Compact C^*-Quantum Groupoids
Definition and Examples

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Overview

What is a quantum groupoid?

- a Hopf bimodule $B \overset{\rho}{\underset{\sigma}{\rightrightarrows}} A \xrightarrow{\Delta} A * A$ with Haar weights, antipode, modular element, ...

Which types and theories of quantum groupoids exist?

- finite (Böhm, Szlachányi, Nikshych, Vainerman, ...)
- algebraic (Schauenburg, ...)
- measurable (Enock, Lesieur, Vallin)
- compact (Enock, T.)

Where do quantum groupoids appear?

- subfactors, quantum field theory, monoidal equivalence of quantum groups, noncommutative differential geometry

Compact C^*-quantum graphs

Example G locally compact Hausdorff groupoid

Definition A *compact C^*-quantum graph* consists of

- a unital C^*-algebra B ... $C(G^0)$
- a faithful KMS-state μ on B ... q.-invariant measure on G^0
- a unital C^*-algebra A ... $C(G)$
- unital embeddings $\rho\colon B \to A$ and $\sigma\colon B^{op} \to A$ ... r^*, s^*
- cond. expectations $\phi\colon A \to \rho(B) \cong B$, $\psi\colon A \to \sigma(B^{op}) \cong B^{op}$

  ... integration w.r.t. Haar systems

such that

- $\nu := \mu \circ \phi$ and $\nu^{-1} := \mu^{op} \circ \psi$ are faithful KMS-states
- there exists $\delta = d\nu/d\nu^{-1} \in A \cap \rho(B)' \cap \sigma(B^{op})'$

Modules and bimodules over KMS-states

Definition A *C^*-module over μ* consists of

- a Hilbert space K
- and a closed subspace $\gamma \subseteq \mathcal{L}(H_\mu, K)$

such that $[\gamma H_\mu] = K$, $[\gamma^* \gamma] = B$, $[\gamma B] = \gamma$.

Lemma If (K, γ) is a C^*-module over μ, then

- γ is a Hilbert C^*-module over B
- $\gamma \otimes_B H_\mu \cong K$ via $\xi \otimes_B \eta \equiv \xi\eta$
- $\exists$ normal nondegenerate faithful representation $\rho_\gamma \colon B' \to \mathcal{L}(K) \equiv \mathcal{L}(\gamma \otimes_B H_\mu)$, $x \mapsto \mathrm{id} \otimes_B x$

Definition A *C^*-bimodule over (μ^{op}, μ)* is a triple (K, γ, ϵ) s.t.

- (K, γ), (K, ϵ) are C^*-modules over μ^{op}, μ
- $\epsilon = [\rho_\gamma(B)\epsilon]$ and $\gamma = [\rho_\epsilon(B^{op})\gamma]$

The relative tensor product $H_\alpha \otimes_\beta H$

Examples $(H, \widehat{\alpha}, \widehat{\beta})$ and (H, β, α), where

- $H = H_\nu$
- $\widehat{\alpha} = [\Lambda_\phi(A)] \subseteq \mathcal{L}(H_\mu, H)$ and $\Lambda_\phi(a) \colon \Lambda_\mu(b) \mapsto \Lambda_\nu(a\rho(b))$
- $\widehat{\beta} = [\Lambda_\psi(A)] \subseteq \mathcal{L}(H_\mu, H)$ is defined similarly for ψ, σ
- $\beta = [\Lambda_{\phi^{op}}(A^{op})] = J_\nu \widehat{\alpha} J_\mu$ and $\alpha = [\Lambda_{\psi^{op}}(A^{op})] = J_\nu \widehat{\beta} J_\mu$

Definition

$$H_\mu \xrightarrow{\beta} H \xleftarrow{\alpha} H_\mu \xrightarrow{\beta} H \xleftarrow{\alpha} H_\mu$$

$$H = \alpha \otimes_B H_\mu \qquad H = H_\mu \, {}_{B^{op}}\!\otimes \beta$$

$$\alpha \otimes_B H_\mu \xrightarrow{r(\beta)} \alpha \otimes_B H_\mu \, {}_{B^{op}}\!\otimes \beta \xleftarrow{l(\alpha)} H_\mu \, {}_{B^{op}}\!\otimes \beta$$

$$\underbrace{\alpha \otimes_B H_\mu \, {}_{B^{op}}\!\otimes \beta}_{=: H_\alpha \otimes_\beta H}$$

where $r(\eta) \colon \omega \mapsto \omega \, {}_{B^{op}}\!\otimes \eta$ and $l(\xi) \colon \omega \mapsto \xi \otimes_B \omega$

The fiber product $A_\alpha *_\beta A$

Definition A *C^*-algebra over* (μ^{op}, μ) consists of

- a C^*-module (K, γ, ϵ) over (μ^{op}, μ)
- a nd. C^*-algebra $C \subseteq \mathcal{L}(K)$

such that $\rho_\gamma(B), \rho_\epsilon(B^{op}) \subseteq M(C) \subseteq \mathcal{L}(K)$

Definition *Morphisms* of C^*-algebras over (μ^{op}, μ) ...

Examples (H, β, α, A), $(H, \hat{\beta}, \hat{\alpha}, A^{op})$

Definition

$$\begin{array}{ccccc} H & \xrightarrow{r(\beta)} & H_\alpha \otimes_\beta H & \xleftarrow{l(\alpha)} & H \\ \downarrow A & & \downarrow A_\alpha *_\beta A & & \downarrow A \\ H & \xrightarrow{r(\beta)} & H_\alpha \otimes_\beta H & \xleftarrow{l(\alpha)} & H \end{array}$$

$$A_\alpha *_\beta A := \{x : x^{(*)} r(\beta) \subseteq [r(\beta)A],\ x^{(*)} l(\alpha) \subseteq [l(\alpha)A]\}$$

Compact C^*-quantum groupoids

Definition A *compact C^*-quantum groupoid* consists of

- a compact C^*-quantum graph $(B, \mu, A, \rho, \sigma, \phi, \psi)$
- a morphism $\Delta\colon A \to A_\alpha {*}_\beta A$ of C^*-algebras over (μ^{op}, μ)
- an anti-automorphism R of A

such that

- Δ is coassociative
- several *bisimplifiability conditions* hold
- ϕ is *left-invariant* and ψ is *right-invariant*
- R satisfies $R \circ R = \mathrm{id}_A$ and flips (ρ, ϕ), (σ, ψ)
- *strong invariance* holds

Properties of compact C^*-quantum groupoids

$\mathcal{G} = (B, \mu, A, \rho, \sigma, \phi, \psi, \Delta, R)$ ... a compact C^*-quantum groupoid

Proposition (Unimodularity) we can replace μ by an equivalent KMS-state and assume $\delta = 1_A$

Theorem (Uniqueness of Haar weights)

If $(B, \mu, A, \rho, \sigma, \tilde{\phi}, \tilde{\psi})$ is a compact C^*-quantum graph, then $\tilde{\phi}$ is left-invariant $\Leftrightarrow$ $\tilde{\phi} = \phi_x$ for some $x \in \sigma(B^{op})$ and $\tilde{\psi}$ is right-invariant $\Leftrightarrow$ $\tilde{\psi} = \psi_y$ for some $y \in \rho(B)$

Integration along orbits $\tau := \psi \circ r \colon B \to B^{op}$ is a conditional expectation onto $\{x \in B \mid \rho(x) = \sigma(x^{op})\} \subseteq Z(B) = B \cap B^{op}$ and satisfies a KMS-condition w.r.t. σ^{μ}

The associated fundamental unitary

Theorem There is a *regular C^*-pseudo-multiplicative unitary*

$$V\colon H_{\widehat{\beta}}\otimes_{\alpha} H \to H_{\alpha}\otimes_{\beta} H, \quad \Lambda_{\psi}(a) \otimes_{B^{op}} \omega \mapsto \Delta(a)(\Lambda_{\psi}(1) \otimes_{B} \omega)$$

This unitary is fundamental and allows us to (re)construct:

- (A, Δ) by $A = [r(\beta)^* V r(\alpha)] = A$ and $\Delta\colon a \mapsto V(a_{\hat{\beta}}\otimes_{\alpha} 1)V^*$

$$\begin{array}{ccccc} H & \xrightarrow{\quad r(\alpha) \quad} & H_{\widehat{\beta}}\otimes_{\alpha} H & \xleftarrow{\quad l(\widehat{\beta}) \quad} & H \\ \downarrow \widehat{A} & & \downarrow V & & A \downarrow \\ H & \xleftarrow{\quad r(\beta)^* \quad} & H_{\alpha}\otimes_{\beta} H & \xrightarrow{\quad l(\alpha)^* \quad} & H \end{array}$$

- a *reduced dual Hopf C^*-bimodule* $(\widehat{A}, \widehat{\Delta})$ by $\widehat{A} := [l(\alpha)^* V l(\widehat{\beta})]$ and $\widehat{\Delta}\colon \hat{a} \mapsto V^*(1_{\alpha}\otimes_{\beta}\hat{a})V$
- a completion that is a *measurable quantum groupoid* in the sense of Enock and Lesieur

Further applications of the fundamental unitary

Duality for coactions on C^*-algebras

We construct *reduced crossed product functors*

$$\{\text{coactions of } (A, \Delta)\} \underset{-\rtimes_r A}{\overset{-\rtimes_r \widehat{A}}{\rightleftarrows}} \{\text{coactions of } (\widehat{A}, \widehat{\Delta})\},$$

and under natural assumptions $(C, \delta) \rtimes_r \widehat{A} \rtimes_r A \sim_M (C, \delta)$

(Co)representations and universal Hopf C^*-bimodules

Have C^*-tensor categories of representations and corepresentations of V with *universal (co)orepresentations* that yield *universal Hopf C^*-bimodules* (A_u, Δ_u) and $(\widehat{A}_u, \widehat{\Delta}_u)$

Question

Can we extend Tannaka-Krein duality to this setting?

Examples of compact C^*-quantum groupoids

Sources of examples

- function algebra of a compact groupoid
- reduced C^*-algebra of an r-discrete groupoid
- principal compact C^*-quantum groupoids
- Morita equivalent base changes
- free products
- monoidal equivalence of compact quantum groupoids and associated linking quantum groupoids

Principal compact C^*-quantum groupoids

$\mathcal{G} = (B, \mu, A, \rho, \sigma, \phi, \psi, \Delta, R)$ ... a compact C^*-quantum groupoid

Recall groupoid $G \rightsquigarrow$ principal groupoid $(r, s)(G) \subseteq G^0 \times G^0$

Definition $\mathcal{G}$ is *principal* $:\Leftrightarrow A = [\rho(B)\sigma(B^{op})]$

Theorem The map $\tau := \psi \circ \rho \colon B \to B^{op}$ (*integration along orbits*) satisfies $\tau^{op} = \phi \circ \sigma, \ \mu_\theta \circ \tau = \mu_\theta, \ \rho \circ \tau = \sigma \circ \tau$, and $\tau \colon B \to \tau(B) \subseteq Z(B)$ is a σ^μ-KMS-cond. expectation

Theorem

$$\{B, \mu, \tau \text{ satisfying the conditions above}\}$$

$$\updownarrow \text{1:1 up to isomorphism}$$

$$\{\text{principal compact } C^*\text{-quantum groupoids}\}$$

$$\uparrow A \mapsto [\rho(B)\sigma(B^{op})]$$

$$\{\text{compact } C^*\text{-quantum groupoids}\}$$

Morita equivalences on the base

$B, \tilde{B}$ unital, full corners in the linking algebra $\begin{pmatrix} B & F \\ E & \tilde{B} \end{pmatrix}$,

$\mu, \tilde{\mu}$ f.f. traces on $B, \tilde{B}$ s.t. $\tilde{\mu}(\eta\xi^*) = \mu(\xi^*\eta)$ for all $\eta, \xi \in E$

Theorem There exist equivalences of categories

- $\{C^*\text{-mod.s over } (\mu^{op}, \mu)\} \to \{C^*\text{-mod.s over } (\tilde{\mu}^{op}, \tilde{\mu})\}$, written $(K, \gamma, \epsilon) \mapsto (\tilde{K}, \tilde{\gamma}, \tilde{\epsilon})$, where $\tilde{K} = E \otimes_{\rho_\gamma} K_{\rho_\epsilon} \otimes F^{op}$
- $\{C^*\text{-alg.s over } (\mu^{op}, \mu)\} \to \{C^*\text{-alg.s over } (\tilde{\mu}^{op}, \tilde{\mu})\}$, written $(K, \gamma, \epsilon, C) \mapsto (\tilde{K}, \tilde{\gamma}, \tilde{\epsilon}, \tilde{C})$, where $\tilde{C} = [XCX^*]$ with $X = [l(E)r(F^{op})] \subseteq \mathcal{L}(K, \tilde{K})$
- $\left\{\begin{matrix}\text{compact } C^*\text{-quantum} \\ \text{groupoids over } (B, \mu)\end{matrix}\right\} \to \left\{\begin{matrix}\text{compact } C^*\text{-quantum} \\ \text{groupoids over } (\tilde{B}, \tilde{\mu})\end{matrix}\right\}$