hopf C^*-bimodule over a KMS-weight?

Definition:

- *Hopf C^*-bimodule over* μ := C^*-algebra $A_H^{\alpha,\beta}$ over (μ, μ^{op}) + *comultiplication* $\Delta \in \mathrm{Mor}\left(A_H^{\alpha,\beta}, A_H^{\alpha,\beta} * A_H^{\alpha,\beta}\right)$ s.t.
  $(\Delta * \mathrm{id}) \circ \Delta = (\mathrm{id} * \Delta) \circ \Delta : A \to \mathcal{L}(H_\alpha \otimes_\beta H_\alpha \otimes_\beta H)$
- *bd. left Haar weight* := cond. exp. $\phi : A \to \rho_\beta(B) \subseteq M(A)$ s.t.
  $\omega(\phi(a)) = \phi((\omega * \mathrm{id})(\Delta(a)))$ for all $\alpha \in A$, $\omega = \xi^*(\cdot)\xi$, $\xi \in \alpha$,
  $$(\omega * \mathrm{id})(x) : H \xrightarrow{l(\xi)} H_\alpha \otimes_\beta H \xrightarrow{x} H_\alpha \otimes_\beta H \xrightarrow{l(\xi)^*} H$$
- *bd. right Haar weight* := . . .

What is a compact Hopf C^*-bimodule?

1)

$$\begin{tikzcd}
 & & A_{\alpha}*_{\beta}A & & \\
\mathbb{C} \arrow[r, shift left] & B \arrow[l, dotted, shift left, "\text{KMS-state } \mu"] \arrow[r, shift left, "\text{unital rep. } \rho"] & A \arrow[l, dotted, shift left, "\text{cond. exp. } \phi"] \arrow[u, "\Delta"] \arrow[r, dotted, shift right, "\text{cond. exp. } \psi"'] \arrow[ll, dotted, bend left, "\text{KMS-state } \nu"] \arrow[rr, dotted, bend right, "\text{KMS-state } \nu^{-1}"'] & B^{op} \arrow[l, shift right, "\text{unital rep. } \sigma"'] \arrow[r, dotted, shift right, "\text{KMS-s. } \mu^{op}"'] & \mathbb{C} \arrow[l, shift right]
\end{tikzcd}$$

$$\delta := d\nu/d\nu^{-1} \in A$$

$\rightsquigarrow$ $H_{\nu} \cong H_{\nu^{-1}} =: H$ Hilbert C^*-module over $(\mu, \mu^{op}, \mu^{op}, \mu)$ w.r.t.

GNS-constructions $\widehat{\alpha}$, β, $\widehat{\beta}$, α for ϕ, ϕ^{op}, ψ, ψ^{op}

$\rightsquigarrow$ $A_H^{\alpha,\beta}$ C^*-algebra over (μ, μ^{op})

2) $(A_H^{\alpha,\beta}, \Delta)$ Hopf C^*-bimod.; $\Delta(\delta) = \delta_{\alpha}\otimes_{\beta}\delta$; ϕ, ψ Haar weights

Example: Tracial center-valued conditional expectation

Given:

$$\begin{tikzcd}
B \arrow[r, "\text{tracial cond. exp. } \tau"] \arrow[rr, bend right, "\text{trace } \mu"'] & Z := Z(B) \arrow[r, "\text{faithful state } \theta"] & \mathbb{C}
\end{tikzcd}$$

Get:

$$\begin{tikzcd}
b \arrow[r, mapsto] & b \underset{Z}{\otimes} 1^{op} & 1 \underset{Z}{\otimes} c^{op} & c^{op} \arrow[l, mapsto] \\
B \arrow[rr, "\rho"] & & B \underset{Z}{\otimes} B^{op} \arrow[ll, dotted, "\phi"] \arrow[r, dotted, "\psi"'] & B^{op} \arrow[l, "\sigma"'] \\
b\tau(c) & & b \underset{Z}{\otimes} c^{op} \arrow[ll, mapsto] \arrow[r, mapsto] \arrow[d, mapsto, "\Delta"] & \tau(b)c^{op} \\
& & (b \underset{Z}{\otimes} 1^{op}) {}_{\alpha}\!\otimes_{\beta} (1 \underset{Z}{\otimes} c^{op}) &
\end{tikzcd}$$

How to proceed?

compact Hopf C^*-bimodule

$\downarrow$

C^*-pseudo-multiplicative unitary

$\downarrow$

- quantum groupoid in the setting of von Neumann algebras [Lesieur], universal C^*-algebra
- dual Hopf C^*-bimodule
- reduced crossed products for coactions, biduality

What is a C^*-pseudo-multiplicative unitary?

Defn: *C^*-pseudo-multiplicative unitary (C^*-p.m.u.) over* μ :=

Hilbert C^*-trimodule $(H, \widehat{\beta}, \alpha, \beta)$ over $(\mu^{op}, \mu, \mu^{op})$

+ unitary $V\colon H_{\widehat{\beta}} \otimes_{\alpha} H \to H_{\alpha} \otimes_{\beta} H$ satisfying

- $V_{12}V_{13}V_{23} = V_{23}V_{12}$ (V_{ij}: op. on $H_{?} \otimes_{?} H_{?} \otimes_{?} H$)
- intertwining relations with regard to $\alpha, \widehat{\beta}, \beta$

Facts:

$$\begin{array}{c}\{\text{p.m.u.s [Vallin, Enock, Lesieur]}\}\\ \cup \\ \{C^*\text{-p.m.u.s}\} \rightsquigarrow \{\text{p.m.u.s on } C^*\text{-modules [T]}\}\\ \cup \\ \{\text{bundles of m.u.s [Blanchard]}\}\end{array}$$

The C^*-p.m.u. of a compact Hopf C^*-bimodule

Given: compact Hopf C^*-bimodule $\begin{matrix} B \\ B^{op} \end{matrix} \underset{\psi}{\overset{\phi}{\rightleftarrows}} A \xrightarrow{\Delta} A_\alpha *_\beta A$,

as before $\nu = \mu \circ \phi$, $H = H_\nu$, $\widehat{\beta} = [\Lambda_\psi(A)]$, $\alpha = [\Lambda_{\psi^{op}}(A^{op})]$

Prop.:

- $\exists$ unitary $H_{\widehat{\beta}} \otimes_\alpha H \cong \widehat{\beta} \otimes_{\rho_\alpha} H \xrightarrow{V} \alpha \otimes_{\rho_\beta} H \cong H_\alpha \otimes_\beta H$
  $$\Lambda_\psi(a) \otimes_{\rho_\alpha} \omega \longmapsto \Delta(a)\big(\Lambda_{\psi^{op}}(1^{op}) \otimes_{\rho_\beta} \omega\big)$$
- V satisfies half of the intertwining relations for a C^*-p.m.u
- if V satisfies all, then $V_{12}V_{13}V_{23} = V_{23}V_{12}$ and V is a C^*-pseudo-multiplicative unitary over μ

Unitary antipode and inversion formula for V

Assume: R anti-automorphism of A ("unitary antipode")

which flips (ρ, ϕ), (σ, ψ) and satisfies $R^2 = \mathrm{id}_A$

$\rightsquigarrow$ involution I on H satisfying $R(a) = Ia^*I$ for all $a \in A$

$\rightsquigarrow$ anti-unitaries $H_{\hat{\beta}} \otimes_{\alpha} H \underset{J_\nu \underset{J_\mu}{\otimes} I}{\overset{J_\nu \underset{J_\mu}{\otimes} I}{\rightleftarrows}} H_{\alpha} \otimes_{\beta} H$

Thm.: $V^* = (J_\nu \underset{J_\mu}{\otimes} I) V (J_\nu \underset{J_\mu}{\otimes} I) \Leftrightarrow$ strong invariance, i.e.

$$\forall x, y \in A : (\tilde{\psi} * \mathrm{id})\big(\Delta(x)(y^{op} \otimes 1)\big) = R\Big((\tilde{\psi} * \mathrm{id})\big((x^{op} \otimes 1)\Delta(y)\big)\Big)$$

Thm.: strong invariance $\Rightarrow$ V is a C^*-p.m.u over μ

The legs of a general C^*-pseudo-multiplicative unitary

$$
\begin{tikzcd}
H \arrow[rr, dashed] \arrow[d, dashed, "r(\alpha)"'] & {\color{red} A(V) := [r(\beta)^* V r(\alpha)]} & H \\
H_{\widehat{\beta}} \otimes_{\alpha} H \arrow[rr, "V"] & & H_{\alpha} \otimes_{\beta} H \arrow[u, dashed, "r(\beta)^*"'] \arrow[d, dashed, "l(\alpha)^*"] \\
H \arrow[u, dashed, "l(\widehat{\beta})"] \arrow[rr, dashed] & {\color{red} \widehat{A}(V) := [l(\alpha)^* V l(\widehat{\beta})]} & H
\end{tikzcd}
$$

Def.: V regular $:\Leftrightarrow [l(\alpha)^* V r(\alpha)] = [\alpha\alpha^*]$

Thm.: V regular $\Rightarrow \left(A(V)_H^{\alpha,\beta}, \Delta\right), \left(\widehat{A}(V)_H^{\widehat{\beta},\alpha}, \widehat{\Delta}\right)$ Hopf C^*-bimod.,

where $\Delta(a) = V(1_{\widehat{\beta}} \otimes_{\alpha} a)V^*$ and $\widehat{\Delta}(\widehat{a}) = V^*(\widehat{a}_{\alpha} \otimes_{\beta} 1)V$

The dual of a compact Hopf C^*-bimodule

Given: compact Hopf C^*-bimodule $\begin{matrix} B \\ B^{op} \end{matrix} \overset{\phi}{\underset{\psi}{\rightleftarrows}} A \xrightarrow{\Delta} A_\alpha *_\beta A$

Thm.: The associated C^*-p.m.u. V is regular

$\rightsquigarrow$ Hopf C^*-bimodule $\left(A(V)_H^{\alpha,\beta}, \Delta_V\right) = (A_H^{\alpha,\beta}, \Delta)$

and "dual" Hopf C^*-bimodule $\left(\widehat{A}(V)_H^{\widehat{\beta},\alpha}, \widehat{\Delta}_V\right)$

Prop.: $\widehat{A}(V) = \overline{\mathrm{span}}\{\text{convolution operators } \varrho(a) \mid a \in A\}$,

$\varrho(a)^* = J_\nu \varrho(R(a)) J_\nu$ for all $a \in A$,

$\varrho(a) \mapsto J_\nu \varrho(a)^* J_\nu$ anti-automorphism $\widehat{R}$ of $\widehat{A}(V)$

Open questions for further investigations

- *proper* instead of *compact* Hopf C^*-bimodules
- duality for Hopf C^*-bimodules that are *proper* and *étale*
- use approach of Kustermans-Vaes instead of Masuda-Nakagami-Woronowicz
- representation theory of compact Hopf C^*-bimodules
- *universal* instead of *reduced* fiber product and Hopf C^*-bimodules
- further examples