Compact braided quantum groups

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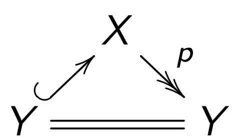
Background

Goal In a given category, describe all projections



Examples

1. In vector spaces, $X \cong Y \oplus \ker p$ (direct sum), where $\ker p$ is just a vector space
2. In discrete groups, $X \cong Y \ltimes \ker p$ (semi-direct product), where $\ker p$ is a group with an action of Y by automorphisms
3. In Hopf algebras, $X \cong Y \boxtimes Z$ (Radford product/Majid's bosonization), where Z is a braided Hopf algebra in ${}_{Y}\mathfrak{YD}^{Y}$
4. In C^*-algebraic quantum groups, also $X \cong Y \boxtimes Z$ (Roy)

This talk Look at the special case of compact quantum groups; then Z is a *braided compact quantum group*

Notation

We fix a *reduced compact quantum group* (A, Δ_A) and write

- $A_0 \subseteq A$ for the dense Hopf $*$-algebra of matrix elements of finite-dimensional corepresentations
- h_A for the Haar state on A
- H_A for the associated GNS-space with cyclic vector ζ_A
- $W_A \in \mathcal{L}(H_A \otimes H_A)$ for the multiplicative unitary, given by
$$a\zeta_A \otimes \xi \mapsto \Delta_A(a)(\zeta_A \otimes \xi)$$
- $(\hat{A}, \hat{\Delta}_A)$ for the reduced dual, so that $W_A \in M(\hat{A} \otimes A)$

Yetter-Drinfeld C^*-algebras over (A, Δ_A)

(Nest, Voigt. *Equivariant Poincaré duality for quantum group actions.* **JFA, 2010)**

Definition A *Yetter-Drinfeld C^*-algebra* over (A, Δ_A) is a C^*-algebra B with continuous coactions α of A and λ of $\hat{A}$ such that

$$B \xrightarrow{(\alpha\otimes\mathrm{id})\circ\lambda} M(B \otimes A \otimes \hat{A}) \quad \circlearrowleft \quad \big\downarrow \mathrm{id} \otimes \mathrm{Ad}(W_A\Sigma) \qquad B \xrightarrow{(\lambda\otimes\mathrm{id})\circ\alpha} M(B \otimes \hat{A} \otimes A)$$

Remark These are just continuous coactions of the codouble $A \otimes \hat{A}$

Compact case: $B_{\mathrm{alg}} := \{b \in B : \alpha(b) \in B \otimes_{\mathrm{alg}} A\} \subseteq B$ is dense and

- a right A_0-comodule algebra by $b \mapsto b_{[0]} \otimes b_{[1]} := \alpha(b)$
- a right A_0-module algebra by $b \triangleleft a := (\mathrm{id} \otimes a)(\lambda(b))$ (as $A_0 \subseteq \hat{A}'$)

satisfying the Yetter-Drinfeld condition
$$(b \triangleleft a)_{[0]} \otimes (b \triangleleft a)_{[1]} = (b_{[0]} \triangleleft a_{(2)}) \otimes S(a_{(1)}) b_{[1]} a_{(3)}$$

The twisted tensor product of Yetter-Drinfeld C^*-algebras

(Nest, Voigt. *Equivariant Poincaré duality for quantum group actions.* **JFA, 2010)**

Theorem (Nest, Voigt) Yetter-Drinfeld C^*-algebras for (A, Δ_A) form a monoidal category, where for all YD-C^*-algebras B and C,

- $B \boxtimes C = \alpha(B)_{13}\lambda(C)_{23}$ in $M(B \otimes C \otimes \mathcal{K}(H_A))$
- the natural embeddings of B and C into $B \boxtimes C$ given by
$$b \mapsto b \boxtimes 1 := \alpha(b)_{13} \quad \text{and} \quad c \mapsto 1 \boxtimes c := \lambda(c)_{23}$$
are morphisms, i.e., equivariant with respect to A and to $\hat{A}$

Compact case: $B_{\text{alg}} \boxtimes C_{\text{alg}} \subseteq B \boxtimes C$ is a dense subalgebra, where
$$(1 \boxtimes c)(b \boxtimes 1) = b_{[0]} \boxtimes (c \triangleleft b_{[1]})$$

Further references

- Meyer, Roy, Woronowicz: *Quantum-group twisted tensor products of C^*-algebras I* and *II* (IJM 2014 and arXiv 2015)
- Roy and T.: *The maximal quantum-group twisted tensor products of C^*-algebras* (submitted)

Compact braided quantum groups over (A, Δ_A)

(Meyer, Roy, Woronowicz. *Quantum group-twisted tensor products of C^*-algebras II.* **JNCG, to appear.)**

Definition A *compact braided quantum group* is a Yetter-Drinfeld C^*-algebra (B, λ_B, α_B) with a morphism $\Delta_B \colon B \to B \boxtimes B$ such that

1. (coassociativity) $(\Delta_B \boxtimes \mathrm{id}) \circ \Delta_B = (\mathrm{id} \boxtimes \Delta_B) \circ \Delta_B$
2. (cancellation) $[(B \boxtimes 1)\Delta_B(B)] = B \boxtimes B = [\Delta_B(B)(1 \boxtimes B)]$

Non-Example The irrational rotation algebra
$$B := A_\theta = C^*(\text{unitaries } u, v : vu = e^{2\pi i\theta} uv)$$
is a Yetter-Drinfeld C^*-algebra over $C^*(\mathbb{Z})$, where

- u has degree 0 and v has degree 1,
- $k \triangleright u = e^{-2\pi i k\theta} u$ and $k \triangleright v = v$ for all $k \in \mathbb{Z}$.

There exists a $*$-homomorphism $\Delta_B \colon B \to B \boxtimes B$, $\begin{cases} u \mapsto u \boxtimes u \\ v \mapsto v \boxtimes v \end{cases}$ since
$$(v \boxtimes v)(u \boxtimes u) = v e^{-2\pi i\theta} u \boxtimes vu = e^{2\pi i\theta} uv \boxtimes uv = e^{2\pi i\theta}(u \boxtimes u)(v \boxtimes v)$$

The large compact quantum group (C, Δ_C)

(Meyer, Roy, Woronowicz. ***Quantum group-twisted tensor products of C^*-algebras II.*** **JNCG, to appear.)**

Let (B, Δ_B) be a braided compact quantum group. Regard A as a Yetter-Drinfeld C^*-algebra s.t. $\alpha_A(a) = \Delta_A(a)$, $\lambda_A(a) = \hat{W}(a \otimes 1)\hat{W}^*$.

Theorem (Meyer, Roy, Woronowicz) Let $C = A \boxtimes B$ with embeddings $A \xrightarrow{j_A} C \xleftarrow{j_B} B$. There exists a $*$-homomorphism $\Delta_C \colon C \to C \otimes C$ s.t.

$$
\begin{array}{ccc}
A & \xrightarrow{\Delta_A} & A \otimes A \\
{\scriptstyle j_A}\downarrow & \circlearrowleft & \downarrow{\scriptstyle j_A \otimes j_A} \\
C & \xrightarrow[\Delta_C]{} & C \otimes C
\end{array}
\qquad
\begin{array}{ccccc}
B & \xrightarrow{\Delta_B} & B \boxtimes B & \xrightarrow{\alpha_B \boxtimes \mathrm{id}} & B \otimes A \boxtimes B \\
{\scriptstyle j_B}\downarrow & & & & \downarrow{\scriptstyle j_B \otimes \mathrm{id}_C} \\
C & & \xrightarrow[\Delta_C]{} & & C \otimes C
\end{array}
$$

and (C, Δ_C) is a compact quantum group.

Remark There exist quantum group morphisms $A \xrightarrow{j_A} C \xrightarrow{p} A$, and every reduced compact quantum group C with projection to A arises this way (Roy).

Examples of compact braided quantum groups

1. B =quantum plane, where $A = C(\mathbb{T})$ and C =simplified $E(2)$ (Roy. *C^*-quantum groups with projections.* PhD thesis, 2013)
2. $B = \mathrm{SU}_q(2)$ for $q \notin \mathbb{R}$, where $A = C(\mathbb{T})$ and $C = \mathrm{U}_q(2)$ (Kasprzak, Meyer, Roy, Woronowicz. *Braided quantum SU(2) groups.* JNCQ, to appear).)
3. B =quantum Minkowski space, where $A = C(\mathbb{C})$ (Kasprzak. *Rieffel deformation of tensor functors and braided quantum groups.* CMP, 2015.)

The space of matrix elements

Proposition Every unitary corepresentation of C has the form

$$Z = X_{12} Y_{13} \in M(\mathcal{K}(H) \otimes A \boxtimes B),$$

where X is a unitary corepresentation of A, $Y \in M(\mathcal{K}(H) \otimes B)$ and

- $(\mathrm{id} \otimes \Delta_B)(Y) = Y_{12} Y_{13}$ in $M(\mathcal{K}(H) \otimes B \boxtimes B)$
- $(\mathrm{id} \otimes \alpha_B)(Y) = X_{13} Y_{12} X_{12}^*$ in $M(\mathcal{K}(H) \otimes B \otimes A)$

From now on we only consider Z as above with H finite-dimensional.
On $C = A \boxtimes B$, have a well-defined slice map $h_A \boxtimes \mathrm{id}\colon C \to B$.

Lemma The following subspaces of B coincide:

1. the span of matrix elements of Y for $Z = X_{12} Y_{13}$ as above;
2. the span of matrix elements of $(P_X \otimes 1) Y$, where $P_X = (\mathrm{id} \otimes h_A)(X)$ is the projection onto the invariant vectors;
3. $(h_A \boxtimes \mathrm{id})(C_0)$, where $C_0 \subseteq C$ is the dense Hopf $*$-algebra.

The space of matrix elements, continued

Proof *Step 1:* Since $C_0 \subseteq C$ is spanned by all matrix elements

$$(\omega \otimes \mathrm{id})(X_{12} Y_{13}) \qquad (\omega \in \mathcal{L}(H)_* \text{ and } Z = X_{12} Y_{13} \text{ as above}),$$

the space $(h_A \boxtimes \mathrm{id})(C_0)$ is spanned by matrix elements of the form

$$(\omega \otimes h_A \boxtimes \mathrm{id})(X_{12} Y_{13}) = (\omega \otimes \mathrm{id})((P_X \otimes 1) Y).$$

Step 2: Let $Z' := X^c \textcircled{\scriptsize T} Z =: X'_{13} Y'_{23}$. Then each

$$(\omega_{\xi,\eta} \otimes \mathrm{id})(Y)$$

is equal to a matrix element of the form

$$(\omega'_{\zeta,\bar{\xi} \otimes \eta} \otimes \mathrm{id})((P_{X'} \otimes \mathrm{id}) Y'). \qquad \square$$

Proposition This subspace $B_0 \subseteq B$ is a dense $*$-subalgebra. It satisfies

$$\Delta_B(B_0) \subseteq B_0 \boxtimes B_0, \quad \alpha_B(B_0) \subseteq B_0 \otimes A_0, \quad B_0 \triangleleft A_0 \subseteq B_0.$$

In particular, B_0 is a braided Hopf algebra over A_0.

The Haar state

Definition A *Haar state* for a braided compact quantum group (B, Δ_B) is a state h_B on B that

- is invariant with respect to α_B and λ_B
- is left- and right-invariant with respect to Δ_B in the sense that

$$(h_B \boxtimes \mathrm{id})(\Delta_B(b)) = h_B(b)1 = (\mathrm{id} \boxtimes h_B)(\Delta_B(b))$$

Proposition Every braided compact quantum group (B, Δ_B) has a unique Haar state h_B. More precisely, the Haar state h_C of the associated large compact quantum group $C = A \boxtimes B$ has the form

$$h_C = h_A \boxtimes h_B.$$

The GNS-space for the Haar state h_B

Consider the Haar state h_B on B, the associated GNS-space H_B with cyclic vector $\zeta_B \in H_B$, and the GNS-representation $B \to \mathcal{L}(H_B)$.

Invariance of h_B with respect to α_B and λ_B implies:

Lemma H_B carries corepresentations U of A and V of $\hat{A}$ s.t.

$$U(b\zeta_B \otimes 1) = \alpha_B(b)(\zeta_B \otimes 1) \text{ and } V(b\zeta_B \otimes 1) = \lambda_B(b)(\zeta_B \otimes 1).$$

The triple (H_B, U, V) forms a *Yetter-Drinfeld Hilbert space*, that is,

$$V_{12}U_{12}W^A_{23} = W^A_{23}U_{13}V_{12} \text{ in } M(\mathcal{K}(H_B) \otimes \hat{A} \otimes A).$$

Proposition (Roy) Yetter-Drinfeld Hilbert spaces form a braided monoidal category. For (H^1, U^1, V^1) and (H^2, U^2, V^2), the braiding is

$$H^1 \otimes H^2 \xrightarrow{\Sigma} H^2 \otimes H^1 \xrightarrow{Z} H^2 \otimes H^1,$$

where $Z = (\pi_{U^2} \otimes \pi_{V^1})(W^{A,\mathrm{u}})$; equivalently, $Z_{12} = V^{1*}_{23}U^{2*}_{12}V^1_{23}U^2_{12}$.

The braided multiplicative unitary

For (H_B, U, V), the associated unitary Z on $H_B \otimes H_B$ is given by

$$Z(b\zeta_B \otimes b'\zeta_B) = b_{[0]}\zeta_B \otimes (b' \triangleleft b_{[1]})\zeta_B$$

for all $b, b' \in B_0$, where $b_{[0]} \otimes b_{[1]} = \alpha_B(b)$, and yields a representation

$$\Delta_B : B \boxtimes B \to H_B \otimes H_B, \quad b \boxtimes b' \mapsto (b \otimes 1)Z(1 \otimes b')Z^*.$$

Theorem The following formula defines a unitary W^B on $H_B \otimes H_B$:

$$W^B(b\zeta_B \otimes b'\zeta_B) = \Delta_B(b)(\zeta_B \otimes b'\zeta_B).$$

This is a *braided multiplicative unitary* in the sense of Roy, that is,

- it is a morphism of the Yetter-Drinfeld Hilbert space $H_B \otimes H_B$,
- (braided pentagon equation) $W^B_{23}W^B_{12} = W^B_{12}Z_{23}W^B_{13}Z^*_{23}W^B_{23}$

Proof. Use right-invariance of h_B and straightforward calculations.

The dual algebra

Remark Let $b \in B_0$ and write $\Delta_B(b) = b_{(1)} \boxtimes b_{(2)}$. Then

$$W^B(b\zeta_B \otimes b'\zeta_B) = b_{(1)}\zeta_B \otimes b_{(2)}b'\zeta_B.$$

From W^B, we can construct a reduced $(B_r, \Delta_{B,r})$ and a dual $(\hat{B}, \hat{\Delta}_B)$.

Let $\xi = b'\zeta_B$, $\eta = b''\zeta_B$ and $\upsilon = b'' \cdot h_B \cdot b'^* : b \mapsto h_B(b'^* b b'')$. Then

$$(\mathrm{id} \otimes \omega_{\xi,\eta})(W^B)b\zeta_B = b_{(1)}\upsilon(b_{(2)})\zeta_B.$$

Proposition The space $\hat{B}_0 = B_0 \cdot h_B \subseteq B'_0$ is an algebra w.r.t.

$$(\upsilon * \omega)(b) = \upsilon(b_{(1)})\omega(b_{(2)})$$

Proof As $b', b'' \in B_0$ vary, the following maps span the same space:

- $b \mapsto h_B(b_{(1)}b')h_B(b_{(2)}b'')$
- $b \mapsto h_B(b_{(1)}b'_{(1)})h_B(b_{(2)}b'_{(2)}b'')$
- $b \mapsto h_B(bb')h_B(b'')$ (beware: $\Delta_B(bb') \neq b_{(1)}b'_{(1)} \boxtimes b_{(2)}b'_{(2)}$)

Proposition The slices $(\mathrm{id} \otimes \omega_{\xi,\eta})(W^B) \in \mathcal{L}(H_B)$ with ξ, η as above span a $*$-algebra that is isomorphic to $\hat{B}_0$.

The structure of the dual algebra

Proposition The dual algebra $\hat{B}_0$ is a direct sum of matrix algebras.

Proof. Choose a representative family $Z^\alpha = X^\alpha_{12} Y^\alpha_{13}$ of unitary irreducible corepresentations of C. Then matrix elements

$$(\omega \otimes \mathrm{id})((P_{X^\alpha} \otimes 1) Y^\alpha) \quad \text{and} \quad (\omega \otimes \mathrm{id})(Y^\alpha)$$

span finite-dimensional subspaces B^α and $\tilde{B}^\alpha$ of B_0 such that

$$B_0 = \bigoplus_\alpha B^\alpha \quad \text{and} \quad \Delta_B(B^\alpha) \subseteq B^\alpha \boxtimes \tilde{B}^\alpha.$$

Fix α and let $I_\alpha = \{\beta : Z^\beta \text{ is contained in } (X^\alpha)^c \textcircled{T} Z^\alpha\}$. Then

$$\tilde{B}^\alpha \subseteq \bigoplus_{\beta \in I_\alpha} B^\beta.$$

Let $\hat{B}^\alpha = \{\phi \in \hat{B}_0 : B^\beta \subseteq \ker \phi \text{ for } \beta \neq \alpha\}$. Then $\hat{B}_0 = \bigoplus_\alpha \hat{B}^\alpha$ and

$$\hat{B}^\alpha \cdot \hat{B}^\beta = 0 \text{ unless } \beta \in I_\alpha.$$

For each $\phi \in \hat{B}_0$, we get $\dim \phi \cdot \hat{B}_0 < \infty$ and $\dim \hat{B}_0 \cdot \phi \cdot \hat{B}_0 < \infty$. □

Advertisement for algebraic quantum groupoids

Extending Van Daele's theory of multiplier Hopf algebras with integrals, a theory of *multiplier Hopf algebroids with integrals* connecting the *algebraic* and *operator-algebraic* approaches to quantum groupoids is developed in

[1] T. and Van Daele. *Regular multiplier Hopf algebroids.*
arXiv:1307.0769, 39 pages

[2] T. *Integration on algebraic quantum groupoids.*
IJM 27(2), 72 pages, 2016

[3] T. *On duality of algebraic quantum groupoids.*
arXiv:1605.06384, 45 pages, submitted

[4] T. *Algebraic quantum groupoids on the level of operator algebras.*
in preparation

The compact case where the basis is classical and discrete and the quantum groupoid is proper is studied in

[5] De Commer and T. *Partial compact quantum groups.*
J. Algebra 438, 41 pages, 2015