QUANTUM GROUPOIDS

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March 25, 2017

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WHAT

IS A QUANTUM GROUPOID?

FROM GROUPS TO QUANTUM GROUPS

Classical

a semigroup is a space Γ with a map $\Gamma \times \Gamma \xrightarrow{m} \Gamma$ s.t.

$$\begin{array}{ccc} \Gamma \times \Gamma \times \Gamma & \xrightarrow{m \times \mathrm{id}} & \Gamma \times \Gamma \\ {\scriptstyle \mathrm{id} \times m} \downarrow & \circlearrowright & \downarrow {\scriptstyle m} \\ \Gamma \times \Gamma & \xrightarrow[m]{} & \Gamma \end{array}$$

a finite group is a semigroup with a unit and inversion map

a locally compact group is a group with a suitable topology or measure (Weil)

Quantum

$\rightsquigarrow$ a bialgebra is an algebra A with a morphism $A \xrightarrow{\Delta} A \otimes A$ s.t.

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle \Delta} \downarrow & \circlearrowright & \downarrow {\scriptstyle \mathrm{id} \otimes \Delta} \\ A \otimes A & \xrightarrow[\Delta \otimes \mathrm{id}]{} & A \otimes A \otimes A \end{array}$$

$\rightsquigarrow$ a f.d. Hopf algebra is a bialgebra with a counit and antipode

$\rightsquigarrow$ a locally compact quantum group is a C^*-/W^*-bialgebra a with left/right Haar weights

Groupoid | Quantum Groupoid

basic ingredients

Groupoid

- a base space G^0
- a total space G
- a target map $G \xrightarrow{t} G^0$
- a source map $G \xrightarrow{s} G^0$
- a multiplication $G_s \times_t G \xrightarrow{m} G$

Quantum Groupoid

- a base algebra B
- a total algebra A
- a target morphism $B \xrightarrow{\alpha} A$
- a source morphism $B^{\mathrm{op}} \xrightarrow{\beta} A$
- a comultiplication $A \to A_\beta \times_\alpha A$

  (if $A \circlearrowright H$, then $A_\beta \times_\alpha A \circlearrowright H_\beta \otimes_\alpha H$)

basic assumptions

Groupoid

- associativity of m
- $t(\gamma\gamma') = t(\gamma)$
- $s(\gamma\gamma') = s(\gamma')$

Quantum Groupoid

- coassociativity of Δ
- $\Delta(\alpha(b)) = \alpha(b) \otimes 1$
- $\Delta(\beta(b^{\mathrm{op}})) = 1 \otimes \beta(b^{\mathrm{op}})$
- $[\alpha(B), \beta(B^{\mathrm{op}})] = 0$

Associated to a finite groupoid G, we have two quantum groupoids:

The Function Algebra

base algebra and total algebra:	$C(G^0)$ and $C(G)$,	where $\delta_g \delta_{g'}$ is $\delta_{g,g'}\delta_g$
target map and source map:	$C(G^0) \rightrightarrows C(G)$,	pull-back along t or s
comultiplication:	$C(G) \to C(G^{(2)})$,	$\delta_\gamma \mapsto \sum_{\gamma=\gamma'\gamma''} \delta_{\gamma'} \otimes \delta_{\gamma''}$

The Groupoid Algebra

base algebra and total algebra:	$\mathbb{C}G^0$ and $\mathbb{C}G$,	where $g \cdot g'$ is gg' or 0
target map and source map:	$\mathbb{C}G^0 \hookrightarrow \mathbb{C}G$,	the natural inclusion
comultiplication:	$\mathbb{C}G \to \mathbb{C}(G * G)$,	$g \mapsto g \otimes g$

($G * G$: all pairs (g, g') with same source, same target)

WHY

STUDY QUANTUM GROUPOIDS?

VARIANTS OF QUANTUM GROUPOIDS AND WHERE APPEARED

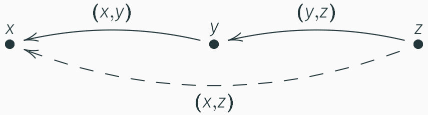
finite quantum groupoids (Nikshych & Vainerman, Böhm, ...)	$\leftrightsquigarrow$	invariants of 3-manifolds (Turaev)
partial compact quantum groups (De Commer & T.)	$\leftrightsquigarrow$	Tannaka-Krein duals of partial fusion C^*-categories
dynamical quantum groups (Etingof & Varchenko, Koelink & Rosengren, ...)	$\leftrightsquigarrow$	dynamical Yang-Baxter equation from physics
measured quantum groupoids (Enock & Lesieur & Vallin)	$\leftrightsquigarrow$	quantum symmetries of inclusions of II_1 factors
algebraic quantum groupoids (Lu, Xu, Böhm & Szlachányi, T. & Van Daele)	$\leftrightsquigarrow$	Pontrjagin duality for (quantum) groupoids

Work in progress: C^*-algebraic theory of locally compact quantum groupoids

How

about examples?

- For every space X, the full equivalence relation $X \times X$ is a groupoid:



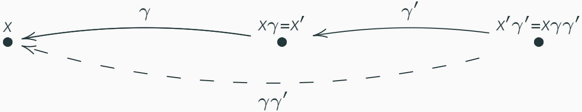
- If Γ acts *freely* on X, then $(X \times X)/_\Gamma$ is a groupoid over $X/_\Gamma$.
- For *any* action of a group Γ on X, we get a quantum groupoid

$$C_0(X) \rightrightarrows C_0(X) \rtimes \Gamma \ltimes C_0(X)$$

  with comultiplication $f \rtimes \gamma \ltimes f' \mapsto (f \rtimes \gamma \ltimes 1) \otimes (1 \rtimes \gamma \ltimes f')$

- Roughly, if a quantum group Γ acts on an algebra B,
  we get a quantum groupoid $B \rtimes \Gamma \ltimes B^{\mathrm{op}}$ with base B.

- If a group Γ acts on a space X, we get a transformation groupoid $X \rtimes \Gamma$



with groupoid algebra $C_0(X) \rtimes \Gamma$ and function algebra $C_0(X) \otimes C_0(\Gamma)$

- If Γ acts on an algebra B, we get crossed product with canonical maps

$$B \to B \rtimes \Gamma \to (B \rtimes \Gamma) \otimes \mathbb{C}\Gamma$$

To obtain a quantum transformation groupoid, we also need a map

$$B^{\mathrm{op}} \to B \rtimes \Gamma, \qquad b^{\mathrm{op}} \mapsto \sum_\gamma b_\gamma \rtimes \gamma$$

whose image commutes with $b' \rtimes e$ for all $b' \in B$, i.e., $b' b_\gamma = b_\gamma \gamma(b')$.

- Roughly, if Γ is a quantum group and B a braided-commutative Yetter-Drinfeld algebra, we obtain quantum transformation groupoids

$$B \rtimes \Gamma \quad \text{and} \quad B \rtimes \hat{\Gamma}.$$

A DEFORMATION OF $S^2 \rtimes \mathrm{SU}(2)$

- An important compact quantum group is $\mathrm{SU}_q(2)$, where $q \in (0,1]$:
  - $C(\mathrm{SU}_q(2)) = C^*\left(\alpha, \gamma : \text{ the matrix } u := \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ is unitary}\right)$
  - $\Delta\colon C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2)) \otimes C(\mathrm{SU}_q(2))$ given by $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$

- We have an inclusion $\mathbb{T} \hookrightarrow \mathrm{SU}_q(2)$ in the form of a $*$-homomorphism

$$C(\mathrm{SU}_q(2)) \xrightarrow{\pi} C(\mathbb{T}) \text{ given by } \alpha \mapsto z \text{ and } \gamma \mapsto 0,$$

  and obtain a quantum homogeneous space $S^2_q = {}_{\mathbb{T}}\backslash \mathrm{SU}_q(2)$ in form of

$$C(S^2_q) = \{f \in C(\mathrm{SU}_q(2)) : (\pi \otimes \mathrm{id})\Delta(f) = 1 \otimes f\},$$

  which is a braided-commutative Yetter-Drinfeld algebra for $\mathrm{SU}_q(2)$.

- We get a measured quantum groupoid $\mathcal{G} = L^\infty(S^2_q) \rtimes \mathrm{SU}_q(2)$, and

$$\mathcal{G} = L^\infty\left({}_{\mathbb{T}}\backslash \mathrm{SU}_q(2)\right) \rtimes \mathrm{SU}_q(2) \sim_M \mathbb{T} \ltimes L^\infty\left(\mathrm{SU}_q(2)/_{\mathrm{SU}_q(2)}\right) = L\mathbb{T}.$$

UNIVERSAL SYMMETRIES

- A f.d. algebra D has a quantum automorphism group $\mathrm{QAut}(D) = A$ with an action, that is, a homomorphism $D \xrightarrow{\delta} D \otimes A$ such that

$$\begin{array}{ccc} D & \xrightarrow{\quad \delta \quad} & D \otimes A \\ {\scriptstyle \delta}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle \mathrm{id} \otimes \Delta} \\ D \otimes A & \xrightarrow[\delta \otimes \mathrm{id}]{} & D \otimes A \otimes A \end{array}$$

  that is universal (every quantum group action on D is a quotient).

- For example, $\mathrm{QAut}(\mathbb{C}^n)$ is called a quantum permutation group.

- For every map $E \to X$, we have an automorphism groupoid

$$\mathrm{Aut}(E \to X) = \coprod_{x,y \in X} \mathrm{Iso}(E_x, E_y).$$

- To an inclusion of f.d. algebras $B \hookrightarrow D$, we can associate a quantum automorphism groupoid $\mathrm{QAut}(B \hookrightarrow D)$ with a universal action on D.