Free dynamical quantum groups and deformations of $SU(2)$

Thomas Timmermann

23.01.2013

Westfälische Wilhelms-Universität Münster

Introduction

Why study dynamical quantum groups?

- Links between special functions and representation theory
- Obtain rich examples of quantum groupoids

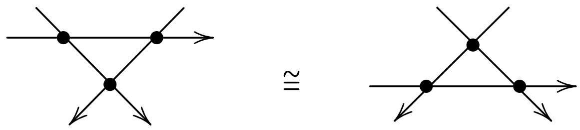
Plan of the talk

1. Dynamical quantum groups and dynamical R-matrices
2. Example: a dynamical analogue $SU_q(2)$
3. Example: the free orthogonal and free unitary ones
4. The passage to operator algebras

R-matrices and the Yang-Baxter equation

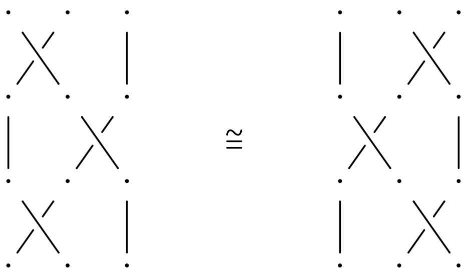
Definition *An R-matrix on V is an $R \in \mathrm{Aut}(V \otimes V)$ such that*

- *R satisfies the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$*



or, equivalenty,

- *$\hat{R} := \Sigma R$ satisfies the braid relations $\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$*



From R-matrices to quantum groups

Proposition (Faddeev-Reshetikhin-Takhtajan construction)
Let R be an R-matrix on $\mathbb{C}^n$ and let A_R be the algebra with

- *generators $(u_{ij})_{i,j}$ and*
- *relations $\hat{R}(u \boxtimes u) = (u \boxtimes u)\hat{R}$, where $u \boxtimes u = (u_{ij}u_{kl})_{(i,j),(k,l)}$.*

Then there exist homomorphisms

$$\Delta\colon A \to A \otimes A,\ u_{ij} \mapsto \textstyle\sum_k u_{ik} \otimes u_{kj}, \qquad \epsilon\colon A \to \mathbb{C},\ u_{ij} \mapsto \delta_{i,j}.$$

Thus, (A_R, Δ, ϵ) is a bialgebra, $u = (u_{ij})_{i,j}$ a corepresentation, and $\hat{R}$ a Yang-Baxter operator on u

Examples

- $A_{\mathrm{id}} = \mathcal{O}(M_n(\mathbb{C}))$ and $A_{\mathrm{id}}/_{(\det -1)} = \mathcal{O}(GL_n(\mathbb{C}))$
- $\mathcal{O}_q(M_2(\mathbb{C}))$ and $\mathcal{O}_q(M_2(\mathbb{C}))/_{(q\det -1)} = \mathcal{O}(\mathrm{SU}_q(2))$

Dynamical R-matrices

Definition Fix a group Γ acting on a commutative algebra B.

- *A dynamical vector space is a Γ-graded B-bimodule $\mathcal{V}$, where $vb = \gamma(b)v$ if v has degree γ*
- *The tensor product of $\mathcal{V}, \mathcal{W}$ is $\mathcal{V}\tilde{\otimes}\mathcal{W} = \bigoplus_{\alpha,\beta} \big(\mathcal{V}_\alpha \underset{B}{\otimes} \mathcal{W}_\beta\big)$*

$\rightsquigarrow$ get notion of dynamical R-matrix $R \in \mathrm{Aut}(\mathcal{V}\tilde{\otimes}\mathcal{V})$

Setup considered by Etingof-Varchenko:

- $\mathcal{V} = V \otimes B$ for some Γ-graded vector space V
- one has an abelian Lie algebra $\mathfrak{h}$ and a $\mathfrak{h}$-module V
  - $\Gamma \subseteq \mathfrak{h}^*$ is the lattice generated by the weights of V
  - B is the field of meromorphic functions on $\mathfrak{h}^*$

Dynamical quantum groups

Definition (following Etingof-Varchenko)

- *A dynamical algebra is a $\Gamma \times \Gamma$-graded algebra $\mathcal{A}$ with $B \otimes B \to \mathcal{A}_{e,e}$ s.t. $a(b \otimes b') = (\gamma(b) \otimes \gamma'(b'))a$ if $a \in \mathcal{A}_{\gamma,\gamma'}$*
- *The tensor product of $\mathcal{A}, \mathcal{C}$ is $\mathcal{A}\tilde{\otimes}\mathcal{C} = \bigoplus_{\alpha,\beta,\gamma} \big(\mathcal{A}_{\alpha,\beta} \underset{B}{\otimes} \mathcal{C}_{\beta,\gamma}\big)$*
- *A dynamical bialgebroid is a dynamical algebra $\mathcal{A}$ with*
  - *a comultiplication $\Delta\colon \mathcal{A} \to \mathcal{A}\tilde{\otimes}\mathcal{A}$ such that $(\Delta\tilde{\otimes}\iota)\Delta = (\iota\tilde{\otimes}\Delta)\Delta$*
  - *a counit $\epsilon\colon \mathcal{A} \to B \rtimes \Gamma$ such that $(\epsilon\tilde{\otimes}\iota)\Delta = \iota = (\iota\tilde{\otimes}\epsilon)\Delta$*
- *A dynamical quantum group is $(\mathcal{A}, \Delta, \epsilon)$ with an antipode S*

Example: The crossed product $B \rtimes \Gamma = \langle B, \Gamma : \gamma \cdot b = \gamma(b) \cdot \gamma \rangle$:

- $B \rtimes \Gamma = \bigoplus_{\gamma} B\gamma$ and $B \otimes B \to B \hookrightarrow B \rtimes \Gamma$ is the multiplication
- $\Delta(b\gamma) = b\gamma\tilde{\otimes}\gamma = \gamma\tilde{\otimes}b\gamma, \quad \epsilon(b\gamma) = b\gamma, \quad S(b\gamma) = \gamma^{-1}b$

A semi-classical example

Start with

- a Lie group G with algebra $\mathcal{O}(G)$ (matrix coeff. of f.d. reps)
- a maximal torus $T \subseteq G$ with Lie algebra $\mathfrak{t}$

Obtain a dynamical quantum group

- $\mathcal{A} \subseteq \mathrm{End}(\mathcal{O}(G))$ generated by
  - $\mathcal{O}(G)$ acting via multiplication
  - $B \otimes B = U\mathfrak{t} \otimes U\mathfrak{t}$ acting via left or right invariant diff-ops
- $\Gamma = \left\{ \begin{matrix} \text{weights of} \\ \text{f.d. reps. of } G \end{matrix} \right\} \subseteq \mathfrak{t}^*$ acting on $B = U\mathfrak{t} \cong \mathcal{O}(\mathfrak{t}^*)$ by shifts
- Δ, ϵ on $B \otimes B$ map $b \otimes b'$ to $(b \otimes 1)\tilde{\otimes}(1 \otimes b')$ or bb'
- Δ, ϵ on $\mathcal{O}(G)$ are transposes of $G \underset{T}{\times} G \xrightarrow{\text{mult}} G$ and $\mathfrak{t} \xrightarrow{\exp} G$

The dynamical quantum group $\mathrm{SU}_q^{\mathrm{dyn}}(2)$ (Koelink-Rosengren)

The dynamical FRT-construction applied to some specific R yields a dynamical quantum group $\mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$, where

- $\Gamma = \mathbb{Z}$ acting on $B = \mathfrak{M}(\mathbb{C})$ meromorphic functions by shifts
- $\mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$ =universal dynamical algebra with generators $\alpha, \alpha^*, \gamma, \gamma^*$ of degree (1,1), (-1,-1), (1,-1), (-1,1) satisfying

  1. $\alpha\gamma = q(F \otimes 1)\gamma\alpha$ and $\alpha\gamma^* = q(1 \otimes F_{-1})\gamma^*\alpha$,
     $\gamma^*\alpha^* = q(F \otimes 1)\alpha^*\gamma^*$ and $\gamma\alpha^* = q(1 \otimes F_{-1})\alpha^*\gamma$,
     where $F(\lambda) = \frac{q^{2(\lambda+1)} - q^{-2}}{q^{2(\lambda+1)} - 1}$ and $F_{-1}(\lambda) = F(\lambda - 1)$
  2. $\alpha\alpha^* - \alpha^*\alpha = H\gamma\gamma^*$ and $(1 \otimes G)\alpha\alpha^* - (G \otimes 1)\alpha^*\alpha = -qH\gamma^*\gamma$,
     where $H(\lambda, \lambda') = \frac{(q - q^{-1})(q^{2(\lambda+\lambda'+2)} - 1)}{(q^{2(\lambda+1)} - 1)(q^{2(\lambda'+1)} - 1)}$
     and $G(\lambda) = \frac{(q^{2(\lambda+1)} - q^2)(q^{2(\lambda+1)} - q^{-2})}{(q^{2(\lambda+1)} - 1)^2}$

Free orthogonal dynamical quantum groups

Fix $\Gamma \circlearrowright B$, $\gamma_1, \dots, \gamma_n \in \Gamma$, $f \in GL_n(B)$ s.t. $f_{ij} = 0$ if $\gamma_i \neq \gamma_j^{-1}$

Define *the free orthogonal dynamical quantum group* $\mathcal{A}_o(f)$ *as the universal dynamical algebra with generators* $(v_{ij})_{i,j}$ *s.t.*

1. $v_{ij}(b \otimes b') = (\gamma_i(b) \otimes \gamma_j(b'))v_{ij}$ *for all* b, b'
2. $f_{(r)} v^{-\top} = v f_{(s)}$, *where* $f_{(s)} = (1 \otimes f_{ij})_{i,j}$ *and* $f_{(r)} = (\gamma_i(f_{ij}) \otimes 1)_{i,j}$

Theorem (T.) $\mathcal{A}_o(f)$ *is a dynamical quantum group w.r.t.*

$$\Delta \colon v_{ij} \mapsto \textstyle\sum_k v_{ik} \tilde{\otimes} v_{kj}, \quad \epsilon \colon v_{ij} \mapsto \delta_{i,j}\gamma_i, \quad S \colon v_{ij} \mapsto (v^{-1})_{ij}$$

Thus, $v, v^{-\top}$ are matrix corepresentations with an intertwiner f

Example Obtain $\mathcal{O}(SU_q^{dyn}(2)) = \mathcal{A}_o(f)$ if
$\Gamma = \mathbb{Z}$, $B = \mathfrak{M}(\mathbb{C})$, $(\gamma_1, \gamma_2) = (1, -1)$, $f = \begin{pmatrix} 0 & -1 \\ F_{-1}^{-1} & 0 \end{pmatrix}$, $v = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$

Free unitary dynamical quantum groups

Fix $\Gamma \circlearrowright B$, $\gamma_1, \dots, \gamma_n \in \Gamma$, $f \in GL_n(B)$ s.t. $f_{ij} = 0$ if $\gamma_i \neq \gamma_j$,
where B is a $*$-algebra

Define *the free unitary dynamical quantum group* $\mathcal{A}_u(f)$ *as the universal dynamical $*$-algebra with generators* $(v_{ij})_{i,j}$ *s.t.*

1. $v_{ij}(b \otimes b') = (\gamma_i(b) \otimes \gamma_j(b'))v_{ij}$ *for all* b, b'
2. $f_{(r)} \bar{v}^{-\top} = v f_{(s)}$, *where* $f_{(s)} = (1 \otimes f_{ij})_{i,j}$ *and* $f_{(r)} = (\gamma_i(f_{ij}) \otimes 1)_{i,j}$

Theorem (T.) $\mathcal{A}_u(f)$ *is a dynamical quantum group w.r.t.*

$$\Delta \colon v_{ij} \mapsto \textstyle\sum_k v_{ik} \tilde{\otimes} v_{kj}, \quad \epsilon \colon v_{ij} \mapsto \delta_{i,j}\gamma_i, \quad S \colon v_{ij} \mapsto (v^{-1})_{ij}$$

Thus, $v, \bar{v}^{-\top}$ are matrix corepresentations with an intertwiner f

The full dynamical quantum group $\mathrm{SU}_q^{\mathrm{dyn}}(2)$

Idea $\mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$ is defined over $\left\langle \left(\lambda \mapsto \frac{q^{\lambda+k}-q^{-\lambda-k}}{q^{\lambda+l}-q^{-\lambda-l}}\right) : k, l \in \mathbb{Z} \right\rangle$

Definition $\mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2)) := \mathcal{A}_o(f)$, *where*

- $B = \left\langle \frac{(1-Q^2)}{Q^n(1-Q^{2m})} \cdot \frac{(Q^kX-Q^{-k}Y)}{(Q^lX-Q^{-l}Y)} \,\middle|\, m, n \in \mathbb{N}, k, l \in \mathbb{Z} \right\rangle \subseteq \mathbb{C}(Q, X, Y)$
- $\Gamma = \mathbb{Z}$ *acting by* $X \xrightarrow{k} Q^kX$, $Y \xrightarrow{k} Q^{-k}Y$ *and* $(\gamma_1, \gamma_2) = (1, -1)$

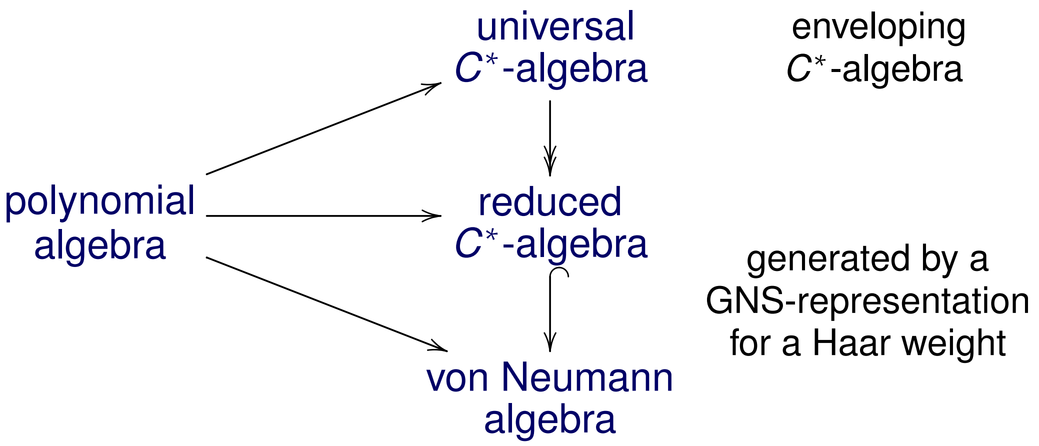
$\mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2))$ "contains" $\mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$ for all q and limit cases:

Base change Given a Γ-equivariant homomorphism $B \xrightarrow{\phi} C$,
get $\left\{ \begin{smallmatrix} \text{dynamical quantum} \\ \text{groups over } \Gamma \circlearrowright B \end{smallmatrix} \right\} \xrightarrow{\phi_*} \left\{ \begin{smallmatrix} \text{dynamical quantum} \\ \text{groups over } \Gamma \circlearrowright C \end{smallmatrix} \right\}$, $\mathcal{A} \mapsto \mathcal{A} \underset{B \otimes B}{\otimes} (C \otimes C)$

Examples **1.** $\phi \colon B \to \mathfrak{M}(C)$ s.t. $Q \mapsto q$, $X \mapsto (\lambda \mapsto q^{\lambda})$,
$Y \mapsto (\lambda \mapsto q^{-\lambda})$ gives $\phi_*\mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2)) \cong \mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$
2. $\phi \colon B \to \mathbb{C}$ s.t. $Q \mapsto q$ and $(X, Y) \mapsto (0, 1)$ or $(X, Y) \mapsto (1, 0)$
gives $\phi_*\mathcal{O}(\mathrm{SU}_Q^{\mathrm{dyn}}(2)) \cong \mathcal{O}(\mathrm{SU}_q(2))$ (limit case $\lambda \to \pm\infty$)

The passage to operator algebras

Aim construct operator-algebraic dynamical quantum groups



Completions in the form of measured quantum groupoids

Assume $\mathcal{A}$ is a dynamical quantum group over $\Gamma \circlearrowright B$ and

- $\mu: B \to \mathbb{C}$ is positive, Γ-quasi-invariant with bounded GNS-rep.
- $h: \mathcal{A} \to B \otimes B$ is a cond. expectation and $\Delta(\ker h) \subseteq \ker h \tilde{\otimes} \ker h$
- $\nu: \mathcal{A} \xrightarrow{h} B \otimes B \xrightarrow{\mu \otimes \mu} \mathbb{C}$ is faithful and positive

Theorem (T.) **1.** *ν has a bounded GNS-rep. $\pi_\nu: \mathcal{A} \to \mathcal{L}(H_\nu)$*

2. *$\phi := (\iota \otimes \mu) \circ h$ is left-invariant: $(\iota \otimes \phi)(\Delta(a)) = \phi(a) \otimes 1$*

$\psi := (\mu \otimes \iota) \circ h$ is right-invariant: $(\psi \otimes \iota)(\Delta(a)) = 1 \otimes \psi(a)$

3. *get $\pi_\nu(\mathcal{A})'' \xrightarrow{\bar{\phi}, \bar{\psi}} \pi_\mu(B)''$ and $\pi_\nu(\mathcal{A})'' \xrightarrow{\bar{\Delta}} \pi_\nu(\mathcal{A})'' * \pi_\nu(\mathcal{A})''$*

4. *$\pi_\nu(\mathcal{A})''$ is a measured quantum groupoid [Enock, Lesieur]*

Proof Use a unitary $V: H_\nu \underset{\mu}{\otimes} H_\nu \to H_\nu \underset{\mu}{\otimes} H_\nu$, $x \otimes y \mapsto \Delta(x)(1 \otimes y)$

Plan Apply this construction to $\mathcal{O}(SU_Q^{dyn}(2))$