Free dynamical quantum groups and deformations of SU(2)

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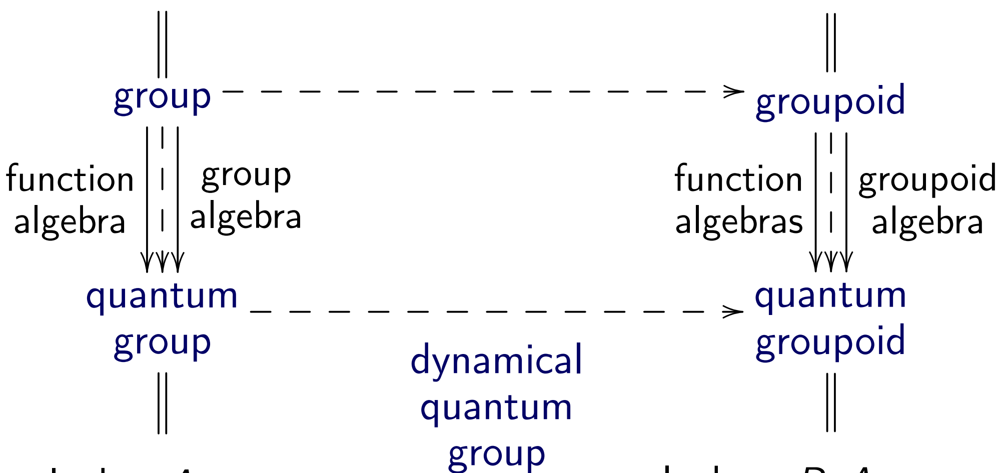
What is a quantum groupoid?

- a set G
- a map $G \times G \xrightarrow{m} G$
- conditions ...

- sets X, G
- maps $G \underset{X}{\times} G \xrightarrow{m} G \rightrightarrows X$
- conditions ...



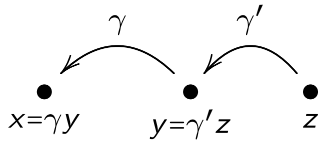
- an algebra A
- a map $A \xrightarrow{\Delta} A \otimes A$
- conditions ...

- algebras B, A
- maps $B \rightrightarrows A \xrightarrow{\Delta} A \underset{B}{\times} A$
- conditions ...

Examples of groupoids

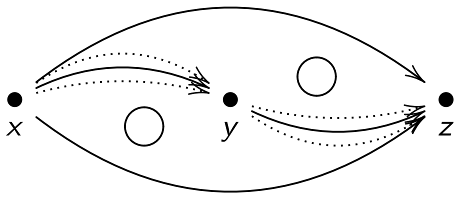
1. dynamical system $\Gamma \circlearrowright X \rightsquigarrow$ transformation groupoid:

$$X \rtimes \Gamma = \{(x, \gamma, y) : x = \gamma y\}$$



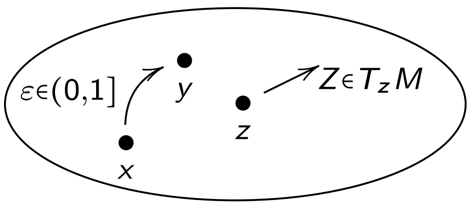
2. topological space $X \Rightarrow$ fundamental groupoid:

$$\Pi_1(X) = [[0,1], X]$$



3. manifold $M \Rightarrow$ tangent groupoid:

blow-up of $\Delta(M)$ in $M \times M$ /
deformation of TM to $M \times M$



Examples of quantum groups

The following are compact matrix quantum groups of the form

$$A = \langle u_{i,j} \mid u = (u_{ij})_{i,j} \text{ is unitary} \rangle, \qquad \Delta(u_{ij}) = \textstyle\sum_k u_{ik} \otimes u_{kj}$$

1. q-deformation of polynomial functions on $\mathrm{SU}_q(2)$:

$$\mathcal{O}(\mathrm{SU}_q(2)) = \left\langle a, c \,\middle|\, u = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \text{ is unitary} \right\rangle$$

2. quantum permutation group:

$$A = \left\langle p_{ij}, 1 \le i,j \le n \,\middle|\, \text{each row/column of } (p_{ij})_{i,j} \text{ consists of pairw. orthog. projections with sum } 1 \right\rangle$$

3. free orthogonal quantum group for parameter $F \in \mathrm{GL}_n(\mathbb{C})$:

$$A_o(F) = \left\langle u_{ij}, 1 \le i,j \le n \,\middle|\, F\bar{u} = uF \right\rangle$$

Dynamical quantum groups (Etingof-Varchenko)

Fix a commutative algebra B with an action by a group Γ

- A dynamical algebra is a $\Gamma \times \Gamma$-graded algebra $\mathcal{A}$ with $B \otimes B \to \mathcal{A}_{e,e}$ s.t. $a(b \otimes b') = (\gamma(b) \otimes \gamma'(b'))a$ for $a \in \mathcal{A}_{\gamma,\gamma'}$
- The tensor product of $\mathcal{A}, \mathcal{C}$ is $\mathcal{A}\tilde{\otimes}\mathcal{C} = \bigoplus_{\gamma,\gamma',\gamma''} \left(\mathcal{A}_{\gamma,\gamma'} \underset{B}{\otimes} \mathcal{C}_{\gamma',\gamma''}\right)$
- A dynamical quantum group is a dynamical algebra $\mathcal{A}$ with
  - a comultiplication $\Delta\colon \mathcal{A} \to \mathcal{A}\tilde{\otimes}\mathcal{A}$ such that $(\Delta\tilde{\otimes}\iota)\Delta = (\iota\tilde{\otimes}\Delta)\Delta$
  - a counit $\epsilon\colon \mathcal{A} \to B \rtimes \Gamma$ such that $(\epsilon\tilde{\otimes}\iota)\Delta = \iota = (\iota\tilde{\otimes}\epsilon)\Delta$
  - an antipode $S\colon \mathcal{A} \to \mathcal{A} \ldots$

Example The crossed product $B \rtimes \Gamma = \langle B, \Gamma : \gamma \cdot b = \gamma(b) \cdot \gamma \rangle$:

- $B \rtimes \Gamma = \bigoplus_{\gamma} B\gamma$ and $B \otimes B \to B \hookrightarrow B \rtimes \Gamma$ is the multiplication
- $\Delta(b\gamma) = b\gamma\tilde{\otimes}\gamma = \gamma\tilde{\otimes}b\gamma, \quad \epsilon(b\gamma) = b\gamma, \quad S(b\gamma) = \gamma^{-1}b$

A "semi-classical" example

Start with

- a Lie group G with algebra $\mathcal{O}(G)$ (matrix coeff. of f.d. reps)
- a toral subgroup $T \subseteq G$ with Lie algebra $\mathfrak{t}$

Obtain a dynamical quantum group, where

- $B = U\mathfrak{t} \cong \mathcal{O}(\mathfrak{t}^*)$ and $\Gamma = \left\{ \begin{matrix} \text{weights of} \\ \text{f.d. reps. of } G \end{matrix} \right\} \subseteq \mathfrak{t}^*$ acting by shifts
- $\mathcal{A} \subseteq \mathrm{End}(\mathcal{O}(G))$ is the algebra generated by
  - $\mathcal{O}(G)$ acting by multiplication
  - $B \otimes B = U\mathfrak{t} \otimes U\mathfrak{t}$ acting by left or right invariant diff.ops
- comultiplication Δ and counit ϵ
  - on $B \otimes B$: map $b \otimes b'$ to $(b \otimes 1)\tilde{\otimes}(1 \otimes b')$ or bb'
  - on $\mathcal{O}(G)$: are transposes of $G \underset{T}{\times} G \xrightarrow{m} G$ and $\mathfrak{t} \xrightarrow{\exp} G$

The example $\mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$ (Koelink-Rosengren)

$\mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$ is a deformation of the last example for $G = \mathrm{SU}(2)$:

- $B = \mathfrak{M}(\mathbb{C})$ (meromorphic functions) $\Gamma = \mathbb{Z}$ acting by shifts
- $\mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$ = universal dynamical algebra with
  - generators a, a^*, c, c^* of degree (1,1), (-1,-1), (1,-1), (-1,1)
  - relations involving special functions $f, g \in B$ and $h \in B \otimes B$:

$$ac = q(f \otimes 1)ca, \qquad ac^* = qc^*(1 \otimes f)a,$$
$$c^*a^* = q(f \otimes 1)a^*c^*, \quad ca^* = qa^*(1 \otimes f)c,$$
$$aa^* - a^*a = hcc^*, \quad (1 \otimes g)aa^* - (g \otimes 1)a^*a = -qhc^*c,$$

where $f(\lambda) = \frac{q^{2(\lambda+1)} - q^{-2}}{q^{2(\lambda+1)} - 1}, \quad g(\lambda) = \frac{(q^{2(\lambda+1)} - q^2)(q^{2(\lambda+1)} - q^{-2})}{(q^{2(\lambda+1)} - 1)^2},$

$$h(\lambda, \lambda') = \frac{(q - q^{-1})(q^{2(\lambda+\lambda'+2)} - 1)}{(q^{2(\lambda+1)} - 1)(q^{2(\lambda'+1)} - 1)}$$

- $\Delta(u_{ij}) = \sum_k u_{ik} \tilde{\otimes} u_{kj}$ for $u = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$

Corepresentations of dynamical quantum groups

Each group G has a tensor category of matrix representations:

- objects: finite set I and $u \in \mathrm{GL}_I(C(G))$ s.t. $u_{ij}(xy) = \sum_k u_{ik}(x)u_{kj}(y)$
- $\mathrm{Hom}((J,v),(I,u)) = \{T \in M_{J\times I}(\mathbb{C}) : Tv = uT\}$
- $(I,u) \otimes (J,v) = (I \times J, w)$, where $w_{(i,k),(j,l)} = u_{ij}v_{kl}$

Each dynamical qtm. group $B \otimes B \to \mathcal{A} \xrightarrow{\Delta} \mathcal{A}\tilde{\otimes}\mathcal{A}$ has an analogue:

- objects: finite set I and $u \in \mathrm{GL}_I(A)$ such that $\Delta(u_{ij}) = \sum_k u_{ik}\tilde{\otimes}u_{kj}$
  and $u_{ij} \in \mathcal{A}_{\gamma_i,\gamma_j}$ for some $\gamma \in \Gamma^I$ (the degree of u)
- $\mathrm{Hom}((J,v),(I,u)) = \{T \in M_{J\times I}(B) : \hat{T}v = u\check{T},\ T_{ij} = 0 \text{ if } \gamma_i^u \neq \gamma_j^v\}$,
  where $\hat{T} = \left(\gamma_i^u(T_{ij}) \otimes 1\right)_{i,j}$ and $\check{T} = \left(1 \otimes T_{ij}\right)_{i,j}$

(Idea: replace vector spaces by dynamical vector spaces:
Γ-graded B-bimodules $\mathcal{V}$ s.t. $vb = \gamma(b)v$ if $v \in \mathcal{V}_\gamma$)

The free orthogonal dynamical quantum groups $\mathcal{A}_o^B(\nabla, F, G)$

Proposition (T.) Let $\nabla \in \Gamma^n$, $F \in \mathrm{GL}_n(B)$ s.t. $F_{ij} = 0$ if $\nabla_i \neq \nabla_j^{-1}$.
Let also $G \in \mathrm{GL}_n(B)$ s.t. $G_{ij} = 0$ if $\nabla_i \neq \nabla_j^{-1}$ and $GF^* = FG^*$.

Then there exists universal $*$-dynamical qtm. group $\mathcal{A}_o(\nabla, F, G)$ with a corepresentation u of degree ∇ s.t. $F \in \mathrm{Hom}(u^{-\top}, u)$ and $G \in \mathrm{Hom}(\bar{u}, u)$.

Proof $\mathcal{A} := \langle B \otimes B, u_{ij} : (\nabla_i(b) \otimes \nabla_j(b'))u_{ij} = u_{ij}(b \otimes b'), \hat{F}u^{-\top} = u\check{F}\rangle$ admits $\Delta\colon \mathcal{A} \to \mathcal{A}\tilde{\otimes}\mathcal{A}$, $\epsilon\colon \mathcal{A} \to B \rtimes \Gamma$, $S\colon \mathcal{A} \to \mathcal{A}^{\mathrm{op,co}}$ such that

$$\Delta(u_{ij}) = \textstyle\sum_k u_{ik}\tilde{\otimes}u_{kj}, \quad \epsilon(u_{ij}) = \delta_{ij}\nabla_i, \quad S(u_{ij}) = (u^{-1})_{i,j}.$$

Example $\mathcal{A}_o^B(\nabla, F, G) = \mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$ if we take

$$B = \mathfrak{M}(\mathbb{C}),\ \Gamma = \mathbb{Z},\ \nabla = (1, -1),\ F = \begin{pmatrix} 0 & -1 \\ f_{-1}^{-1} & 0 \end{pmatrix},\ G = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix}$$

Glueing in $\mathcal{O}(\mathrm{SU}_q(2))$ as a limit case of $\mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$

Observations Let $B = \mathfrak{M}(\mathbb{C})$, $\Gamma = \mathbb{Z}$ and $T = \begin{pmatrix} f^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(B)$.

- $\mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$ is the universal $*$-dynamical quantum group with a corepresentation $u = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$ such that $T \in \mathrm{Hom}(u^{-*}, u)$
- For $\lambda \to -\infty$, have $f(\lambda) = \frac{q^{2(\lambda+1)} - q^{-2}}{q^{2(\lambda+1)} - 1} \to 1$ and get $\mathcal{O}(\mathrm{SU}_q(2))$
- The relations involve only $\left\langle \left(\lambda \mapsto \frac{q^{\lambda+k} - q^{-\lambda-k}}{q^{\lambda+l} - q^{-\lambda-l}}\right) : k, l \in \mathbb{Z} \right\rangle \subseteq B$

Definition $\mathcal{O}(\mathrm{SU}_q^{\mathrm{full}}(2)) := \mathcal{A}_o(\nabla, F, G)$, where

- $B = \left\langle \frac{q^k X - q^{-k} Y}{q^l X - q^{-l} Y} \,\middle|\, k, l \in \mathbb{Z} \right\rangle \subseteq \mathbb{C}(X, Y)$
- $\Gamma = \mathbb{Z}$ acting by $X \xrightarrow{k} q^k X$, $Y \xrightarrow{k} q^{-k} Y$
- $\nabla = (1, -1)$ and F, G are appropriately chosen

Glueing in $\mathcal{O}(\mathrm{SU}_q(2))$ as a limit case of $\mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$

Definition $\mathcal{O}(\mathrm{SU}_q^{\mathrm{full}}(2)) := \mathcal{A}_o(\nabla, F, G)$, where

- $B = \left\langle \frac{q^k X - q^{-k} Y}{q^l X - q^{-l} Y} \,\middle|\, k, l \in \mathbb{Z} \right\rangle \subseteq \mathbb{C}(X, Y)$
- $\Gamma = \mathbb{Z}$ acting by $X \xrightarrow{k} q^k X$, $Y \xrightarrow{k} q^{-k} Y$

$\mathcal{O}(\mathrm{SU}_q^{\mathrm{full}}(2))$ "contains" $\mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$ for all q and limit cases:

Base change Given a Γ-equivariant homomorphism $B \xrightarrow{\phi} C$, get

$$\left\{ \begin{matrix} \text{dynamical quantum} \\ \text{groups over } \Gamma \circlearrowright B \end{matrix} \right\} \xrightarrow{\phi_*} \left\{ \begin{matrix} \text{dynamical quantum} \\ \text{groups over } \Gamma \circlearrowright C \end{matrix} \right\}, \ \mathcal{A} \mapsto \mathcal{A} \underset{B \otimes B}{\otimes} (C \otimes C)$$

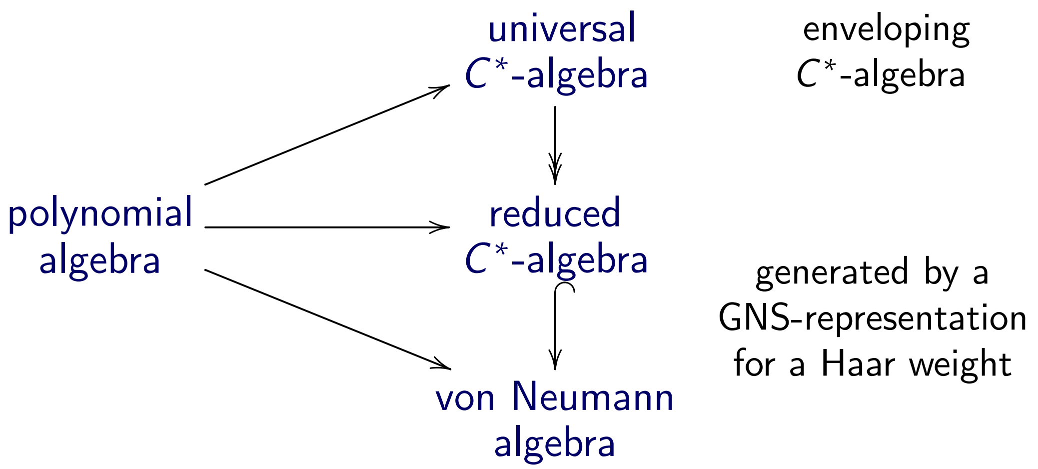
Examples **1.** $\phi_* \mathcal{O}(\mathrm{SU}_q^{\mathrm{full}}(2)) \cong \mathcal{O}(\mathrm{SU}_q(2))$ if $\phi \colon B \to \mathbb{C}$, $\begin{cases} X \mapsto 1 \\ Y \mapsto 0 \end{cases}$

2. $\phi_* \mathcal{O}(\mathrm{SU}_q^{\mathrm{full}}(2)) \cong \mathcal{O}(\mathrm{SU}_q^{\mathrm{dyn}}(2))$ if $\phi \colon B \to \mathfrak{M}(\mathbb{C})$, $\begin{cases} X \mapsto (\lambda \mapsto q^{\lambda}) \\ Y \mapsto (\lambda \mapsto q^{-\lambda}) \end{cases}$

3. (treat q as variable) $\phi \colon B \to \mathbb{C}(\lambda)$, $q \mapsto 1$, $\frac{q^k X - q^{-k} Y}{q^l X - q^{-l} Y} \mapsto \frac{\lambda - k}{\lambda - l}$

The passage to operator algebras

Aim construct operator-algebraic dynamical quantum groups



Completions in the form of measured quantum groupoids

Assume $\mathcal{A}$ is a dynamical quantum group over $\Gamma \circlearrowright B$ and

- $\mu: B \to \mathbb{C}$ is positive, Γ-quasi-invariant with bounded GNS-rep.
- $h: \mathcal{A} \to B \otimes B$ is a cond. expectation and $\Delta(\ker h) \subseteq \ker h \tilde{\otimes} \ker h$
- $\nu: \mathcal{A} \xrightarrow{h} B \otimes B \xrightarrow{\mu\otimes\mu} \mathbb{C}$ is faithful and positive

Theorem (T.) **1.** *ν has a bounded GNS-rep. $\pi_\nu: \mathcal{A} \to \mathcal{L}(H_\nu)$*

2. *$\phi := (\iota \otimes \mu) \circ h$ is left-invariant: $(\iota \otimes \phi)(\Delta(a)) = \phi(a) \otimes 1$*
$\psi := (\mu \otimes \iota) \circ h$ is right-invariant: $(\psi \otimes \iota)(\Delta(a)) = 1 \otimes \psi(a)$

3. *get $\pi_\nu(\mathcal{A})'' \xrightarrow{\bar{\phi},\bar{\psi}} \pi_\mu(B)''$ and $\pi_\nu(\mathcal{A})'' \xrightarrow{\bar{\Delta}} \pi_\nu(\mathcal{A})'' * \pi_\nu(\mathcal{A})''$*

4. *$\pi_\nu(\mathcal{A})''$ is a measured quantum groupoid [Enock, Lesieur]*

Proof Use a unitary $V: H_\nu \underset{\mu}{\otimes} H_\nu \to H_\nu \underset{\mu}{\otimes} H_\nu,\ x \otimes y \mapsto \Delta(x)(1 \otimes y)$

Plan Apply this construction to $\mathcal{O}(\mathrm{SU}_q^{\mathrm{full}}(2))$