Quantum Transformation Groupoids in the Setting of Operator Algebras
(work in progress)

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Question: Are there “Quantum” Transformation Groupoids?

Recall:

- action $X \circlearrowleft \Gamma \quad \leadsto \quad$ transformation groupoid $X \rtimes \Gamma$
- action $C_0(X) \circlearrowleft \Gamma \quad \leadsto \quad C_0(X) \rtimes \Gamma = C^*(X \rtimes \Gamma)$

Question:

- Given an action $B \circlearrowleft \Gamma$, is $B \rtimes \Gamma$ still a quantum groupoid
  - if B is a noncommutative C^*-algebra or W^*-algebra?
  - if Γ is a locally compact quantum group?
  - if $B \circlearrowleft \Gamma$ is a deformation of a classical action?

Plan of the Talk and some Relation to the Audience

What we shall do...

... and some buzzwords related to conference participants:

1. Warming up in the algebraic setting
   - Quantum transformation groupoid (Brzeziński)
   - Multiplier Hopf $*$-algebroids (Van Daele)
2. Passage to the setting of operator algebras
   - Measured quantum groupoids (Enock-Vallin)
   - Yetter-Drinfeld algebras (Nest-Voigt)
   - Crossed product (Vaes)
   - Example: $SU_q(2) \ltimes S_q^2$ (Woronowizc)

Tentative Constituents of a Quantum Transformation Groupoid

Which structure should be there? What do we have for $B \circlearrowleft \Gamma$?

- a *base algebra* B and a *total algebra* A
  $A = B \rtimes \Gamma$
- commuting *range* and *source maps* $r\colon B \to A$ and $s\colon B^{op} \to A$
  $r\colon b \mapsto b \rtimes 1$; $s\colon b^{op} \mapsto b \rtimes 1$ not possible unless B commutes
- a *comultiplication* Δ from A to a fiber product $A_s{*}_r A$
  should correspond to the dual coaction on $A = B \rtimes \Gamma$
- *an antipode* $S\colon (A, \Delta) \to (A^{op}, \Delta^{co})$
  expect $S(b \rtimes 1) = S(r(b)) = s(b^{op})$ and $S(1 \rtimes f) = 1 \rtimes S_\Gamma(f)$
- *weight* μ on B
  should be (quasi-)invariant with respect to $B \circlearrowleft \Gamma$
- *Haar weights* $\Phi\colon A \to r(B) \equiv B$ and $\Psi\colon A \to s(B^{op}) \equiv B^{op}$
  expect $\Phi(b \rtimes f) = b\phi_\Gamma(f)$; for Ψ need to s or S

“Quantum” necessitates a Dual Pair of Transformations

Assume $B \circlearrowleft \Gamma$, where B is an algebra an Γ a (quantum) group

Thought experiment:

- If $B \rtimes \Gamma$ is a quantum groupoid, it should have a dual $\widehat{B \rtimes \Gamma}$
- In the special case where $B = C_0(X)$ and Γ is a group,

$$\widehat{C_0(X) \rtimes \Gamma} = C^*\widehat{(X \rtimes \Gamma)} \\ = C_0(X \rtimes \Gamma) = C_0(X) \otimes C_0(\Gamma)$$

- Expect $\widehat{B \rtimes \Gamma} = B \rtimes \widehat{\Gamma}$ for a suitable action $B \circlearrowleft \widehat{\Gamma}$

Conclusion:

- Assume a second action $B \circlearrowleft \hat{\Gamma}$, *compatible* with $B \circlearrowleft \Gamma$

Compatibility Assumptions on the Pair of Transformations

Assume $B \circlearrowleft \Gamma$ and $B \circlearrowleft \hat{\Gamma}$

Thought experiment:

- Assume Γ is a discrete group
- Then for all $b, c \in B$,

$$[r(b), s(c^{op})] = \left[b \rtimes 1,\ \textstyle\sum_\gamma c_\gamma \rtimes \gamma\right] \\ = \sum_\gamma bc_\gamma \rtimes \gamma - \sum_\gamma c_\gamma(\gamma \triangleright b) \rtimes \gamma$$

should be 0

- ...and $s(b^{op})s(c^{op}) = s((cb)^{op})$ ......

Conclusion:

- Assume B is a braided-commutative Yetter-Drinfeld algebra

The Algebraic Version of Quantum Transformation Groupoids

Theorem (Lu '96; Brzeziński, Militaru '01)

Assume

- *H is a Hopf algebra*
- *B is a braided-commutative Yetter-Drinfeld algebra over H.*

Then the following ingredients form a Hopf algebroid:

- $A = B \rtimes H$ *(the crossed product for the action)*
- $r\colon B \to A,\ b \mapsto b \rtimes 1_H$
- $s\colon B^{op} \to A,\ b^{op} \mapsto \sum b_{(0)} \rtimes b_{(1)}$ *($b \mapsto \sum b_{(0)} \otimes b_{(1)}$ the coaction)*
- $\Delta\colon A \to A_s \underset{B}{\times}{}_r A,\quad b \rtimes h \mapsto \sum (b \rtimes h_{(1)}) \otimes (1 \rtimes h_{(2)})$.

Passage to Operator Algebras

Assumptions

- $(L^\infty\mathbb{G}, \Delta)$ and $(L\mathbb{G}, \hat{\Delta})$ are dual l.c. quantum groups with
  - right Haar weights $\phi, \hat{\phi}$ and GNS-space $\mathbb{H}$
  - multiplicative unitary $\mathbb{W} \in \mathcal{L}(\mathbb{H} \otimes \mathbb{H})$ of $(L^\infty\mathbb{G}, \Delta)$
- B is a W^*-algebra with right coactions α, λ of $L^\infty\mathbb{G}, L\mathbb{G}$ s.t.
  - *Yetter-Drinfeld condition*, Nest-Voigt '09:

$$\begin{array}{ccc} B & \xrightarrow{(\mathrm{id}\,\bar{\otimes}\alpha)\circ\lambda} & L^\infty\mathbb{G}\bar{\otimes}L\mathbb{G}\bar{\otimes}B \\ \Big\| & \circlearrowleft & \Big\downarrow \mathrm{Ad}_{\Sigma W}\bar{\otimes}\,\mathrm{id} \\ B & \xrightarrow[(\mathrm{id}\,\bar{\otimes}\lambda)\circ\alpha]{} & L\mathbb{G}\bar{\otimes}L^\infty\mathbb{G}\bar{\otimes}B \end{array}$$

  - *braided commutativity*:
    $[\alpha(B), \mathrm{Ad}_{U\otimes 1}(\lambda(B))] = 0$ in $\mathcal{L}(\mathbb{H})\bar{\otimes}B$, where $U = J_\phi J_{\hat{\phi}}$
- μ is a weight on B which is invariant for α, λ (w.r.t. δ^{-1}, $\hat{\delta}^{-1}$)

The Hopf-von Neumann Bimodule

To obtain a Hopf-von Neumann bimodule, define

- $A := L\mathbb{G} \ltimes_\alpha B = ((L\mathbb{G} \otimes 1)\alpha(B))'' \subseteq \mathcal{L}(\mathbb{H})\bar{\otimes}B \hookrightarrow \mathcal{L}(\mathbb{H} \otimes H_\mu)$
- the range map $r\colon B \to A,\ b \mapsto \alpha(b)$
- a source map $s\colon B^{op} \to \mathcal{L}(\mathbb{H} \otimes H_\mu),\ b^{op} \mapsto \mathrm{Ad}_{(J_\phi \otimes J_\mu)}(\lambda(b)^*)$

Proposition

1. $s(B^{op}) \subseteq A$
2. $\exists$ *specific unitary* $\Xi\colon \mathbb{H} \otimes H_\mu \otimes \mathbb{H} \to (\mathbb{H} \otimes H_\mu)_s \underset{\mu}{\otimes} {}_r(\mathbb{H} \otimes H_\mu)$
3. $\Delta_A\colon A \xrightarrow{\hat{\alpha}} L\mathbb{G}\bar{\otimes}A \hookrightarrow \mathcal{L}(\mathbb{H}\otimes\mathbb{H}\otimes H_\mu) \xrightarrow{\sigma_{(321)}} \mathcal{L}(\mathbb{H}\otimes H_\mu \otimes \mathbb{H}) - \cdots$
   $\cdots \xrightarrow{\mathrm{Ad}_\Xi} \mathcal{L}((\mathbb{H} \otimes H_\mu)_s \underset{\mu}{\otimes} {}_r(\mathbb{H} \otimes H_\mu))$
   satisfies $\Delta_A(A) \subseteq A_s * {}_rA$ *and* $(\Delta_A * \mathrm{id}) \circ \Delta_A = (\mathrm{id} * \Delta_A) \circ \Delta_A$
   whence $(B, A, r, s, \mu, \Delta_A)$ *is a Hopf* W^**-bimodule (Vallin '96).*

Guessing the Unitary Antipode

The proof uses the *unitary implementations* of the coactions α, λ: $X, \hat{X} \in \mathcal{U}(\mathbb{H} \otimes H_\mu)$ s.t. $\alpha = \mathrm{Ad}_X(1 \otimes -)$ and $\lambda = \mathrm{Ad}_{\hat{X}}(1 \otimes -)$

How to proceed?

- a left Haar weight on A is easy to guess, but not a right one
- try to guess an anti-unitary I_A on $\mathbb{H} \otimes H_\mu$ such that
  $I_A r(b)^* I_A = s(b^{op}),\quad I_A(f^* \otimes 1)I_A = \hat{R}(f) \otimes 1$ for $f \in L\mathbb{G}$ (*)

Proposition

1. $I_A := \mathrm{Ad}_X(\hat{X}^*(J_\phi \otimes J_\mu))$ *satisfies (*), whence* $I_A A I_A = A$.
2. $R_A\colon A \to A, a \mapsto I_A a^* I_A$, *satisfies* $R_A(r(b)) = R_A(s(b^{op}))$ *for all* $b \in B$ *and* $\Delta_A \circ R_A = (R_A * R_A) \circ \mathrm{Ad}_\Sigma \circ \Delta_A$

The Haar Weights and the Measured Quantum Groupoid

There exists an expectation $\Phi_A \colon A \xrightarrow{\hat{\alpha}} L\mathbb{G}\bar{\otimes}A \xrightarrow{\hat{\phi}\bar{\otimes}\,\mathrm{id}\,\bar{\otimes}\,\mathrm{id}} \alpha(B) \equiv B$;
if $\hat{\phi}$ is bounded, $\Phi_A((f \otimes 1)\alpha(b)) = \hat{\phi}(f)\alpha(b)$ for $f \in L\mathbb{G}, b \in B$.

Claims

1. *Φ_A is left-invariant with respect to Δ_A*

   (checked if $\hat{\phi}$ bounded)

2. *$\Psi_A := R_A \circ \Phi_A \circ R_A$ is right-invariant with respect to Δ_A*

   (immediate from **1.***)*

3. *$(B, A, r, s, \Delta_A, \mu, \Phi_A, \Psi_A)$ is a measured quantum groupoid*

   (immediate from **1.** *and* **2.***)*

Application to $\mathrm{SU_q}(2) \ltimes \mathrm{S^2_q}$

Example

Let $\mathbb{G} = \mathrm{SU}_q(2)$ (Woronowicz) and let S^2_q be the Podleś sphere.
Then:

- $C(\mathrm{S}^2_q)$ is a braided-commutative $\mathbb{G}$-Yetter-Drinfeld C^*-algebra
  (for the natural coaction of $C(\mathbb{G})$ and the adjoint coaction of $C^*(\mathbb{G})$)
- $L(\mathrm{SU}_q(2)) \ltimes L^\infty(\mathrm{S}^2_q)$ is a measured quantum groupoid and $C^*(\mathrm{SU}_q(2)) \ltimes_r C(\mathrm{S}^2_q)$ is a "reduced C^*-quantum groupoid"

Recall: G compact, $K, N \triangleleft G \Rightarrow (K\backslash G) \rtimes N \sim_M K \ltimes (G/N)$.

Question: Is $L(\mathbb{G}) \ltimes L^\infty(\mathbb{G}/\mathbb{T}) \sim_M L^\infty(\mathbb{G}/\mathbb{G}) \rtimes L(\mathbb{T}) = L(\mathbb{T})$?