Compact C^*-quantum groupoids

Thomas Timmermann

timmermt@math.uni-muenster.de

University of Münster

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Quantum groupoids/Hopf bimodules — Introduction

Ingredients of a quantum groupoid:

- a Hopf bimodule

$$
\begin{array}{ccccc}
B & \overset{\rho}{\underset{\sigma}{\rightrightarrows}} & A & \xrightarrow{\quad\Delta\quad} & A_\sigma *_\rho A \\
 & & \circlearrowleft & & \circlearrowleft \\
 & & H & & H_\sigma \otimes_\rho H
\end{array}
$$

- left/right Haar weights, antipode, modular element, . . .

Flavours of quantum groupoids:

- finite ones [Böhm, Szlachányi, Nikshych,. . .]
- cnt. bundles of l.c. quantum groups [Blanchard, Enock]
- measurable quantum groupoids in the setting of von Neumann algebras [Enock, Lesieur, Vallin]
- locally compact quantum groupoids in the setting of C^*-algebras?

What is a Hilbert module over a KMS-weight?

Given: C^*-algebra B with faithful KMS-weight μ

$\rightsquigarrow$ GNS-rep. $B \hookrightarrow \mathcal{L}(H_\mu)$, commutant $B' \subseteq \mathcal{L}(H_\mu)$

Def.: *Hilbert C^*-module over μ* := (H, α), where

- H Hilbert space, α closed subspace of $\mathcal{L}(H_\mu, H)$,
- $[\alpha H_\mu] = H$, $[\alpha^* \alpha] = B$, $[\alpha B] = \alpha$

Lemma:

- α is a Hilbert C^*-module over B
- $\alpha \otimes_B H_\mu \cong H$ via $\xi \otimes_B \eta \equiv \xi\eta$
- $\exists$ normal nd. representation $\rho_\alpha \colon B' \to \mathcal{L}(H)$ s.t.

$$\rho_\alpha(x)\xi\eta = \xi x \eta \text{ for all } \xi \in \alpha, \eta \in H_\mu$$

Examples of Hilbert C^*-modules over KMS-weights

1. If B commutative, then

$$\begin{array}{ccc} \{\text{Hilbert } C^*\text{-modules over } \mu\} & \to & \{\text{n. nd. representations of } B'\} \\ \uparrow\downarrow & & \uparrow\downarrow \\ \{\text{cont. Hilbert bundles on } \widehat{B}\} & \to & \{\mu\text{-mb. Hilbert bundles on } \widehat{B}\} \end{array}$$

2. If $B \subseteq A$ and $\phi\colon A \to B$ is a conditional expectation s.t. $\nu := \mu \circ \phi$ is a KMS-weight on A, then:
    - $\exists$ GNS-map $\Lambda_\phi\colon A \to \mathcal{L}(H_\mu, H_\nu)$ s.t. $\Lambda_\phi(a)\Lambda_\mu(b) = \Lambda_\nu(ab)$
    - $(H_\nu, \overline{\Lambda_\phi(A)})$ is a Hilbert C^*-module over μ

What is a Hilbert bimodule over KMS-weights?

Given: C^*-algebras B, C with faithful KMS-weights μ, ν

$\rightsquigarrow$ opposite KMS-weight μ^{op} on B^{op} via $b^{op} \mapsto \mu(b)$

$\rightsquigarrow$ GNS-representations $B^{op} \circlearrowright H_{\mu^{op}} \cong H_\mu \circlearrowleft B$

Def.: *Hilbert C^*-bimodule over* $(\mu^{op}, \nu) := (H, \alpha, \beta)$, where

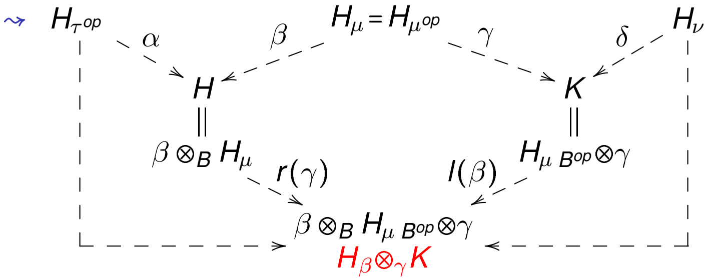
- (H, α), (H, β) Hilbert C^*-modules over μ^{op}, ν
- $\alpha = [\rho_\beta(C^{op})\alpha]$ and $\beta = [\rho_\alpha(B)\beta]$

Ex.: If $B \subseteq A$, $\phi: A \to B$ and $\nu = \mu \circ \phi$ are as before, then

- $\exists$ opposite cond. expectation $\phi^{op}: A^{op} \to B^{op}$ with GNS-map $\Lambda_{\phi^{op}}: A^{op} \to \mathcal{L}(H_{\mu^{op}}, H_\nu)$
- $\left(H_\nu, \overline{\Lambda_{\phi^{op}}(A^{op})}, \overline{\Lambda_\phi(A)}\right)$ is H.-C^*-bimodule/(μ^{op}, μ)

The relative tensor product of Hilbert C^*-bimodules

Given: H.-C^*-bimod. (H,α,β), (K,γ,δ) over (τ^{op},μ), (μ^{op},ν)



Hilbert C^*-bimodule over (τ^{op},ν)

Thm:

- this tensor product is functorial, unital, associative
- $\exists$ natural iso. $H_\beta \otimes_\gamma K \cong H_{\rho_\beta} \underset{\tilde{\mu}}{\otimes}{}_{\rho_\gamma} K$ (Connes' fusion)
- $(H,\alpha,\beta) \mapsto (H,\rho_\alpha,\rho_\beta)$ is a monoidal functor from H.-C^*-bimodules/(μ^{op},μ) to H.-bimodules/(B'',B')

What is a Hopf C^*-bimodule?

Def.: *C^*-algebra over μ* := (H, α, A), where

- (H, α) Hilbert C^*-module over μ
- $A \subseteq \mathcal{L}(H)$ nd. C^*-algebra and $\rho_\alpha(B^{op}) \subseteq M(A)$

Ex.: $B \subseteq A$, $\phi\colon A \to B$ and $\nu = \mu \circ \phi$ as before

$\rightsquigarrow$ $(H_\nu, \overline{\Lambda_{\phi^{op}}(A^{op})}, A)$ is a C^*-algebra over μ^{op}

Def.: category of C^*-algebras over (μ^{op}, τ) ...

Def.: *Hopf C^*-bimodule over μ* := $(H, \alpha, \beta, \Delta)$, where

- (H, α, β, A) – briefly $A_H^{\alpha,\beta}$ – is a C^*-algebra/(μ, μ^{op})
- a *coassociative* $\Delta \in \mathrm{Mor}\left(A_H^{\alpha,\beta}, A_H^{\alpha,\beta} * A_H^{\alpha,\beta}\right)$

Lemma: $(H, \rho_\alpha, \rho_\beta, A'', \tilde{\Delta})$ is a Hopf-von Neumann-bimodule

The fiber product of C^*-algebras over KMS-weights

Given: (H, α, β, A), (K, γ, δ, B) C^*-algebras/(μ^{op}, τ), (τ^{op}, ν)

$$\rightsquigarrow \quad \begin{array}{ccccc} H & \xrightarrow{r(\gamma)} & H_\beta \otimes_\gamma K & \xleftarrow{l(\beta)} & K \\ A \downarrow & & \downarrow {\color{red} A_\beta *_\gamma B} & & \downarrow B \\ H & \xrightarrow{r(\gamma)} & H_\beta \otimes_\gamma K & \xleftarrow{l(\beta)} & K \end{array}$$

$$A_\beta *_\gamma B := \left\{ T \,:\, T^{(*)} l(\beta) \subseteq [l(\beta)B],\ T^{(*)} r(\gamma) \subseteq [r(\gamma)A] \right\}$$

is a C^*-algebra over (μ^{op}, ν) if it is nondegenerate

Lemma:

- this fiber product is functorial, not associative, unital only on certain subcategories
- $A_\beta *_\gamma B \subseteq A''{}_{\rho_\beta} *_{\rho_\gamma} B''$ (fiber product of v.N.-alg.)

Compact Hopf C^*-bimodules I

Axioms:

- B unital C^*-algebra with faithful KMS-state μ
- A unital C^*-algebra with faithful KMS-states ν, ν^{-1}
- $\rho: B \to A$, $\sigma: B^{op} \to A$ unital embeddings
- $\phi: A \to \rho(B) \cong B$, $\psi: A \to \sigma(B^{op}) \cong B^{op}$ faithful cond. expectations s.t. $\nu = \mu \circ \phi$, $\nu^{-1} = \mu^{op} \circ \psi$
- $\exists\ \delta = d\nu/d\nu^{-1} \in A \cap \rho(B)' \cap \sigma(B^{op})'$

Lemma:

- $H := H_\nu \cong H_{\nu^{-1}}$ is a Hilbert C^*-module over $(\mu, \mu^{op}, \mu^{op}, \mu)$ w.r.t. $\widehat{\alpha} := \overline{\Lambda_\phi(A)}$, $\widehat{\beta} := \overline{\Lambda_\psi(A)}$, $\beta := \overline{\Lambda_{\phi^{op}}(A^{op})}$, $\alpha := \overline{\Lambda_{\psi^{op}}(A^{op})}$
- (H, α, β, A) is a C^*-algebra over (μ, μ^{op})

Compact Hopf C^*-bimodules II

Axioms:

- Δ s.t. $(H, \alpha, \beta, A, \Delta)$ is a Hopf C^*-bimodule
- ϕ is *left-* and ψ *right-invariant* w.r.t. Δ
- $\Delta(\delta) = \delta_\alpha \otimes_\beta \delta$
- R anti-automorphism of A that flips ρ, σ and ϕ, ψ
- *strong invariance* relating ϕ, ψ and R

Theorem:

- $\exists$ *regular C^*-pseudo-multiplicative unitary*

$$H_{\widehat{\beta}} \otimes_\alpha H \cong \widehat{\beta} \otimes_{\rho_\alpha} H \xrightarrow{V} \alpha \otimes_{\rho_\beta} H \cong H_\alpha \otimes_\beta H$$

$$\Lambda_\psi(a) \otimes_{\rho_\alpha} \omega \longmapsto \Delta(a)\big(\Lambda_{\psi^{op}}(1^{op}) \otimes_{\rho_\beta} \omega\big)$$

- $\exists$ anti-unitary $I: H \to H$, $\Lambda_{\nu^{-1}}(a) \mapsto \Lambda_\nu(R(a)^*)$, $(V, \lambda^{i/4} I J_\phi)$ is a *weak C^*-pseudo-Kac system*

C^*-pseudo-multiplicative unitaries I

Def.: *C^*-pseudo-multiplicative unitary*:= $(H, \widehat{\beta}, \alpha, \beta, V)$, where

- $(H, \widehat{\beta}, \alpha, \beta)$ is a Hilbert C^*-trimodule/$(\mu^{op}, \mu, \mu^{op})$
- $V\colon H_{\widehat{\beta}} \otimes_{\alpha} H \to H_{\alpha} \otimes_{\beta} H$ is a unitary and a morphism of Hilbert C^*-modules over $(\mu, \mu^{op}, \mu, \mu^{op})$
- $V_{12} V_{13} V_{23} = V_{23} V_{12}$ (V_{ij}: op. on $H_{?} \otimes_{?} H_{?} \otimes_{?} H$)

Prop.: $(H, \rho_{\widehat{\beta}}, \rho_{\alpha}, \rho_{\beta}, V)$ is a *pseudo-multiplicative unitary* in the sense of [Vallin, Enock, Lesieur]

Ex.
- cont. bundles of multiplicative unitaries [Blanchard]
- locally compact groupoids
- tracial conditional expectations
- external tensor product, direct sum, restriction, ...

C^*-pseudo-multiplicative unitaries II

$$
\begin{array}{ccccc}
H & \xrightarrow{\;r(\alpha)\;} & H_{\widehat{\beta}}\otimes_{\alpha} H & \xleftarrow{\;l(\widehat{\beta})\;} & H \\
{\color{red}A(V)}\big\downarrow & & \big\downarrow V & & \big\downarrow \widehat{A}(V) \\
H & \xleftarrow{\;r(\beta)^*\;} & H_{\alpha}\otimes_{\beta} H & \xrightarrow{\;l(\alpha)^*\;} & H
\end{array}
$$

Def.: V regular $:\Leftrightarrow [l(\alpha)^* V r(\alpha)] = [\alpha\alpha^*]$

Thm.: V regular $\Rightarrow \big(H, \alpha, \beta, A(V), \Delta\big)$ and $\big(H, \widehat{\beta}, \alpha, \widehat{A}(V), \widehat{\Delta}\big)$ are Hopf C^*-bimodules, where $\Delta, \widehat{\Delta}$ are given by $\Delta\colon a \mapsto V(1_{\widehat{\beta}}\otimes_{\alpha} a)V^*$, $\widehat{\Delta}\colon \widehat{a} \mapsto V^*(\widehat{a}_{\alpha}\otimes_{\beta} 1)V$

Ex.: If V is the unitary of $(B, \mu, A, \rho, \sigma, \phi, \psi, \delta, \Delta, R)$, then $A(V) = A$, $\big(H, \widehat{\beta}, \alpha, \widehat{A}(V), \widehat{\Delta}\big)$ is the *dual* Hopf C^*-bimod.

Rmk.: representations/corepresentations of V lead to *universal* C^*-algebras $A_u(V)$, $\widehat{A}_u(V)$

C^*-pseudo-Kac systems and duality for coactions

Def.: *Coaction* of a Hopf C^*-bimodule $(H, \alpha, \beta, A, \Delta)$ over μ :=

- C^*-algebra C_K^γ over μ with
- $\delta \in \mathrm{Mor}(C_K^\gamma, C_K^\gamma * A_H^{\alpha,\beta})$ s.t. $(\delta * \mathrm{id}) \circ \delta = (\mathrm{id} * \Delta) \circ \delta$

Ex.: actions on and Fell bundles of l.c. Hausdorff groupoids

Def.: *C^*-pseudo-Kac system* := C^*-pseudo-multiplicative unitary $(H, \widehat{\beta}, \alpha, \beta, V)$ and unitary $U: H \to H$ s.t. ...

Def./Prop.: $\exists$ *"red. crossed product/dual coaction"* functors

$$\left\{\text{coactions of } (A(V)_H^{\alpha,\beta}, \Delta)\right\} \rightleftarrows \left\{\text{coactions of } (\widehat{A}(V)_H^{\widehat{\beta},\alpha}, \widehat{\Delta})\right\}$$

Thm.: Every *well-behaved* coaction is *equivariantly Morita equivalent* to its bidual