The partial Bernoulli shift of a discrete quantum group

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Partial actions of discrete groups

Partial actions of discrete groups

Definition A *partial action* of a discrete group G on a set X is given by

- a subset $D_g \subseteq X$ for each $g \in G$
- a bijection $\theta_g \colon D_{g^{-1}} \to D_g$ for each $g \in G$

such that

- $D_e = X$ and $\theta_e = \mathrm{id}_X$,
- $\theta_g \circ \theta_h \subseteq \theta_{gh}$ for all $g, h \in G$ as partial maps.

Example The restriction of an action $G \circlearrowright Y$ to a subset $X \subseteq Y$:

- $D_{g^{-1}} := \{x \in X : g(x) \in X\} = X \cap g^{-1}(X)$ for each $g \in G$
- $\theta_g(x) = g(x)$ for each $g \in G$ and $x \in D_{g^{-1}}$

Proposition Every partial action is the restriction of some action.

The partial Bernoulli shift and its transformation groupoid

Example The *Bernoulli shift* of G is the action on the power set $\mathcal{P}(G)$ by left multiplication, $gA = \{gy : y \in A\}$. Restriction to the subset

$$\mathcal{P}_e(G) = \{A \subseteq G : e \in A\}$$

yields the *partial Bernoulli shift*, where each $g \in G$ acts as

$$\{A \subseteq G : e, g^{-1} \in A\} = D_{g^{-1}} \xrightarrow{g-} D_g = \{A \subseteq G : g, e \in A\} .$$

Construction The partial shift gives rise to a transformation groupoid

$$\mathcal{P}_e(G) \rtimes G$$

which consists of all labelled arrows of the form

$$gA \xleftarrow{g} A \qquad (g \in G, A \in D_{g^{-1}}).$$

Proposition Partial actions of G correspond to actions of $\mathcal{P}_e(G) \rtimes G$.

Idea of correspondence

The groupoid $\mathcal{P}_e(G) \rtimes G$ consists of all labelled arrows of the form

$$gA \xleftarrow{g} A \quad (g \in G, A \subseteq G, e, g^{-1} \in A).$$

Proposition Partial actions of G correspond to actions of $\mathcal{P}_e(G) \rtimes G$.

Idea of proof Let $((D_g)_g, (\theta_g)_g)$ be a partial action on a set X.

We obtain an action of $\mathcal{P}_e(G) \rtimes G$ as follows.

1. Define $\pi: X \to \mathcal{P}_e(G)$ by $\pi(x) = \{g \in G : x \in D_{g^{-1}}\}$. Then
$$\{(g,x) : x \in D_{g^{-1}}\} = \{(g,x) : g \in \pi(x)\} \subseteq G \times X.$$
2. Let $gA \xleftarrow{g} A$ be an arrow in $\mathcal{P}_e(G) \rtimes G$ and $x \in \pi^{-1}(A)$.
As $g^{-1} \in A$, we have $x \in D_{g^{-1}}$. Now, let the arrow act by $x \mapsto \theta_g(x)$.

The converse construction is similar.

Variants of partial actions

Definition A *(disconnected) partial action* on a
(i) top. space, **(ii)** vector space, **(iii)** algebra

is a partial action on the underlying set such that

- **(i)** each D_g is open (and closed) and each θ_g is a homeomorphism
- **(ii)** each D_g a subspace and each θ_g a linear isomorphism
- **(iii)** each D_g a (unital) two-sided ideal and each θ_g an isomorphism.

Example The partial Bernoulli shift is a disc. part. action on a space:
$\mathcal{P}(G) = \{0,1\}^G$ is compact space and $\mathcal{P}_e(G)$ and each D_g are clopen.

Proposition (Disc.) partial actions of G on top. spaces/vector spaces/
algebras correspond to actions of the groupoid $\mathcal{P}_e(G) \rtimes G$.

Partial actions of Hopf algebras

Partial representations of Hopf algebras

($\theta_g \circ \theta_h \subseteq \theta_{gh}$ iff $\theta_{gh} \circ \theta_{h^{-1}} = \theta_g \circ \theta_h \circ \theta_{h^{-1}}$ and $\theta_{g^{-1}} \circ \theta_{gh} = \theta_{g^{-1}} \circ \theta_g \circ \theta_h$.)

Definition A *partial representation* of a Hopf algebra H on a vector space V is a linear map $\pi\colon H \to \mathrm{End}(V)$ such that for all $h, k \in H$,

1. $\pi(1_H) = \mathrm{id}_V$;
2. $\pi(kh_{(1)})\pi(S(h_{(2)})) = \pi(k)\pi(h_{(1)})\pi(S(h_{(2)}))$,
   $\pi(S(k_{(1)}))\pi(k_{(2)}h) = \pi(S(k_{(1)}))\pi(k_{(2)})\pi(h)$;
3. like **2.** but with S replaced by S^{-1}.

Example Partial actions of a group G on k-vector spaces correspond to partial representations of the Hopf algebra $H = kG$.

Lemma (Alves-Batista-Vercruysse)
There exists an algebra H_{par} such that partial representations of H on a vector space V correspond to unital homomorphisms $H_{\mathrm{par}} \to \mathrm{End}(V)$.

The Hopf analogue of the Bernoulli shift

H_{par} is the analogue of the transformation groupoid $\mathcal{P}_e(G) \rtimes G$:

Theorem (Alves-Batista-Vercruysse)
There exists a subalgebra $A \subset H_{\text{par}}$ with a partial action of H such that $H_{\text{par}} \cong A\#H$. Moreover, H_{par} is a *Hopf algebroid* with base algebra A.

Definition A *partial action* of H on a unital algebra A is a linear map $H \otimes A \to A$, written $h \otimes a \mapsto h \cdot a$, such that for all $h, k \in H$ and $a, b \in A$,

1. $1_H \cdot a = a$;
2. $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$;
3. $h \cdot (k \cdot a) = (h_{(1)} \cdot 1_A)(h_{(2)}k \cdot a) = (h_{(1)}k \cdot a)(h_{(2)} \cdot 1_A)$.

(This is an action if and only if $h \cdot 1_A = \epsilon(h)1_A$ for all $h \in H$.)

Example Partial actions of a group algebra kG correspond to disc. partial actions of G on A (each $D_g = g \cdot A$ is an ideal with unit $g \cdot 1_A$).

The partial Bernoulli shift of a discrete quantum group

C^*-algebras and C^*-bialgebras

The partial Bernoulli shift of a discrete quantum group has an underlying *quantum space* given by a non-commutative C^*-algebra.

Definition A *C^*-algebra* is a Banach *-algebra s.t. $\|a^*a\| = \|a\|^2$ for all a.

Example $C_0(X)$: all cont. functions on a l.c. Hd. space X vanishing at ∞

$$(\text{l.c. Hd. spaces}) \xleftrightarrow{\textit{Gelfand duality}} (\text{commutative } C^*\text{-algebras})$$

Definition A *C^*-bialgebra* is a C^*-algebra $\mathscr{H}$ with a *-homomorphism $\Delta\colon \mathscr{H} \to M(\mathscr{H} \otimes \mathscr{H})$ s.t. $\Delta(\mathscr{H})(\mathscr{H} \otimes \mathscr{H}) \subseteq \mathscr{H} \otimes \mathscr{H}$ is lin. dense and

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

($\otimes$ is the *minimal tensor product* of C^*-algebras, ,
$M(-)$ is the *multiplier C^*-algebra*, a max. unitisation.)

Example Let G be a l.c. Hd. group. Then $C_0(G)$ is a C^*-bialgebra with $\Delta\colon C_0(G) \to M(C_0(G) \otimes C_0(G)) \cong C_b(G \times G)$ given by $(\Delta(f))(g, g') = f(gg')$.

Partial coactions of C^*-bialgebras

Definition A *partial coaction* of a C^*-bialgebra $(\mathscr{H}, \Delta)$ on a C^*-algebra $\mathscr{A}$ is a $*$-homomorphism $\delta\colon \mathscr{A} \to M(\mathscr{A} \otimes \mathscr{H})$ such that

1. $\delta(\mathscr{A})(1 \otimes \mathscr{H}) \subseteq \mathscr{A} \otimes \mathscr{H}$,
2. δ extends to a strict $*$-homomorphism from $M(\mathscr{A})$ to $M(\mathscr{A} \otimes \mathscr{H})$,
3. $(\delta \otimes \mathrm{id})(\delta(a)) = (\delta(1) \otimes 1)(\mathrm{id} \otimes \Delta)(\delta(a))$ for all $a \in A$.

It is a *coaction* if the extension in **2.** is unital.

Example Given a coaction on a C^*-algebra $\mathscr{B}$ and an ideal $\mathscr{A} \subseteq \mathscr{B}$, the coaction restricts to a partial coaction on $\mathscr{A}$.

Theorem (Kraken, Quast, T.)
If $(\mathscr{H}, \Delta)$ is a *C^*-quantum group* and $\mathscr{H}$ is *nuclear*, then every *regular* partial coaction of $(\mathscr{H}, \Delta)$ is the restriction of a coaction which can be chosen to be *minimal* and then is unique up to iso.

Partial coactions of discrete quantum groups

We now fix a discrete quantum group, which is a C^*-bialgebra $(\mathscr{H}, \Delta)$ subject to certain conditions. Suggestively, one writes $\mathscr{H}$ as $C_0(\mathbb{G})$, but the "quantum space $\mathbb{G}$" is just virtual. Then

- $\mathscr{H} = C_0(\mathbb{G})$ is a c_0-sum of matrix algebras: $C_0(\mathbb{G}) \cong \overline{\bigoplus}_\alpha M_{n_\alpha}(\mathbb{C})$
- $\bigoplus_\alpha^{\text{alg}} M_n(\mathbb{C}) =: \mathcal{O}(\mathbb{G})$ is a *multiplier Hopf algebra*
- $\bigoplus_\alpha^{\text{alg}} \text{Hom}(M_n(\mathbb{C}), \mathbb{C}) =: \mathbb{C}\mathbb{G}$ is a *Hopf algebra*
- $C_0(\mathbb{G})$ has a *counit*, a $*$-homomorphism $\varepsilon\colon C_0(\mathbb{G}) \to \mathbb{C}$.

Proposition Let δ be a partial coaction of $(C_0(\mathbb{G}), \Delta)$ on a C^*-algebra $\mathscr{A}$ s.t. $(\text{id} \otimes \varepsilon)\delta = \text{id}$. Then we obtain a partial action of $\mathbb{C}\mathbb{G}$ on $\mathscr{A}$ via

$$h \cdot a := (\text{id} \otimes h)(\delta(a)) \qquad (a \in \mathscr{A}, h \in \mathbb{C}\mathbb{G}).$$

Proposition In case of a classical discrete group G, counital partial coactions of $(C_0(G), \Delta)$ correspond to disconn. partial actions of G.

The Bernoulli shift of a discrete quantum group I

We want a quantum analogue of the space $\mathcal{P}(G)$. The latter comes with a "tautological subset" of $\mathcal{P}(G) \times G$ which corresponds to a projection in $C_b(\mathcal{P}(G) \times G) \cong M(C(\mathcal{P}(G)) \otimes C_0(G))$.

Write $C_0(\mathbb{G}) \cong \overline{\bigoplus}_\alpha M_{n_\alpha}(\mathbb{C})$ and choose matrix units $e_{ij}^\alpha \in C_0(\mathbb{G})$.

Notation Let $\mathscr{B}$ be the universal unital C^*-algebra with generators

$$p_{i,j}^\alpha \text{ for each } \alpha \text{ and } 1 \le i, j \le n_\alpha \tag{1}$$

and relations such that

$$p := \textstyle\sum_{\alpha,i,j} p_{ij}^\alpha \otimes e_{ij}^\alpha \in M(\mathscr{B} \otimes C_0(\mathbb{G})) \text{ is a projection} \tag{2}$$

$$[p \otimes 1, (\text{id} \otimes \hat{\Delta})(p)] = 0 \text{ in } M(\mathscr{B} \otimes C_0(\mathbb{G}) \otimes C_0(\mathbb{G})). \tag{3}$$

Example Suppose $\mathbb{G} = G$ is a classical discrete group. Then
(1) $\Rightarrow$ $\mathscr{B}$ is generated by 1 and elements p^g for each $g \in G$,
(2) $\Rightarrow$ each p^g is a projection, (3) $\Rightarrow$ all p^g commute $\Rightarrow \mathscr{B} \cong C(\mathcal{P}(G))$.

The Bernoulli shift of a discrete quantum group II

Theorem (T.)
There exists a unique coaction $\delta\colon \mathscr{B} \to M(\mathscr{B} \otimes C_0(\mathbb{G}))$ such that

$$(\delta \otimes \mathrm{id})(p) = (\mathrm{id} \otimes \hat{\Delta})(p)$$

and $\mathscr{B}$ is a *braided-commutative Yetter-Drinfeld C^*-algebra* over $\mathbb{G}$.

Lemma

1. The projection $p_\varepsilon := (\mathrm{id} \otimes \varepsilon)(p) \in M(\mathscr{B} \otimes \mathbb{C}) \cong \mathscr{B}$ is central
   (we apply $\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}$ to $[p \otimes 1, (\mathrm{id} \otimes \Delta)(p)] = 0$ and get $[(p_\varepsilon \otimes 1), p] = 0$).
2. $\mathscr{B}_\varepsilon := p_\varepsilon \cdot \mathscr{B} \subseteq \mathscr{B}$ is a unital ideal, that is, a direct summand.
3. δ restricts to a partial coaction δ_ε on $\mathscr{B}_\varepsilon$ s.t. $b \mapsto (p_\varepsilon \otimes 1)\delta(b)$.

Definition We call $(\mathscr{B}_\varepsilon, \delta_\varepsilon)$ the *partial Bernoulli shift* of $\mathbb{G}$.

Note All these constructions descend to the algebraic level, giving a braided-commutative YD- algebra $B \subseteq \mathscr{B}$ over the Hopf algebra $\mathbb{C}G$.