K-theory and representations

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Introduction

Subject of the essay The subject of this essay is a theorem of M.F. Atiyah and G.B. Segal which relates representation theory and K-theory:

Let G be a compact Lie group. There is a *universal* principal G-bundle $E_G \to B_G$ characterised by the property that every other principal G-bundle on any compact space X pulls back from E_G via an essentially unique map $X \to B_G$. Any G-module V gives rise to an associated vector bundle $(V \times E_G)/G \to B_G$ with fibre V. The set of all representations of G is classified by the representation ring $R(G)$, the set of all vector bundles on a space X is classified by the Grothendieck group $K(X)$, and the previous construction induces a homomorphism $\alpha : R(G) \to K(B_G)$. The space B_G is quite large and bundles on it can be "infinitely twisted". We account for this by extending the map α to the completion $R(G)\hat{}$ of $R(G)$ with respect to the ideal of zero-dimensional characters:

Main Theorem (M.F. Atiyah, G.B. Segal [6])**.** For any compact Lie group G, the map $\alpha : R(G)\hat{} \to K(B_G)$ is an isomorphism.

History of the theorem In [2], M.F. Atiyah proved the theorem for finite groups G. His approach uses a spectral sequence and the Chern character to relate the K-group of B_G to the (topological) cohomology $H(B_G)$, which in turn is closely related to the (algebraic) cohomology of G. Then, for cyclic G the theorem follows from explicit calculations. An induction step going over from a normal subgroup with prime index to the whole group extends the result to all solvable groups. Finally, $R(G)\hat{}$ and $K(B_G)$ of an arbitrary group G are represented as quotients of the sum of the corresponding objects for all solvable subgroups of G.
The paper is very long and mostly algebraical in nature.

In [4], M.F. Atiyah and F. Hirzebruch extended the result to compact connected Lie groups. For a torus T, one can use the Chern character and group cohomology to compute $K(B_T)$ just as in the case of finite cyclic groups. For general G one has an inclusion of a maximal torus $T \hookrightarrow G$. One can show that $E_G \to E_G/T$ is a universal bundle for T. This yields a projection $B_T \cong E_G/T \to B_G$ and an action of the Weyl group W on B_T. These maps induce restrictions $R(G) \to R(T)$ and $K(B_G) \to K(B_T)$. It is well-known that $R(G)$ is just the subring of $R(T)$ fixed by W. Using homotopy-invariance of pull-backs and connectedness of G, one can also show that the image of $K(B_G)$ in $K(B_T)$ is invariant under W. The map α_T commutes with the action of the Weyl group, and now the theorem follows easily from diagram

$$\begin{array}{ccc} R(G) & \xrightarrow{\alpha_G} & K(B_G) \\ \downarrow & & \downarrow \\ R(T) & \xrightarrow{\alpha_T} & K(B_T) \end{array}$$

if one can show that the restriction $K(B_G) \to K(B_T)$ is injective. This is the most difficult step of the proof and involves a generalized Riemann-Roch theorem and an investigation of the geometry of the space G/T (which is the fibre of the projection $E_G/T \to B_G$).

Finally, in [6] G.B. Segal and M.F. Atiyah employed equivariant K-theory to extend the statement to arbitrary compact Lie groups. The idea of equivariant K-theory is to consider vector bundles with a group action – then the representation ring of a group G is just the K-group of vector bundles with a G-action on a one-point space. Now the K-group remains fixed when simultaneously shrinking the group and enlarging the base space in specific ways.
The proof given in [6] is the subject of the essay. Its hardest part are several Main Lemmata which are equivalent to the difficult step in the previous paper. For a discussion of the structure of the proof, proceed to section 3.

The Main Theorem can be formulated for other generalised equivariant cohomology theories besides K-theory, see [12].

Structure of the essay The first chapter introduces the "mathematical context" of the theorem: fibre bundles, classifying spaces, and equivariant K-theory. The last section investigates algebraic properties of the representation ring of a compact Lie group and is rather technical[1].

The second chapter establishes the Main Lemmata, which are the most difficult part of the proof and involves families of elliptic differential operators on manifolds. The general theory developed yields a proof of a generalized Bott periodicity theorem and indicates that elliptic differential operators provide a "homological counterpart" to the "cohomological" K-groups.

Finally, in the last chapter we prove the Main Theorem.

On the whole, sections 2.5 and 3 should be the most rewarding parts of the essay.

Prerequesites The subject touches many areas of mathematics and requires basic knowledge of general topology, Lie groups and their representations, functional analysis, complex manifolds, "topological" cohomology, and commutative algebra. The only theorems which will be assumed without any indication of the proof are standard results about Cartan subgroups of compact Lie groups. However, for space reasons, the exposition of most of the proofs relies on the willingness and ability of the reader to provide or skip minor details.

Conventions In the following, by "space" we mean a topological space, and maps between topological spaces are always assumed to be continuous. Groups will always be compact topological groups unless stated otherwise. Subgroups of Lie groups are always asummed to be closed.

1 Basics

This chapter is fundamental to an understanding of the formulation of the Main Theorem, except for the section on the representation ring of a compact Lie group (which I suggest to skip at a first reading).

The material on bundles can be found in [16], [1], and [22], the section on classifying spaces follows [22], and the section on equivariant K-theory follows [19] with additions from [16] and [1]. The relation between classifying spaces and group cohomology has been worked out using [9] and [14]. Finally, the subsection on the representation ring is based on [20]; supplementary material on Cartan subgroups can also be found in [8]. The commutative algebra involved is contained in [5].

1.1 Bundles

This section is "compulsory".

Definitions and constructions

1.1.1 Definition. i) A *(left) action* of a group G on a space X is a map $G \times X \to X$ such that $g(g'x) = (gg')x$ for any $g, g' \in G$ and $x \in X$. Similarly, a *right action* of G on X is a map $X \times G \to X$ such that $(xg)g' = x(gg')$. A left action defines a right action via $(x, g) \mapsto g^{-1}x$, and vice versa.

A space with a G-action is called a *G-space*. The action is called *free* if the stabilizer $G_x = \{g \in G : gx = x\}$ of each point $x \in X$ is trivial.

ii) A map $f : X \to Y$ of G-spaces is a *G-map* or *equivariant* if it commutes with the G-actions on X and Y.

iii) The *orbit space* X/G is the quotient of X by the equivalence relation $x \sim x' :\Leftrightarrow x \in Gx'$.

[1] actually it should not appear in a chapter called "Basics"

1.1.2 Definition. i) A *fibre bundle* $\xi = (p, E, B, F)$ is a map $p : E \to B$ with fibre $p^{-1}(x) \cong F$ for all $x \in B$. We call E the total space, B the base space, and p the projection of ξ. We loosely denote ξ by $E \to B$ or just E.

ii) A *bundle map* between two bundles $E \to B$ and $E' \to B'$ is a pair of maps $E \to E'$ and $B \to B'$ commuting with the projections. When considering bundles on a fixed space $B = B'$ we require that $B \to B'$ is the identity.

iii) A bundle isomorphic to $B \times F \to B$ is called trivial. It is *locally trivial* if there are bundle isomorphisms $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ for an open cover $\{U_\alpha\}$ of B. A choice of $\{U_\alpha, h_\alpha\}$ is called a *trivialisation*. The bundle is *numerable* if it admits a countable trivialisation with a subordinate partition of unity. It is well-known that any locally trivial bundle on a compact space is numerable.

iv) A group S acting on F is called a *structure group* of a bundle if the latter has a trivialisation where the maps $h_\alpha \circ h_\beta^{-1} : (U_{\alpha\beta}) \times F \to p^{-1}(U_{\alpha\beta}) \to U_{\alpha\beta} \times F$ have the form $(x, f) \mapsto (x, g_{\alpha\beta}(x) f)$ for continuous *transition functions* $g_{\alpha\beta} : U_{\alpha\beta} \to S$ (here $U_{\alpha\beta}$ denotes $U_\alpha \cap U_\beta$).

v) A *complex vector bundle of rank r* is a locally trivial fibre bundle with fibre $\mathbb{C}^n$ and structure group $\mathrm{GL}(n, \mathbb{C})$ (the action being the natural one).

vi) A *G-equivariant bundle* is a fibre bundle where G acts equivariantly on the total and the base space. In the case of vector bundles G is required to act linearly on fibres.

vii) A *principal S-bundle* is a fibre bundle $p : E \to B$ with a free right S-action on E and an S-equivariant trivialisation $\{U_i, \phi_i\}$, $\phi_i : p^{-1}(U_i) \to U_i \times S$, where S acts on $U_i \times S$ via right multiplication.

viii) A *section of a bundle over a an open set $U \subset B$* is a map $\sigma : U \to E$ such that $p \circ \sigma = \mathrm{id}_U$. The set of global sections is denoted by $\Gamma(E)$.

Example. Let H be a subgroup of a Lie group G, acting on G via right multiplication. Then the bundle $G \to G/H$ is a principal H-bundle (for local triviality, see [8] theorem 4.3).

1.1.3 Construction. Given an S-space F, an open cover $\{U_\alpha\}$ of a space B, and functions $g_{\alpha\beta} : U_{\alpha\beta} \to S$ satisfying the cocycle condition $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ for all α, β, γ, define E to be the quotient of $\bigcup_\alpha (U_\alpha \times F \times \{\alpha\})$ by the equivalence relation $(x, f, \beta) \sim (x, g_{\alpha\beta}(x) f, \alpha)$. The projection $E \to B$ onto the first component yields a locally trivial fibre bundle with fibre F and structure group S.

1.1.4 Construction. Given a principal S-bundle $p : E \to B$ and an S-space F, let $E' = E \times_S F = E \times F/\sim$ where $(es, f) \sim (e, sf)$. Then p induces a projection which turns $E' \to B$ into a locally trivial fibre bundle with fibre F and structure group S.

Remark. If F is a free S-space, there is a bijection between i) isomorphism classes of locally trivial fibre bundles on B with fibre F, ii) isomorphism classes of principal S-bundles, and iii) elements of the first Čech cohomology group $\check{\mathrm{H}}^1(X, S)$ which classifies transition functions satisfying the cocycle condition.

1.1.5 Lemma. Given a fibre bundle $E \to B$ and a map $f : B' \to B$, the diagram on the right has a pull-back which is unique up to isomorphism. It is denoted by $f^* E$. Pull-back preserves triviality, local triviality, structure group, principal bundles and G-bundles in the obvious sense.

$$\begin{array}{ccc} & & E \\ & & \downarrow \\ B' & \to & B \end{array}$$

Proof. Easy and omitted. □

1.1.6 Definition. Let ξ and ξ' be complex vector bundles on B with a common trivializing cover $\{U_\alpha\}$ and transitions functions $g_{\alpha\beta}, g'_{\alpha\beta}$. We construct:

the vector bundle	using the transition functions	obtained from
Whitney sum $\xi \oplus \xi'$	$g_{\alpha\beta} \oplus g'_{\alpha\beta}$	diagonal sum,
tensor product $\xi \otimes \xi'$	$g_{\alpha\beta} \otimes g'_{\alpha\beta}$	exterior product,
dual ξ^*, *conjugate* $\bar{\xi}$	$(g_{\alpha\beta}^{-1})^T$, $\quad \bar{g}_{\alpha\beta}$	in the obvious way.

Homotopy and pull-back of vector bundles

1.1.7 Lemma. Let E be a G-vector bundle on a compact space B. Then any equivariant section σ over a closed G-subspace $A \subset B$ extends to an equivariant section over B.

Proof. Choose a trivialising cover U_i and a subordinate partition of unity $\{\phi_i\}$. Over each intersection $A \cap U_i$ one can extend σ to a section σ_i on U_i by the extension theorem of Tietze. Then $\sum \phi_i \sigma_i$ is a global extension of σ. Now average over G. □

1.1.8 Lemma. Let $E \to B$ be a G-vector bundle, and B' a compact G-space. If $f, g : B' \to B$ are G-homotopic G-maps, then $f^*E \cong g^*E$ as G-vector bundles.

Proof. Let $H : B' \times [0,1] \to B$ be a G-homotopy between f and g. Write $H_t = H(\cdot, t)$, and denote the projection $B' \times [0,1] \to B'$ by π. We prove that every $t \in [0,1]$ has a neighbourhood U such that $(H_t)^*\xi \cong (H_{t'})^*\xi$ for all $t' \in U$. Then the same holds for $t = 0$ and $t' = 1$.

Consider the bundle $E_t = (H_t \circ \pi)^*\xi$ on $B' \times [0,1]$. The G-isomorphism $E_t|_{B' \times t} \cong H^*\xi|_{B' \times t}$ corresponds to an equivariant section of $E_t^* \otimes H^*\xi$ over $B' \times t$, which can be extended to an equivariant global section σ by the previous lemma. By compactness of B' there is a neighbourhood $U \subset [0,1]$ of t on which σ is nonsingular and represents an isomorphism. The equivariance of σ implies that this is a G-isomorphism. □

1.1.9 Construction (Clutching)**.** Let X be a G-space covered by two open G-subsets U_0 and U_1 such that $U = U_0 \cap U_1$ is G-contractible to $A \subset U$. Denote the inclusion $A \hookrightarrow U$ by j. Given two G-bundles ξ_i on U_i with a G-bundle isomorphism $\alpha : \xi_0|_A \xrightarrow{\sim} \xi_1|_A$, we define a new G-bundle ξ on X by the following *clutching construction*: From lemma 1.1.8 we get an isomorphism $\xi_0|_U \cong j^*\xi_0|_A \xrightarrow{j^*\alpha} j^*\xi_1|_A \cong \xi_1|_U$ which can be used to "glue" ξ_0 and ξ_1 on together. We obtain a G-bundle ξ which on U_i restricts to ξ_i. If ξ_0 and ξ_1 are trivial, the isomorphism α is called a clutching function of ξ.

If ξ_i is the restriction of a G-bundle ξ' to U_i, $i = 1, 2$, then clearly $\xi \cong \xi'$.

The link between algebraic, analytical and topological K-theory

1.1.10 Definition. A *hermitian metric* on a complex vector bundle ξ is a section of $\xi \otimes \bar{\xi}$ which induces a hermitian metric on each fibre. If ξ is a G-bundle, the metric is called *invariant* if the corresponding section is equivariant.

Any complex G-vector bundle on a paracompact space admits an invariant hermitian metric: Choose metrics on trivialising open sets, use a partition of unity to glue these together, and average over G. Hence we can reduce the structure group of every vector bundle on a paracompact space from $\mathrm{GL}(n, \mathbb{C})$ to U_n.

1.1.11 Theorem (Swan)**.** For any vector bundle ξ with compact base there is a vector bundle ξ' such that $\xi \oplus \xi'$ is trivial.

Proof. Let $\{U_i, h_i\}_1^m$ be a finite trivialisation of E, and $\{\phi_i\}$ a subordinate partition of unity. Denote $\tilde{h}_i : p^{-1}(U_i) \xrightarrow{h_i} U_i \times \mathbb{C}^n \to \mathbb{C}^n$ the obvious composition and let $\chi_i = \phi_i \circ p : p^{-1}(U_i) \to [0,1]$. Now embed E into $B \times \bigoplus^m \mathbb{C}^n$ via the map $e \mapsto (p(e), \chi_1(e)\tilde{h}_1(e), \ldots, \chi_m(e)\tilde{h}_m(e))$. Take $\xi' = \xi^\perp$ for some hermitian metric on $B \times \bigoplus^m \mathbb{C}^n$ (for a proof of local triviality of ξ', cf. to section 2.2). □

Let $C(B)$ denote the algebra of complex valued functions on B. Since $\Gamma(\xi \oplus \xi') = \Gamma(\xi) \oplus \Gamma(\xi')$, the theorem says that for each vector bundle ξ on a compact space B, the set $\Gamma(\xi)$ is a finitely generated projective $C(B)$-module. In fact, Γ is a contravariant functor which induces an equivalence between the category of vector bundles on B and the category of finitely generated $C(B)$-modules. If B is compact, $C(B)$ is a C^*-algebra. Introducing a hermitian metric on E induces a sesquilinear pairing $\Gamma(E) \times \Gamma(E) \to C(B)$ which turns $\Gamma(E)$ into a Hilbert-module over $C(B)$. This relates algebraic, analytical and topological K-theory.

1.2 Classifying spaces

The main output of the following section will be the definition of classifying spaces, their uniqueness (theorem 1.2.4), and ways to construct them (lemma 1.2.5 and 1.2.6).

Definition, construction and examples

Let $[X, Y]$ denote the set of homotopy classes of maps $X \to Y$.

1.2.1 Definition. A numerable principal S-bundle $E \to B$ is *classifying* or *universal*, if for every numerable principal S-bundle $E' \to B'$ there is a unique homotopy class $f \in [B', B]$ such that $f^*E \cong E'$. The space B is called a *classifying space* for S.

Categorically speaking, B represents the functor that associates to a space B' the set of numerable principal S-bundles with base B'.

1.2.2 Theorem. Each group S has a classifying bundle.

Proof. Construction of the classifying bundle (Milnor construction): The join $X \star Y$ of two spaces X and Y is defined as $X \times Y \times [0,1]/\sim$, where

$$\begin{array}{lll} (x,y,0) \sim (x,y',0) & \text{for all} & x \in X,\ y,y' \in Y, \\ (x,y,1) \sim (x',y,1) & \text{for all} & x,x' \in X,\ y \in Y. \end{array}$$

Consider the space

$$E := S \star S \star S \star \cdots := \Big\{ (s_i, \lambda_i)_i \in \textstyle\prod_{i=1}^{\infty} (S \times [0,1]) \ :\ \sum \lambda_i = 1 \text{ is a finite sum} \Big\} / \sim,$$

where $(s_i, \lambda_i)_i \sim (s_i', \lambda_i')_i :\Leftrightarrow \lambda_i = \lambda_i'$ for all i, and $s_i = s_i'$ whenever $\lambda_i > 0$. Let $\phi_i : E \to [0,1]$ denote the obvious projection and put $U_i := \phi_i^{-1}((0,1])$. Then the obvious projection $t_i : U_i \to S$ is well-defined. Give E the weakest topology which makes all these maps continuous. Then U_i is open.

Componentwise right multiplication defines a free right action of S on E. Consider the quotient map $p : E \to E/S =: B$ onto the orbit space. Over the open set U_j we have a trivialisation $p^{-1}(U_j) \to U_j \times S$ defined by $(s_i, \lambda_i)_i \mapsto ((s_i s_j^{-1}, \lambda_i), s_j)$. Choose a countable partition of unity $\{\tau_n\}$ on $(0,1]$ such that $\operatorname{supp} \tau_n \subset (1/n, 1)$. Put $U_{j,n} = U_j$ and $\phi_{j,n} = \phi_j \cdot (\tau_n \circ \phi_j)$. Then $\{\phi_{j,n}\}$ is a partition of unity subordinate to the trivialising cover $\{U_{j,n}\}$. Hence $E \to B$ is a numerable principal S-bundle.

Each principal S-bundle pulls back from γ: Let $p' : E' \to B'$ be a numerable principal S-bundle with a countable trivialisation $\{V_n, h_n\}$ and a subordinate partition of unity $\{\psi_n\}$. Denote $\tilde{h}_n : p'^{-1}(V_n) \to V_n \times S \to S$ the obvious composition, and let $\chi_n = \psi_n \circ p'$. Then $e \mapsto (\tilde{h}_n(e), \chi_n(e))_n$ defines a bundle map $\tilde{f} : E' \to E$ (compare the proof of theorem 1.1.11). This commutes with the action of S and hence induces a map $f : B' \to B$. By construction, $f^*E \cong E'$.

Each principal S-bundle pulls back in a unique way: We show that any two S-equivariant maps $\tilde{f}, \tilde{g} : E' \to E$ are S-homotopic. In coordinates, leaving out the arguments, $\tilde{f} =$

$(s_1\lambda_1,\, s_2\lambda_2,\, \ldots)$ and $g = (s'_1\lambda'_1,\, s'_2, \lambda'_2,\, \ldots)$. Consider the map $e \mapsto (s_1\lambda_1,\, \cdot 0,\, s_2\lambda_2,\, \cdot 0, \ldots)$. The S-homotopy $(e,t) \to (s_1\lambda_1,\, s_2 t\lambda_2,\, s_2(1-t)\lambda_2,\, s_3 t\lambda_3,\, s_3(1-t)\lambda_3,\, \ldots)$ removes the first 0. Because of the coarse topology of E, we can iterate homotopies to remove one 0 after the other with ever increasing speed to get an S-homotopy to $\tilde{f}$. Similarly, $\tilde{g}$ is S-homotopic to the map $e \mapsto (\cdot 0,\, s'_1\lambda'_1,\, \cdot 0,\, s'_2\lambda'_2,\, \cdot 0, \ldots)$. We finish with the S-homotopy $(e,t) \mapsto (s_1 t\lambda_1,\, s'_1(1-t)\lambda'_1,\, s_2 t\lambda_2,\, s'_2(1-t)\lambda'_2,\, \ldots)$. □

1.2.3 Lemma. The universal Milnor bundle is contractible.

Proof. By the last part of the previous proof, the identity map on E is S-homotopic to the map $(s_1\lambda_1,\, s_2\lambda_2,\, \ldots) \mapsto (\cdot 0,\, s_1\lambda_1,\, s_2\lambda_2,\, \ldots)$. Now contract via $t \mapsto (1t,\, s_1(1-t)\lambda_1,\, s_2(1-t)\lambda_2,\, \ldots)$. □

1.2.4 Theorem. The base spaces B and B' of any two classifying S-bundles $E \to B$ and $E' \to B'$ are homotopy-equivalent.

Proof. By definition there are maps $f : B \to B'$ and $g : B' \to B$ such that $E \cong f^*E'$ and $E' \cong g^*E$. Now $g^*f^*E' \cong E'$ together with the uniqueness property imply that $g \circ f$ is homotopic to the identity on B', and similarly for $f \circ g$. □

The classifying bundle of a group S is denoted $E_S \to B_S$, knowing that it is defined only up to homotopy equivalence.

Remark. One can extend the notion of a classifying space to

- numerable G-equivariant principal S-bundles and prove its existence and uniqueness up to homotopy equivalence in much the same way, see [22];
- an (arbitrary) category, see [21]. The space B_S constructed above is then obtained as the classifying space of the category $\mathcal{S}$ which consists of one object and has S as its morphism set.

1.2.5 Lemma. Let S be a subgroup of T. Then $E_T \to E_T/S$ is a classifying S-bundle.

Proof. Let $E \to B$ be a numerable principal S-bundle. Then $E \times_S T \to B$ is a numerable principal T-bundle and hence pulls back from E_T. This gives a bundle map $E \times_S T \to E_T$. Composing with the inclusion $E \to E \times_S T$ defined by $e \mapsto [e, 1]$, we get an S-equivariant map $E \to E \times_S T \to E_T$ which factorizes to a map $f : B \to E_T/S$. By construction, $f^*E_T = E$, and hence any numerable principal S-bundle pulls back from $E_T \to E_T/S$.

The proof of theorem 1.2.2 shows that any two S-equivariant maps $f, g : E \to E_T$ are S-homotopic. □

1.2.6 Lemma. Let S and T be two groups with classifying bundles $E_S \to B_S$ and $E_T \to B_T$. Then $E_S \times E_T \to B_S \times B_T$ is a classifying bundle for $S \times T$.

Proof. Any principal $S \times T$-bundle $E \to B$ decomposes as the product $E = E/S \times_B E/T$ of the principal T-bundle $E/S \to B$ and the principal S-bundle $E/T \to B$. Clearly $[B, B_S \times B_T] \cong [B, B_S] \times [B, B_T]$, and both decompositions can be shown to be compatible. □

Remark. It is well-known that any compact Lie group G can be embedded into a unitary group U. In the proof of the Main Theorem, we will use the bundle $E_U \to E_U/G$ as a classifying bundle for G. Observe that from the Milnor construction we obtain an embedding $E_G \hookrightarrow E_U$. By the proof of theorem 1.2.2, E_U can be homotoped G-equivariantly to $E_G \subset E_U$.

Examples

The circle group T: Let S^{2m-1} denote the unit sphere in $\mathbb{C}^m$. Then $S^{2m-1} \star T \cong S^{2m+1}$ via $(s_1, s_2, \lambda) \mapsto (s_1 \cos \frac{\lambda\pi}{2}, s_2 \sin \frac{\lambda\pi}{2})$, and hence $\star^m T \cong S^{2m-1}, (\star^m T)/T \cong \mathbb{C}P^m$. The universal bundle for T is $S^\infty \to \mathbb{C}P^\infty$.
The torus T^k: By the previous lemma and example the universal bundle is $\times^k S^\infty \to \times^k \mathbb{C}P^\infty$.
The reflection group $\mathbb{Z}/2$: For any space X, the join $X \star (\mathbb{Z}/2)$ is the suspension $SX := X \times [0,1]/X \times \{0,1\}$ of X. Now $S(S^m) \cong S^{m+1}$ and hence $\star^m \mathbb{Z}/2 \cong S^{m-1}, (\star^m \mathbb{Z}/2)/(\mathbb{Z}/2) \cong \mathbb{R}P^{m-1}$. The universal bundle is $S^\infty \to \mathbb{R}P^\infty$.
The unitary group U_k: Instead of trying to understand what $U_k \star \cdots \star U_k$ looks like, let us indicate another construction which relies on the bijection between principal U_k-bundles and vector bundles of rank k. Swan's theorem showed that any vector bundle $E \to B$ can be embedded into a trivial bundle $B \times \mathbb{C}^n$. Looking at the fibre at each point, this gives a map of B into the *Grassmannian* $Gr(n,k) := \{k - \text{dimensional subspaces of } \mathbb{C}^n\}$. It is easy to see that $Gr(n,k) \cong \mathrm{GL}(n)/(\mathrm{GL}(n-k) \times \mathrm{GL}(k))$ and that introduction of a metric gives $Gr(n,k) \cong U_n/(U_{n-k} \times U_k)$. There is a *tautological bundle* on $Gr(n,k)$ which corresponds to the principal U_k-bundle $U_n/U_{n-k} \to U_n/(U_{n-k} \times U_k)$ and has at each point of $Gr(n,k)$ as its fibre the k-dimensional subspace of $\mathbb{C}^n$ represented by this point.

One can show that there is an "infinite-dimensional" Grassmannian $Gr(\infty, k)$ which is the classifying space of U_k.

The Borel construction and group cohomology

1.2.7 Definition. Given a G-space X, let $X_G = E_G \times_G X$ (cf. construction 1.1.4). X_G is a numerable bundle on B_G with fibre X.

In this section we indicate a link between group cohomology, classifying spaces and the Borel construction. We will not use this later, but it is necessary to understand (and assumed in) the proof of the Main Theorem given in [2]. Basic terminology of homological algebra is summarised in the appendix A.4.

For a discrete group G the classifying bundle can also be constructed as a bundle of simplicial complexes:

1.2.8 Construction. Consider the simplicial complex (K_*, ∂) with $K_n := G^{n+1}$ as the set of n-simplexes, and face operators $\partial_n^i : K_n \to K_{n-1}, (g_0, \ldots, g_n) \mapsto (g_0, \ldots, \widehat{g_i}, \ldots, g_n)$. Let E'_G be its geometric realisation.

We define a right G-action on K_n by $(g_0, \ldots, g_n)g = (g_0 g, \ldots, g_n g)$. This commutes with the face operators, and we obtain a simplicial complex $(K_*/G, \partial)$. Denote B'_G its geometric realisation. For later use, let $[g_0, \ldots, g_n] = (g_0, \ldots, g_n)G$.

The G-bundle $E'_G \to B'_G$ can be shown to be universal by either copying the proof given for the Milnor construction or by considering the map $E'_G \to E_G$ defined by $K_n \times \Delta^n \to E_G, (g_0, \ldots, g_n, \lambda_0, \ldots, \lambda_n) \mapsto (g_0, \lambda_0, g_1, \lambda_1, \ldots)$ which is clearly G-equivariant.

Let $\mathbb{Z}G$ be the group ring of G. Throughout this section, all modules $\mathbb{Z}G$-modules will be *right* modules.

1.2.9 Definition. Let M be a $\mathbb{Z}G$-module. Given a projective resolution $P_\star$ of the $\mathbb{Z}G$-module $\mathbb{Z}$ (with trivial G-action), we obtain a complex $\mathrm{Hom}_{\mathbb{Z}G}(P_\star, M)$. The *$i$-th cohomology of G with coefficients M* is defined as $H^i(G, M) := H^i(\mathrm{Hom}_{\mathbb{Z}G}(P_\star, M))$.[2]

[2]Different resolutions give isomorphic $H^i(G, M)$ because the additive function $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$ preserves homotopy equivalence of complexes, see A.4.

If the action of G on M is trivial, construction 1.2.8 reveals a close relation between the algebraically defined cohomology of G and the simplicial cohomology of the classifying space B_G:

For a set K with a G-action, let $\mathbb{Z}K$ denote the $\mathbb{Z}G$-module of finite formal linear combinations $\sum_{k\in K}\lambda_k k$ with integer coefficients λ_k. For a simplicial space X, let $K_n(X)$ denote the set of its n-simplices.

A standard check shows that $\mathbb{Z}K_*(E_G)\to 0$ is a free resolution of $\mathbb{Z}$. The module $C^n(B_G, M) = \mathrm{Hom}_{Set}(K_n(B_G), M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}K_n(B_G), M)$ of n-cochains equals $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}K_n(E_G), M)$ because the G-action on the $\mathbb{Z}G$-module M is trivial. Hence

1.2.10 Lemma. For any $\mathbb{Z}G$-module M with trivial G-action, $H^*(B_G, M) \cong H^*(G, M)$. □

What if the action of G on M is not trivial? In simplicial cohomology, the "twisting of coefficients" can be modelled by "local coefficient systems":

1.2.11 Definition. i) A *local coefficient system* on a simplicial complex K_* is a direct system $(\mathcal{M}, \rho)$ of abelian groups $\mathcal{M}_\sigma$ indexed by the simplices $\sigma \in K_*$. The set of simplices is ordered by inclusion, i.e. for each inclusion $\sigma' \subset \sigma$ we have a map $\rho_{\sigma'\sigma} : \mathcal{M}_{\sigma'} \to \mathcal{M}_\sigma$ such that $\rho_{\sigma'\sigma}\circ\rho_{\sigma''\sigma'} = \rho_{\sigma''\sigma}$ whenever $\sigma'' \subset \sigma' \subset \sigma \in K$. We write $\rho_{\sigma,i}$ for $\rho_{(\partial^i\sigma)\sigma}$.

ii) For each $n \geq 0$, the group of *simplicial n-cochains on K_* with coefficients in $\mathcal{M}$* is $C^n(K_*, \mathcal{M}) := \prod_{\sigma\in K^n}\mathcal{M}_\sigma$. The boundary operator $d : C^n(K_*, \mathcal{M}) \to C^{n+1}(K_*, \mathcal{M})$ is defined by $(d\omega)(\sigma) = \sum_i(-1)^i\rho_{\sigma,i}(\omega(\partial^i\sigma))$. A standard check shows that $(C^*(K_*, \mathcal{M}), d)$ forms a complex. Its n-th cohomology group is denoted by $H^n(K_*, \mathcal{M})$.

To a right $\mathbb{Z}G$-module M we associate the following coefficient system $\mathcal{M}$ on B_G as follows: For each $\sigma = [g_0, \ldots, g_n] \in K_n(B_G)$, put $\mathcal{M}_\sigma = M$, $\rho_{\sigma,i} = \mathrm{id}_M$ for $i < n$, and $\rho_{\sigma,n}(m) = mg_{n-1}g_n^{-1}$. Note that this is well-defined.

1.2.12 Theorem. $H^*(B_G, \mathcal{M}) \cong H^*(G, M)$ for any $\mathbb{Z}G$-module M.

Proof. Write K_n for $K_n(E_G)$. We construct chain maps $S : \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}K_*, M) \to C^*(K_*, \mathcal{M})$ and $T : C^*(K_*, \mathcal{M}) \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}K_*, M)$ that are inverses.

For $f \in \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}K_n, M)$ and $\sigma = [g_0, \ldots, g_n]$ put $(Sf)(\sigma) = f(g_0, \ldots, g_n)g_n^{-1}$. Note note that this is well-defined. For $\omega \in C^n(K_*, \mathcal{M})$ and $\tau = (g_0, \ldots, g_n)$ put $(T\omega)(\tau) = \omega([\tau])g_n$. Then T defines a $\mathbb{Z}G$-homomorphism since $(T\omega)(\tau g) = \omega([\tau g])g_n g = ((T\omega)(\tau))g$ for any $g \in G$ (recall $[\tau g] = [\tau]$). Clearly T and S are inverses.

The following computations show that S and T are chain maps:

$$\begin{aligned}
(dSf)(\sigma) &= \sum_i(-1)^i\rho_{\sigma,i}((Sf)(\partial^i\sigma))\\
&= \sum_{i<n}(-1)^i f(\partial^i\sigma)g_n^{-1} + (-1)^n f(\partial^n\sigma)g_{n-1}^{-1}\cdot(g_{n-1}g_n^{-1})\\
&= (df)(\sigma)g_n^{-1} = (Sdf)(\sigma),
\end{aligned}$$

$$\begin{aligned}
(dT\omega)(\tau) &= \sum_{i<n}(-1)^i\omega(\partial^i[\tau])g_n + (-1)^n\omega(\partial^n[\tau])g_{n-1}\\
&= \left(\sum_i(-1)^i\rho_{[\tau],i}(\omega(\partial^i[\tau]))\right)g_n = (Td\omega)(\tau).
\end{aligned}$$

□

We can visualise the local coefficient system $\mathcal{M}$ as follows: Applying construction 1.1.4 to the principal bundle $E_G \to B_G$ and the discrete G-space M yields an M-bundle $M_G := E_G \times_G M \xrightarrow{\pi} B_G$ with discrete fibres. Its total space M_G has a natural structure of a simplicial complex with n-simplices $(K_n(E_G) \times M)/G$. A simplicial cochain on B_G with coefficients in $\mathcal{M}$ associates to each n-simplex $\sigma \in K_n(B_G)$ an element of $\mathcal{M}_\sigma = M \cong \pi^{-1}(\sigma)$. We identify the "simplicial fibre" $\pi^{-1}(\sigma)$ with M via $([g_0, \ldots, g_n], m) \mapsto m g_n^{-1}$. The map $\rho_{\sigma'\sigma} : \mathcal{M}_{\sigma'} \to \mathcal{M}_\sigma$ is determined by the inclusion $\pi^{-1}(\sigma') \to \pi^{-1}(\sigma)$.

1.3 K_G-Theory

K-Theory was invented by Grothendieck in the context of algebraic geometry as a tool to prove the Riemann-Roch theorem. It has been rephrased in topological, algebraic and analytical language. A major success of topological K-theory is the celebrated Atiyah-Singer index theorem on the index of elliptic differential operators on manifolds, which is a vast generalisation of the Riemann-Roch theorem.

Definition of $K_G(X)$

1.3.1 Definition. Let X be a compact G-space. The set $\mathrm{Vect}_G(X)$ of all isomorphism classes of G-vector bundles on X is a monoid under $\oplus$. Let $K_G(X)$ denote its associated universal group[3]. A G-map $f : X \to Y$ induces a map $f^* : K_G(Y) \to K_G(X)$ via pull-back. The tensor product of G-vector bundles turns $K_G(X)$ into a ring.

We omit the index G if the group is trivial.

Thus K_G is a functor from the category of compact G-spaces and G-homotopy classes of G-maps to the category of abelian rings. Later we will define "higher" K_G-groups K_G^{-n} and "relative" K_G-groups $K_G(X, A)$, and show that these functors share many properties of ordinary cohomology (formally, K^* defines an *extraordinary cohomology theory*, see [9]). Computationally, K-theory is related to ordinary cohomology by the following tools (which we do not to discuss and do not need; however, the proof of the Main Theorem given in [2] relies heavily on this):

i) To a vector bundle E on X one can associate its *Chern character* $ch(E) \in \bigoplus_k H^{2k}(X)$ $=: H^{ev}(X)$ which measures the twisting of E in terms of the shape of X. One can show that ch induces a ring homomorphism from $K(X)$ to H^{ev}. For compact X, after tensoring with $\mathbb{Q}$ this becomes an isomorphism $K(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^{ev}(X, \mathbb{Q})$.

ii) If X is a finite CW-complex, there is a converging *spectral sequence* relating $K(X)$ to $H^*(X)$.

Definition of $\tilde{K}_G(X)$ and representability

Our aim is theorem 1.3.5, which says that the functor K is representable. This result will be used, but is not crucial.

In the following we often will consider G-spaces with a chosen base point. This base point is always assumed to be fixed by G.

1.3.2 Definition. Let X be a compact G-space with base point x_0, or connected with $x_0 \in X$ an arbitrary point fixed by G. Let $\tilde{K}_G(X) := \ker(K_G(X) \to K_G(x_0))$, the map being induced by the inclusion $x_0 \hookrightarrow X$.

Obviously, for a point x_0, $\mathrm{Vect}_G(\{x_0\})$ is the set of all finite-dimensional representations of G, and $K_G(\{x_0\}) = R(G)$. In particular, $K(\{x_0\}) = \mathbb{Z}$ (counting the dimension). We denote the map $K(X) \to K(\{x_0\})$ by dim.

1.3.3 Definition. Two bundles ξ and ξ' on X are *stably equivalent*, written $\xi \sim \xi'$, iff $\xi \oplus \eta_k \cong \xi' \oplus \eta_k$ for some $k \in \mathbb{N}$, where η_k denotes the trivial bundle $X \times \mathbb{C}^k$. An element of $\mathrm{Vect}(X)/\!\sim$ is a *stable vector bundle*.

[3] also called the Grothendieck group

1.3.4 Lemma. For a compact connected space X we have $\tilde{K}(X) \cong \mathrm{Vect}(X)/\!\sim$.

Proof. By Swans Lemma each stable vector bundle is invertible, so $\mathrm{Vect}(X)/\sim$ is a group. Define a homomorphism $\alpha : \mathrm{Vect}(X)/\!\sim \to \tilde{K}(X)$ by $\xi \mapsto \xi - \eta_k$, where $k = \dim(\xi)$.

If $\alpha(\xi) = 0$, by definition of $K(X)$ there is a bundle ξ' such that $\xi + \xi' = \eta_k + \xi'$. Again by Swans Lemma, $\xi' + \xi'' = \eta_l$ for some bundle ξ'' and some $l \in \mathbb{N}$, showing that $\xi + \eta_l = \eta_{k+l}$. Hence ξ is 0 in $\mathrm{Vect}(X)/\!\sim$, and α is injective.

Given any bundles $\xi - \xi' \in \tilde{K}(X)$, with ξ'' and l as before, we have $\xi - \xi' = (\xi + \xi'') - \eta_l$. Now $\dim(\xi + \xi'') = l$ because $\dim(\xi - \xi') = 0$. □

Let $\mathrm{Vect}_n(X) \subset \mathrm{Vect}(X)$ denote the set of isomorphism classes of vector bundles of rank n. Consider the direct system formed by the groups $\mathrm{Vect}_n(X)$ with the maps $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ sending ξ to $\xi \oplus \eta_1$. Then $\mathrm{Vect}(X)/\!\sim \cong \varinjlim \mathrm{Vect}_n(X) \cong \varinjlim [X, B_{U_n}]$. The maps $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ correspond to specific embeddings $B_{U_n} \to B_{U_{n+1}}$, and from lemma A.2.1 we get $\varinjlim [X, B_{U_n}] \cong [X, \varinjlim B_{U_n}]$. Clearly $K(X) = \tilde{K}(X) \oplus \mathbb{Z}$, and so we get

1.3.5 Theorem. $\tilde{K}(X) \cong [X, BU]$ and $K(X) = [X, BU \times \mathbb{Z}]$ for compact X. □

Topological induction

The following fundamental theorem summarises the most important topological induction steps used in explicit calculations of K_G-groups.

1.3.6 Theorem. Let X be a compact space and H a closed subgroup of a compact Lie group G.

i) $K_G(G/H) \cong R(H)$.

ii) $K_G(G \times_H X) \cong K_H(X)$ for any H-action on X.

iii) $K_G(G/H \times X) \cong K_H(X)$ for any G-action on X.

iv) $K_G(X) \cong K(X/G)$ for any free G-action on X.

v) $K_G(X) \cong K(X) \otimes R(G)$ for any trivial G-action X.

Proof. i) $G \to G/H$ is a principal H-bundle which is locally trivial (see [8]). We show that any G-vector bundle $\pi : E \to G/H$ arises by construction 1.1.4.

Since $H/H \subset G/H$ is fixed by H, the fibre $E_0 := \pi^{-1}(H/H)$ is an H-module. The action of G on E induces a map $G \times E_0 \to E$ which factorises to a map $\alpha : G \times_H E_0 \to E$. This is equivariant when G acts on $G \times_H E_0$ by multiplication on the first component. We prove that α is a homeomorphism by constructing its inverse:

Consider the homeomorphism $\beta : G \times E \to G \times E$ defined by $(g, e) \mapsto (g, g^{-1}e)$. Let $F = \beta^{-1}(G \times E_0) = \{(g, e) : \pi(e) = gH\}$. The composite $F \xrightarrow{\beta} G \times E_0 \to G \times_H E_0$ factorises through E: Given $(g, e) \in F$, for any $g' \in \pi(e)$ we have $(g', g'^{-1}e) = (g, g^{-1}e)$ in $G \times_H E_0$. Now the composite map $F \to G \times_H E_0$ is continuous because the projection $F \to E$ is an open map.

ii) The proof of i) carries over.

iii) Consider the map $G \times X \to G \times X$ defined by $(g, x) \mapsto (g, gx)$. Since $(gh^{-1}, hx) \mapsto (gh^{-1}, gx)$, this map factorises to $G \times_H X \to G/H \times X$. It is easy to check that this is a G-homeomorphism, yielding $K_G(G/H \times X) \cong K_G(G \times_H X) \cong K_H(X)$.

iv) If E is a G-vector bundle on X, then E/G is a vector bundle on X/G (for local triviality, see [22]). It is easy to check that $E \to E/G$ is inverse to the map π^*.

v) There is a natural map $\mu : K(X) \otimes R(G) \to K_G(X)$. We construct an inverse to μ, which generalises the standard decomposition of G-modules: The map $P : E \to E$ defined by

$e \to \int_G ge$ projects E onto the subspace E^G of points fixed by G. Hence E^G is the kernel of $(\mathrm{id}_E - P)$ and thus itself locally trivial (cf. section 2.2). The map $E \mapsto E^G$ defines a functor $\epsilon : K_G(X) \to K(X)$. For any G-module M, let $\mathbb{M}$ denote the trivial G-bundle $X \times M$. The group G acts on the vector bundle $\mathrm{Hom}(\mathbb{M}, E)$ and fixes exactly the G-equivariant maps $\mathrm{Hom}_G(\mathbb{M}, E)$. The functor $E \mapsto \mathrm{Hom}_G(\mathbb{M}, E)$ induces a functor $\epsilon_M : K_G(X) \to K(X)$. Let $\hat{G}$ denote the set of finite-dimensional irreducible representations of G. The map $\nu : K_G(X) \to R(G) \otimes K(X)$ defined by $\nu(E) = \sum_{[M] \in \hat{G}} [M] \otimes \epsilon_M(E)$ is the inverse of μ:

a) For any G-vector bundle E on X, the natural map $\bigoplus_{[M] \in \hat{G}} \mathbb{M} \otimes \mathrm{Hom}_G(\mathbb{M}, E) \to E$ is an isomorphism because it is one on fibres, and hence $\mu \circ \nu = \mathrm{id}$.

b) If G acts trivially on E, then for any G-modules M_1 and M_2 we have $\mathrm{Hom}_G(\mathbb{M}_1, \mathbb{M}_2 \otimes E) \cong \mathrm{Hom}_G(\mathbb{M}_1, \mathbb{M}_2) \otimes E$. If M_1 and M_2 are irreducible, then this is E if $[M_1] = [M_2]$, and $\emptyset$ otherwise. Hence $\nu \circ \mu = \mathrm{id}$. □

Remark. The previous examples can be generalised as follows: A homomorphism $\alpha : G \to G'$ of compact Lie groups induces an isomorphism $K_{G'}(G' \times_G X) \to K_G(X)$ if $\ker \alpha$ acts freely on X, see [19].

Higher K_G-groups, relative K_G-groups, and the long exact sequence

The main purpose of this subsection is to establish theorem 1.3.8, which allows to calculate the K-groups of X, $A \subset X$, and the pair (X, A) if two of the groups are given. This is fundamental for explicit calculations. In order to introduce the necessary higher K-groups and relative K-groups, we have to recall some topological constructions:

For a space X and a subset $A \subset X$, denote $X/A := X/\!\sim$ where $x \sim x' :\Leftrightarrow x = x'$ or $x, x' \in A$. The space X/A has a natural base point A/A.

The *cone* of a space X is $CX = X \times [0,1]/X \times 1$. If X has a base point x_0, the *reduced cone* on X is $\tilde{C}X = CX/Cx_0$. The *suspension* of X is $SX = X \times [0,1]/X \times \{0,1\}$, and the *reduced suspension* is $\tilde{S}X = SX/Sx_0$.

The *push-out* of two base-point preserving maps $i_1 : X \to Y_1$ and $i_2 : X \to Y_2$ is $Y_1 \coprod_X Y_2 = Y_1 \coprod Y_2/\!\sim$, where $y_1 \sim y_2 :\Leftrightarrow i_1(y_1) = i_2(y_2)$.

The *mapping cone* of a base-point preserving map $f : Y \to X$ is the push-out $\tilde{C}f := \tilde{C}Y \coprod_Y X$ of f and the inclusion $Y \to Y \times 0 \to \tilde{C}Y$. Note that $f(Y) \subset X \subset \tilde{C}f$ is contractible.

If X is locally compact, its one-point compactification is denoted by X^+ and has a natural base point. If X is compact, we let $X^+ = X \cup \{+\}$. Generally, $\tilde{S}(X^+) = (SX)^+$.

Remark. If the spaces we started with were G-spaces, and all maps equivariant, the spaces constructed all carry a natural G-action.

1.3.7 Definition. i) Let X be a compact G-space with base point, and $A \overset{i}{\hookrightarrow} X$ a closed G-subspace containing the base point. We define

$$\tilde{K}_G^{-q}(X) := \tilde{K}_G(\tilde{S}^q X), \quad \tilde{K}_G^{-q}(X, A) := \tilde{K}_G(\tilde{S}^q \tilde{C}i), \quad q \in \mathbb{N}.$$

ii) Let X be a locally compact G-space and $A \overset{i}{\hookrightarrow} X$ a closed G-subspace. We define

$$K_G^{-q}(X) := \tilde{K}_G^{-q}(X^+), \quad K_G^{-q}(X, A) := \tilde{K}_G^{-q}(X^+, A^+), \quad q \in \mathbb{N}.$$

Remarks. i) For a compact G-space X we have $K_G^{-q}(X^+) = K_G(S^q X)$.

ii) Generally, $K_G^{-q}(X) = K_G(X \times \mathbb{R}^q)$ since $\tilde{S}^q(X^+) = (X \times \mathbb{R}^q)^+$.

iii) For ordinary cohomology with compact support, $H_c^q(\mathbb{R}^q) = \mathbb{Z}$ is generated by a "bump volume element" ω. For nice locally compact spaces X, there is a *Thom isomorphism* $H^n(X) \xrightarrow{\sim} H^{n+q}(X \times \mathbb{R}^q)$ which is taking the cup product with ω. This might motivate the definition of the higher K-groups.

iv) The definition of the relative K-groups will be justified by the long exact sequence 1.3.8, and more intuitively by the excision theorem 1.3.12.

1.3.8 Theorem. For a compact G-space X with a base point, and a closed G-subspace A, there is a long exact sequence

$$\cdots \to \tilde{K}_G^{-q-1}(X,A) \to \tilde{K}_G^{-q-1}(X) \to \tilde{K}_G^{-q-1}(A) \to \tilde{K}_G^{-q}(X,A) \to \tilde{K}_G^{-q}(X) \to \tilde{K}_G^{-q}(A) \to \\ \cdots \to \tilde{K}_G^{0}(X,A) \to \tilde{K}_G^{0}(X) \to \tilde{K}_G^{0}(A). \quad (1)$$

The same holds for locally compact G-spaces with K instead of $\tilde{K}$.

Proof. We give two sketches:
i) If G is trivial, $\tilde{K}_G = \tilde{K}$ is representable by 1.3.5, and the theorem follows from

1.3.9 Lemma (Puppe)**.** For any base point preserving map $f : A \to X$ and any space V with a base point, applying the functor $[\cdot, V]$ to the sequence

$$A \to X \to \tilde{C}f \to \tilde{S}^1 A \to \tilde{S}^1 X \to \tilde{S}^1 \tilde{C}f \to \tilde{S}^2 A \to \cdots$$

gives an exact sequence of sets. □

Drawing pictures of the spaces involved makes this obvious. One can show that $\tilde{K}_G$ is representable, and the lemma generalizes to the equivariant setting.

ii) Another proof is given in [19]. The first step is

1.3.10 Lemma. The sequence $\tilde{K}_G(\tilde{C}i) \to \tilde{K}_G(X) \to \tilde{K}_G(A)$ is exact.

Proof. The composition is 0 because $A \to \tilde{C}i$ is G-homotopic to the constant map that sends A to the vertex of $\tilde{C}A$ in $\tilde{C}i$. Let the bundle E on X represent an element of $\tilde{K}_G(X)$ which vanishes in $\tilde{K}_G(A)$. Then $E|_A \oplus A \times M \cong A \times N$ for some G-modules M, N. Form a bundle E' on $\tilde{C}i$ by clutching $E \oplus A \times M$ to $A \times N$ on $\tilde{C}A$ using this isomorphism. Then E' represents the desired element of $\tilde{K}_G(\tilde{C}i)$. □

The exactness of the other terms of the long sequence can then be shown by reducing each of its short sequences to one of the form above using suitable G-homotopies. □

Remark. From the previous theorem one can easily deduce the existence of a long exact Mayer-Vietoris sequence. However, this is not as useful as in ordinary cohomology – just look what happens if you try to compute the K-groups of spheres (which is easy for cohomology).

In the following, we use "-" to denote set-theoretical subtraction.

1.3.11 Lemma. If A is a closed G-contractible subspace of a compact G-space X, then $K_G(X/A) \xrightarrow{\sim} K_G(X)$.

Proof. Given a G-vector bundle E on X we construct a bundle $\tilde{E}$ on X/A as follows: Since A is contractible, $E|_A \cong M \times A$ for some G-module M. Extend this isomorphism to an open G-neighbourhood U of A in X. Now $X - A \cong X/A - A/A$. Construct $\tilde{E}$ by clutching $E|_{X-A}$ and $M \times (U/A)$ using the isomorphism between them on $(X/A - A/A) \cap (U/A) \cong U - A$. One must check that the isomorphism class of $\tilde{E}$ depends only on E; then $E \mapsto \tilde{E}$ defines a map $K_G(X) \to K_G(X/A)$ which is obviously additive and inverse to the natural map $K_G(X/A) \to K_G(X)$. □

1.3.12 Theorem (Excision)**.** If A is a closed G-subspace of a locally compact G-space X, the natural map $K_G^{-q}(X - A) \to K_G^{-q}(X, A)$ is an isomorphism.

Proof. Let $i : A^+ \hookrightarrow X^+$ and $j : \tilde{S}^q A^+ \to \tilde{S}^q X^+$ denote the inclusions. Then $\tilde{S}^q \tilde{C} i \cong \tilde{C} j$: for $q = 1$ this is easy to check, and for general q it follows by induction. Furthermore, $(X - A)^+ \cong (X^+ - A^+)^+ \cong X^+/A^+$, the subset $\tilde{S}^q A^+ \subset \tilde{C} j$ is contractible, and $S^q(X^+/A^+) \cong \tilde{C} j / \tilde{S}^q A^+$. Summarising we see that the map in the theorem is induced from $\tilde{S}^q \tilde{C} i \cong \tilde{C} j \to \tilde{C} j / \tilde{S}^q A^+ \cong \tilde{S}^q (X - A)^+$. But this is an isomorphism by the previous lemma. $\square$

Products for higher and relative K-groups

For compact G-spaces X and Y we have a product $K_G(X) \otimes K_G(Y) \to K_G(X \times Y)$ induced by the product in $K_G(X \times Y)$ and the natural maps $K_G(X) \to K_G(X \times Y) \leftarrow K_G(Y)$. It is easy to show that this product extends uniquely to the case of locally compact G-spaces.

1.3.13 Proposition. Let X, Y be compact G-spaces with closed G-subspaces A, B. Then there is a unique product $K_G(X, A) \otimes K_G(Y, B) \to K_G(X \times Y, X \times B \cup A \times Y)$ which for $A, B = \emptyset$ coincides with the one previously defined.

Proof. Observe that $(X \setminus A) \times (Y \setminus B) = X \times Y \setminus (X \times B \cup A \times Y)$ and apply excision. $\square$

Finally, we get a product of higher K_G-groups

$$\begin{array}{ccccc} K_G^{-p}(X, A) & \otimes & K_G^{-q}(Y, B) & \to & K_G^{-(p+q)}(X \times Y, X \times B \cup A \times Y) \\ \| & & \| & & \| \\ K_G(X \times \mathbb{R}^p, A \times \mathbb{R}^p) & \otimes & K_G(Y \times \mathbb{R}^q, B \times \mathbb{R}^q) & \to & K_G(X \times Y \times \mathbb{R}^{p+q}, (X \times B \cup A \times Y) \times \mathbb{R}^{p+q}) \end{array}$$

for any locally compact G-spaces X and Y with closed G-subspaces $A \subset X$ and $B \subset Y$.

1.3.14 Corollary. i) Let X be a compact connected G-space which can be covered by n closed G-contractible G-subspaces $A_1, \dots, A_n$. Then $\xi^n = 0$ for any $\xi \in \tilde{K}_G(X)$.

ii) For any compact space X, every $\xi \in \tilde{K}(X)$ is nilpotent.

Proof. i) For a closed G-subspace $A \subset X$, let $K_G^A(X) = \ker(K_G(X) \to K_G(A))$. By the previous proposition, $K_G^{A_1}(X) \otimes K_G^{A_2}(X) \to K_G^{A_1 \cup A_2}(X)$, and by induction we get $K_G^{A_1}(X) \otimes \cdots \otimes K^{A_n}(X) \to K_G^X(X) = 0$. But $\tilde{K}_G(X) \subset K_G^{A_i}(X)$ for each i by contractibility and connectedness.

ii) Given $\xi \in \tilde{K}(X)$ there is a finite closed cover $X = A_1 \cup \ldots \cup A_m$ which trivializes ξ. Then $\xi \in K^{A_i}(X)$ for all i, and we conclude $\xi^m = 0$ as before. $\square$

Sketch of Bott periodicity

We now sketch one half of an elementary proof of the fundamental Bott periodicity theorem. A more general theorem will be given with complete details in section 2. Technically, this subsection may be skipped, but some of the manipulations carried out here translate directly to topological operations and thus provide an intuitive picture which is hardly perceivable in the proof which we are going to give later on.

First we indicate another characterisation of $\tilde{K}^{-1}(X) = \tilde{K}(SX)$:

1.3.15 Lemma. For any X there is an isomorphism $\mathrm{Vect}_n(SX) \cong [X, \mathrm{GL}(n, \mathbb{C})]$, where SX denotes the unreduced suspension of X.

Proof. Write $SX = C^+X \cup C^-X$, where $C^+X = X \times [0, \frac{1}{2}]/X \times 0$ and $C^-X = X \times [\frac{1}{2}, 1]/X \times 1$. Then $C^+X \cap C^-X = X \times \frac{1}{2} \cong X$. Any vector bundle E on SX of rank n admits trivialisations $\alpha^\pm : E|_{C^\pm X} \xrightarrow{\sim} C^\pm X \times \mathbb{C}^n$ because C^+X and C^-X are contractible. For the same reason, $\alpha := \alpha^+|_X \cdot \alpha^-|_X^{-1} : X \to \mathrm{GL}(n, \mathbb{C})$ is well-defined up to homotopy. Clutching $C^+X \times \mathbb{C}^n$ to $C^-X \times \mathbb{C}^n$ via α gives back E. $\square$

Combining this with our previous investigations in section 1.3 we get

1.3.16 Corollary. For any compact space X we have $K^{-1}(X) = K(SX) \cong \varinjlim[X, \mathrm{GL}(n,\mathbb{C})] = [X, \varinjlim \mathrm{GL}(n,\mathbb{C})]$, where $[X, \mathrm{GL}(n,\mathbb{C})] \to [X, \mathrm{GL}(n+1,\mathbb{C})]$ is induced by $-\oplus 1 : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(n+1,\mathbb{C})$. $\square$

In the case $X = S^1$ we have $SX = S^2$. The line bundle on S^2 obtained from the clutching function $z \mapsto 1/z$, $z \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is called the *Hopf bundle*. We denote its class in $K(S^2)$ by H.

1.3.17 Theorem. (Bott Periodicity) $K(S^2) = \mathbb{Z} \oplus \mathbb{Z}H$ and $\tilde{K}(S^2) = \mathbb{Z}(1-H)$.

Proof. We only sketch that 1 and H generate $K(S^2)$ (additively): Let E be a vector bundle E on S^2 represented by the clutching function $\alpha : S^1 \to \mathrm{GL}(n,\mathbb{C})$.

Step 1: α can be approximated arbitrarily close by a Laurent polynomial clutching function of the form $\alpha'(z) = \sum_{-r}^{s} A_k z^k$ with matrices $A_k \in M(n,\mathbb{C})$, (see [15]), and hence is homotopic to such an α'. The bundle $F := E \otimes H^{-r}$ has a polynomial clutching function of the form $\beta(z) = \sum_0^t B_k z^k$, where $t = r+s$ and $B_k = A_{k-r}$.

Step 2: The bundle $G := F \oplus t\eta_n$ has a clutching function of the form

$$\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \left(\sum_0^t B_k z^k\right) \end{pmatrix}$$

which can be homotoped to the linear clutching function

$$\begin{pmatrix} 1 & -z & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & -z \\ B_t & \dots & B_1 & B_0 \end{pmatrix}$$

using elementary row and column operations. We can write this clutching function as $\gamma(z) = C_1 z + C_0$, with matrices $C_1, C_0 \in M(m,\mathbb{C})$ where $m = tn$.

Step 3: We now reduce to the case $C_1 = 1_m$. Since $\gamma(1) \in \mathrm{GL}(m,\mathbb{C})$, we can find $\lambda \in (0,1)$ such that $C_1' := C_1 + C_0\lambda \in \mathrm{GL}(m,\mathbb{C})$. Since $|(z+t\lambda)/(1+zt\lambda)| = |z| = 1$ for $t \in [0,1]$, γ can be homotoped inside $\mathrm{GL}(n,\mathbb{C})$ to the clutching function $\gamma_1(z) := C_1(z+\lambda)/(1+z\lambda) + C_0$. This in turn can be homotoped to $\gamma_2(z) := (1+z\lambda)\gamma_1(z) = (C_1+C_0\lambda)z + (C_1\lambda + C_0)$. Therefore G can be obtained from the clutching function $z \mapsto C_1' z + C_0$ and hence also from $\gamma_3(z) := 1_m z + {C'}_1^{-1} C_0$.

Now $C_0' := {C'}_1^{-1} C_0$ has no eigenvalues of absolute value 1 because γ_3 is non-singular. Therefore we can split $G = G^+ \oplus G^-$ where $G^+ = \bigoplus_{|\lambda|<1} G_\lambda$ and $G^- = \bigoplus_{|\lambda|>1} G_\lambda$ are the sums of the generalised eigenspaces $G_\lambda = \{x \in G | \exists n : (\lambda - C_0')^n x = 0\}$. These subbundles are stable under $\gamma_3(z)$ for any z. On G^+ we can homotop γ_3 to $z \mapsto 1_m z$ via $(z,t) \mapsto 1_m z + tC_0'$, and on G^- to C_0' via $(z,t) \mapsto 1_m tz + C_0'$. This shows $G^- \cong \eta_{\dim(G^-)}$ and $G^+ \cong \dim(G^+)H^{-1}$.

Step 4: Summarizing, $E \cong F \otimes H^r$, and $(F \oplus \eta_m) \otimes H^r \cong G \otimes H^r \cong \left(\dim(G^+)H^{-1} \oplus \eta_{\dim(G^-)}\right) \otimes H^r \cong \dim(G^+)H^{r-1} \oplus \dim(G^-)H^r$.

By corollary 1.3.14, we have $(1-H)(1-H^n) = 0 \in \tilde{K}(S^2)$, for any n, hence $H + H^n = 1 + H^{n+1}$. By induction on n we deduce that $H^n \sim nH$. Hence $K(S^2)$ is generated by η_1 and H, and $\tilde{K}(S^2)$ is generated by $1-H$. We do not show that this happens freely. $\square$

1.3.18 Theorem. i) For any locally compact space X, multiplication by $1 - H^{-1}$ induces an isomorphism $K(X) \to \tilde{K}(S^2 \times X)$.

ii) For any closed subspace A there is an isomorphism $K^{-q}(X,A) \cong K^{-q-2}(X,A)$.

Proof. One first proves both results for compact spaces and deduces the general case by a standard argument.

i) The strategy of the previous sketch carries over, where $\mathrm{GL}(n, \mathbb{C})$ has to be replaced by the Banach algebra of continuous maps from X to $\mathrm{GL}(n, \mathbb{C})$ which requires more work in step 1.

ii) As pairs of spaces, $(S^2 \times X, \{pt\})$ and $((X \times \mathbb{R}^2)^+, +)$ are homotopic. Thus $\tilde{K}(S^2 \times X) = K(S^2 \times X, \{pt\}) \cong K(X \times \mathbb{R}^2) = K^{-2}(X)$. Replacing X by $\tilde{S}^q X$ we obtain the statement for all $q \in \mathbb{N}$. The relative version follows from the long exact sequence and the 5-lemma. $\square$

1.3.19 Definition. For a locally compact space X we define $K^*(X) = K^0(X) \oplus K^{-1}(X)$, and similarly $K^*(X, A)$, $\tilde{K}^*(X)$ whenever this makes sense.

Now the long exact sequence of theorem 1.3.8 can be compressed (we omit the obvious commutativity checks):

1.3.20 Theorem. For any locally compact space X with a closed subspace A, there is an exact triangle

$$\begin{tikzcd} & K^*(X,A) \arrow[dr] & \\ K^*(A) \arrow[ur] & & K^*(X) \arrow[ll] \end{tikzcd}$$

$\square$

1.4 The representation ring of a compact Lie group

The purpose of this section is to prove two theorems on algebraic properties of the representation ring of a compact Lie group. Both results can be found in [20]; however, in that paper the second theorem is obtained as a corollary to a more general statement which involves heavy machinery[4].

Before we actually start, let us recall the following standard result:

1.4.1 Theorem. Any compact Lie group has a faithful representation and hence can be embedded into a unitary group. $\square$

This follows easily from the Peter-Weyl theorem.

Now, let G be a compact Lie group, G^0 its identity component, and $\Gamma = G/G^0$ its group of components. In the following, "subgroup" will mean a closed subgroup, and "cyclic group" will mean a compact Lie group containing an element whose powers are dense. It is elementary that every cyclic group is a product of a torus and a finite cyclic group.

1.4.2 Definition. A subgroup $S \subset G$ is a *Cartan subgroup* if it is cyclic and of finite index in its normaliser $N(S)$.

Cartan subgroups take over the role played by maximal tori in the connected case (see [8],[20]):

1.4.3 Proposition. i) Each element of G is contained in a Cartan subgroup S.

ii) The projection $\{\text{Cartan subgroups of } G\} \to \{\text{cyclic subgroups of } \Gamma\}$ induces a bijection of conjugacy classes. In particular, there are only finitely many conjugacy classes of Cartan subgroups.

iii) The restriction $R(G) \to \prod_S R(S)$, where S runs through the conjugacy classes of Cartan subgroups, is an injection. $\square$

[4] a generalized Lefschetz fixed point formula for elliptic complexes and thus the Atiyah-Singer index theorem

The following results are well-known:

1.4.4 Proposition. i) The representation rings of unitary groups and tori are noetherian.

ii) Let $T \hookrightarrow U$ be the inclusion of a maximal torus into a unitary group. Then the restriction $R(G) \to R(T)$ is injective and turns $R(T)$ into a finite ring extension of $R(U)$. $\square$

We can now prove the first theorem:

1.4.5 Theorem. i) Let G be a subgroup of a compact Lie group H. Then $R(G)$ is finite over $R(H)$.[5]

ii) The representation ring of any compact Lie group is finitely generated and hence noetherian.

Proof. i) We embed H in a unitary group U and obtain restriction maps $R(U) \to R(H) \to R(G)$. Then we clearly may assume $U = H$.

Since $R(G) \to \prod_S R(S)$ is injective and $R(U)$ is noetherian, it is enough to show that for any Cartan subgroup S, the ring $R(S)$ is finite[6] over $R(U)$. Any such group is cyclic and hence contained in a maximal torus T of U. Since S and T are abelian, $R(T) = \mathbb{Z}[\hat{T}]$ and $R(S) = \mathbb{Z}[\hat{S}]$, where $\hat{S}$ denotes the character group. It is easy to see that the inclusion $S \hookrightarrow T$ induces a surjection $\hat{T} \to \hat{S}$. Now $R(U)$ is finite over $R(T)$ which is finite over $R(S)$. $\square$

Let I_G denote the kernel of the natural augmentation $R(G) \to \mathbb{Z}$. Again, we generalise a standard result:

1.4.6 Proposition. Let $T \hookrightarrow U$ be the inclusion of a maximal torus in a unitary group. Then some power I_T^n is contained in the ideal generated by the image of I_U in $R(T)$. $\square$

The second theorem we need is:

1.4.7 Theorem. If G is a subgroup of a unitary group U, then for any $R(G)$ - module M, its I_G-adic completion and its I_U-adic completion coincide: $\varprojlim M/I_G^n M \cong \varprojlim M/I_U^n M$.

Proof. Let $J_U := R(G)I_U$ denote the ideal in $R(G)$ generated by the image of I_U under the restriction map $R(U) \to R(G)$. Clearly $J_U \subset I_G$. It is enough to show that $I_G^n \subset J_U$ for some n.

For any noetherian ring R and any ideal $J \subset R$, the radical $rad(J) := \{x \in R : \exists n : x^n \in J\}$ equals the intersection of all primes containing J. Thus, given ideals $I, J \subset R$, we have $I^n \subset J$ for some n if and only if every prime containing J also contains I (note that I is finitely generated). We will use both directions of this equivalence:

The restriction map $R(G) \to \prod_S R(S)$ induces a map $\mathrm{Spec}(\prod_S R(S)) \to \mathrm{Spec}(R(G))$. By the previous theorem, $\prod_S R(S)$ is finite over $R(G)$, and so the map is surjective by the theorem of "lying over". Furthermore, $\mathrm{Spec}(\prod_S R(S)) = \coprod_S \mathrm{Spec}(R(S))$. This means that every prime $p \subset R(G)$ is the preimage of some prime $q \subset R(S)$ under the restriction map $r : R(G) \to R(S)$;

$$\begin{array}{ccc} R(G) & \to & R(S) \\ \cup & & \cup \\ p = r^{-1}(q) & \to & q \end{array}$$

Now, $J_U = R(G)I_U \subset p$ implies $R(S)I_U \subset q$, and $I_S \subset q$ implies $I_G \subset p$ since $r(I_G) \subset I_S$. Hence it is enough to consider the case $G = S$.

So, let S be any Cartan subgroup, and T a maximal torus containing S. Since the surjective restriction $R(T) \to R(S)$ maps $R(T)I_U$ to $R(S)I_U$ and I_T onto I_S, proposition 1.4.6 shows that some power I_S^n of I_S is contained in $R(S)I_U$. $\square$

[5] i.e. finitely generated as an $R(H)$-ring

[6] i.e. finitely generated as an $R(U)$-module

2 The Main Lemmata

2.1 Introduction

Subject: In this chapter we will prove the main result of equivariant K-theory – the Thom isomorphism – and deduce from it the equivariant splitting principle.

Both have analogues in ordinary cohomology theory. There the splitting principle states that for each vector bundle $\pi : E \to B$ there is a space F and a map $f : F \to B$ such that f^*E splits into a sum of line bundles (i.e. the structure group reduces from a unitary group to a torus), and $f^* : H^*(B) \to H^*(F)$ is injective. In fact one can choose $f : F \to B$ to be the flag bundle of $E \to B$ which has as fibre over $b \in B$ the set of subspaces of $\pi^{-1}(b)$. This result if fundamental for establishing identities for the Chern character like e.g. multiplicativity.

The Main Lemma we are going to prove states that for any compact space B the map $j^* : K_U(B) \to K_T(B)$ induced by the inclusion of a maximal torus $j : T \hookrightarrow U$ into a unitary group has a left-inverse and hence is injective.

Strategy of the proof: By theorem 1.3.6, we have $K_T(X) \cong K_U(U/T \times X)$, and under this identification, the projection $U/T \times X \to X$ induces a map $K_U(X) \to K_U(U/T \times X)$ which coincides with j^*. In order to construct a left-inverse j_* to j^*, we consider a specific family of elliptic differential operators on the fibres U/T parameterised by X, which "compute the cohomology of U/T" in each fibre. Any vector bundle on $U/T \times X$ gives rise to an extension of this family of differential operators which again is elliptic. Now the kernel and cokernel of any elliptic operator have finite dimension, and so we get a "positive" and "negative" finite-dimensional vector space at each point $x \in X$. These glue together to give locally trivial vector bundles on X. This construction induces a map $j_* : K_U(U/T \times X) \to K_U(X)$, which will be shown to be a left-inverse to j^*.

Structure of the chapter: To carry out this programme, we first consider the following general problem: given two bundles of function spaces over X, and a bundle map which has fibrewise finite-dimensional kernel and cokernel, can we construct an element of $K(X)$ which represents "[kernel] - [cokernel]"?

Then we introduce elliptic differential operators on manifolds and show that their kernel and cokernel always have finite dimension. This requires some local theory of pseudo-differential operators which is included primarily because I did not know it before and wanted to provide a concise approach. Afterwards we consider families of elliptic differential operators and show how to apply the results of the first subsection.

The particular elliptic differential operators we are going to consider are derived from the Dolbeault operator which is intimately related to the cohomology of complex manifolds. We summarise briefly the basic results and hide some explicit calculations in the appendix.

Finally, we prove the Main Lemmata.

References: The material of section 2.2 is based on [1] and needed to establish theorem 2.3.17 which is the main result of section 2.3. For the theory of pseudo-differential operators, I follow [11], which also contains theorem 2.4.5. Families of differential operators are treated in [17][7]; the original papers of M.F. Atiyah and G.B. Segal are oriented towards utmost generality and do not employ Hilbert space theory. Dolbeault cohomology is treated in [13]. Finally, the exposition of the Main Lemmata closely follows [3].

[7] which I only got hold of after having finished this section

2.2 K-Theory and Fredholm operators

2.2.1 Definition. Let H_1, H_2 be complex Hilbert spaces, and $\mathcal{B}(H_1, H_2)$ the normed space of bounded operators $H_1 \to H_2$. An operator $T \in \mathcal{B}(H_1, H_2)$ is a *Fredholm operator* if its kernel and cokernel have finite dimension. The integer $\dim\ker T - \dim\operatorname{coker} T$ is the *index* of T. The space of Fredholm operators is denoted by $\mathcal{F}(H_1, H_2)$.

2.2.2 Lemma. An operator $T \in \mathcal{B}(H_1, H_2)$ is Fredholm if there is are $S_1, S_2 \in \mathcal{B}(H_2, H_1)$ such that $S_1T - 1_{H_1}$ and $TS_2 - 1_{H_2}$ are compact.

Proof. $S_1T - 1_{H_1}|_{\ker T} = 1_{\ker T}$, and eigenspaces of compact operators have finite dimension. $S_2^*T^* - 1_{H_2}$ is also compact, and hence $\operatorname{coker} T \cong \ker T^*$ has finite dimension. □

2.2.3 Definition. Let X be a compact space. A *Hilbert bundle on* X is a locally trivial fibre bundle with a complex seperable Hilbert space as fibre. For two Hilbert bundles $\mathcal{H}_1, \mathcal{H}_2 \to X$, let $\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ denote the space of all bundle maps $T : \mathcal{H}_1 \to \mathcal{H}_2$ that are fibrewise Fredholm, i.e. $T_x \in \mathcal{F}(\mathcal{H}_{1,x}, \mathcal{H}_{2,x})$ for all $x \in X$. A *Fredholm bundle on* X is a pair of Hilbert bundles $\mathcal{H}_1, \mathcal{H}_2$ on X together with a map $T \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$.

Given a Fredholm bundle $T : \mathcal{H}_1 \to \mathcal{H}_2$ on X, at each point $x \in X$ we have the finite dimensional vector spaces $\ker T_x$ and $\operatorname{coker} T_x$. We want to associate to T an index $[\ker T] - [\operatorname{coker} T] \in K(X)$. But the example $X = [-1, 1], \mathcal{H}_1 = \mathcal{H}_2 = X \times \mathbb{C}$ and $T_x(\lambda) = x\lambda$ shows that $\ker T$ and $\operatorname{coker} T$ need not have constant dimension. However, we will see that "dimension jumps" in the kernel and cokernel occur simultaneously and "cancel out".

2.2.4 Lemma. i) Given a Fredholm bundle $T : \mathcal{H}_1 \to \mathcal{H}_2$, there is a finite dimensional Hilbert space W and a map $K : X \times W \to \mathcal{H}_2$ such that $T \oplus K : \mathcal{H}_1 \oplus (X \times W) \to \mathcal{H}_2$ is surjective.

ii) The bundle $\ker(T \oplus K)$ is a finite-dimensional vector bundle on X.

Proof. i) For $x \in X$ let $T' : U \times H_1 \to U \times H_2$ be a local representation of T with respect to a trivialisation of $\mathcal{H}_1$ and $\mathcal{H}_2$ over a neighbourhood U of x.

Choose a finite dimensional subspace $i_x : V_x \hookrightarrow H_2$ for which $T'_x| \oplus i_x : (\ker T'_x)^\perp \oplus V_x \to H_2$ is a bijection. By the closed graph theorem this is an isomorphism, and hence on some neighbourhood of x, the map $T'_y| \oplus i_x$ remains an isomorphism for all y. Using a cut-off function we find a finite-dimensional subspace $W_x \subset \Gamma(\mathcal{H}_2)$ of sections that span V_x in the fibre at x. Then we have an obvious map $K_x : X \times W_x \to \mathcal{H}_2$, and $T \oplus K_x : \mathcal{H}_1 \oplus (X \times W_x) \to \mathcal{H}_2$ is surjective on a neighbourhood of x.

Now take $W = \bigoplus W_{x_i}$ and $K = \bigoplus K_{x_i}$ for some finite open cover $\{U_{x_i}\}$ of X.

ii) Let $K' : U \times W \to U \times H_2$ be a local representation of K. Let T' be as above, and $R' = T' \oplus K'$. Denote the orthogonal projection from $H_1 \oplus W$ onto $\ker R'_x$ by π and define $S' : U \times (H_1 \oplus W) \to U \times (H_2 \oplus \ker R'_x)$ by $S'_y(h, w) := (R'_y(h, w), \pi(h, w))$. Then S' is a bijection and hence an isomorphism by the closed graph theorem. Again, on a neighbourhood V of x the map $S'_y : H_1 \oplus W \to H_2 \oplus \ker R'_x$ remains an isomorphism for all y, and the inverse of S'_y gives a trivialisation $(\ker R')|_V \cong V \times (\ker R'_x)$. □

We now define $\operatorname{index} : \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2) \to K(X)$ by putting $\operatorname{index} T := [\ker(T \oplus K)] - [X \times W]$.

2.2.5 Lemma. This definition does not depend on the choice of K and W.

Proof. If $L : X \times V \to \mathcal{H}_2$ is another choice, so is $L \oplus K : X \times (V \oplus W) \to \mathcal{H}_2$, hence it suffices to assume $W \subset V$ and $L|_W = K$. Then the snake lemma applied to

$$\begin{array}{ccccccc} 0 \to & \ker(T \oplus K) & \to & \mathcal{H}_1 \oplus X \times W & \to & \mathcal{H}_2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \ker(T \oplus L) & \to & \mathcal{H}_1 \oplus X \times V & \to & \mathcal{H}_2 & \to 0 \end{array}$$

shows that $[\ker(T \oplus K)] - [\ker(T \oplus L)] = [X \times (V/W)] = [X \times V] - [X \times W]$. □

2.2.6 Lemma. index is homotopy-invariant: Given a family $H : [0,1] \to \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ of Fredholm bundles we have $\operatorname{index} H(0) = \operatorname{index} H(1)$.

Proof. Denote the projection $X \times [0,1] \to X$ by π, and the inclusion $i_t : X \to X \times t \to X \times [0,1]$ by i_t. Consider H as a Fredholm bundle $\pi^*\mathcal{H}_1 \to \pi^*\mathcal{H}_2$ on $X \times [0,1]$. Then by homotopy invariance of pull-back, $i_0^* \operatorname{index} H = i_1^* \operatorname{index} H$. $\square$

Remarks. i) Given Fredholm operators S and T, their composition $S \circ T$ is clearly Fredholm. Furthermore, $\operatorname{index}(S \circ T) = \operatorname{index} S + \operatorname{index} T$. This generalizes to Fredholm bundles.

ii) Fix a seperable complex Hilbert space H. Then a Fredholm bundle $T : X \times H \to X \times H$ is just a map $T : X \to \mathcal{F}(H)$, and index induces a homomorphism $[X, \mathcal{F}(H)] \to K(X)$. Using Swan's theorem it is easy to see that this is surjective. Injectivity follows from Kuiper's theorem which states that the group of invertible elements in $\mathcal{B}(H)$ is contractible. Thus $\mathcal{F}(H)$ is a "classifying space" for K.

2.3 Families of elliptic differential operators

Elliptic differential operators on manifolds - 1

Let M be a compact smooth manifold of dimension m with smooth vector bundles E and F of rank r and s. Let $\mathcal{D}(E)$ and $\mathcal{D}(F)$ denote the spaces of smooth sections of E and F.

2.3.1 Definition. A linear map $L : \mathcal{D}(E) \to \mathcal{D}(F)$ is a *differential operator* :$\Leftrightarrow$ for any chart $\tau : U \xrightarrow{\sim} \tilde{U} \subset \mathbb{R}^m$, $U \subset M$, and any trivialisations $h : E|_U \to \tilde{U} \times \mathbb{C}^r$, $g : F|_U \to \tilde{U} \times \mathbb{C}^s$ compatible with τ, the induced operator

$$\begin{array}{ccccccc} C^\infty(\tilde{U})^r \supset & h_*(\mathcal{D}(E)|_U) & \xrightarrow{\tilde{L}} & g_*(\mathcal{D}(F)|_U) & \subset C^\infty(\tilde{U})^s \\ & \uparrow & & \uparrow & \\ & \mathcal{D}(E) & \xrightarrow{L} & \mathcal{D}(F) & \end{array}$$

is a linear partial differential operator, i.e. of the form $\tilde{L} = \sum_{|\alpha| \le k} A_\alpha D^\alpha$ with smooth matrix functions $A_\alpha : \tilde{U} \to \mathbb{C}^{s \times r}$. L has *order* k if each of its local representations $\tilde{L}$ is of order k. Let $\mathrm{Diff}_k(E, F)$ denote the vector space of differential operators of order k.

2.3.2 Definition. Let T_M^* denote the cotangent bundle of M, and T_M^0 denote T_M^* without the zero section, with projection $\pi : T_M^0 \to M$. Let $\mathrm{Symb}_k(E, F) = \{\sigma \in \mathrm{Hom}(\pi^*E, \pi^*F) : \sigma(x, \lambda v) = \lambda^k \sigma(x, v)\ \forall (x, v) \in T_M^0, \lambda > 0\}$ be the *symbols of order* k. Given $L \in \mathrm{Diff}_k(E, F)$ with a local representation as above, the map

$$(x, v) \mapsto \sum_{|\alpha| = k} A_\alpha(x) v^\alpha$$

defines an element of $\mathrm{Symb}_k(E|_U, F|_U)$. It is easy to see that these local definitions are compatible with coordinate changes and give a well-defined element $\sigma_k(L) \in \mathrm{Symb}_k(E, F)$ which we call the *leading symbol of* L. The *local symbol* of L is the map $(x, v) \mapsto \sum_\alpha A_\alpha(x) v^\alpha$.

A symbol is *elliptic* if it is an isomorphism. An operator $L \in \mathrm{Diff}_k(E, F)$ is *elliptic* if its leading symbol $\sigma_k(L)$ is.

Using local coordinates it is easy to see that

- Given smooth vector bundles E, F and H, we have inclusions and commutativity in

$$\begin{array}{ccccc} \mathrm{Diff}_k(F,H) & \circ & \mathrm{Diff}_l(E,F) & \subset & \mathrm{Diff}_{k+l}(E,H) \\ \downarrow \sigma_k & & \downarrow \sigma_l & & \downarrow \sigma_{k+l} \\ \mathrm{Symb}_k(F,H) & \circ & \mathrm{Symb}_l(E,F) & \subset & \mathrm{Symb}_{k+l}(E,H) \end{array}$$

- if we fix a metric on M that induces a measure μ, and metrics on E and F, then each $L \in \mathrm{Diff}_k(E,F)$ has a *formal adjoint* $L^* \in \mathrm{Diff}_k(F,E)$ satisfying $(Lf,g)_{L^2} = \int_m ((Lf),g)d\mu = (f, L^*g)_{L^2}$ for all $f \in \mathcal{D}(E)$ and $g \in \mathcal{D}(F)$. Furthermore, $\sigma_k(L^*) = (-1)^k \sigma_k(L)^*$.

Local Theory of Pseudo-differential operators

Now we introduce the class of pseudo-differential operators which allow us to "invert elliptic differential operators up to smoothing terms". All results of this section will be needed to establish theorems 2.3.11 and 2.3.12. First we recall some well-known facts and introduce notation.

Let $\mathcal{S}$ denote the *Schwartz space* of rapidly decreasing functions on $\mathbb{R}^m$, i.e. $\mathcal{S} = \{f \in C^\infty(\mathbb{R}^m) | \forall \alpha, \beta \in \mathbb{N}^m : \sup |x^\alpha D^\beta f(x)| < \infty\}$, where $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$.

In the following, dx, dy, etc. denote the Lebesgue measure on $\mathbb{R}^m$ divided by $(2\pi)^{m/2}$.

The *Fourier transform* of a function $f \in \mathcal{S}$ is $(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int e^{-ix\xi} f(x)dx$. It is well-known that $\mathcal{F}$ is a bijection of $\mathcal{S}$ with inverse given by $f(x) = \int e^{ix\xi}\widehat{f}(\xi)d\xi$. Furthermore, we have

$$D^\alpha \mathcal{F} = (-1)^{|\alpha|} m^\alpha \mathcal{F}, \qquad m^\alpha \mathcal{F} = \mathcal{F} D^\alpha, \qquad \text{where } (mf)(x) = xf(x),$$
$$\widehat{f}\widehat{g} = \widehat{f \star g}, \qquad \widehat{f} \star \widehat{g} = \widehat{fg}, \qquad (f,g)_{L^2} = (\widehat{f}, \widehat{g})_{L^2}.$$

For an open subset $U \subset \mathbb{R}^m$ and a number $s \in \mathbb{R}$, the *Sobolev norm* $\|\cdot\|_s$ on $C_c^\infty(U)$ is defined as $\|f\|_s = \int (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi$, and the *Sobolev space* $H_s(U)$ is the completion of $C_c^\infty(U)$ with respect to this norm. $\mathcal{S}$ is densely embedded in $H_s(\mathbb{R}^m)$.

Remarks. It is easy to see that:

i) For any $\alpha \in \mathbb{N}^m$, D^α extends to a bounded operator $H_s \to H_{s-|\alpha|}$.

ii) For $s \in \mathbb{N}$ we can expand $(1+|\xi|^2)^s$ and deduce fom the identity $\|\xi^\alpha \widehat{f}(\xi)\|_{L^2} = \|D^\alpha f\|_{L^2}$ that $\|\cdot\|_s$ is equivalent to the norm $f \mapsto \sum_{|\alpha|\leq s} \|D^\alpha f\|_{L^2}$.

iii) The pairing $(f,g)_s := \int \widehat{f}(\xi)\widehat{g}(\xi)(1+|\xi|^2)^s d\xi$ turns H_s into a Hilbert space.

iv) The Cauchy-Schwartz inequality shows that the L^2-product on $\mathcal{S} \times \mathcal{S}$ extends to a dual pairing $H_s \times H_{-s} \to \mathbb{C}$.

The following two lemmas will be used to show that elliptic differential operators are Fredholm and that their kernel is smooth.

2.3.3 Lemma (Sobolev)**.** For $k \in \mathbb{N}$ and $s > k + m/2$ we have an embedding $H_s \hookrightarrow C^k$ with $\|f\|_{\infty,k} := \sum_{|\alpha|\leq k} \|D^\alpha f\|_\infty < C\|f\|_s$ for some C depending on k and s.

Proof. For $|\alpha| \leq k$ and $f \in \mathcal{S}$ we have $\int |\widehat{f}(\xi)||\xi|^{|\alpha|}d\xi \leq \|f\|_s \left(\int |\xi|^{2|\alpha|}/(1+|\xi|^2)^s d\xi\right)^{1/2}$, and the second integral exists by the assumption on s and k. Recalling $D^\alpha f(x) = \int e^{ix\xi}\widehat{f}(\xi)\xi^\alpha d\xi$ we see that any $\|\cdot\|_s$-Cauchy sequence in $\mathcal{S}$ is a $\|\cdot\|_{\infty,k}$-Cauchy sequence in C^k. □

2.3.4 Lemma (Rellich)**.** Let $K \subset \mathbb{R}^m$ be compact, $f_n \in C_c^\infty(K)$, and $\|f_n\|_s < C$. Then for any $t < s$ there is a subsequence $\{f_{n_k}\}$ converging in H_t.

Proof. We may assume $f_n \in \mathcal{S}$ for all n. Choose $\chi \in C_c^\infty(\mathbb{R}^m)$ with $\chi|_K \equiv 1$. Then $\chi f_n = f_n$ implies $\widehat{\chi} \star \widehat{f_n} = \widehat{f_n}$ and thus by Cauchy-Schwarz $|\widehat{f_n}(\xi)| \leq \int |\widehat{\chi}(\xi - \eta)\widehat{f_n}(\eta)| d\eta \leq \|f_n\|_s \left(\int |\widehat{\chi}(\xi - \eta)|^2/(1+|\eta|^2)^s\right)^{1/2} d\eta$. Observing that $\partial_i(\widehat{\chi} \star \widehat{f_n}) = (\partial_i \widehat{\chi}) \star \widehat{f_n}$ we get a similar estimate for $|\partial_i \widehat{f_n}(\xi)|$. By Arzela-Ascoli, going over to a subsequence we may assyme that $\widehat{f_n}$ converges uniformly on any compact set. Now

$$\begin{aligned} \|f_i - f_j\|_t &= \int |\widehat{f}_i(\xi) - \widehat{f}_j(\xi)|^2 (1+|\xi|^2)^s/(1+|\xi|^2)^{s-t} d\xi \\ &\leq \left(\int_{|\xi| \leq r} |\widehat{f}_i(\xi) - \widehat{f}_j(\xi)|^2 (1+|\xi|^2)^t d\xi\right) + 2C/(1+|r|^2)^{s-t}. \end{aligned}$$

Choosing r large makes the second summand arbitrarily small, and the uniform convergence of the $\widehat{f_n}$ on $\{\xi : |\xi| \leq r\}$ makes the first summand arbitrarily small. □

2.3.5 Lemma. Let $\tau : U \to V$ be a diffeomorphism of open sets U and $V \subset \mathbb{R}^m$ that extends to $\bar{U}$, where $\bar{U}$ is compact. Then $\tau_* : C_c^\infty(U) \to C_c^\infty(V)$ extends to an isomorphism $\tau_* : H_s(U) \to H_s(V)$ for any $s \in \mathbb{Z}$.[8]

Proof. The transformation formula shows that $\tau_* : L_2(U) \to L_2(V)$ is bounded. Now for $s \in \mathbb{N}$ the claim follows easily from remark ii), and for $-s \in \mathbb{N}$ from remark iv) and the case $s \in \mathbb{N}$. □

We now introduce the class of pseudo-differential operators.

2.3.6 Definition. Let $U \subset \mathbb{R}^m$ and $d \in \mathbb{Z}$. Let S^d be the set of all $p \in C^\infty(U \times \mathbb{R}^m)$ satisfying

- $\operatorname{supp} p \subset K \times \mathbb{R}^m$ for some compact $K \subset U$,
- for all $\alpha, \beta \in \mathbb{N}^m$ there is a $C_{\alpha\beta}$ such that $|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{d-\beta}$.

For $p \in S^d(U)$ we define the associated *pseudo-differential operator* $P : \mathcal{S} \to C_c^\infty(U)$ by $Pf(x) := \int e^{ix\xi} p(x,\xi)\widehat{f}(\xi) d\xi$.

2.3.7 Lemma. P extends to a continuous operator $H_s(\mathbb{R}^m) \to H_{s-d}(\mathbb{R}^m)$.

Proof. It is sufficient to show that $|(Pf, g)| \leq C\|f\|_s \|g\|_{d-s}$ for all $f, g \in \mathcal{S}$ because of remark iv).

Changing the order of integration we can write

$$\widehat{Pf}(\xi) = \int e^{ix\xi} e^{-ix\zeta} p(x,\xi) dx \hat{f}(\xi) d\xi = \int q(\zeta - \xi, \xi)\widehat{f}(\xi) d\xi, \tag{2}$$

where $q(\cdot,\xi) = \widehat{p(\cdot,\xi)}$. Put $K(\zeta,\xi) = q(\zeta-\xi,\xi)(1+|\xi|^2)^{-s/2}(1+|\zeta|^2)^{(s-d)/2}$. Then

$$\begin{aligned} |(Pf, g)| &= |(\widehat{Pf}, \widehat{g})| = \left|\int q(\zeta - \xi)\widehat{f}(\xi)\widehat{g}(\zeta) d\xi d\zeta\right| \\ &= \left|\int K(\zeta,\xi)\widehat{f}(\xi)(1+|\xi|^2)^{s/2}\widehat{g}(\zeta)(1+|\zeta|^2)^{(d-s)/2} d\xi d\zeta\right| \\ &\leq \left(\|f\|_s \sup_\xi \int |K(\zeta,\xi)| d\zeta\right)^{1/2} \left(\|g\|_{d-s} \sup_\zeta \int |K(\zeta,\xi)| d\xi\right)^{1/2} \end{aligned}$$

Now $|D_x^\alpha p(x,\xi)| \leq C_\alpha(1+|\xi|)^d$ implies that $f := p(\cdot,\xi)/(1+|\xi|)^d \in \mathcal{S}$, hence $\hat{f} = q(\cdot,\xi)/(1+|\xi|)^d \in \mathcal{S}$, and we find $|(\zeta-\xi)^\alpha q(\zeta-\xi,\xi)| \leq C'_\alpha(1+|\xi|)^d$ for any α. Using this bound with $|\alpha|$ sufficiently large, an easy calculation shows that the suprema above are finite. □

[8]The lemma holds for arbitrary $s \in \mathbb{R}$ which we do not need and do not prove because this requires (much) more work.

2.3.8 Theorem. Let $p \in S^d(U)$ and $q \in S^e(U)$ define P and Q. Then $P \circ Q$ is a pseudo-differential operator defined by a symbol $s \in S^{d+e}(U)$ with $pq - s \in S^{d+e-1}(U)$.

To prove the theorem we need a larger class of symbols: let $\tilde{S}^d(U)$ be the set of all $r \in C^\infty(U \times \mathbb{R}^m \times \mathbb{R}^m)$ satisfying

- $\operatorname{supp} r \subset K \times \mathbb{R}^m \times L$ for some compact $K \subset U$ and $L \subset \mathbb{R}^m$,
- for all $\alpha, \beta, \gamma \in \mathbb{N}^m$ there is a $C_{\alpha\beta\gamma}$ such that $|D_x^\alpha D_\xi^\beta D_y^\gamma r(x,\xi,y)| \le C_{\alpha\beta\gamma}(1+|\xi|)^{d-\beta}$.

We associate to r an operator $R : \mathcal{S} \to C_c^\infty(U)$ by $Rf(x) = \int e^{i(x-y)\xi} r(x,\xi,y) f(y) dy\, d\xi$.

2.3.9 Lemma. There is a symbol $s \in S^d(U)$ whose associated operator is R. Furthermore, $s(x,\xi) - r(x,\xi,x) \in S^{d-1}(U)$.

Proof. Let $q(x,\xi,\cdot) = \widehat{r(x,\xi,\cdot)}$. In the definition of R, integrating with respect to y is taking the Fourier transform $\mathcal{F}(r(x,\xi,\cdot)f) = \widehat{r(x,\xi,\cdot)} \star \widehat{f}$, hence $Rf(x) = \int e^{ix\xi} q(x,\xi,\xi-\zeta)\widehat{f}(\zeta) d\zeta\, d\xi$. If we can change order of integration, then

$$s(x,\zeta) := \int e^{ix(\xi-\zeta)} q(x,\xi,\xi-\zeta) d\xi = \int e^{ix\xi} q(x,\zeta+\xi,\xi) d\xi \tag{3}$$

"does the job". Let us justify the application of Fubini: The assumption on r implies by the same argument as before that $|q(x,\xi,\zeta)| \le C_N(1+|\xi|)^d(1+|\zeta|)^{-N}$. Furthermore, $|\widehat{f}(\zeta)| \le C'_N(1+|\zeta|)^{-N}$ for any $N \in \mathbb{N}$, and hence $|q(x,\xi,\xi-\zeta)\widehat{f}(\zeta)| \le C''_N(1+|\xi|)^d(1+|\xi-\zeta|)^{-N}(1+|\zeta|)^{-N}$ which can be shown to be integrable.

It is easy to show that $s \in S^d(U)$.

For some $\lambda \in [0,1]$ we have

$$q(x,\zeta+\xi,\xi) = q(x,\zeta,\xi) + \sum_{|\alpha|=1} \partial_\zeta^\alpha q(x,\zeta+\lambda\xi,\xi)\xi^\alpha. \tag{4}$$

For any $N \in \mathbb{N}$,

$$|\partial_\zeta^\alpha q(x,\zeta+\lambda\xi,\xi)| \le C_N(1+|\zeta+\lambda\xi|)^{d-1}(1+|\xi|)^{-N} \le C'_N(1+|\zeta|)^{d-1}(1+|\xi|)^{-N+d-1},$$

and substituting (4) into (3) shows that $s(x,\zeta) - r(x,\zeta,x) \in S^{d-1}$. □

Proof of the theorem. With $v(\cdot,\xi) = \widehat{q(\cdot,\xi)}$ we have $\widehat{Qf}(\zeta) = \int e^{-iy\xi} v(\zeta-\xi,\xi) f(y) dy\, d\xi$ (compare (2)). Hence we can write

$$\begin{aligned} PQf(x) &= \int e^{ix\xi} p(x,\zeta) e^{-iy\xi} v(\zeta-\xi,\xi) f(y) dy\, d\xi\, d\zeta \\ &= \int e^{i(x-y)\zeta} \left[p(x,\zeta) t(\zeta,y)\phi(y)\right] f(y) dy\, d\zeta \end{aligned}$$

where $t(\zeta,y) = \int e^{iy\eta} v(\eta,\zeta-\eta) d\eta$ (think $\eta = \zeta - \xi$), and $\phi \in C_c^\infty(\mathbb{R}^m)$ is chosen such that $\phi|_U \equiv 1$ (we omit the justification for applying Fubini). Then $p(x,\zeta)t(\zeta,y)\phi(y) \in \tilde{S}^{d+e}(U)$, and we can apply the lemma.

Writing $v(\eta,\zeta-\eta) = v(\eta,\zeta) - \sum_{|\alpha|=1} \partial_\zeta^\alpha v(\eta,\zeta-\lambda\eta)\eta^\alpha$ with a suitable $\lambda \in [0,1]$ we deduce $t(\zeta,x) - q(x,\zeta) \in S^{e-1}(U)$ as above. Thus $p(x,\zeta)t(\zeta,x)\phi(x) - p(x,\zeta)q(x,\zeta) \in S^{d+e-1}(U)$. □

Elliptic differential operators on manifolds - 2

Choosing smooth compatible vector and matrix norms, all definitions and results of the previous section carry over to vector-valued functions and matrix-valued symbols.

2.3.10 Definition. Let M be a compact smooth manifold with a finite atlas $\{U_i, \tau_i\}$ and a subordinate partition of unity $\{\phi_i\}$. For $s \in \mathbb{Z}$ and $f \in C^\infty(M)$ we define $\|f\|_s := \sum_i \|\tau_{i*}(\phi_i f)\|_s$. Up to equivalence this norm does not depend on the choices made: Given another finite atlas $\{V_j, \sigma_j\}$ with subordinate partition of unity $\{\psi_j\}$, we have a bound

$$\sum_j \|\sigma_{j*}(\psi_j f)\|_s \leq \sum_{i,j} \|\sigma_{j*}(\psi_j f \phi_i)\|_s \leq C \sum_{i,j} \|\tau_{i*}(\psi_j f \phi_i)\|_s \leq C' \sum_i \|\tau_{i*}(f\phi_i)\|_s$$

using the finiteness of the cover, lemma 2.3.5 and the fact that multiplication by ψ_j is a pseudo-differential operator of order 0. Let $H_s(M)$ denote the completion of $C^\infty(M)$ with respect to $\|\cdot\|_s$. Similarly we define the Sobolev spaces $H_s(E)$ for any smooth vector bundle E on M.

One can also carry over the theory of pseudo-differential operators onto manifolds. We avoid the tedious work that is required to show independence of the choice of local coordinates and take a short-cut to prove the main theorem of this section:

2.3.11 Theorem. Let M be a compact smooth manifold with smooth vector bundles E and F. Let $P : \mathcal{D}(E) \to \mathcal{D}(F)$ be an elliptic differential operator of order d. Then for any integer $s \in \mathbb{Z}$, P extends to a bounded operator $H_s(E) \to H_{s-d}(F)$. This extension is a Fredholm operator.

Proof. Considering geodesically convex coverings of M and using compactness we can find a finite partition of unity $\{\phi_i\}$, an atlas $\{U_{ij}, \tau_{ij} : U_{ij} \xrightarrow{\sim} V_{ij} \subset \mathbb{R}^m\}$ with $(\operatorname{supp}\phi_i \cup \operatorname{supp}\phi_j) \subset U_{ij}$, and trivialisations $\alpha_{ij} : E|_{U_{ij}} \to V_{ij} \times \mathbb{C}^r$, $\beta_{i,j} : F|_{U_{ij}} \to V_{ij} \times \mathbb{C}^r$ compatible with τ_{ij}. Then we find cut-off functions χ_i with $\chi_i|_{\operatorname{supp}\phi_i} \equiv 1$ and $(\operatorname{supp}\chi_i \cup \operatorname{supp}\chi_j) \subset U_{ij}$. Denote $\widetilde{\phi_{ij}} := \tau_{ij*}\phi_i$, $\widetilde{\psi_{ij}} := \tau_{ij*}\phi_j$ and $\widetilde{\chi_{ij}} := \tau_{ij*}\chi_i$.

Let $\widetilde{P_{ij}} : C^\infty(V_{ij}, \mathbb{C}^r) \to C^\infty(V_{ij}, \mathbb{C}^r)$ be the local representation of P, and $p_{ij} : V_{ij} \times \mathbb{R}^m \to \mathbb{C}^{r\times r}$ its symbol. By ellipticity, the leading symbol of $\widetilde{P_{ij}}$ has an inverse $q_{ij} : V_{ij} \times \mathbb{R}^m \to \mathrm{GL}(r, \mathbb{C})$. Then $p_{ij}\widetilde{\chi_{ij}} \in S^d(V_{ij}, \mathbb{C}^r)$ and $q_{ij}\widetilde{\chi_{ij}} \in S^{-d}(V_{ij}, \mathbb{C}^r)$ define pseudo-differential operators P_{ij} and Q_{ij}. Since $\widetilde{\phi_{ij}} - q_{ij}\widetilde{\phi_{ij}}p_{ij} \in S^{-1}(V_{ij}, \mathbb{C}^r)$, the operator $(\widetilde{\phi_{ij}} - Q_{ij}\widetilde{\phi_{ij}}P_{ij})$ has order -1 and is compact as a map $H_s(V_{ij}, \mathbb{C}^r) \to H_{s+1} \to H_s$ by the lemma of Rellich.

Put $\tilde{H}_s(E) = \bigoplus H_s(V_{ij}, \mathbb{C}^r)$. Define $\alpha : H_s(E) \to \tilde{H}_s(E)$ by $e \mapsto \oplus\alpha_{ij*}(\chi_i e)$, $\tilde{\alpha} : \tilde{H}_s(E) \to H_s(E)$ by $\oplus e_{ij} \mapsto \sum \chi_i \alpha_{ij}^*(e_{ij})$, and similarly β and $\tilde{\beta}$. Note that these are continuous. Put $\widetilde{\Phi} = \bigoplus \widetilde{\phi_{ij}}$, $\widetilde{\Psi} = \bigoplus \widetilde{\psi_{ij}}$, $\widetilde{P} := \bigoplus P_{ij}$, $\widetilde{Q} = \bigoplus Q_{ij}$, and $Q = \tilde{\alpha}\widetilde{\Psi}\widetilde{Q}\widetilde{\Phi}\beta$. Then Q is an operator of order $-d$, and the commutative diagrams

$$\begin{array}{ccccc} H_s(E) & \xrightarrow{P} & H_{s-d}(F) & \xrightarrow{Q} & H_s(E) \\ \alpha\downarrow & & \widetilde{\Phi}\beta\downarrow & & \uparrow\tilde{\alpha} \\ \tilde{H}_s(E) & \xrightarrow{\widetilde{\Phi}\widetilde{P}} & \tilde{H}_{s-d}(F) & \xrightarrow{\widetilde{\Psi}\widetilde{Q}} & \tilde{H}_s(E) \end{array} \quad \text{and} \quad \begin{array}{ccc} H_s(E) & \xrightarrow{\sum\phi_i\phi_j} & H_s(E) \\ \alpha\downarrow & & \uparrow\tilde{\alpha} \\ \tilde{H}_s(E) & \xrightarrow{\widetilde{\Psi}\widetilde{\Phi}} & \tilde{H}_s(E) \end{array}$$

show that $1 - QP$ factorises through a compact operator. Similarly one can show that for $Q' = \tilde{\alpha}\widetilde{\Phi}\widetilde{\Psi}\widetilde{Q}\beta$, the operator $1 - PQ'$ is compact. By lemma 2.2.2, P is Fredholm. $\square$

2.3.12 Theorem. i) In the notation of the previous proof, the kernel of the extension P_s of P to $H_s(E)$ is in $\mathcal{D}(E)$ and independent of $s \in \mathbb{Z}$.

ii) The cokernel of P_s does not depend on s in the sense that $\operatorname{coker} P_s \cong \operatorname{coker}(P : \mathcal{D}(E) \to \mathcal{D}(F))$ for all $s \in \mathbb{Z}$.

Proof. i) The map constructed in the previous proof shows $\ker P_s \subset H_{s+1}(E)$, and iterating we find $\ker P_s \subset H_\infty(E)$. Sobolevs lemma (applied locally on $\bigoplus H_s(V_{ij}, \mathbb{C}^r)$) shows that $H_\infty(E) = \mathcal{D}(E)$.

ii) For any $f \in H_s(E)$ we have $f = (1 - PQ')f + PQ'f \in H_{s-1}(E) + \mathrm{im} P_s$, hence $\operatorname{coker} P_s$ does not depend on $s \in \mathbb{Z}$. However, to show that this coincides with the "smooth cokernel", one has to construct a better pseudoinverse Q called *parametrix* for which $PQ - 1$ and $QP - 1$ are infinitely smoothing, i.e. operators of order $-\infty$ (instead of -1) (see [11]). □

Families of elliptic differential operators

Throughout this section, let M be a compact smooth manifold and X a compact space. Given a family of smooth vector bundles E_x, F_x on M, and a family of elliptic differential operators $P_x : \mathcal{D}(E_x) \to \mathcal{D}(F_x)$ parameterized by $x \in X$, we want to construct a map $\mathrm{index}_P : K(M \times X) \to K(X)$ employing the index map constructed in section 2.2.

Before going into the details, let us indicate the "mathematical context" of this construction:

In ordinary (co-)homology theory one has a slant product $H^n(M \times X) \otimes H_m(M) \to H^{n-m}(X)$. Intuitively, this corresponds to integrating a closed differential form on $M \times X$ along a closed submanifold in the fibre M to get a closed differential form on X. We have seen that the K-groups resemble the cohomology groups. The map we are going to construct will show that we can consider elliptic differential operators as representing elements of a group $K_0(M)$ which resembles homology.

Formulating this in the language of C^*-algebras has lead to a bivariant KK-theory which combines cohomology and homology (see [7] and especially [18]).

2.3.13 Definition. Let N be a smooth manifold. A map $M \times X \to N$ is *continuously smooth* or just *c-smooth* if it is smooth along M and all of its derivatives along M are continuous along X. A *vector bundle* $E \to M \times X$ is —em c-smooth if it has a trivialisation with c-smooth transition functions.

2.3.14 Lemma. Any vector bundle E on $M \times X$ is homotopic to a c-smooth vector bundle E' on $M \times X$, and any two such c-smooth vector bundles are c-smooth homotopic.

Proof. Using Swan's theorem we can write E as the kernel of a map $(M \times X) \to \mathbb{C}^{r \times r}$. It is easy to see that any such map can be approximated by functions in $C^\infty(M, \mathbb{C}^{r\times r}) \otimes C(X)$, and these evidently are c-smooth. Similarly we can approximate each homotopy this way. □

2.3.15 Definition. A *family* $P_x : \mathcal{D}(E_x) \to \mathcal{D}(F_x)$ *of elliptic differential operators* of order d on c-smooth vector bundles E and F on $M \times X$ is *c-smooth* if any local representation of P with respect to c-smooth trivialisations of E and F has the form $\sum_{|\alpha| \le d} A_\alpha(m, x) D_m^\alpha$ with c-smooth matrix functions A_α.

2.3.16 Lemma. For each $s \in \mathbb{Z}$, the sets $H_s^X(E) := \coprod_{x \in X} H_s(E_x)$ and $H_s^X(F)$ carry a natural topology as a Hilbert bundle. P extends to a Fredholm bundle $P : H_s^X(E) \to H_s^X(F)$ on X.

Proof. By compactness of M we find[9] for each $x_0 \in X$

- a neighbourhood V,
- an atlas $\{\tau_{ij} : U_{ij} \to V_{ij} \subset \mathbb{R}^m\}$ of M
- a subordinate partition of unity $\{\phi_i\}$ on M,
- cut-off functions $\{\chi_i\}$ on M with $\chi_i|_{\operatorname{supp}\phi_i} \equiv 1$ and $(\operatorname{supp}\chi_i \cup \operatorname{supp}\chi_j) \subset U_{ij}$,
- and c-smooth trivialisations $\alpha_{ij} : E|_{U_{ij}\times V} \to (V_{ij} \times V) \times \mathbb{C}^r$, $\quad \beta_{ij} : F|U_{ij} \times V \to (V_{ij} \times V) \times \mathbb{C}^r$ compatible with τ_{ij}.

[9] compare the proof of theorem 2.3.11

In the following we submit summation indices i and j.

Define $\alpha_x : H_s(E_x) \to \bigoplus H_s(V_{ij}, \mathbb{C}^r)$ by $e \mapsto \oplus\alpha_{ij,x*}(\phi_i e)$ and $\tilde{\alpha}_x : \bigoplus H_s(V_{ij}, \mathbb{C}^r) \to H_s(E_x)$ by $\oplus e_{ij} \mapsto \sum \alpha^*_{ij,x}(\tau_{ij*}(\phi_j)e_{ij})$. Then $\tilde{\alpha}_x \circ \alpha_x = 1$, and $\operatorname{im}(\alpha_x)$ is independent of $x \in V$. The norm on $H_s(E_x)$ is equivalent to $e \mapsto \sum \|\alpha_{ij,x*}(\phi_i e)\|_s$. Using c-smoothness of the α_{ij} and the equivalence of the norms

$$\|\cdot\|_s \sim \sum_{|\alpha|\leq s} \|D^\alpha \cdot\|_{L^2} \text{ for } s \in \mathbb{N}, \tag{5}$$

it is easy to see that the topology induced on $\coprod_{x\in V} H_s(E_x)$ does not depend on the choice of the trivialisation α_{ij}. Thus we obtain Hilbert bundles $H^X_s(E)$ and $H^X_s(F)$ as claimed.

By theorem 2.3.11, $P_x : H_s(E_x) \to H_s(F_x)$ is Fredholm for each $x \in X$. It remains to show that $P : H^X_s(E) \to H^X_s(F)$ is continuous. For this it is sufficient to show that for each i and j the local representation P_{ij} of P varies continuously, i.e. that the map $V \to \mathcal{B}(H_s(V_{ij}, \mathbb{C}^r), H_s(V_{ij}, \mathbb{C}^r))$ sending x to $P_{ij,x}$ is continuous. This again follows from the c-smoothness of the matrix coefficient functions A_α and the equivalence of norms (5). □

Given a c-smooth family $P : E \to F$ and a c-smooth vector bundle G, we can extend P to a family $P_G : E \otimes G \to F \otimes G$ using c-smooth local coordinates and a c-smooth partition of unity. It is easy to see that this is again a c-smooth family, so we get a Fredholm bundle $P_G : H^X_s(E \otimes G) \to H^X_{s-d}(F \otimes G)$. The leading symbol of P_G does not depend on the choices made, hence for another extension P'_G the operator $P'_{G,x} - P_{G,x}$ has order -1 and is compact as a map $H_s(E_x \otimes G_x) \to H_s(F_x \otimes G_x)$ for each $x \in X$. Therefore, P_G and P'_G are homotopic inside $\mathcal{F}(H^X_s(E \otimes G), H^X_{s-d}(F \otimes G))$ via $t \mapsto P_G + t(P'_G - P_G)$.

2.3.17 Theorem. i) There is a well-defined homomorpism $\operatorname{index}_P : K(X \times M) \to K(X)$ defined by $G \mapsto \operatorname{index}(P_G : H_s(E \otimes G) \to H_{s-d}(F \otimes G))$, which is independent of $s \in \mathbb{Z}$.

ii) index_P is a homomorphism of $K(X)$-modules, i.e $\operatorname{index}_P([V] \otimes [W]) = (\operatorname{index}_P[V]) \otimes [W]$ for all $[V] \in K(M \times X)$ and $[W] \in K(X)$.

iii) index_P is functorial in X, i.e. given a map $f : Y \to X$, the diagram

$$\begin{array}{ccc} K(M \times X) & \xrightarrow{\operatorname{index}_P} & K(X) \\ (\mathrm{id} \times f)^* \downarrow & & \downarrow f^* \\ K(M \times Y) & \xrightarrow{\operatorname{index}_{f^*P}} & K(Y) \end{array}$$

commutes.

Proof. i) We have shown that the map is well-defined, and additivity is clear. Theorem 2.3.12 says that $\ker P_G$ and $\operatorname{coker} P_G$ are independend of s. If $\dim\ker P_G$ is constant, this shows that $\operatorname{index} P_G = [\ker P_G] - [\operatorname{coker} P_G]$ is independent of s. For the general case the claim follows from the construction of index.

ii) Let $\pi : M \times X \to X$ denote the projection. For any vector bundle W on X, it is clear that $\ker P_{\pi^*W} = (\ker P) \otimes W$ and $\operatorname{coker} P_{\pi^*W} = (\operatorname{coker} P) \otimes W$. If $\dim\ker P_{\pi^*W}$ is constant we see $\operatorname{index} P_{\pi^*W} = [\ker P_{\pi^*W}] - [\operatorname{coker} P_{\pi^*W}]$. For the general case the claim also follows from the construction of index.

iii) Easy and omitted. □

2.4 Dolbeault cohomology

This section is meant to be not more than a short summary.

2.4.1 Definition. Let M be a complex manifold. Its tangent bundle T_M decomposes as the direct sum of the holomorphic and the antiholomorphic tangent bundles T'_M and T''_M. The sheaf $\mathcal{A}^{p,q}$ of *smooth* (p,q)*-forms on* M is the sheaf of sections of the smooth vector bundle $\Lambda^{p,q}T^*_M := \Lambda^p {T'_M}^* \otimes \Lambda^q {T''_M}^*$. The *Dolbeault operator* $\bar\partial : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M)$ is defined by

$$\bar\partial(f dz_I \wedge d\bar z_J) := \sum_k (\partial f/\partial \bar z_k) d\bar z_k \wedge dz_I \wedge d\bar z_J.$$

Given a holomorphic vector bundle E on M, the Dolbeault operator extends uniquely to an operator $\bar\partial_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$, where $\mathcal{A}^{p,q}(E)$ denotes the sheaf of sections of the bundle $\Lambda^{p,q}T^*_M \otimes E$.

Let $\mathcal{O}(E)$ denote the sheaf of holomorphic sections of E. Write $\Omega^p = \mathcal{O}(\Lambda^p {T'_M}^*)$ for the sheaf of holomorphic p-forms, and let $\Omega^p(E) = \mathcal{O}(\Lambda^p {T'_M}^* \otimes E)$.

2.4.2 Definition. An *elliptic complex* on M is a finite sequence $V_0 \xrightarrow{P_0} \cdots \xrightarrow{P_{k-1}} V_k$ of smooth vector bundles V_i on M and elliptic differential operators P_i of such that

i) all P_i have the same order d,

ii) $P_i \circ P_{i-1} = 0$,

iii) the associated sequence of leading symbols $0 \to \pi^* V_0 \xrightarrow{\sigma_d(P_0)} \cdots \xrightarrow{\sigma_d(P_{k-1})} \pi^* V_k \to 0$ is exact (recall that $\pi : T^0_M \to M$ is the cotangent bundle without the zero section).

We denote the elliptic complex by (V_*, P_*), and its i-th cohomology $\ker P_i/\mathrm{im} P_{i-1}$ by $H^i(P,V)$.

2.4.3 Lemma. $(\mathcal{A}^{p,*}, \bar\partial_E)$ is an elliptic complex for each p.

Proof. Conditions i) and ii) are obviously satisfied. An easy calculation shows that the symbol of $\bar\partial_E$ acts as $\sigma_1(\bar\partial_E)_{(x,v)} \omega \otimes e = (v^{0,1} \wedge \omega) \otimes e$, where $(x,v) \in T^0_M$, $\omega \otimes e \in (\Lambda^{p,q} T^*_{M,x}) \otimes E_x$ and $v = v^{1,0} + v^{0,1}$ is the decomposition into holomorphic and antiholomorphic part. This implies iii). $\square$

Example. Another example of an elliptic complex is provided by the de Rham-complex. In the fibre above $(x,v) \in T^0_M$, its associated sequence of symbols is $\cdots \to \Lambda^i T^*_{M,x} \xrightarrow{v\wedge\cdot} \Lambda^{i+1} T^*_{M,x} \to \cdots$, the symbol acting as $\sigma_1(d)_{(x,v)} \omega = v \wedge \omega$.

The (p,q)-Dolbeault cohomology group of E is $H^q(\mathcal{A}^{p,*}(E), \bar\partial_E) =: H^{p,q}_{\bar\partial}(M,E)$. It is not too hard[10] to prove

2.4.4 Theorem (Dolbeault)**.** $H^{p,q}_{\bar\partial}(M,E) \cong H^q(M, \omega^p(E))$.

2.4.5 Theorem (Hodge Decomposition)**.** Let (V_*, P_*) be an elliptic complex. Choose smooth metrics on V_k, and let P^*_k be the formal adjoint to P_k with respect to the L^2-product. Define the Laplacian $\Delta_k := P_{k-1}P^*_k + P^*_k P_k : \mathcal{D}(V_k) \to \mathcal{D}(V_k)$, and put $D = \bigoplus (P_{2k} + P^*_{2k-1}) : \bigoplus \mathcal{D}(V_{2k}) \to \bigoplus \mathcal{D}(V_{2k+1})$. Then

i) Δ_k and D are elliptic,

ii) $\mathcal{D}(V_k) = \ker \Delta_k \oplus \mathrm{im} P_{k-1} \oplus \mathrm{im} P^*_k$, and this decomposition is orthogonal with respect to the L^2-product,

iii) $\ker \Delta_k \cong H^k(P,V)$,

iv) $\mathrm{index}\, D := \dim\ker D - \dim \mathrm{coker}\, D = \sum_k (-1)^k \dim H^k(P,V)$.

Proof. The hardest part needed are theorems 2.3.11 and 2.3.12. We omit the proof of i) and iv). Δ_k being formally self-adjoint implies $V_{k+1} = cl(\mathrm{im}\Delta_k) \oplus \ker(\Delta_k)$. But $\mathrm{im}\Delta_k$ is closed because it has finite codimension, hence we get ii). This in turn implies $\ker P_k = \ker \Delta_k \oplus \mathrm{im} P_{k-1}$ which gives iii). $\square$

[10] basically, any sheaf of $C^\infty(M)$-modules is acyclic, and a generalized Poincare-Lemma shows that $\Omega^p \to \mathcal{A}^{p,*}$ is an exact sequence of sheaves and hence a resolution of Ω^p

2.5 The Main Lemmata

After the painstaking preparations of the previous sections, we are now ready to harvest the central theorems of equivariant K-theory.

2.5.1 Lemma. Assume that for each compact space X we have a map $\alpha_X : K^{-2}(X) \to K(X)$ which is a $K(X)$-module homomorphism and functorial in X. Then α can be extended to a functorial homomorphism $\alpha_X : K^{-q-2}(X) \to K^{-q}(X)$ which commutes with right multiplication by elements of $K^{-p}(X)$.

Proof. First define α_X for a locally compact space X using the diagram

$$\begin{array}{ccccccc} 0 & \to & K^{-2}(X) & \to & K^{-2}(X^+) & \to & K^{-2}(+) \\ & & & & \downarrow & & \downarrow \\ 0 & \to & K(X) & \to & K(X^+) & \to & K(+) \end{array} \tag{6}$$

which commutes by functoriality of α. For any $q \in \mathbb{N}$ we get an extension $\alpha : K^{-q-2}(X) = K^{-2}(\mathbb{R}^q \times X) \to K(\mathbb{R}^q \times X) = K^{-q}(X)$ which is clearly functorial.

We now check multiplicativity: For compact spaces X and Y, the diagram

$$\begin{array}{ccc} K^{-2}(X) \otimes K(Y) & \xrightarrow{\phi} & K^{-2}(X \times Y) \\ \alpha_x \otimes 1 \downarrow & & \downarrow \alpha_{X \times Y} \\ K(X) \otimes K(Y) & \xrightarrow{\psi} & K(X \times Y) \end{array}$$

commutes: all maps are $K(Y)$-module homomorphisms, and denoting $\pi : X \times Y \to X$ the projection we have $\psi((\alpha_X \otimes 1)(u \otimes 1)) = \pi^*\alpha_X(u) = \alpha_{X\times Y}(\pi^* u) = \alpha_{X \times Y}(\phi(u \otimes 1))$ by functoriality of α ($u \in K^{-2}(X)$). The diagram also commutes for locally compact spaces X and Y: Consider the square above for X^+ and Y^+ and add on the right the commuting square

$$\begin{array}{ccc} K^{-2}(X^+ \times Y^+) & \to & K^{-2}((X \times Y)^+) \\ \downarrow & & \downarrow \\ K(X^+ \times Y^+) & \to & K((X \times Y)^+). \end{array}$$

Now replace X and Y by $X \times \mathbb{R}^q$ and $X \times \mathbb{R}^p$, and use the diagonal map $X \times \mathbb{R}^{p+q} \to X \times \mathbb{R}^q \times X \times \mathbb{R}^p$ to get commutativity in

$$\begin{array}{ccc} K^{-q-2}(X) \otimes K^{-p}(X) & \to & K^{-p-q-2}(X) \\ \downarrow & & \downarrow \\ K^{-q}(X) \otimes K^{-p}(X) & \to & K^{-p-q}(X). \end{array}$$

□

2.5.2 Lemma. Let $\beta : K(X) \to K^{-2}(X)$ be left multiplication by some $b \in K^{-2}(+)$. Assume that α additionally satisfies $\alpha(b) = 1$. Then β is an isomorphism and α is its inverse.

Proof. First, $\alpha\beta = 1$ since $(\alpha\beta)(x) = \alpha(bx) = \alpha(b)x = x$ for any $x \in K(X)$.

To show $\beta\alpha = 1$, we investigate the commutativity of multiplication: Let θ denote the automorphism of $K^{-4}(X) = K(\mathbb{R}^2 \times \mathbb{R}^2 \times X)$ obtained by switching the two copies of $\mathbb{R}^2$. Then for any $y, z \in K^{-2}(X)$ we have $\theta(yz) = zy$. The involution $(u, v) \mapsto (v, -u)$ of $\mathbb{R}^2 \times \mathbb{R}^2$ is homotopic to the identity via rotation by $-\pi/2$ and hence induces the identity on $K^{-4}(X)$. Let σ denote the involution of $K^{-2}(X)$ induced by $u \mapsto -u$ on $\mathbb{R}^2$. Then $yz = z\sigma(y)$, and $(\beta\alpha)(y) = \alpha(y)b = \alpha(yb) = \alpha(b\sigma(y)) = \sigma(y)$, i.e. $\beta\alpha = \sigma$.

Now, since σ is an automorphism, α and β are inverse isomorphisms and $\sigma = 1$. □

2.5.3 Proposition (Bott periodicity)**.** $K^{-q}(X, A) \cong K^{-q-2}(X, A)$ for any locally compact space X with a closed subspace $A \subset X$ and any $q \in \mathbb{N}$.

Proof. We assume $A = \emptyset$ and X to be compact. The general result follows from the long exact sequence 1.3.8, the 5-lemma, and from diagram (6).

Identify the one-point compactification of $\mathbb{R}^2$ with S^2, and S^2 with the complex projective line $\mathbb{P}^1$. Take $b = 1 - H^{-1}$, where H is the Hopf bundle on S^2, which corresponds to the standard bundle on $\mathbb{P}^1$. On $\mathbb{P}^1$ we have the Dolbeault operator $\bar{\partial} : C^\infty(\mathbb{P}^1) \to \mathcal{A}^{0,1}(\mathbb{P}^1)$ which is clearly elliptic. Let α be the composition $K^{-2}(X) = K(\mathbb{R}^2 \times X) \to K(\mathbb{P}^1 \times X) \xrightarrow{\mathrm{index}_{\bar{\partial}}} K(X)$. We compute $\alpha(b)$: The Dolbeault lemma 2.4.4 tells us that for any holomorphic vector bundle Q over $\mathbb{P}^1$, $\ker \bar{\delta}_Q \cong H^0(\mathbb{P}^1, \mathcal{O}(Q))$ and $\operatorname{coker} \bar{\delta}_Q \cong H^1(\mathbb{P}^1, \mathcal{O}(Q))$. Proposition A.3.1 says that for the trivial bundle $1 = \mathbb{C}$, $H^0 = \mathbb{C}$ and $H^1 = 0$, whereas for the Hopf bundle $H^0 = H^1 = 0$. Thus $\alpha(b) = 1$. $\square$

As in section 1.3, we put $K_G^*(X) = K_G^0(X) \oplus K_G^{-1}(X)$ and define similarly $K_G^*(X, A)$.

2.5.4 Theorem. For any compact G-space X with closed G-subspace A, there is an exact triangle

$$\begin{array}{ccc} & K_G^*(X,A) & \\ \nearrow & & \searrow \\ K_G^*(A) & \longleftarrow & K_G^*(X) \end{array} \tag{7}$$

$\square$

2.5.5 Proposition (Equivariant periodicity theorem)**.** For any compact G-space X and any complex G-module V, there is an isomorphism $K_G(X) \cong K_G(V \times X)$.

Proof. We follow the same formal lines as in the proof of Bott periodicity:

Let $\mathbb{C}$ denote the trivial G-module. Consider the projective space $\mathbb{P}(V \oplus \mathbb{C})$, which inherits the G-action from $V \oplus \mathbb{C}$. We can identify the space $\mathbb{P}(V \oplus \mathbb{C})/\mathbb{P}(V)$ with the one-point compactification V^+ of V.[11] The natural map $\mathbb{P}(V \oplus \mathbb{C}) \to V^+$ is equivariant and induces a homomorphism $j : K_G(V \times X) = \tilde{K}_G(V^+ \times X) \to K_G(\mathbb{P}(V \oplus \mathbb{C}) \times X)$.

We denote the trivial G-bundle $V \times \mathbb{P}(V \oplus \mathbb{C}) \to \mathbb{P}(V \oplus \mathbb{C})$ by $\mathbb{V}$, and the standard bundle on $\mathbb{P}(V \oplus \mathbb{C})$ by H. Consider the element $\lambda := \sum_i (-1)^i H^i \otimes \Lambda^i \mathbb{V} \in K_G(\mathbb{P}(V \oplus \mathbb{C}))$. We want to show that it lifts under j to an element λ_V of $K_G(V)$, i.e. that its restriction to the inverse image of $\{+\}$ under the map $\mathbb{P}(V \oplus \mathbb{C}) \to V^+$ is 0. This preimage is just $\mathbb{P}(V)$, and the restriction of H is the standard bundle H_V on $\mathbb{P}(V)$, hence $\lambda|_{\mathbb{P}(V)} = \sum_i (-1)^i H_V^i \otimes \Lambda^i \mathbb{V}|_{\mathbb{P}(V)}$.

We have a natural inclusion $H_V^{-1} \hookrightarrow \mathbb{V}|_{\mathbb{P}(V)}$. Tensoring with H_V we get a nowhere vanishing section $s : \mathbb{P}(V) \to \{1\} \times \mathbb{P}(V) \hookrightarrow H_V \otimes H_V^{-1} \to H_V \otimes \mathbb{V}|_{\mathbb{P}(V)}$. It is easy to check that the sequence

$$\cdots \xrightarrow{s_y \wedge} H_{V,y}^{i-1} \otimes \Lambda^{i-1} V \xrightarrow{s_y \wedge} H_{V,y}^{i} \otimes \Lambda^{i} V \xrightarrow{s_y \wedge} \cdots$$

is exact for all $y \in \mathbb{P}(V)$, and hence $\bigoplus_i H^{2i} \otimes \Lambda^{2i} \mathbb{V}|_{\mathbb{P}(V)} \cong \bigoplus_i H^{2i+1} \otimes \Lambda^{2i+1} \mathbb{V}|_{\mathbb{P}(V)}$. Thus $\lambda|_+ = 0$, and λ lifts to an element $\lambda_V \in K_G(V)$.

We now put $b = \lambda_V^*$ and define $\beta : K_G(X) \to K_G(V \times X)$ by $x \mapsto bx$. We use a slight generalisation of proposition 2.5.2 and finish the proof by constructing an inverse α to β. Choose a G-invariant metric on $\mathbb{P}(V \oplus \mathbb{C})$ and consider the elliptic operator $D = \bar{\partial} + \bar{\partial}^* : \bigoplus_i \mathcal{A}^{0,2i}(\mathbb{P}(V \oplus \mathbb{C})) \to \bigoplus \mathcal{A}^{0,2i+1}(\mathbb{P}(V \oplus \mathbb{C}))$ (cf section 2.4). Its kernel and cokernel carry a natural G-action. It is easy to check that the construction of the index map of section 2.2 generalizes to the equivariant setting to give a map $K_G(\mathbb{P}(V \oplus \mathbb{C}) \times X) \xrightarrow{\mathrm{index}_D} K_G(X)$. Let

[11] Going backwards, $\mathbb{P}(V \oplus \mathbb{C})$ is obtained from V^+ by blowing up the point $+$.

$\alpha = \mathrm{index}_D \circ j$. We compute $\alpha(b)$: The Hodge theorem 2.4.5 tells us that

$$\begin{aligned}\mathrm{index}_D(\lambda) &= \sum_{i,j}(-1)^{i+j}H^i(\mathbb{P}(V\oplus\mathbb{C}),\mathcal{O}(H^{-j}\otimes\Lambda^j\mathbb{V}^*)) \\ &= \sum_{i,j}(-1)^{i+j}H^i(\mathbb{P}(V),\mathcal{O}(-j))\otimes\Lambda^jV^*.\end{aligned}$$

Now $H^i(\mathbb{P}(V\oplus\mathbb{C}),\mathcal{O}(-j))=0$ for $1\le j\le\dim V$ by proposition A.3.1. Therefore,

$$\mathrm{index}_D(\lambda) = \sum_i(-1)^iH^i(\mathbb{P}(V\oplus\mathbb{C}),\mathcal{O})\otimes\Lambda^0V^* = \mathbb{C}$$

and $\alpha(b)=1$. □

Before going further, let us comment on the element $\lambda_v\in K_G(V)$ (this paragraph will not be needed later on and can be skipped):

If there were no group action present, we could prove the theorem by iterating the Bott isomorphism $K(X)\xrightarrow{\sim}K(\mathbb{R}^2\times X)\xrightarrow{\sim}\cdots\xrightarrow{\sim}K(\mathbb{R}^{2\dim_{\mathbb{C}}V}\times X)$, mutliplying by one copy of $(1-H^{-1})$ "along each complex dimension". We show that this corresponds to multipication by λ_V^*.

Let us reinterprete the construction of the standard bundle H on $\mathbb{P}^1$. Let $[z_0:z_1]$ be homogeneous coordinates for $\mathbb{P}^1$, and let η denote the trivial bundle on $\mathbb{P}^1$. The kernel of the map $z_0:\eta\to H\otimes\eta$ defined by $([z_0:z_1],\lambda)\mapsto([z_0:z_1],z_0\lambda)$ is a "skyscraper" supported only on $[0:1]$. Now, to construct the Hopf bundle, we took two trivial bundles ξ_0 and ξ_1 on U_0 and U_1, which we can think of as the cokernels of the inclusions $\ker z_{0,1}\hookrightarrow\eta$. Then we identified them on $U_0\cap U_1$ in a way that corresponds to taking the kernel of the map $\xi_0\oplus\xi_1\to H\otimes\eta$ defined by $([z_0:z_1],(a,b))\mapsto([z_0:z_1],z_0\otimes a-z_1\otimes b)$. Hence we can think of the Hopf bundle as the second cohomology of the complex $0\to\ker z_0\oplus\ker z_1\to\eta\oplus\eta\to H\otimes\eta\to 0$, which is the same as its Euler characteristic. Let $\mathbb{C}$ denote the trivial line bundle on $\mathbb{R}^2$. Restricting to $\mathbb{R}^2\cong U_1\hookrightarrow\mathbb{P}^1$, we have $\ker z_1|_{\mathbb{R}^2}=0$ and a trivialisation $H|_{\mathbb{R}^2}\cong\mathbb{C}$ defined fibrewise by $\lambda\mapsto\lambda/z_1$. Thus we can think of $H|_{\mathbb{R}^2}$ as the Euler characteristic of the complex $0\to\ker z_0\oplus 0\to\mathbb{C}\oplus\mathbb{C}\xrightarrow{\phi}\mathbb{C}\to 0$, where $\phi(a,b)=za-b$, $z=z_0/z_1$. But this complex splits as

$$\begin{array}{ccccccc} 0 & \to & \ker z_0 & \to & \ker\phi & \to & 0 & & \\ & & \oplus & & \oplus & & \oplus & & \\ & & 0 & \to & D & \to & \mathbb{C} & \to & 0,\end{array}$$

where $D=\mathbb{C}\oplus\mathbb{C}/\ker\phi\cong\mathbb{C}$. The Euler characteristic of the first row is $\mathbb{C}$, and so we can think of the element $1-H\in K(\mathbb{R}^2)$ as the Euler characteristic of the complex $\mathbb{C}\xrightarrow{z}\mathbb{C}$. Let us rewrite this as $0\to\mathbb{C}\to\mathbb{C}e_1\to 0$ with a symbol e_1.

Now, multiplying copies of $(1-H)$ should correspond to forming the tensor product of the corresponding complexes. As an example, for $\dim_{\mathbb{C}}V=2,3$ we would get the complexes

$$0\to\mathbb{C}\otimes\mathbb{C}\to\mathbb{C}e_1\otimes\mathbb{C}\oplus\mathbb{C}\otimes\mathbb{C}e_2\to\mathbb{C}e_1\otimes\mathbb{C}e_2\to 0 \quad\text{and}$$
$$0\to\mathbb{C}\to\mathbb{C}e_1\oplus\mathbb{C}e_2\oplus\mathbb{C}e_3\to\mathbb{C}e_1e_2\oplus\mathbb{C}e_1e_3\oplus\mathbb{C}e_2e_3\to\mathbb{C}e_1e_2e_3\to 0.$$

But this can be rewritten as

$$0\to\Lambda^0V_v\xrightarrow{v\wedge}\Lambda^1V_v\xrightarrow{v\wedge}\cdots\xrightarrow{v\wedge}\Lambda^{\dim_{\mathbb{C}}V}V_v\to 0 \tag{8}$$

which is exact for $v\neq 0$.[12] Collapsing we get two vector bundles and a bundle map $\bigoplus_i\Lambda^{2i}\pi^*V\to\bigoplus_i\Lambda^{2i+1}\pi^*V$ which is an isomorphism except at $v=0$ (here π^*V denotes the trivial bundle

[12]From the viewpoint of homological algebra, this iterated construction of the complex (8) is formalised as forming a *Koszul complex*; (8) is the Koszul complex of the tensor algebra $\bigotimes^* V$.

$V \times V \to V$). These bundles carry a natural G-action, for which the map is equivariant, and so this complex defines an element of $K_G(V)$. It is easy to check that this is exactly λ_V. The "intuitive reason" for tensoring with H in λ is that the isomorphism outside $v = 0$ is "twisted at infinity" and can not be extended to an isomorphism on $\mathbb{P}(V \oplus \mathbb{C})$, whereas the bundle H is "twisted the other way around".

2.5.6 Theorem (Equivariant Thom isomorphism)**.** Let G be a compact Lie group, Y a G-space and E a G-bundle on Y. Then multiplication by λ_E^* induces an isomorphism $K_G(Y) \xrightarrow{\sim} K_G(E)$.

Proof. Let $X \to Y$ be the principal U_n-bundle associated to E. Then $E = (\mathbb{C}^n \times X)/U_n$ and $Y = X/U_n$. By lemma 1.3.6 we get $K_{G\times U_n}(X) \cong K_G(Y)$ and $K_{G\times U_n}(\mathbb{C}^n \times X) \cong K_G(E)$. Under this isomorphism, $\lambda_{\mathbb{C}^n}^*$ corresponds to λ_E^* since taking the quotient and forming the alternating product commute. Now the theorem follows from 2.5.5. $\square$

Now we approach the splitting principle in equivariant K-theory:

2.5.7 Lemma. If X is a compact G-space and V a complex G-module, then the $K_G(X)$-homomorphism $K_G(X) \to K_G(\mathbb{P}(V) \times X)$ is injective.

Proof. We proceed as in the proof of theorem 2.5.5: Put $D = \bar\partial + \bar\partial^* : \bigotimes_i \mathcal{A}^{0,2i}(\mathbb{P}(V)) \to \bigotimes_i \mathcal{A}^{0,2i+1}(\mathbb{P}(V))$ for some G-invariant metric on V, and let $\alpha = \mathrm{index}_D$. Then $\mathrm{index}_D(\mathbb{C} \times \mathbb{P}(V)) = \sum_i (-1)^i H^i(\mathbb{P}(V), \mathcal{O}) = \mathbb{C}$ by proposition A.3.1, and hence $\alpha(1) = 1$. $\square$

2.5.8 Theorem (Equivariant splitting principle)**.** Let $j : T \to U$ be the inclusion of a maximal torus into the unitary group $U = U_n$. For any compact U-space X, the induced map $j^* : K_U(X) \to K_T(X)$ has a left-inverse $j_* : K_T(X) \to K_U(X)$.

Proof. Let $H = (U_1)^k$ act trivially on $\mathbb{C}^n$. By the previous lemma, the map $K_{U_n\times H}(X) \to K_{U_n\times H}(\mathbb{P}(\mathbb{C}^n) \times X)$ is injective. $\mathbb{P}(\mathbb{C}^n) \cong U_n/(U_{n-1} \times U_1)$ and hence $K_{U_n\times H}(\mathbb{P}(\mathbb{C}^n) \times X) = K_{U_n\times H}(U_n/U_{n-1} \times X) = K_{U_{n-1}\times U_1\times H}(X)$ by lemma 1.3.6. It is easy to check that the composite $K_{U_n\times H}(X) \to K_{U_{n-1}\times U_1\times H}(X)$ is just the map induced by the inclusion $U_{n-1} \times U_1 \to U_n$. Now iterate with $k = 0, \dots, n-1$. $\square$

2.5.9 Theorem. With the notation as above, $K_T^*(X)$ if finite over $K_U^*(X)$.

Proof. It is sufficient to show that $K_G(\mathbb{P}(V) \times X)$ is finite over $K_G(X)$.

Choosing a G-invariant metric on V, we can write $\mathbb{P}(V)$ as the quotient of the unit sphere $S(V)$ of V by the circle group S^1 which acts by componentwise multiplication. Then $K_G(\mathbb{P}(V)\times X) \cong K_{G\times S^1}(S(V) \times X)$. Let $D(V)$ denote the unit disc in V. We have an exact triangle

$$\begin{array}{ccc} & K_{G\times S^1}(D(V) \times X,\, S(V) \times X)) & \\ \nearrow & & \searrow \alpha \\ K_{G\times S^1}(S(V) \times X) & \longleftarrow & K_{G\times S^1}(D(V) \times X) \end{array} \quad .$$

Now the restriction map $K_{G\times S^1}(D(V) \times X) \to K_{G\times S^1}(X)$ induced by the zero section $X \to D(V) \times X$ is an isomorphism because $D(V)$ is contractible. Furthermore, multiplication by $\lambda_V^* = \sum(-1)^i \Lambda^i \mathbb{V}^*$ gives an isomorphism $K_{G\times S^1}(X) \xrightarrow{\to} K_{G\times S^1}(D(V) \times X, S(V) \times X)$ by theorem 2.5.6. A generalisation of theorem 1.3.6 shows that $K_{G\times S^1}(X) \cong K_G(X) \otimes R(S^1) = K_G(X)[\rho, \rho^{-1}]$, where ρ denotes the standard representation of S^1. The restriction of λ_V^* to X decomposes as $\sum_i (-1)^i \Lambda^i \mathbb{V}^* \otimes \rho^{-i} =: f \in K_G(X)[\rho, \rho^{-1}]$. Regarding f as a polynomial in ρ^{-1} with coefficients in $K_G(X)$, its constant term is 1, and hence it is not a zero divisor.

Summarising, from the exact triangle above we get an exact sequence

$$0 \to K_G(X)[\rho, \rho^{-1}]/(f) \to K_{G\times S^1}(S(V) \times X) \to 0,$$

and so $K_G(\mathbb{P}(V) \times X)$ is generated over $K_G(X)$ by $1, \rho^{-1}, \dots, \rho^{-n}$, with $n = \dim V - 1$. $\square$

Remark. In fact one can show that the ρ^i, $i = 0, \ldots, n$, generate the module freely. Thus $K_T(X)$ is a free $K_U(X)$-module of rank $n!$.

3 The Main Theorem

Now, everything done so far is put together.

3.1 Formulation of the Main Theorem

The right-hand side In the formulation of the Main Theorem as given in the introduction, we consider the K-group of the classifying space B_G. Now in general, B_G need not be compact (otherwise it could not classify principal G-bundles that are numerable but have no finite trivialisation) or locally compact, so this group still has to be defined.

3.1.1 Definition. Let X be a G-space with compact G-subspaces X^n such that $X = \bigcup_n X^n$. We put $K_G(X) := \varprojlim K_G(X^n)$.

For another sequence of compact G-subspaces Y^n with $Y = \bigcup_n Y^n$, we have $Y^n \subset X^k$ and $X^n \subset Y^l$ for some $k, l > n$ by lemma A.2.1. Hence $\varprojlim K_G(Y^n) = \varprojlim K_G(X^n)$ by lemma A.1.3, and $K_G(X)$ is well-defined.

At least for compact spaces this definition coincides with the old one. For locally compact, spaces, the old definition resembled cohomology with compact support, whereas now we allow for infinite twists.

The universal space E_G is the union of its compact G-subspaces $E_G^n := \star^n G$, and B_G is the union of the compact G-subspaces $B_G^n := E_G^n/G$.

The left-hand side Let $I_G \subset R(G)$ be the kernel of the augmentation $R(G) \to \mathbb{Z}$ (counting the dimension). Then the I_G-adic completion of $R(G)$ is $R(G)\hat{} := \varprojlim R(G)/I_G^n$.

The isomorphism of the theorem By lemma 1.3.6 we can write $K_G(+)$ and $K_G(\{+\} \times E_G)$ for $R(G)$ and $K(B_G)$. There is a natural homomorphism $\alpha_n : R(G) = K_G^*(+) \to K_G^*(E_G^n) = K^*(B_G^n)$ which corresponds to applying the construction 1.1.4 to obtain a vector bundle on B_G^n from a G-module V and the principal G-bundle $E_G^n \to B_G^n$.

3.1.2 Lemma. α_n factorizes through $R(G)/I_G^n$.

Proof. Let $U_i/G \subset B_G^n$ be the open subset where the i-th coordinate does not vanish (cf. the proof theorem 1.2.2). This is G-contractible to $(\cdot 0, \ldots, 1, \cdot 0, \ldots)G$. Now $B_G^n = \bigcup_{i=1}^n U_i/G$, and hence the product of any n elements of $\tilde{K}(B_G^n)$ is 0 by lemma 1.3.14. But $\tilde{K}^*(B_G^n)$ is the kernel of the augmentation of $K^*(B_G^n) \to \mathbb{Z}$, and the composite $R(G) \xrightarrow{\alpha_n} K^*(B_G^n) \to \mathbb{Z}$ is the usual augmentation of $R(G)$, whose kernel is I_G. Hence $\alpha_n(I_G^n) \subset \tilde{K}(B_G^n)^n = 0$. □

Now the $\alpha_n : R(G)/I_G^n \to K^*(B_G^n)$ induce a homomorphism $\alpha : R(G)\hat{} \to K^*(B_G)$ which the Main Theorem claims to be an isomorphism.

One step of the proof requires a slight generalisation of the statement: Let X be any compact G-space. Then the projection $X \times E_G^n \to X$ induces a map $\alpha_n : K_G^*(X) \to K_G^*(X \times E_G^n)$.

3.1.3 Lemma. α_n factorizes through $K_G^*(X)/I_G^n K_G^*(X)$.

Proof. This follows from the previous lemma and commutativity of the diagram

$$\begin{array}{ccc} K_G^*(+) & \to & K_G^*(\{+\} \times E_G^n) \\ \downarrow & & \downarrow \\ K_G^*(X) & \to & K_G^*(X \times E_G^n). \end{array}$$

□

Main Theorem. Let X be a compact G-space such that $K_G^*(X)$ is finite over $R(G)$. Then the homomorphisms

$$\alpha_n : K_G^*(X)/I_G^n K_G^*(X) \to K_G^*(X \times E_G^n)$$

induce an isomorphism

$$\boxed{\alpha : \varprojlim K_G^*(X)/I_G^n K_G^*(X) \xrightarrow{\sim} K_G^*(X \times E_G) \cong K^*(X \times B_G).}$$

Actually, α is continuous with respect to specific natural topologies on these rings.

3.2 Proof of the Main Theorem

The proof proceeds in four steps:

- the case $G = U_1 = T$ is settled by explicit calculation – here already we use the "heavy" equivariant Thom isomorphism,
- an induction argument extends the result to tori,
- the most difficult step is to extend from maximal tori to unitary groups – this requires the equivariant splitting principle
- for an arbitrary group one uses an embedding into a unitary group and finishes by considering an auxiliary space - this requires the generalisation $X \neq \{+\}$.

Step 1) We will need a slightly more general statement for the next steps.

3.2.1 Lemma. Let G be a compact Lie group and X a compact G-space such that $K_G^*(X)$ is finite over $R(G)$. Let $\theta : G \to T$ be a homomorphism by which G acts on E_T. Then the homomorphisms $\alpha_n : K_G^*(X)/I_T^n K_G^*(X) \to K_G^*(X \times E_T^n)$ induce an isomorphism of the inverse limits (here $K_G^*(X)$ is regarded as an $R(T)$-module via θ).

Proof. We can identify $E_T^n = T \star \cdots \star T$ with S^{2n-1}, the unit sphere in $\mathbb{C}^n$, on which T acts as a subgroup of the multiplicative group of $\mathbb{C}$ by componentwise multiplication (see example (1) in section 1.2).

Consider the exact sequence of K_G-groups for the pair $(X \times D^{2n}, X \times S^{2n-1})$ (compare the proof of proposition 2.5.9):

$$\begin{array}{ccc} & K_G^*(X \times D^{2n}, X \times S^{2n-1}) & \\ \nearrow & & \searrow \\ K_G^*(X \times S^{2n-1}) & \longleftarrow & K_G^*(X \times D^{2n}) \end{array}$$

Since D^{2n} is contractible, we have $K_G^*(X \times D^{2n}) \cong K_G^*(X)$. By the Thom isomorphism, $K_G^*(X) \xrightarrow{\sim} K_G^*(X \times D^{2n}, X \times S^{2n-1}) \cong K_G^*(X \times \mathbb{C}^n)$ via multiplication by $\lambda_{\mathbb{C}^n}^*$, where G acts on $\mathbb{C}^n$ via θ. As a T-module, $\lambda_{\mathbb{C}^n}^*$ decomposes as $\sum(-1)^i \Lambda^i(\mathbb{C}^n) = \sum(-1)^i \binom{n}{i} \rho^{-i} = (1-\rho^{-1})^n$, where ρ denotes the one-dimensional standard representation of T. Let $\xi = (1 - \rho^{-1})$.

With these identifications, the map $K_G^*(X) \xrightarrow{\sim} K_G^*(X \times D^{2n}, X \times S^{2n-1}) \to K_G^*(X \times D^{2n}) \xrightarrow{\sim} K_G^*(X)$ becomes multiplication by ξ^n, and we obtain an exact sequence

$$0 \to K/\xi^n K \xrightarrow{\alpha_n} K_G^*(X \times S^{2n-1}) \to K_{\xi^n} \to 0, \tag{9}$$

where $K = K_G^*(X)$ and $K_{\xi^n} = \{x \in K : \xi^n x = 0\}$.

Now ξ generates I_T, and so to prove the lemma it is sufficient to find for each n a k and a homomorphism $\beta_n : K_G^*(X \times S^{2n+2k-1}) \to K/\xi^n K$ which makes (the left square in) the diagram

$$\begin{array}{ccccccc} 0 \to & K/\xi^{n+k} K & \xrightarrow{\alpha_{n+k}} & K_G^*(X \times S^{2n+2k-1}) & \to & K_{\xi^{n+k}} & \to 0 \\ & \downarrow & \swarrow \beta_n & \downarrow \tau_{n,k} & & \downarrow \xi^k & \\ 0 \to & K/\xi^n K & \xrightarrow[\alpha_n]{} & K_G^*(X \times S^{2n-1}) & \to & K_{\xi^n} & \to 0 \end{array}$$

commute. If ξ^k annihilates $K_{\xi^{n+k}}$ for some k, then $\mathrm{im}(\tau_{n,k})$ lifts to a submodule of $K/\xi^n K$ via α_n, and we get β_n. But K is finite over $R(G)$ which is noetherian by proposition 1.4.5, and hence the ascending chain K_{ξ^n} stabilizes, i.e. $K_{\xi^k} = K_{\xi^{k+n}}$ for some k and all n.[13] $\square$

Step 2) The previous lemma holds with T replaced by T^m:

Put $H = T^{m-1}$. Then $E_{T^m} \cong E_T \times E_H$ by lemma 1.2.5, and $K_G^*(E_{T^m}) = \varprojlim_{p,q} K_G^*(E_T^p \times E_H^q)$. Furthermore, $R := R(T^m) = R(T) \otimes R(H)$. Hence $I_{T^m} = a + b$, where $a = I_T \otimes R(H)$ and $b = R(T) \otimes I_H$. From $a^n + b^n \subset (a+b)^n$ and $(a+b)^{(p+q-1)} \subset a^p + b^q$ we see that $\varprojlim_n K/I_{T^m}^n = \varprojlim_{p,q} K/(a^p + b^q)K$. Thus we have reduced to proving that the homomorphisms $\alpha_{p,q} : K/(a^p + b^q)K \to K_G^*(X \times E_T^p \times E_H^q)$ induce an isomorphism of the inverse limits.

Now $K/(a^p+b^q)K \cong K \otimes_R (R/a^p) \otimes_R (R/b^q)$, and the homomorphism $\alpha_{p,q}$ can be factorized as

$$K \otimes (R/a^p) \otimes (R/b^q) \xrightarrow{\phi_{p,q}} K_G^*(X \times E_T^p) \otimes (R/b^q) \xrightarrow{\psi_{p,q}} K_G^*(X \times E_T^p \times E_H^q).$$

For fixed q, the homomorphisms $\phi_{p,q}$ define an isomorphism of the inverse subsystems indexed by p. Therefore they define an isomorphism of the inverse limits of the (entire) inverse systems on the left and in the middle by lemma A.1.4.

Similarly, for fixed p, the $\psi_{p,q}$ define an isomorphism of the inverse subsystems indexed by q in virtue of the induction hypothesis. Note that from the exact sequence (9) we see that $K_G^*(X \times E_T^p)$ is finite over $R(G)$ because $K_G^*(X)$ is. $\square$

Step 3) Let U be a unitary group and $T \subset U$ a maximal torus. Consider the diagram

$$\begin{array}{ccc}
K_U^*(X)/I_U^n K_U^*(X) & \xrightarrow{\alpha_{n,U}} & K_U^*(X \times E_U^n) \\
j^* \downarrow\uparrow j_* & & j^* \downarrow\uparrow j_* \\
K_T^*(X)/I_U^n K_T^*(X) & \xrightarrow{\eta_n} & K_T^*(X \times E_U^n).
\end{array}$$

Here j_* is the left inverse to j^* constructed in the proof of the splitting principle. Going over to inverse limits, it is easy to see that the $\alpha_{n,U}$ define an isomorphism of inverse limits if the η_n do so.

Consider the diagram

$$\begin{array}{ccc}
K_T^*(X)/I_U^n K_T^*(X) & \xrightarrow{\eta_n} & K_T^*(X \times E_U^n) \\
\lambda_n \downarrow & & \downarrow \rho_n \\
K_T^*(X)/I_T^n K_T^*(X) & \xrightarrow{\alpha_{n,T}} & K_T^*(X \times E_T^n).
\end{array}$$

We show that $\alpha_{n,T}$, ρ_n and λ_n define an isomorphism of inverse limits. Then the same holds for η_n.

For $\alpha_{n,T}$, we apply step 2: By assumption, $K_U^*(X)$ is finite over $R(U)$, and by proposition 2.5.9, $K_T^*(X)$ is finite over $K_U^*(X)$, hence it is also finite over $R(U)$ and thus over $R(T)$. The λ_n define an isomorphism because the I_U-adic and the I_T-adic topologies on $K_T^*(X)$ coincide by lemma 1.4.7. Finally, E_U is a universal space for T by lemma 1.2.5, and it is T-retractible to $E_T \subset E_U$. Hence the ρ_n arising from the inclusion $X \times E_T^n \subset X \times E_U^n$ also induce an isomorphism. $\square$

Step 4) The general case: Let G be an arbitrary compact Lie group. We can embed it into a unitary group U by lemma 1.4.1. If X is a compact G-space, then $\bar{X} := U \times_G X$ is a compact U-space, and $K_U^*(\bar{X})$ is naturally isomorphic to $K_G^*(X)$ by lemma 1.3.6.

Now $R(G)$ is finite over $R(U)$ by lemma 1.4.5, and $K_G^*(X)$ is finite over $R(G)$ by assumption, hence $K_U^*(\bar{X}) = K_G^*(X)$ is finite over $R(U)$.

[13] The fact that we can choose k independent of n accounts for the continuity of α.

Furthermore, $\bar{X} \times E_U^n = (U \times_G X) \times E_U^n \cong U \times_G (X \times E_U^n)$, and so again by lemma 1.3.6 we find $K_U^*(\bar{X} \times E_U^n) \cong K_G^*(X \times E_U^n)$.

Now step 3, applied to the U-space $\bar{X}$, says that the maps $K_G^*(X)/I_U^n K_G^*(X) \to K_G^*(X \times E_U^n)$ define an isomorphism of inverse limits. But E_U is also a universal space for G, and since the I_U-adic and the I_G-adic completion coincide for any $R(G)$-module by 1.4.7, we can finish as in step 3. $\square$

We are finished.

Conclusion

Outlook The relevance of the Main Theorem lies in the fact that it allows to compute the equivariant K_G-group of a G-space X via the ordinary K-group of an auxiliary space X_G. However, I do not know of striking applications of this result.

The general theory developed in this essay can be pursued in many directions: Algebraical K-theory, K- and KK-theory of C^*-algebras, the Atiyah-Singer index formula, equivariant K-theory in algebraical geometry and its relation to representation theory, and so on. For the first two developments, the topological constructions treated can provide some motivation for definitions which otherwise might appear to be purely formal and abstract. The most noteworthy instance thereof is the connection between the index map for families of elliptic differential operators and the definition of KK-groups.

Personal comment This is my first essay of a length of more than 15 pages. Looking back, the worst part seems to be the section on the local theory of pseudo-differential operators, which presumably should have been omitted anyway. The topic least adequately covered in literature accessible to me was the construction of the index map for families of elliptic differential operators. The details of the proofs of the theorems 1.2.12, 1.4.7, 2.3.11, 2.5.9 and A.3.1 have been worked out following only hints in the literature. The derivation of the Thom element from the Bott element and the connection with the Koszul complex has been worked out by myself and looks quite clumsy.

> *Jōshū sees the hermits* Jōshū went to a hermit's cottage and asked, "Is the master in? Is the master in?". The hermit raised his fist. Jōshū said, "The water is too shallow to anchor here", and he went away.
> Coming to another hermit's cottage, he asked again, "Is the master in? Is the master in?". This hermit, too, raised his fist. Jōshū said, "Free to give, free to take, free to kill, free to save", and he made a deep bow.
>
> *Mumonkan, Case 11*
> *"Two Zen Classics: Mumonkan and Hegikanroku"*
> *transl. by K. Sekida*

A Appendix

A.1 Inverse systems

The following could be summarised shortly in categorical language, which probably would be of no use to the reader who happens to look at this.

A.1.1 Definition. An *inverse system of modules* consists of a directed set T, modules $\{A_t\}_{t\in T}$ and homomorphisms $\alpha_{tt'} : A_t \to A_{t'}$ for all $t, t' \in T$ with $t > t'$, such that $\alpha_{t't''} \circ \alpha_{tt'} = \alpha_{tt''}$ whenever $t > t' > t''$.

The *inverse limit* of an inverse system is the module $\varprojlim A_t = \{(a_t)_{t\in T} : a_t \in A_t,\ \alpha_{tt'}(a_t) = a_{t'} \forall t > t'\}$.

A *map between inverse systems* $\{A_*, \alpha_*\}_T$ and $\{B_*, \beta_*\}_S$ consists of a monotonous map $\sigma : S \to T$ and a collection of maps $\phi_s : A_{\sigma(s)} \to B_s$ such that $\beta_{ss'} \circ \phi_{s'} = \phi_s \alpha_{\sigma(s)\sigma(s')}$. We denote the map by (σ_*, ϕ_*). One could extend this to non-monotonous index maps σ which we do not need.

A.1.2 Lemma. Let $\{A_*, \alpha_*\}_T$ and $\{B_*, \beta_*\}_S$ be inverse systems with maps (σ_*, ϕ_*) and (τ_*, ψ_*) such that the diagrams

$$\begin{array}{ccc} A_{\sigma\tau(t)} & \to & B_{\tau(t)} \\ \downarrow & \swarrow & \\ A_t & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A_{\sigma(s)} & \leftarrow & B_{\tau\sigma(s)} \\ & \searrow & \downarrow \\ & & B_s \end{array}$$

commute. Then $A := \varprojlim A_t \cong \varprojlim B_s =: B$.

Proof. By the commutativity assumption on maps between inverse systems, we get maps $\phi : A \to B$, $(a_t)_t \mapsto (\phi_s(a_{\sigma(s)}))_s$ and $\psi : B \to A$, $(b_s)_s \mapsto (\psi_t(b_{\tau(t)}))_t$. Now the commutativity assumption of the lemma implies that ϕ and ψ are inverse isomorphisms. $\square$

A.1.3 Lemma. If $T' \subset T$ is cofinal, then $\varprojlim_{t\in T} A_t = \varprojlim_{t'\in T'} A_{t'}$.

Proof. Easy or obvious. $\square$

A.1.4 Lemma. Let $\{A_{(s,t)}, \alpha_*\}_{S\times T}$ and $\{B_{(s,t)}, \beta_*\}_{S\times T}$ be inverse systems with an index map (θ_*, ϕ_*) such that $\theta(s,t) = (\sigma(s,t), t)$ for some $\sigma : S \times T \to S$. Assume that for all t, the map $\phi_{*,t}$ induces an isomorphism $\phi_t : A_t := \varprojlim_s A_{s,t} \xrightarrow{\sim} \varprojlim_s B_{s,t} := B_t$. Then ϕ_* induces an isomorphism $\varprojlim_{s,t} A_{s,t} \xrightarrow{\sim} \varprojlim_{s,t} B_{s,t}$.

Proof. From the definition it is clear that the $\{A_t\}_T$ form an inverse system such that $\varprojlim_t A_t \cong \varprojlim_{s,t} A_{s,t}$. Now the diagrams

$$\begin{array}{ccc} A_t & \xrightarrow{\phi_t} & B_t \\ \downarrow & & \downarrow \\ A_{t'} & \xrightarrow{\phi_{t'}} & B_{t'} \end{array}$$

commute whenever $t > t'$ by assumption on ϕ, and so the claim follows. $\square$

A.2 A topological lemma

A.2.1 Lemma. Let X be the union of its compact subspaces $\{X^t\}$, $t \in T$, where T is a directed set with $X^{t'} \subset X^t$ whenever $t > t'$. Then for any compact $Y \subset X$ there is a t such that $Y \subset X^t$.

Proof. Assume the contrary. Let $U = \bigcup_t int(X^t)$. Replacing X by $Y \cap X \setminus U$, X^t by $Y \cap X^t \setminus U$ and Y by $Y \setminus U$ we may assume that $Y = X = \bigcup_t X^t$ and $int(X^t) = \emptyset$. Now we get a contradiction using Baire's category theorem for compact (instead of complete) spaces. $\square$

A.2.2 Corollary. Let $\{X^t\}_T$ be a direct system of compact spaces, i.e. T is a directed set, and for any $t, t' \in T$ with $t < t'$ we have a map $\phi_{tt'} : X^t \to X^{t'}$ such that $\phi_{t't''} \circ \phi_{tt'} = \phi_{tt''}$ whenever $t < t' < t''$. Then for any compact space Y, $\varinjlim [Y, X^t] = [Y, \varinjlim X^t]$.

Proof. By the previous lemma, the image of Y or $Y \times [0,1]$ in $\varinjlim X^t$ under a continuous map is contained in some X^t. $\square$

A.3 Cohomology of the standard line bundle on $\mathbb{P}^n$

Let $[z_0 : \cdots : z_n]$ be homogeneous coordinates on $\mathbb{P}^n$. For $s \subset \{0, \ldots, n\}$, let $U_s = \{[z_0 : \cdots : z_n] \mid z_i = 0\, \forall i \in s\}$. On $U_i := U_{\{i\}}$ we have local coordinates $\{z_i/z_j\}_i$. The standard bundle H on $\mathbb{P}^n$ is the line bundle given by the transition functions $\phi_{ij} = z_j/z_i$. It corresponds to the divisor $z_0 = 0$. The dual of H is the tautological line bundle which corresponds to the principal bundle $U_n/U_{n-1} \to U_n/(U_{n-1} \times U_1) = \mathbb{P}^n$. If $n = 1$, H is the Hopf bundle. Let $\mathcal{O}(m)$ denotes the sheaf of holomorphic sections of H^m.

Since a holomorphic function on a complex compact manifold must be constant, we have $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}) = \mathbb{C}$.

A.3.1 Theorem. [14] $H^p(\mathbb{P}^n, \mathcal{O}) = 0$ for $p > 0$, $H^p(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for $1 \leq -m \leq n$ and $p \in \mathbb{N}$.

Proof. We compute the cohomology as the Cech cohomology of the standard (acyclic) open cover $\{U_i\}$. For $p > n$ the claim is obvious since the cover consists of just n sets.

Let $p = 0$. If c were a global section of $\mathcal{O}(m)$, then on U_i we could write c as a Laurent series in the variables $\{z_j/z_i\}_j$, and on $U_{\{i,j\}}$ we would have two Laurent series representations of c, one with only z_i in the denominator and the other with only z_j in the denominator, which differ by the transition function $\phi_{ij} = z_j/z_i$. This shows $c = 0$.

Consider now the case $p \leq n$. A p-cocycle c is a set $c = \{c(s_0, \ldots, s_p) : 0 \leq s_0 < \cdots < s_p \leq n\}$ of holomorphic functions $c(s) : U_s \to \mathbb{C}$. For convenience, put $c(s_{\sigma(0)}, \ldots, s_{\sigma(p)}) = \mathrm{sgn}(\sigma) c(s_0, \ldots, s_p)$ for any permutation $\sigma \in S_{p+1}$. In local coordinates, we may regard $c(s)$ as a holomorphic function on $U_s \subset \mathbb{C}^{n+1}$ which is homogeneous of degree m. Each such function can be expanded into a Laurent series $c(s)(z) = \sum_{|\alpha|=m} c(s)_\alpha z^\alpha$ with $c(s)_\alpha = 0$ if any $\alpha_j < 0$ for some $j \notin s$ (see [10]). We construct a $(p-1)$-cochain a with $Da = c$: Denote $\alpha_+ = |\{i : \alpha_i \geq 0\}|$ for $\alpha \in \mathbb{N}^{n+1}$, and put

$$\begin{aligned} c_j(s)(z) &= \sum_{|\alpha|=m,\ \alpha_j \geq 0} c(s)_\alpha z^\alpha / \alpha_+, \\ a(s') &= \sum_{0 \leq j \leq n} c_j(j, s'), \qquad s' = (s'_0, \ldots, s'_{p-1}), \end{aligned}$$

If $p \leq n$, then $Dc = 0$ implies $c(s_0, \ldots, s_p) = \sum_i (-1)^i c(j, s_0, \ldots, \widehat{s_i}, \ldots, s_p)$ for any j, and hence $c(s_0, \ldots, s_p)_\alpha = 0$ whenever $\alpha_+ = 0$. If $p = n$, $-m \leq n$ and $\alpha_+ = 0$, then $|\alpha| \leq -n < m$ and hence $c(s)_\alpha = 0$.

[14] The idea of the proof of this theorem is taken from [10]. The proof given in [10] seems to be wrong.

It is easy to see that the Laurent series defined above converge.

We compute

$$(Da)(s) = \sum_i (-1)^i a(s_0, \ldots, \widehat{s_i}, \ldots, s_p) = \sum_{i,j} (-1)^i c_j(j, s_0, \ldots, \widehat{s_i}, \ldots, s_p).$$

Since c is a cocycle, for any j we have

$$\sum_i (-1)^i c_j(j, s_0, \ldots, \widehat{s_i}, \ldots, s_p)(z) = \sum_{|\alpha|=m,\ \alpha_j \geq 0} c(s_0, \ldots, s_p)_\alpha z^\alpha / \alpha_+$$

and hence

$$(Da)(s)(z) = \sum_{|\alpha|=m} \sum_{j \text{ with } \alpha_j \geq 0} c(s_0, \ldots, s_p)_\alpha z^\alpha / \alpha_+ = c(s)(z).$$

□

A.4 Complexes

The following makes sense in any abelian category, like e.g. modules over a ring and vector bundles or sheaves on a fixed space X.

A.4.1 Definition. A *complex* is a sequence of objects C^k and morphisms (called *differentials*) $\partial_k : C^k \to C^{k+1}$ such that $\partial_{k+1} \circ \partial_k = 0$. The complex is *exact* if $\mathrm{im}\partial_k = \ker \partial_{k+1}$. We omit the index for ∂ and write (C^*, ∂_*) or just C^* for the complex.

The *k-th cohomology* of (C^*, ∂) is $H^k(C^*) := \ker \partial_k / \mathrm{im}\partial_{k-1}$. The *Euler characteristic* of the complex is the expression $\sum_k (-1)^k H^k(C^*)$ which formally is an element of the K-group of the category.

A *chain map* $f : C^* \to D^*$ of order n is a sequence of maps $f^k : C^k \to D^{k+n}$ which commute with the differentials. A chain map induces a map of cohomology objects.

A *chain homotopy* between two chain maps $f, g : C^* \to D^*$ (of order 0) is a chain map $H : C^* \to D^*$ of order -1 such that $f - g = \partial H - H \partial$. Homotopic chain maps induce the same maps on cohomology.

A.4.2 Definition. A module P is *projective* if each map of P to a quotient of any module M can be lifted to M. A *projective resolution* of a module M is a complex P^* with $P^k = 0$ for $k > 0$, $H^k(P^*) = 0$ for $k < 0$ and $H^0(P^*) \cong M$. Each free module is projective, and hence each module has a projective resolution.

It is easy to show that

- any two projective resolutions of a fixed module are chain homotopy equivalent,
- chain homotopy equivalence is preserved when applying additive functors,
- chain homotopy equivalent complexes have isomorphic cohomology.

References

[1] M. F. Atiyah. *K-Theory.* W. A. Benjamin, 1967.

[2] M.F. Atiyah. Characters and cohomology of finite groups. *Publ. Math. IHES*, 9:23–64, 1961.

[3] M.F. Atiyah. Bott periodicity and the index of elliptic operators. *Quart. Journal Math. Oxford*, 2(19):113–140, 1968.

[4] M.F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. *Differential Geometry, Proc. Symp. Pure Math.*, 3:7–38, 1961. AMS.

[5] M.F. Atiyah and I.G. MacDonald. *Introduction to commutative algebra.* Addison-Wesley, 1969.

[6] M.F. Atiyah and G.B. Segal. Equivariant K-theory and completion. *J. Diff. Geom.*, 3:1–18, 1969.

[7] B. Blackadar. *K-Theory for operator algebras.* Springer, 1986.

[8] T. Broecker and T. tom Dieck. *Representations of compact Lie groups.* Springer, 1985.

[9] E. Dyer. *Cohomology theories.* W.A. Benjamin, 1969.

[10] M. Field. *Several Complex Variables and Complex Manifolds I, II.* London Mathematical Society Lecture Notes 66. Cambridge University Press, 1982.

[11] P. B. Gilkey. *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem.* CRC Press, 1995.

[12] J.P.C. Greenslees and J.P. May. *Equivariant stable homotopy theory.* Handbook of algebraic topology. Elsevier, North Holland, 1995.

[13] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Addison-Wesley, 1978.

[14] P.J. Hilton and U. Stammbach. *A course in homological algebra.* Springer, 1971.

[15] D. Husemoller. *Fibre bundles.* McGraw-Hill, 1966.

[16] M. Karoubi. *K-Theory. An Introduction.* Springer, Berlin New York Heidelberg, 1978.

[17] R.S. Palais. *Foundations of global nonlinear analysis.* Brandeis University, 1967.

[18] J. Rosenberg. Elements of KK-theory, by K. Knudsen Jensen and K. Thomsen. *Book review, Bulletin of the AMS*, 28(2):342–347, April 1993. http://www.ams.org/bull/pre-1996-data/199328-2/199328-2TOC.html.

[19] G. B. Segal. Equivariant K-theory. *Publ. Math. IHES*, 34:129–151, 1968.

[20] G. B. Segal. The representation ring of a compact lie group. *Publ. Math. IHES*, 34:113–128, 1968.

[21] G.B. Segal. Classifying spaces and spectral sequences. *Publ. Math. IHES*, 34, 1968.

[22] Tammo tom Dieck. *Transformation Groups.* de Gruyter, 1987.